MA 206 notes: Review of math 205

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January 20, 2019

Algebtraic geometry

- At its core, algebraic geometry is the study of varieties,
- namely the zero sets of collections of polynomials in $\mathbb{A}^n$ or $\mathbb{P}^n$,
- assumed irreducible (and reduced).
- We are interested in intrinsic properties (dimension, smoothness ...)
- We are interested in ways to embed a variety in projective space,
- We are interested in classifying: telling things apart, similarities, parameters ...
- Birational geometry is a special topic of algebraic geometry
- moduli spaces are a phenomenon best studied in algebraic geometry.

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Affine Schemes

- The current language is alertschemes.
- An affine scheme $\operatorname{Spec} A$ is the set of primes in the commutative ring A,
- which is a departure from varieties, where maximal ideals are taken.
- $\operatorname{Spec} A$ is provided a topology by declaring $V(f) = \{\mathfrak{p} : f \in \mathfrak{p}\}$ to be closed
- (or $D(f) = \{\mathfrak{p} : f \notin \mathfrak{p}\}$ to be open).
- $X = \operatorname{Spec} A$ is made a locally ringed space by declaring $\mathcal{O}_X(D(f)) = A[f^{-1}]$
- (and taking $\mathcal{O}_X$ the sheaf determined by this $\mathcal{B}$-sheaf).

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Schemes

- Schemes are locally ringed sapces which are locally affine schemes.
- Arrows are arrows of locally ringed spaces (so $Sch \subset LRS$ a full subcategory).
- $AffSch \simeq ComRings^{op}$.
- Then starts a barrage of adjectives:
  reduced, irreducible, integral, quasicompact, noetherian, regular, ...
- Further adjectives for morphisms (or S-schemes).
- Important: *separated* and *proper* morphisms.
- A variety over $k = \bar{k}$ is a separated integral scheme of finite type over k.

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Reminder: sheaves and sections

- We are working with schemes X.
- The structure is governed by sheaves of abelian groups, such as $\mathcal{O}_X$.
- Most important are Sheaves of $\mathcal{O}_X$-modules.
- Particularly useful are Quasi-coherent sheaves of $\mathcal{O}_X$-modules.
- We want to understand their sections.
- For instance: we classified morphisms $X \to \mathbb{P}^n$ through sections of an invertible sheaf.[1]
- Understanding sections is a fundamental question of varieties.

[1] also related to divisors and lienar systems

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Reminder: failure of right-exactness

- Recall the sheaf axiom $0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_{ij})$.
- If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact then $0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X)$ exact...
- but right exactness fails in general:
- say $Y =$ two points in $X = \mathbb{P}^1_k$;
- then $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$, but
- $0 \to 0 \to k \to k^2 \to 0$ is not.

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Measuring the failure

- Failure of right exactness is a fact of life[2].
- We want to understand it, measure it, control it, interpret it in geometric terms,
- We need to study cohomology of sheaves.

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Comments on how this is resolved

- We'll follow Hartshorne, who follows GROTHENDIECK, *Sur quelques points d'algèbre homologique*[3], to resolve this using derived functors. This works in the context of left-exact additive functors on abelian categories with enough injective objects.
- Liu follows SERRE, *Faisceax algébriques cohérents*, to resolve using Čech cohomology. This works for sections of quasi-coherent sheaves, and will be subsumed in Hartshorne's treatment.
- An important modern approach uses derived categories (GELFAND–MANIN, WEIBEL), still in the additive realm.
- Homotopy theory has even loftier approaches (model categories, . . .)

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