

EXPANDED DEGENERATIONS AND PAIRS

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ABSTRACT. Since Jun Li's original definition, several other definitions of expanded pairs and expanded degenerations have appeared in the literature. We explain how these definitions are related and introduce several new variants and perspectives. Among these are the twisted expansions used by Abramovich and Fantechi as a basis for orbifold techniques in degeneration formulas.

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1. INTRODUCTION

1.1. Expansions in Gromov–Witten theory. Let $X \rightarrow B$ be a flat morphism from a smooth variety to a smooth curve, with a unique critical value $b_0 \in B$. Suppose that the critical fiber X_0 is the union of smooth varieties Y_1 and Y_2 along a closed subscheme D that is a smooth divisor in each Y_i . Working over \mathbb{C} and assuming that X is projective over B , Jun Li associated a family of *expanded degenerations*

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$X_{\text{Li}}^{\text{exp}} \rightarrow \mathfrak{X}_{\text{Li}}(X/B, b_0)$ to X/B (the symbol \mathfrak{X} is the Gothic letter T). The fibers of $X_{\text{Li}}^{\text{exp}}$ over $\mathfrak{X}_{\text{Li}}(X/B, b_0)$ are different semistable models of X_0 . A typical degenerate fiber of $X_{\text{Li}}^{\text{exp}}$ over $\mathfrak{X}_{\text{Li}}(X/B, b_0)$ consists of a chain

$$(1) \quad Y_1 \sqcup_D P \sqcup_D \cdots \sqcup_D P \sqcup_D Y_2$$

where $P := \mathbb{P}_D(\mathcal{O} \oplus N_{D/Y_1})$ is a \mathbb{P}^1 bundle over D (see Definition 2.3.1).

The copies of D used in this gluing are called *splitting divisors*.

Li also defined related families of *expanded pairs* (see Definition 2.1.1), denoted here by $(Y_i^{\text{exp}}, D^{\text{exp}})_{\text{Li}} \rightarrow \mathcal{T}_{\text{Li}}(Y_i, D)$. These parameterize different models for the pair (Y_i, D) , obtained by splitting the chain above along a splitting divisor:

$$(2) \quad Y_1 \sqcup P \sqcup \cdots \sqcup P.$$

We record Li's definitions of $\mathfrak{X}_{\text{Li}}(X/B, b_0)$ and $\mathcal{T}_{\text{Li}}(Y_i, D)$ in Definition 6.1.2 and Section 6.2.

These stacks form the foundation for Li's approach to the *degeneration formula*. This formula originated in symplectic geometry in the work of A.-M. Li and Y. Ruan [LR01] (see also [IP03]) relating Gromov–Witten invariants of the generic fiber of $X \rightarrow V$ to appropriately defined *relative* Gromov–Witten invariants of the pairs (Y_i, D) .

The stacks $\mathfrak{X}_{\text{Li}}(X/B, b_0)$ and $\mathcal{T}_{\text{Li}}(Y, D)$ continue to be important in Gromov–Witten theory, with new applications and generalizations emerging in the orbifold theory [AF11, ACW10] and in related theories, e.g. [Tze10, Sec 3.3].

It has been observed (in [GV05, Section 2.5], for example) that $\mathfrak{X}_{\text{Li}}(X/B, b_0)$ and $\mathcal{T}_{\text{Li}}(Y_i, D)$ should be algebraic stacks and that $\mathfrak{X}_{\text{Li}}(X/B, b_0)$ and $\mathcal{T}_{\text{Li}}(Y_i, D)$ should be independent, respectively, of $(X/B, b_0)$ and (Y_i, D) . However, we are not aware of any explicit demonstration of these facts in the literature. They are consequences of Theorems 1.3.1 and 1.3.2, below.

1.2. The definition of Graber and Vakil. Li's definitions are geometrically appealing, but are necessarily subtle for reasons explained below. Graber and Vakil avoided these subtleties by anticipating that the stack of expanded pairs is independent of the pair and describing it explicitly when the pair is a point on \mathbb{P}^1 . We recall their definition in Definition 3.1.1 and call their moduli space \mathcal{T}_{GV} . One appealing feature of Graber's and Vakil's definition is its close relationship to the already well-understood moduli space of Deligne–Mumford pre-stable curves: the algebraicity of \mathcal{T}_{GV} is immediate, for example.

Graber and Vakil define \mathcal{T}_{GV} so that it clearly parametrizes expansions of the pair $(\mathbb{P}^1, 0)$; at least intuitively, one can construct an expansion of any pair (Y, D) from an expansion of $(\mathbb{P}^1, 0)$ by “doing the same thing” to (Y, D) as was done to $(\mathbb{P}^1, 0)$. However, we do not know of anywhere that this procedure is spelled out, nor of a demonstration of its equivalence to Li’s definition. This equivalence follows from our Theorem 1.3.2.

A central impetus for this paper is the need to generalize expansions to *twisted expansions*, as applied in [AF11, ACW10]. Taking the point of view of Graber and Vakil, twisted expansions replace the semistable curves appearing in \mathcal{T}_{GV} with *twisted semistable curves* [AV02, AOV11a]. Our approach to the stacks of expansions \mathcal{T} and \mathfrak{I} affords an immediate generalization to stacks \mathcal{T}^{tw} and \mathfrak{I}^{tw} of twisted expansions (see Definition 2.4.2).

1.3. Results: algebraicity and comparison. In Definitions 2.1.5 and 2.3.5 we introduce stacks \mathfrak{I} and \mathcal{T} of expanded degenerations and expanded pairs, along with their universal families, with no restrictions on X or Y . For pairs, our definition of \mathcal{T} is similar to Graber’s and Vakil’s, but the pair $(\mathbb{P}^1, 0)$ is replaced by the *universal pair* $(\mathcal{A}, \mathcal{D})$, where $\mathcal{A} = [\mathbb{A}^1/\mathbb{G}_m]$ and $\mathcal{D} = [0/\mathbb{G}_m]$. The universal virtues of \mathcal{A} are explained in [Cad07, Lemma 2.1.1]: any pair (Y, D) admits a canonical morphism to $(\mathcal{A}, \mathcal{D})$ such that $D = Y \times_{\mathcal{A}} \mathcal{D}$. An expansion $(\mathcal{A}', \mathcal{D}')$ of $(\mathcal{A}, \mathcal{D})$ induces an expansion of (Y, D) by base change: $(Y', D') = (Y \times_{\mathcal{A}} \mathcal{A}', D \times_{\mathcal{D}} \mathcal{D}')$.

For degenerations, we also use base change to reduce to the universal example. The universal degeneration is the multiplication map $\mathcal{A}^2 \rightarrow \mathcal{A}$, in the sense that every degeneration $(X/B, b_0)$ admits a canonical map to $(\mathcal{A}^2/\mathcal{A}, \mathcal{D})$ (see Section 2.2.2). Expansions of $(X/B, b_0)$ are defined by pulling back expansions of $(\mathcal{A}^2/\mathcal{A}, \mathcal{D})$.

To prove that \mathfrak{I} and \mathcal{T} are algebraic we provide an isomorphism with stacks of curves similar to \mathcal{T}_{GV} .

We prove the following:

Theorem 1.3.1 (Algebraicity theorem). *The stacks \mathfrak{I}^{tw} and \mathcal{T}^{tw} are algebraic stacks, locally of finite type over \mathbb{Z} , containing the stacks \mathfrak{I} and \mathcal{T} as open and dense substacks.*

Theorem 1.3.2 (Comparison theorem). (1) *We have an isomorphism $\mathcal{T} \simeq \mathcal{T}_{\text{GV}}$.*

(2) *Assume Y is smooth and projective over a field, and D is irreducible. Then we have a canonical isomorphism $\mathcal{T}_{\text{Li}}(Y, D) \simeq \mathcal{T}$.*

- (3) Assume X above is projective over B and D is connected. Then we have a canonical isomorphism $\mathfrak{I}_{\text{Li}}(X/B, b_0) \simeq \mathfrak{I} \times_{\mathcal{A}} B$.

1.4. Split expansions and gluing maps. One ingredient in the degeneration formula that could easily escape one's attention is the *gluing map* relating the stacks \mathfrak{I} and \mathcal{T} . Its twisted variant adds an extra flavor. Let us first describe the untwisted situation.

Define $\mathfrak{I}_{\text{spl}} = \mathcal{T} \times \mathcal{T}$. We can view it as parametrizing pairs consisting of an expansion of (Y_1, D) together with an expansion of (Y_2, D) . Gluing these along the isomorphism

$$(3) \quad N_{D/Y_1} \otimes N_{D/Y_2} \simeq O_D$$

we get an expansion of $X_0 = Y_1 \sqcup_D Y_2$ together with a choice of a splitting divisor. In fact $\mathfrak{I}_{\text{spl}}$ is canonically the stack parametrizing expansions of X_0 together with a choice of a splitting divisor. Forgetting this choice gives an object of $\mathfrak{I}_0 = 0 \times_{\mathcal{A}} \mathfrak{I}$. We obtain the following:

Proposition 1.4.1. *There is a canonical morphism $\mathfrak{I}_{\text{spl}} \rightarrow \mathfrak{I}_0$ of pure degree 1.*

This morphism is implicit in Li's treatment of the degeneration formula; see [Li01, Propositions 4.12 and 4.13].

The situation is slightly more subtle in the twisted case. First, in order to glue twisted expanded pairs the twisting along the marked divisor must coincide. Denoting by $\mathfrak{r} : \mathcal{T}^{\text{tw}} \rightarrow \mathbb{Z}_{>0}$ the locally constant twisting function, we consider the stack $\mathcal{T}^{\text{tw}} \times_{\mathbb{Z}_{>0}} \mathcal{T}^{\text{tw}}$ of pairs of twisted expansions with the same twisting at the marking. Second, we do not have a morphism from this stack to \mathfrak{I} : the isomorphism (3) must be lifted to the \mathfrak{r} -th roots. Denoting by $\mathfrak{I}_{\text{spl}}^{\text{tw}}$ the stack parametrizing such liftings, we obtain a pair of morphisms $\mathfrak{I}_{\text{spl}}^{\text{tw}} \rightarrow \mathcal{T}^{\text{tw}} \times_{\mathbb{Z}_{>0}} \mathcal{T}^{\text{tw}}$ and $\mathfrak{I}_{\text{spl}}^{\text{tw}} \rightarrow \mathfrak{I}_0^{\text{tw}} := 0 \times_{\mathcal{A}} \mathfrak{I}^{\text{tw}}$. The analogous gluing result is the following proposition.

Proposition 1.4.2. (1) *The morphism $\mathfrak{I}_{\text{spl}}^{\text{tw}} \rightarrow \mathcal{T}^{\text{tw}} \times_{\mathbb{Z}_{>0}} \mathcal{T}^{\text{tw}}$ is a gerbe banded by $\mu_{\mathfrak{r}}$, and thus has degree $1/\mathfrak{r}$*
 (2) *The morphism $\mathfrak{I}_{\text{spl}}^{\text{tw}} \rightarrow \mathfrak{I}_0^{\text{tw}}$ has degree $1/\mathfrak{r}$.*

This is used to prove the orbifold degeneration formula of [AF11]. The second morphism is not a gerbe but rather more like the embedding of a scheme in an $(r-1)$ -st infinitesimal neighborhood.

1.5. Labels and twists. Another result of [AF11] shows that relative orbifold Gromov–Witten invariants are independent of twisting. This relies on Costello's formalism of labeling by a set \mathcal{S} [Cos06] applied to expanded degenerations. For this purpose we define algebraic stacks $\mathcal{T}^{\mathcal{S}}$

and $\mathfrak{T}^{\mathcal{S}}$ of expansions with splitting divisors labeled by \mathcal{S} (Section 7.1) and show that they are algebraic and representable over the respective \mathcal{T} and \mathfrak{T} (Proposition 7.1.2). Furthermore given a function $\mathfrak{r} : \mathcal{S} \rightarrow \mathbb{Z}_{>0}$ we define algebraic stacks $\mathcal{T}^{\mathfrak{r}}$ and $\mathfrak{T}^{\mathfrak{r}}$ of \mathfrak{r} -twisted \mathcal{S} -labeled expansions, where the twisting of a splitting divisor labeled by $\delta \in \mathcal{S}$ is determined as $\mathfrak{r}(\delta)$; accordingly the function \mathfrak{r} is called a *twisting choice*. We prove

Proposition 1.5.1. *If $\mathfrak{r}, \mathfrak{r}'$ are twisting choices and \mathfrak{r} divides \mathfrak{r}' , then the structure map $\mathcal{T}^{\mathfrak{r}'} \rightarrow \mathcal{T}^{\mathcal{S}}$ factors canonically as $\mathcal{T}^{\mathfrak{r}'} \rightarrow \mathcal{T}^{\mathfrak{r}} \rightarrow \mathcal{T}^{\mathcal{S}}$. The same holds for $\mathfrak{T}^{\mathfrak{r}'} \rightarrow \mathfrak{T}^{\mathfrak{r}} \rightarrow \mathfrak{T}^{\mathcal{S}}$. If further \mathfrak{r}' divides \mathfrak{r}'' then the resulting triangle is canonically commutative.*

The map $\mathcal{T}^{\mathfrak{r}'} \rightarrow \mathcal{T}^{\mathfrak{r}}$ is named the *partial untwisting map*.

1.6. Naïve expansions. The definitions employed here may not be the first one would consider. Why not simply define a family of expansions to be an arbitrary flat family with fibers of the forms (1) or (2)? We refer to these families as *naïve expansions* and write $\mathcal{T}_{\text{naïve}}(Y, D)$ for pairs and $\mathfrak{T}_{\text{naïve}}(X/B, b_0)$ for degenerations. The problem with these definitions is that in any reasonable generality the automorphism group and deformation space of a naïve expansion are inappropriate; it is false, for example, that naïve expansions are independent of the target. See Section 5.4 for more about the trouble with the naïve definition.

1.7. Expansions and combinatorial structures. Jun Li's treatment also requires a number of properness and connectedness assumptions. Under these assumptions one can also use logarithmic structures on expanded degenerations. We do not follow this approach in this paper, as it is studied in some detail elsewhere [Kim10, AMW11]. However, in Section 8 we reinterpret our definitions of \mathcal{T} and \mathfrak{T} in more combinatorial terms, using logarithmic structures and configurations of line bundles.

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2. DEFINITIONS OF THE STACKS

2.1. Expanded pairs: naïve and universal approaches. Let Y be either a scheme or an Artin stack over a field k , and $D \subset Y$ a Cartier divisor. We call such (Y, D) a *pair*.

Fix a pair (Y, D) and write $\mathcal{O}_Y(D)|_D = \mathcal{O}_D(D)$. Let P be the projective completion $\mathbb{P}(\mathcal{O}_D(D) \oplus \mathcal{O})$ of the normal bundle of D . It is a \mathbb{P}^1 -bundle over D with two sections, denoted D_+ and D_- with normal

bundles $N_{D_-/P} \simeq \mathcal{O}_D(-D)$ and $N_{D_+/P} \simeq \mathcal{O}_D(D)$. These isomorphisms are *canonical* since the complement of D_+ (resp. D_-) in P is canonically isomorphic to the total space of the line bundle $\mathcal{O}_D(-D)$ (resp. $\mathcal{O}_D(D)$) and D_- (resp. D_+) is the image of the zero-section.

The action of $\mathbb{G}_m(D)$ on the line bundle $\mathcal{O}_D(D)$ induces an action of $\mathbb{G}_m(D)$ on P whose fixed loci are D_- and D_+ .

Definition 2.1.1. The *standard expansion of length ℓ* of (Y, D) is obtained by gluing ℓ copies of P to Y along the distinguished divisors:

$$Y(\ell) := Y \sqcup_{D=D_-} P \sqcup_{D_+=D_-} \cdots \sqcup_{D_+=D_-} P.$$

We allow $\ell = 0$ by setting $Y[0] = Y$.

The notation is meant to indicate that the first copy of P is joined to Y along D_- and to the second copy of P along D_+ , etc. Thus the normal bundles of each copy of D in the two components containing it are dual to each other. The divisor D_+ of the last copy of P (or in Y if $\ell = 0$) is contained in the smooth locus of $Y(\ell)$. Denoting this divisor $D(\ell)$, there is a morphism of pairs

$$(Y(\ell), D(\ell)) \rightarrow (Y, D)$$

collapsing all of the copies of P onto D . This maps $D(\ell)$ isomorphically onto D .

Such expansions have appeared also under the name *accordions* [GV05]. A closely related notion is *Zollstöcke* [Kau08]. We note that even when Y is a stack, the morphism $Y(\ell) \rightarrow Y$ is representable, and in fact projective.

Here is the naïve notion of expanded pairs over a base scheme:

Definition 2.1.2. (1) Let (Y, D) be a pair. A *naïve expansion of (Y, D) over a scheme S* is a flat family of pairs $(Y', D') \rightarrow S$, locally of finite presentation over S , with a proper map $(Y', D') \rightarrow (Y, D)$ such that each geometric fiber of (Y', D') over S is isomorphic over (Y, D) to a standard expansion of some length $\ell \geq 0$.

(2) A morphism from a naïve expansion $(Y', D')/S'$ to $(Y'', D'')/S''$ is a fiber diagram

$$\begin{array}{ccc} Y' & \longrightarrow & Y'' \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S'' \end{array}.$$

We note that when Y is a stack, this construction a priori gives a 2-category, whose objects are naïve expansions $(Y', D')/S$ of (Y, D) , arrows given as above, and 2-arrows are isomorphisms of arrows. But

Since the morphism $Y' \rightarrow Y$ is representable, this 2-category is isomorphic to a category, since 2-isomorphisms are unique when they exist; see [AGV08, Lemma 3.3.3]. In fact it suffices to know that $Y' \rightarrow Y$ restrict to a representable morphism on a schematically dense substack $Y'_0 \subset Y'$, see [AV02, Lemma 4.2.3]; this is used when discussing twisted expansions (Definition 2.4.2).

The functor $(Y', D')/S \mapsto S$ makes this a fibered category.

Definition 2.1.3. Denote the category of naïve expansions of the pair (Y, D) by $\mathcal{T}_{\text{naïve}}(Y, D)$.

We note that even when Y is a scheme, an expansion Y' is only guaranteed to be an algebraic space. An example that is not a scheme is given in Remark 6.1.5.

We have the following:

Lemma 2.1.4. *The category $\mathcal{T}_{\text{naïve}}(Y, D)$ is a stack.*

Proof. Since the pullback of an expansion is an expansion, and since arrows are fiber diagrams, the category $\mathcal{T}_{\text{naïve}}(Y, D)$ is fibered in groupoids over the category of schemes. The descent properties are automatic since we allow a naïve expansion Y' to be an algebraic space or a stack. \square

In general, Definition 2.1.2 above is problematic for a number of reasons (detailed in Section 5.4). Fortunately, we will be able to lean on a particular case where it behaves more predictably:

As discussed in the introduction, there is a remarkable stack $\mathcal{A} := [\mathbb{A}^1/\mathbb{G}_m]$ with a Cartier divisor $\mathcal{D} := [0/\mathbb{G}_m] \simeq \mathcal{B}\mathbb{G}_m$, which together form the *universal pair* $(\mathcal{A}, \mathcal{D})$: given a pair (Y, D) , the sheaf homomorphism $\mathcal{O}_X(-D) \rightarrow \mathcal{O}$ induces a morphism $Y \rightarrow \mathcal{A}$ and D is the pre-image of \mathcal{D} . The morphism $Y \rightarrow \mathcal{A}$ is flat since D is a Cartier divisor.

Definition 2.1.5. Denote by $\mathcal{T} := \mathcal{T}_{\text{naïve}}(\mathcal{A}, \mathcal{D})$ the category of naïve expansions of the pair $(\mathcal{A}, \mathcal{D})$. Its universal family is accordingly denoted $(\mathcal{A}^{\text{exp}}, \mathcal{D}^{\text{exp}}) \rightarrow \mathcal{T}$.

We can use this to redefine expansions for any pair:

Definition 2.1.6. Let (Y, D) be a pair with associated flat morphism $Y \rightarrow \mathcal{A}$. An *expansion of (Y, D) over S* is a commutative diagram with

fiber square

$$\begin{array}{ccc}
 (Y', D') & \longrightarrow & (\mathcal{A}', \mathcal{D}') \\
 \downarrow & & \downarrow \\
 (Y, D)_S & \longrightarrow & (\mathcal{A}, \mathcal{D})_S \\
 & \searrow & \downarrow \\
 & & S,
 \end{array}$$

where the right column is a naïve expansion of $(\mathcal{A}, \mathcal{D})$, and the arrow $(Y, D)_S \rightarrow (\mathcal{A}, \mathcal{D})_S$ in the middle is the canonical arrow.

Remark 2.1.7. We could define a stack $\mathcal{T}(Y, D)$ parametrizing expansions of (Y, D) but the universal property of fibered products gives a morphism $\mathcal{T} \rightarrow \mathcal{T}(Y, D)$, sending $(\mathcal{A}', \mathcal{D}') \rightarrow \mathcal{A}$ to the diagram above, which is evidently an isomorphism. This means that the stack \mathcal{T} is the stack of expansions of any pair (Y, D) . Note that Definition 2.1.6 makes no properness or connectedness assumptions, and readily applies to any geometry where \mathcal{A} is the moduli stack of line bundles with sections. Part (1) of Theorem 1.3.2 says in particular that \mathcal{T} is an *algebraic stack*.

Having the universal case $Y = \mathcal{A}$ is one reason to allow Y to be a stack rather than an algebraic space.

Note that the arrow $(Y', D') \rightarrow (Y, D)_S$ on the left of the diagram above is a naïve expansion. This follows since naïve expansions satisfy a basic compatibility for flat morphisms:

Lemma 2.1.8. *Let $f : Z \rightarrow Y$ be a flat morphism and $(Y', D') \rightarrow (Y, D)$ a naïve expansion. Let $Z' = Z \times_Y Y'$, $E = Z \times_Y D$ and $E' = Z \times_Y D'$. Then $(Z', E') \rightarrow (Z, E)$ is a naïve expansion.*

Proof. This follows since $f^*O_Y(D) = O_Z(E)$. □

2.2. Types of degenerations. Before considering expanded degenerations, we consider which types of degenerations we want to study.

2.2.1. Basic degenerations. The most basic case of degeneration considered here is a flat morphism $p : X \rightarrow B$ from a smooth variety to a smooth curve, with unique critical value $b_0 \in B$, where the fiber is the union of two smooth varieties, Y_1 and Y_2 , meeting along a smooth subvariety D which is itself a divisor in Y_1 and Y_2 . Note that we have an isomorphism of line bundles $\mathcal{O}_X(Y_1) \otimes \mathcal{O}_X(Y_2) \simeq p^*\mathcal{O}_B(b_0)$ such that the defining sections are compatible: $\mathbf{1}_{Y_1} \otimes \mathbf{1}_{Y_2} = p^*\mathbf{1}_{b_0}$.

2.2.2. *Degenerate fiber.* Along with the above case one might consider the degenerate fiber abstractly: we have a scheme $X = Y_1 \sqcup_D Y_2$, with a specified isomorphism $\mathcal{O}_{Y_1}(D)|_D \otimes \mathcal{O}_{Y_2}(D)|_D \simeq \mathcal{O}_D$.

2.2.3. *Letting the degeneration vary.* Still more generally, we can remove the assumption that B is a curve or b_0 is a point: assume B is a scheme or algebraic stack, \mathcal{L} an invertible sheaf and $s : \mathcal{L} \rightarrow \mathcal{O}_B$ a sheaf homomorphism, with associated closed subscheme $B_0 = \text{Spec } \mathcal{O}_B/s(\mathcal{L})$. Consider a flat surjective morphism $p : X \rightarrow B$, two invertible sheaf homomorphisms $s_1 : \mathcal{L}_1 \rightarrow \mathcal{O}_X$ and $s_2 : \mathcal{L}_2 \rightarrow \mathcal{O}_X$ with associated closed subschemes Y_1 and Y_2 intersecting along a subscheme D , and an isomorphism $\mathcal{L}_1 \otimes \mathcal{L}_2 \xrightarrow{\sim} p^*\mathcal{L}$ carrying $s_1 \otimes s_2$ to p^*s . We impose the following non-degeneracy assumption: $D \subset Y_i$ should be a Cartier divisor. It is clear that cases 2.2.1 and 2.2.2 are special cases of 2.2.3.

2.2.4. *The universal case.* A special case of 2.2.3 is the following situation, which is the reason why we want to allow X to be a stack: let $X = \mathcal{A}^2$, $B = \mathcal{A}$ and $p : X \rightarrow B$ induced by the multiplication map $\mathbb{A}^2 \rightarrow \mathbb{A}^1$ sending (x_1, x_2) to x_1x_2 . Let $Y_1 = \mathcal{D}_1 \times \mathcal{A}$, $Y_2 = \mathcal{A} \times \mathcal{D}_2$ and $B_0 = \mathcal{D}$.

Clearly this satisfies the non-degeneracy assumption in 2.2.3. On the other hand, given a situation as in 2.2.3, the data (\mathcal{L}_i, s_i) and (\mathcal{L}_i, s_i) provide a canonical commutative diagram

$$(4) \quad \begin{array}{ccc} X & \longrightarrow & \mathcal{A}^2 \\ \downarrow & & \downarrow \\ B & \longrightarrow & \mathcal{A}. \end{array}$$

Nondegeneracy amounts to flatness of the map $X \rightarrow B \times_{\mathcal{A}} \mathcal{A}^2$. So we can think of $\mathcal{A}^2 \rightarrow \mathcal{A}$ as the universal case of a degeneration.

In the following discussion we allow $X \rightarrow B$ to be any of the above.

2.3. **Naïve and universal expanded degenerations.** Consider the situation in 2.2.3. We make definitions analogous to Definitions 2.1.1 and 2.1.2.

Definition 2.3.1. Let b be a geometric point of $B_0 \subset B$ with degenerate fiber $X_b = Y_1 \sqcup_D Y_2$. The *standard expansion* of length $\ell \geq 0$ of X_b is a morphism obtained by gluing ℓ copies of P to Y along the distinguished divisors:

$$X_b(\ell) := Y_1 \sqcup_{D=D_-} P \sqcup_{D_+=D_-} \cdots \sqcup_{D_+=D_-} P \sqcup_{D_+=D} Y_2$$

where $P = \mathbb{P}(\mathcal{O}_{Y_1}(D)|_D \oplus \mathcal{O})$. There is a natural morphism $X_b(\ell) \rightarrow X_b$ contracting all the copies of P to D . We include the case where X_b is

smooth, declaring the standard expansion of such X_b (necessarily of length 0) to be the identity $X_b \rightarrow X_b$.

Definition 2.3.2. (1) A naïve expansion of $X \rightarrow B$ over a base scheme S is a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \longrightarrow & B \end{array}$$

where $X' \rightarrow S$ is flat and locally of finite presentation and such that each geometric fiber is isomorphic over X to a standard expansion.

(2) We denote by $\mathfrak{T}_{\text{naïve}}(X/B, B_0)$ the category of naïve expansions of $X \rightarrow B$, with arrows given by pullback diagrams.

As with Definition 2.1.2 this can be viewed as a category even when X or B is a stack. As with Lemma 2.1.8, one can pull back naïve expanded degenerations along étale morphisms $Z \rightarrow X$. As with Lemma 2.1.4, the definition of arrows in terms of fibered diagrams immediately gives:

Lemma 2.3.3. *The category $\mathfrak{T}_{\text{naïve}}(X/B, B_0)$ is a stack.*

As with expanded pairs, this is a problematic category in general: see Section 5.4. We instead rely on the universal situation. The following lemma, which follows directly from the definition, allows us to work with an arbitrary base.

Lemma 2.3.4. *Given a morphism $\phi : \tilde{B} \rightarrow B$, denote $\tilde{B}_0 = \phi^{-1}B_0$ and $\tilde{X} = X \times_B \tilde{B}$. Then*

$$\mathfrak{T}_{\text{naïve}}(\tilde{X}/\tilde{B}, \tilde{B}_0) \simeq \mathfrak{T}_{\text{naïve}}(X/B, B_0) \times_B \tilde{B}.$$

Our definition is:

Definition 2.3.5. Denote $\mathfrak{T} := \mathfrak{T}_{\text{naïve}}(\mathcal{A}^2/\mathcal{A}, \mathcal{D})$, with universal family

$$(\mathcal{A}^2)^{\text{exp}} \rightarrow \mathfrak{T}.$$

Definition 2.3.6. For a general degeneration $X \rightarrow B$ as in 2.2.3, let $B \rightarrow \mathcal{A}$ be the morphism associated to (\mathcal{L}, s) and $X \rightarrow \mathcal{A}^2$ associated to (\mathcal{L}_1, s_1) and (\mathcal{L}_2, s_2) . Given a B -scheme S , an expansion of X/B

over S is a commutative diagram with cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & (\mathcal{A}^2)' \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{A}^2 \times_{\mathcal{A}} S \\ & \searrow & \downarrow \\ & & S \end{array}$$

in which $(\mathcal{A}^2)'$ is a naïve expansion of \mathcal{A}^2 over S .

Remark 2.3.7. Again we could define a stack of expansions $\mathfrak{T}(X/B, B_0)$, but by the universal property of fibered products we have a canonical isomorphism $\mathfrak{T}(X/B, B_0) := \mathfrak{T} \times_{\mathcal{A}} B$, with universal expansion

$$\mathcal{X}^{\text{exp}} := \mathcal{X} \times_{\mathcal{A}^2} (\mathcal{A}^2)^{\text{exp}} \rightarrow \mathfrak{T}(X/B, B_0).$$

This makes $\mathfrak{T} \times_{\mathcal{A}} B$ into the stack of expansions of any degeneration over B .

Remark 2.3.8. There is a similarity between expanded pairs and expanded degenerations which can be misleading. Consider a smooth pair (Y, D) over a field k . We can define a morphism $Y \rightarrow \mathcal{A}^2$ whose first factor is defined by $(\mathcal{O}_Y(D), \mathbf{1}_D)$ and second factor defined by $(\mathcal{O}_Y(-D), 0)$. This gives a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{A}^2 \\ \downarrow & & \downarrow m \\ \text{Spec } k & \longrightarrow & \mathcal{A}. \end{array}$$

It is tempting therefore to try to view an expanded pair as a special kind of expanded degeneration over $\mathfrak{T}_0 = \text{Spec } k \times_{\mathcal{A}} \mathfrak{T}$. Indeed, the geometric fibers of $Y' = Y \times_{\mathcal{A}} (\mathcal{A}^2)^{\text{exp}}$ are expansions of (Y, D) . However, since $Y \rightarrow \text{Spec } k \times_{\mathcal{A}} \mathcal{A}^2$ is not flat, the resulting map $Y' \rightarrow \mathfrak{T}_0$ is not flat, so this is not an expanded pair.

2.4. Twisted expansions of pairs and degenerations. The paper [AF11] uses auxiliary stack structures on expansions in order to circumvent some of the difficult deformation theory in the degeneration formula. This formalism is used in [ACW10] to compare orbifold Gromov–Witten invariants to relative Gromov–Witten invariants. Here we lay the foundations, which become straightforward using our universal approach. We build on the notation P and D_{\pm} of 2.1.1.

Definition 2.4.1. (1) Let (Y, D) be a pair. Fix a non-negative integer ℓ and a tuple of positive integers $\mathbf{r} := (r_0, \dots, r_\ell)$. Consider the root stacks $Y_{r_0} = Y(\sqrt[r_0]{D})$ and for $i = 1, \dots, \ell$ write $P_i = P(\sqrt[r_i]{D_-}, \sqrt[r_i]{D_+})$. We use the notation $\mathcal{D}_i \pm$ for the resulting root divisors; for $i = 1, \dots, \ell - 1$ consider the unique isomorphism $\phi_i : \mathcal{D}_i + \simeq \mathcal{D}_{(i+1)} -$ which is band-reversing, in the sense that it identifies the normal bundle of $\mathcal{D}_i +$ with the dual of the normal bundle of $\mathcal{D}_{(i+1)} -$.

The standard \mathbf{r} -twisted expansion of length ℓ of (Y, D) is the stack

$$Y(\ell, \mathbf{r}) := Y_{r_0} \sqcup_{\substack{\mathcal{D} = \mathcal{D}_- \\ \phi_0}} P_1 \sqcup_{\substack{\mathcal{D}_+ = \mathcal{D}_- \\ \phi_1}} \cdots \sqcup_{\substack{\mathcal{D}_+ = \mathcal{D}_- \\ \phi_\ell}} P_\ell.$$

The notation means to convey that Y_{r_0} is joined to P_1 , and P_i to P_{i+1} , by identifying $\mathcal{D}_i +$ with $\mathcal{D}_{(i+1)} -$ through ϕ_i .

(2) Similarly, given a degeneration and $b \in B_0$, the standard \mathbf{r} -twisted expansion of length ℓ of X_b is the stack

$$X_b(\ell, \mathbf{r}) := (Y_1)_{r_0} \sqcup_{\substack{\mathcal{D} = \mathcal{D}_- \\ \phi_0}} P_1 \sqcup_{\substack{\mathcal{D}_+ = \mathcal{D}_- \\ \phi_1}} \cdots \sqcup_{\substack{\mathcal{D}_+ = \mathcal{D}_- \\ \phi_\ell}} (Y_2)_{r_\ell}$$

where $(Y_2)_{r_\ell} = Y_2(\sqrt[r_\ell]{D})$ and the rest of the notation is as above.

We use the notation $\mathfrak{T}_{\text{naive}}^{\text{tw}}(X/B, B_0)$ and $\mathcal{T}_{\text{naive}}^{\text{tw}}(Y, D)$ for the stacks of flat families whose fibers are isomorphic to twisted standard expansions. We will use it only briefly in the next section.

We can now define stacks of twisted expansions:

Definition 2.4.2. (1) Denote by \mathcal{T}^{tw} the category whose objects over a scheme S are flat families of pairs $(\mathcal{A}', \mathcal{D}') \rightarrow S$ with a morphism $(\mathcal{A}', \mathcal{D}') \rightarrow (\mathcal{A}, \mathcal{D})$, such that the geometric fibers over S are isomorphic over $(\mathcal{A}, \mathcal{D})$ to standard \mathbf{r} -twisted expansions of $(\mathcal{A}, \mathcal{D})$. We call this the stack of twisted expanded pairs, the universal expansion denoted by $(\mathcal{A}^{\text{exp tw}}, \mathcal{D}^{\text{exp tw}})$.

For an arbitrary pair, the stack of twisted expansions of (Y, D) is again $\mathcal{T}^{\text{tw}}(Y, D) := \mathcal{T}^{\text{tw}}$, with the universal expansion $(Y \times_{\mathcal{A}} \mathcal{A}^{\text{exp tw}}, D \times_{\mathcal{D}} \mathcal{D}^{\text{exp tw}})$.

(2) Denote by \mathfrak{T}^{tw} the category whose objects over a scheme S are flat families of pairs $(\mathcal{A}^2)' \rightarrow S$ with a morphism $(\mathcal{A}^2)' \rightarrow \mathcal{A}^2$, such that the geometric fibers over S are isomorphic over \mathcal{A}^2 to standard \mathbf{r} -twisted expansions of the degeneration $\mathcal{A}^2 \rightarrow \mathcal{A}$ of 2.2.4. We call this the stack of twisted expanded degenerations, with universal expansion $(\mathcal{A}^2)^{\text{exp tw}}$.

For an arbitrary degeneration as in 2.2.3 the stack of twisted expansions of X is again $\mathfrak{T}^{\text{tw}} \times_{\mathcal{A}} B$, with the universal expansion $X \times_{\mathcal{A}^2} (\mathcal{A}^2)^{\text{exp tw}}$.

Remark 2.4.3. When $(Y, D) = (\mathcal{A}, \mathcal{D})$ it is not hard to see that P_i are actually isomorphic to P , and the stack underlying the standard twisted expansion is $\mathcal{A}' := \mathcal{A} \sqcup P \sqcup \cdots \sqcup P$, not depending on the twisting. What varies with the twisting is the morphism $\mathcal{A}' \rightarrow \mathcal{A}$.

3. THE STACKS OF EXPANSIONS ARE ALGEBRAIC

3.1. Semistable maps and the stack of Graber–Vakil. Recall that a pre-stable marked curve C is called *semistable* if any unstable component is rational and has precisely two special geometric points, counting nodes and marked points of C .

In [GV05], Graber and Vakil introduce a stack of expanded pairs as follows:

Definition 3.1.1. Consider the stack $\mathfrak{M}_{0,3}$ of 3-pointed genus 0 pre-stable curves and its substack $\mathfrak{M}_{0,3}^{\text{ss}}$ of semistable curves. We denote the markings by $0, 1, \infty$. Denote by \mathcal{T}_{GV} the substack where the points marked $1, \infty$ lie on the same irreducible component of the curve.

The stack $\mathfrak{M}_{0,3}$ is well known to be algebraic and locally of finite type over \mathbb{Z} ; the substacks $\mathfrak{M}_{0,3}^{\text{ss}}$ and \mathcal{T}_{GV} are open, so they are also algebraic and locally of finite type over \mathbb{Z} . Graber and Vakil noted that Li’s construction of expanded pairs should be independent of the choice of Y . Moreover the objects of the stack \mathcal{T}_{GV} are (as we see below) expansions of $(\mathbb{P}^1, 0)$. The idea is that once we know what to do to the pair $(\mathbb{P}^1, 0)$ we should just “do the same” to an arbitrary pair. Since an arbitrary pair does not map to $(\mathbb{P}^1, 0)$, we preferred to go by way of the pair $(\mathcal{A}, \mathcal{D})$. This will require just one more step to show that the stack is algebraic.

In order to work in closer parallel with expanded pairs and degenerations, we go by way of a variant of \mathcal{T}_{GV} :

Definition 3.1.2. Consider the stack $\mathfrak{M}_{0,1}^{\text{tw}}(\mathbb{P}^1, 1)$ of (not necessarily representable) maps of degree 1 from 1-pointed twisted prestable curves of genus 0 to \mathbb{P}^1 . The marked point is permitted to have a nontrivial orbifold structure. This stack has a substack $\mathfrak{M}_{0,1}^{\text{tw,ss}}(\mathbb{P}^1, 1)$ where the maps are semistable. Consider the evaluation map $e : \mathfrak{M}_{0,1}^{\text{tw,ss}}(\mathbb{P}^1, 1) \rightarrow \mathbb{P}^1$. Define

$$\mathcal{T}_{\text{maps}}^{\text{tw}} := e^{-1}\{0\}$$

and $\mathcal{T}_{\text{maps}} \subset \mathcal{T}_{\text{maps}}^{\text{tw}}$ the open substack of untwisted curves.

The reader interested only in the untwisted case can safely ignore all references to twisted curves.

Note again that the stack $\mathfrak{M}_{0,1}^{\text{tw}}(\mathbb{P}^1, 1)$ is well known to be algebraic and locally of finite type over \mathbb{Z} (see [AOV11a] for the general case over \mathbb{Z}). The stack $\mathfrak{M}_{0,1}^{\text{tw,ss}}(\mathbb{P}^1, 1)$ is algebraic as it is an open substack of $\mathfrak{M}_{0,1}^{\text{tw}}(\mathbb{P}^1, 1)$, and $\mathcal{T}_{\text{maps}}^{\text{tw}} = e^{-1}\{0\}$ is algebraic as it is closed in $\mathfrak{M}_{0,1}^{\text{tw,ss}}(\mathbb{P}^1, 1)$. The substack $\mathcal{T}_{\text{maps}}$ is open in the latter, so is algebraic as well.

Lemma 3.1.3. *We have isomorphisms $\mathcal{T}_{\text{naïve}}^{\text{tw}}(\mathbb{P}^1, D) \simeq \mathcal{T}_{\text{maps}}^{\text{tw}}$ and $\mathcal{T}_{\text{naïve}}(\mathbb{P}^1, D) \simeq \mathcal{T}_{\text{maps}}$. Hence $\mathcal{T}_{\text{naïve}}^{\text{tw}}(\mathbb{P}^1, D)$ is an algebraic stack, locally of finite type over \mathbb{Z} and $\mathcal{T}_{\text{naïve}}(\mathbb{P}^1, D)$ an open substack.*

Proof. We provide a base-preserving equivalence of categories. Consider an object $(Y', D') \rightarrow (\mathbb{P}^1, 0)$ in $\mathcal{T}_{\text{naïve}}^{\text{tw}}(\mathbb{P}^1, D)(S)$. Then every geometric fiber is by definition a semistable map, since any unstable component is a root stack of $P \simeq \mathbb{P}^1$ along its two special points, D_- and D_+ . Since D' maps to D we have that e is identically 0. This gives an object of $\mathcal{T}_{\text{maps}}^{\text{tw}}$. This identification is clearly compatible with pullbacks, giving a functor $\mathcal{T}_{\text{naïve}}^{\text{tw}}(\mathbb{P}^1, D) \rightarrow \mathcal{T}_{\text{maps}}^{\text{tw}}$, preserving the twisting.

Similarly, an object of $\mathcal{T}_{\text{maps}}^{\text{tw}}$ is a flat family of twisted pointed curves (Y', D') with a map to (\mathbb{P}^1, D) . We need to verify that the fibers are expansions. We apply induction on the number of irreducible components. There is a unique component of Y' mapping with degree 1 onto \mathbb{P}^1 ; if it is the unique component the map is a root stack at 0 and we are done. If there is another component, consider an end component on the same fiber of $Y' \rightarrow \mathbb{P}^1$. Since the map is semistable, that end component must contain the marked divisor D' and a unique node, which we identify as \mathbb{P}_D^1 . Pruning that component off, we get a new semistable map $(Y'', D'') \rightarrow (\mathbb{P}^1, D)$ which is a twisted expansion by induction. Since Y' is obtained by gluing the end component to Y'' , and since twisted curves are by definition balanced, we conclude that (Y', D') is a twisted expansion. Again this identification is compatible with pullbacks, giving a functor $\mathcal{T}_{\text{maps}}^{\text{tw}} \rightarrow \mathcal{T}_{\text{naïve}}^{\text{tw}}(\mathbb{P}^1, D)$.

The compositions of these functors in either way are the identity, giving the required equivalence. \square

We relate these to \mathcal{T}_{GV} using the following well known result.

Lemma 3.1.4. *We have an isomorphism $\mathcal{T}_{\text{GV}} \simeq \mathcal{T}_{\text{maps}}$.*

Proof. Given an object $f : (Y', D') \rightarrow \mathbb{P}^1$ in $\mathcal{T}_{\text{maps}}(S)$ we obtain an object

$$(Y', D', f^{-1}(1), f^{-1}(\infty))$$

of $\mathcal{T}_{\text{GV}}(S)$, which clearly gives a base-preserving functor $\mathcal{T}_{\text{maps}} \rightarrow \mathcal{T}_{\text{GV}}$. Given an object $(Y', p, q, r) \in \mathcal{T}_{\text{GV}}(S)$ the invertible sheaf $\mathcal{O}_{Y'}(r)$ is

base-point-free relative to S . We obtain an associated morphism $Y' \rightarrow \mathbb{P}_S^1$ which maps (p, q, r) to $(0, 1, \infty)$, which is an object of $\mathcal{T}_{\text{maps}}$. It is easy to see that these functors are quasi-inverses, giving the required base-preserving equivalence. \square

We now pass to degenerations, where the replacement for the pair $(\mathbb{P}^1, 0)$ requires more work.

Definition 3.1.5. Consider X , the blowup of $\mathbb{A}^1 \times \mathbb{P}^1$ at the origin, and $B = \mathbb{A}^1$, with the projection morphism $X \rightarrow B$. The unique singular fiber lies over the origin $0 \in \mathbb{A}^1$ and has two components Y_i intersecting at a unique point D .

Consider the stack $\mathfrak{M}_{0,0}^{\text{tw}}(X/\mathbb{A}^1, 1)$ of maps of prestable curves of genus 0 to the fibers of X/\mathbb{A}^1 , again of degree 1—that is, with image class equal to the class of the fiber. Let $\mathfrak{M}_{0,0}^{\text{tw,ss}}(X/\mathbb{A}^1, 1)$ be the substack of $\mathfrak{M}_{0,0}^{\text{tw}}(X/\mathbb{A}^1, 1)$ parameterizing maps that are semistable and are isomorphisms away from D .

As before, $\mathfrak{M}_{0,0}^{\text{tw}}(X/\mathbb{A}^1, 1)$ is well known to be algebraic and locally of finite type over k , and $\mathfrak{M}_{0,0}^{\text{tw,ss}}(X/\mathbb{A}^1, 1)$ an open substack.

Repeating the arguments above we have:

Lemma 3.1.6. *We have an isomorphism $\mathfrak{T}_{\text{naïve}}^{\text{tw}}(X/\mathbb{A}^1, 1) \simeq \mathfrak{M}_{0,0}^{\text{tw,ss}}(X/\mathbb{A}^1, 1)$. In particular $\mathfrak{T}_{\text{naïve}}^{\text{tw}}(X/\mathbb{A}^1, 1)$ is an algebraic stack, locally of finite type over k .*

3.2. Algebraicity of \mathcal{T}^{tw} and \mathfrak{T}^{tw} . We show that \mathcal{T}^{tw} and \mathfrak{T}^{tw} are algebraic by producing isomorphisms to the stacks $\mathcal{T}_{\text{naïve}}^{\text{tw}}(\mathbb{P}^1, 0)$ and $\mathfrak{T}_{\text{naïve}}^{\text{tw}}(X/\mathbb{A}^1, 0)$ (Propositions 3.2.4 and 3.2.5) that were shown to be algebraic in the last section (Lemma 3.1.3). First we define the morphism in one direction in complete generality.

Definition 3.2.1. For each pair (Y, D) , let $\Phi^{(Y,D)}$ be the morphism of stacks $\Phi^{(Y,D)} : \mathcal{T}^{\text{tw}} \rightarrow \mathcal{T}_{\text{naïve}}^{\text{tw}}(Y, D)$ sending an expansion $(\mathcal{A}', \mathcal{D}')/S$ of $(\mathcal{A}, \mathcal{D})$ to the fiber product $(Y', D') := (Y \times_{\mathcal{A}} \mathcal{A}', D \times_{\mathcal{D}} \mathcal{D}')$. For a degeneration $(X/B, B_0)$ we similarly define a morphism $\Phi^{X/B} : \mathfrak{T} \times_{\mathcal{A}} B \rightarrow \mathfrak{T}_{\text{naïve}}^{\text{tw}}(X/B, B_0)$ by sending an expansion $(\mathcal{A}^2)'$ of $\mathcal{A}^2 \times_{\mathcal{A}} B$ to the fibered product $X \times_{\mathcal{A}^2 \times_{\mathcal{A}} B} (\mathcal{A}^2)'$.

Definition 3.2.2. (1) Let (Y, D) be a smooth pair. Define $\Phi = \Phi^{(Y,D)} : \mathcal{T}^{\text{tw}} \rightarrow \mathcal{T}_{\text{naïve}}^{\text{tw}}(Y, D)$ by sending an expansion $(\mathcal{A}', \mathcal{D}')/S$ of $(\mathcal{A}, \mathcal{D})$ to $(Y', D') := (Y \times_{\mathcal{A}} \mathcal{A}', D \times_{\mathcal{D}} \mathcal{D}')$. We also use $\Phi^{(Y,D)}$ to denote the restricted map $\mathcal{T} \rightarrow \mathcal{T}_{\text{naïve}}(Y, D)$.

(2) Let $(X/B, B_0)$ be a degeneration. Define $\Phi = \Phi^{X/B} : \mathfrak{T}^{\text{tw}} \times_{\mathcal{A}} B \rightarrow \mathcal{T}_{\text{naïve}}^{\text{tw}}(X/B, B_0)$ by sending an expansion $(\mathcal{A}^2)'/S$ of $(\mathcal{A}^2 \times_{\mathcal{A}} B)/B$

to $X' := X \times_{\mathcal{A}^2 \times_{\mathcal{A}} B} (\mathcal{A}^2)'$. We also use $\Phi^{X/B}$ for the restriction $\mathfrak{T} \rightarrow \mathfrak{T}_{\text{naïve}}(X/B, B_0)$.

Lemma 3.2.3. (1) *Let (Y, D) be a pair and $U \subset Y$ open, containing D . Then $\mathcal{T}_{\text{naïve}}^{\text{tw}}(Y, D) \simeq \mathcal{T}_{\text{naïve}}^{\text{tw}}(U, D)$.*

(2) *Let $X \rightarrow V$ be a degeneration and $U \subset X$ open, containing D . Then $\mathfrak{T}_{\text{naïve}}^{\text{tw}}(X/B, B_0) \simeq \mathfrak{T}(U/B_0)$.*

Proof. Given a naïve expansion $f : (Y', D') \rightarrow (Y, D)$ over S , we canonically obtain a naïve expansion $(U', D') \rightarrow (U, D)$ by taking $U' = f^{-1}U$. Given a naïve expansion $f_U : (U', D') \rightarrow (U, D)$ over S we have a canonical isomorphism $U' \setminus f_U^{-1}D \simeq U \setminus D$. We obtain a naïve expansion $(Y', D') \rightarrow (Y, D)$ by gluing

$$Y' = U' \sqcup_{U \setminus D} (Y \setminus D).$$

These provide functors inverse to each other. The construction for expanded degenerations is identical. \square

Proposition 3.2.4. *The morphism $\Phi^{(\mathbb{P}^1, 0)} : \mathcal{T}^{\text{tw}} \rightarrow \mathcal{T}_{\text{naïve}}^{\text{tw}}(\mathbb{P}^1, 0)$ is an isomorphism. In particular \mathcal{T}^{tw} is algebraic and locally of finite type over \mathbb{Z} .*

Proof. We provide a quasi-inverse $\Psi^{(\mathbb{P}^1, 0)} : \mathcal{T}_{\text{naïve}}^{\text{tw}}(\mathbb{P}^1, 0) \rightarrow \mathcal{T}^{\text{tw}}$ as follows. Let $f : (Y', D') \rightarrow (\mathbb{P}^1, 0)$ be a twisted expansion over a scheme S . Marking \mathbb{P}^1 at $(0, \infty)$ and Y' at $(D', f^{-1}(\infty))$ we can view f as a morphism of 2-marked rational curves. By Proposition A.4 the morphism f is \mathbb{G}_m -equivariant with respect to the balanced \mathbb{G}_m action. Write

$$\mathcal{Y}' := [(Y' \setminus \{\infty\})/\mathbb{G}_m]; \quad \mathcal{D}' := [D'/\mathbb{G}_m].$$

Since $[(\mathbb{P}^1 \setminus \{\infty\})/\mathbb{G}_m] = \mathcal{A}$ we obtain an expansion $(\mathcal{Y}', \mathcal{D}') \rightarrow (\mathcal{A}, \mathcal{D})$ over S . The procedure is clearly compatible with pullbacks, giving a morphism $\Psi^{(\mathbb{P}^1, 0)} : \mathcal{T}_{\text{naïve}}^{\text{tw}}(\mathbb{P}^1, 0) \rightarrow \mathcal{T}^{\text{tw}}$.

To check that these maps are quasi-inverses, note that it is enough by Lemma 3.2.3 (1) to show that the induced maps between $\mathcal{T}_{\text{naïve}}^{\text{tw}}(\mathbb{A}^1, 0)$ and \mathcal{T}^{tw} are quasi-inverses. This, however, is immediate. \square

Here is the analogue of Proposition 3.2.4 for twisted degenerations:

Proposition 3.2.5. *Consider the degeneration $X \rightarrow \mathbb{A}^1$ of Definition 3.1.5. The morphism $\Phi^{X/\mathbb{A}^1} : \mathfrak{T}^{\text{tw}} \times_{\mathcal{A}} \mathbb{A}^1 \rightarrow \mathfrak{T}_{\text{naïve}}^{\text{tw}}(X/\mathbb{A}^1, 0)$ is an isomorphism.*

The stack \mathfrak{T} is algebraic and locally of finite type over \mathbb{Z} .

Proof. We again provide a quasi-inverse $\Psi^{X/\mathbb{A}^1} : \mathfrak{T}_{\text{naïve}}^{\text{tw}}(X/\mathbb{A}^1, 0) \rightarrow \mathfrak{T}^{\text{tw}} \times_{\mathcal{A}} \mathbb{A}^1$, as follows. The proper transforms s_0 of $0 \times \mathbb{A}^1$ and s_∞ of

$\infty \times \mathbb{A}^1$ are disjoint markings of $X \rightarrow \mathbb{A}^1$. Thus $X \rightarrow \mathbb{A}^1$ is a family of 2-pointed twisted semistable curves, and an object of $\mathfrak{T}_{\text{naive}}^{\text{tw}}(X/\mathbb{A}^1, 0)$ gives a morphism of families of semistable curves $X' \rightarrow X$. By Proposition A.4, this map is equivariant with respect to the canonical actions of \mathbb{G}_m on X' and on X . By Lemma 3.2.3 (2), the family $X' \setminus (s_0 \cup s_\infty)$ is a twisted expansion of $X \setminus (0 \cup \infty)$. Passing to the quotient by the \mathbb{G}_m -action, $\mathcal{X} = [(X' \setminus (s_0 \cup s_\infty))/\mathbb{G}_m]$ gives an expansion of $[(X \setminus \{0, \infty\})/\mathbb{G}_m] = \mathcal{A}^2 \times_{\mathcal{A}} \mathbb{A}^1$.

By Lemma 3.1.6 the stack $\mathfrak{T}_{\text{naive}}^{\text{tw}}(X/\mathbb{A}^1, 0) \simeq \mathfrak{T}^{\text{tw}} \times_{\mathcal{A}} \mathbb{A}^1$ is algebraic and locally of finite type over k . Since $\mathbb{A}^1 \rightarrow \mathcal{A}$ is smooth and surjective it follows that \mathfrak{T}^{tw} is algebraic as well [AOV11a, Lemma C.5]. \square

With Propositions 3.2.4 and 3.2.5 we have completed the proofs of Theorem 1.3.1 and Part (1) of Theorem 1.3.2.

3.3. Exotic isomorphisms.

3.3.1. *The isomorphism between expanded pairs and expanded degenerations.* An expanded pair induces an expanded degeneration by deleting the distinguished divisor. The only information forgotten this way is the order of twisting along the distinguished divisor: a distinguished divisor with any desired order of twisting can always be glued onto one end of an expanded degeneration to make an expanded pair. We make these processes precise in the following proposition.

Proposition 3.3.1. *There is an isomorphism $\mathcal{T}^{\text{tw}} \simeq \mathfrak{T}^{\text{tw}} \times \mathbb{Z}_{>0}$.*

To construct the isomorphism, it is convenient first to study expanded pairs of length 1. Let \mathcal{T}' be the moduli space of naïve expansions of $(\mathcal{A}, \mathcal{D})$ of $(\mathcal{A}, \mathcal{D})$ having at most one node. Twisting is allowed in \mathcal{T}' , but only along the distinguished divisor, and *not along the node*. We can identify \mathcal{T}' as $\mathcal{T}'_{\text{GV}} \times \mathbb{Z}_{>0}$, where the first factor is the open substack of \mathcal{T}_{GV} parameterizing curves in \mathcal{T}_{GV} having at most one node, and the second factor records the order of twisting along the distinguished divisor. We can identify \mathcal{T}'_{GV} with \mathcal{A} and $\mathcal{D} \subset \mathcal{A}$ is the divisor of singular curves.

We can describe \mathcal{T}' and its universal object very explicitly. The stack \mathcal{T}' itself is isomorphic to $\mathcal{A} \times \mathbb{Z}_{>0}$. The universal object can be constructed as an open substack of the quotient of the universal curve on $\mathcal{T}'_{\text{GV}} \times \mathbb{Z}_{>0}$ by the canonical action of \mathbb{G}_m (Proposition A.3). It can be given even more explicitly in coordinates as an open substack of $\mathcal{A}^3 \times \mathbb{Z}_{>0}$: Letting (x, y, z, n) be coordinates on $\mathcal{A}^3 \times \mathbb{Z}_{>0}$, define \mathcal{Z} to be the open substack where y and z do not vanish simultaneously. Equip

\mathcal{Z} with the projection to the base $\mathcal{T}' \times \mathbb{Z}_{>0}$ by mapping $(x, y, z, n) \mapsto (xy, n)$. Writing \mathcal{E} for the divisor defined by the vanishing of z , we have a map of pairs $(\mathcal{Z}, \mathcal{E}) \rightarrow (\mathcal{A}, \mathcal{D})$ by xz^n .

One can verify easily by considering the fibers over $\mathcal{D} \times \mathbb{Z}_{>0} \subset \mathcal{T}'$ and its complement that $(\mathcal{Z}, \mathcal{E})$ is a family of expansions of the pair $(\mathcal{A}, \mathcal{D})$. That it is universal among expansions of length 1 that are untwisted at the node can be seen, for example, by comparison with expansions of $(\mathbb{P}^1, 0)$.

There are maps

$$(5) \quad \begin{array}{ccc} & \mathfrak{T}_{\text{naive}}^{\text{tw}}(\mathcal{Z}/\mathcal{A}, \mathcal{D}) & \\ & \swarrow \quad \searrow & \\ \mathfrak{T}^{\text{tw}} \times \mathbb{Z}_{>0} & & \mathcal{T}^{\text{tw}}. \end{array}$$

To describe the map on the left, note that \mathcal{Z} can be viewed as a degeneration over \mathcal{A} , the map $\mathcal{Z} \rightarrow \mathcal{A}^2$ coming from $(x, y, z) \mapsto (xz^n, y)$. By deleting the distinguished divisor, an expansion of \mathcal{Z}/\mathcal{A} becomes an expansion of $\mathcal{A}^2/\mathcal{A}$; the map to $\mathbb{Z}_{>0}$ simply records the order of twisting along the distinguished divisor. As for the map on the right, note that if \mathcal{X} is an expansion of the degeneration \mathcal{Z}/\mathcal{A} then by composition we obtain a map $\mathcal{X} \rightarrow \mathcal{Z} \rightarrow \mathcal{A}$. Comparing the fibers, one sees that these are all expansions of the pair $(\mathcal{A}, \mathcal{D})$.

Applying Part (2) of Lemma 3.2.3 to the inclusion $\mathcal{A}^2 \subset \mathcal{Z}$ (as the locus where $z \neq 0$) implies that the map $\mathfrak{T}_{\text{naive}}^{\text{tw}}(\mathcal{Z}/\mathcal{A}, \mathcal{D}) \rightarrow \mathfrak{T}^{\text{tw}} \times \mathbb{Z}_{>0}$ is an isomorphism.

Lemma 3.3.2. *The map $\mathfrak{T}_{\text{naive}}^{\text{tw}}(\mathcal{Z}/\mathcal{A}, \mathcal{D}) \rightarrow \mathcal{T}^{\text{tw}}$ is an isomorphism.*

Proof. To prove the lemma, we must check that any expanded pair admits a contraction onto an expansion of length 2. For this, it is easiest to use the Graber–Vakil perspective on expansions. Viewing an expanded pair over S as a family of twisted curves C/S with untwisted markings 1 and ∞ and a potentially twisted marking 0, we use the linear system $\mathcal{O}_C(0 + 1 + \infty)$ to produce a contraction onto a curve C' that has a marking on every component. Since 1 and ∞ must lie on the same irreducible component in each fiber of C' , the fibers of C' can each have at most one node. \square

3.3.2. Moduli of curves and rigidification. We describe a second relationship between expansions and semistable curves, this time by *rigidification* (see [Gir71, Proposition IV.2.3.18] and [AOV08, Appendix A]).

Denote by $\mathfrak{M}' \subset \mathfrak{M}_{0,2}^{\text{tw,ss}}$ the open and closed substack of twisted semistable marked curves where the second marking is not twisted.

Proposition A.4 says that the group scheme \mathbb{G}_m is a normal subgroup of the inertia stack of $\mathfrak{M}_{0,2}^{\text{tw,ss}}$. We can consider the rigidification $\mathfrak{M}'/\mathcal{B}\mathbb{G}_m$ (sometimes denoted $\mathfrak{M}' // \mathbb{G}_m$) of \mathfrak{M}' .

Proposition 3.3.3. *There is a canonical isomorphism $\mathfrak{M}' \xrightarrow{\sim} \mathcal{T}^{\text{tw}} \times \mathcal{B}\mathbb{G}_m$.*

Proof. Let (C, D, E) be an S -point of \mathfrak{M}' . That is, C is a family of genus zero curves over S with a potentially twisted marking D and an untwisted marking E . The complement of the zero section in the normal bundle $N_{E/C}$ is a \mathbb{G}_m -torsor on S . This gives a map $\mathfrak{M}' \rightarrow \mathcal{T}^{\text{tw}}$. Denote this \mathbb{G}_m -torsor by Q .

Since C has an action of \mathbb{G}_m (Proposition A.3), we can twist C by the opposite torsor of Q to produce a family of semistable curves $C' = C \times_{\mathbb{G}_m} Q^\vee$, with sections D' and E' , such that $N_{E'/C'}$ is trivial. But $N_{E'/C'}$ is isomorphic to the complement of the node in the component of C' containing E' . A trivialization of $N_{E'/C'}$ yields a section of C' confined to the component containing E' and not meeting either E' nor a node. Calling this section 1 and letting $\infty = E'$ and $0 = D'$, we get an S -point of \mathcal{T}^{tw} .

The pair $(Q, (C', 0, 1, \infty))$ gives the map claimed by the proposition.

The inverse sends $(Q, (C', 0, 1, \infty))$ to $(Q \times_{\mathbb{G}_m} C', 0, \infty)$. \square

Corollary 3.3.4. *There is a canonical isomorphism $\mathcal{T}^{\text{tw}} \simeq \mathfrak{M}'/\mathcal{B}\mathbb{G}_m$. Restricting to untwisted objects gives $\mathcal{T} \simeq \mathfrak{M}_{0,2}^{\text{ss}}/\mathcal{B}\mathbb{G}_m$.*

We could now repeat the argument for \mathfrak{T}^{tw} , but it is faster to make use of the isomorphism proved in Proposition 3.3.1. Letting \mathfrak{M}'' denote the locus in $\mathfrak{M}_{0,2}^{\text{tw,ss}}$ where *both* markings are untwisted and restricting to the locus where the distinguished divisor is untwisted in Proposition 3.3.3, we obtain the following corollary.

Corollary 3.3.5. *There is a canonical isomorphism $\mathfrak{M}'' \simeq \mathfrak{T}^{\text{tw}} \times \mathcal{B}\mathbb{G}_m$. Therefore $\mathfrak{T}^{\text{tw}} \simeq \mathfrak{M}''/\mathcal{B}\mathbb{G}_m$ and $\mathfrak{T} \simeq \mathfrak{M}_{0,2}^{\text{ss}}/\mathcal{B}\mathbb{G}_m$.*

4. STACKS OF NON-RIGID EXPANSIONS

In [GV05] Graber and Vakil showed that localization in relative Gromov–Witten theory necessitates one more stack of expansions. This time we fix only the “divisor” D and a line bundle N , standing in for the normal bundle $N_{D/Y}$. We allow twisting:

Definition 4.1. A standard non-rigid twisted expansion of (D, N) is of the form

$$P_0 \begin{array}{c} \sqcup \\ \mathcal{D}_+ = \mathcal{D}_- \\ \phi_1 \end{array} \cdots \begin{array}{c} \sqcup \\ \mathcal{D}_+ = \mathcal{D}_- \\ \phi_\ell \end{array} P_\ell$$

where $P = \mathbb{P}_D(N \oplus \mathcal{O}_D)$ is the projective completion of N , and the notation P_i , \mathcal{D}_\pm and ϕ_i is as in Definition 2.4.1.

We define $\mathcal{T}_\sim^{\text{tw}}$ to be the stack whose objects over S are flat families of twisted non-rigid expansions of $(\mathcal{B}\mathbb{G}_m, \mathcal{L})$, where \mathcal{L} is the universal line bundle.

For an arbitrary (D, N) , we define an expansion over S to be a commutative diagram with cartesian squares

$$\begin{array}{ccc} Z & \longrightarrow & \mathcal{P}' \\ \downarrow & & \downarrow \\ D & \longrightarrow & \mathcal{B}\mathbb{G}_m \times S \\ & \searrow & \downarrow \\ & & S \end{array}$$

where the right column is a non-rigid twisted expansion of $(\mathcal{B}\mathbb{G}_m, \mathcal{L})$ and in the middle arrow the factor $D \rightarrow \mathcal{B}\mathbb{G}_m$ comes from the line bundle N .

Again $\mathcal{T}_\sim^{\text{tw}}$ is identified as the stack of non-rigid twisted expansions of (D, N) , by the universal property of fibered products.

Graber and Vakil identified the stack of flat families of non-rigid expansions of $(\text{Spec } k, \mathcal{O})$ as $\mathfrak{M}_{0,2}^{\text{ss}}$. As in the proof of Lemma 3.1.3 we have

Proposition 4.2. *There is a canonical isomorphism $\mathcal{T}_\sim^{\text{tw}} \simeq \mathfrak{M}_{0,2}^{\text{tw,ss}}$. In particular it is an algebraic stack, locally of finite type over \mathbb{Z} , and $\mathcal{T}_\sim \simeq \mathfrak{M}_{0,2}^{\text{ss}}$ is an open substack.*

5. DIAGONALS AND AUTOMORPHISMS: NAÏVE EXPANSIONS

In preparation for the proof of the remainder of Theorem 1.3.2, as well as for the discussion the shortcomings of naïve expansions, we need some information on the diagonals of stacks of naïve expansions and automorphisms of naïve expansions. Since we will not need twisted versions of these we only discuss the untwisted situation.

5.1. The diagonals of the stacks of naïve expansions.

Proposition 5.1.1. (1) *Let (X, D) be a smooth pair with X proper. The diagonal of $\mathcal{T}_{\text{naïve}}^{\text{tw}}(X, D)$ is representable by algebraic spaces and is locally of finite presentation.*

(2) *Let $X \rightarrow B$ be a proper degeneration. The diagonal of $\mathfrak{X}_{\text{naïve}}(X/B, B_0)$ is representable by algebraic spaces and is locally of finite presentation.*

Proof. Let T denote $\mathcal{T}_{\text{naïve}}(X, D)$ in the first case of the proposition, and $\mathfrak{T}_{\text{naïve}}(X/B, B_0)$ in the second case. In the first case, let B be a point. Write E for the universal expansion over T . Let $Y = X \times_B T$, let $T^2 = T \times_B T$, and let $Z = X \times_B T^2$. Note that there is a proper projection $E \rightarrow Y$. Let $E^{(i)}$ be the pullback of E to T^2 via the i -th projection $T^2 \rightarrow T$.

To see that the diagonal of T is representable, we must see that the functor

$$\underline{\text{Isom}}_Z(E^{(1)}, E^{(2)}) : S \mapsto \text{Isom}_{Z_S}(E_S^{(1)}, E_S^{(2)})$$

is representable by an algebraic space. Each $E^{(i)}$ is proper over Z and Z is proper over T^2 , so $E^{(i)}$ is proper, and also flat, over T^2 . Therefore there is a relative algebraic space $\underline{\text{Isom}}_{T^2}(E^{(1)}, E^{(2)})$ whose S -points are isomorphisms between $E_S^{(1)}$ and $E_S^{(2)}$. We can identify

$$\underline{\text{Isom}}_Z(E^{(1)}, E^{(2)}) = \underline{\text{Isom}}_{T^2}(E^{(1)}, E^{(2)}) \times_{\underline{\text{Hom}}_{T^2}(E^{(1)}, Z \times_{T^2} Z)} \underline{\text{Hom}}_{T^2}(E^{(1)}, Z).$$

In the fiber product the map on the right is induced by the diagonal of Z and the map on the left by the projection $E^{(1)} \rightarrow Z$ and the composition $E^{(1)} \rightarrow E^{(2)} \rightarrow Z$. Everything appearing above is an algebraic space over T^2 because $E^{(i)}$ are proper and flat over T^2 (cf. [Art69, Theorem 6.1]).

It follows immediately from the fact that $E^{(1)}$ and $E^{(2)}$ are locally of finite presentation over Z that $\underline{\text{Isom}}_Z(E^{(1)}, E^{(2)})$ is locally of finite presentation over T (see [Gro66, Proposition (8.13.1)]). \square

5.2. Automorphisms of naïve expansions.

Lemma 5.2.1 ([Li01, Corollary 1.4]). *(1) An isomorphism $(Y'_1, D'_1) \rightarrow (Y'_2, D'_2)$ of expansions of (Y, D) over an algebraically closed field is given by an order preserving bijection between the sets of irreducible components and an isomorphism on each component isomorphic to $\mathbb{P}(N_{D/Y} \oplus \mathcal{O}_D)$ preserving the fixed points of the \mathbb{G}_m action and structure map. The same holds for the case of expanded degenerations.*

(2) If each irreducible component of an expansion of (Y, D) , except Y , is identified with $\mathbb{P}(N_{D/Y} \oplus \mathcal{O}_D)$ then automorphisms of that component preserving fixed loci and structure map are in bijection with $\mathbb{G}_m(D)$. The same holds for the case of expanded degenerations.

(3) The automorphism group of an expansion of length ℓ is $\mathbb{G}_m(D)^\ell$.

Proof. Any isomorphism of pairs induces a bijection between the sets of irreducible components. In this case, the order must be preserved because the divisors D_1 and D_2 must be respected, as must the components isomorphic to Y . The automorphism group of $\mathbb{P}(N_{D/Y} \oplus \mathcal{O}_D)$ over

D is the same as the automorphism group of the underlying \mathbb{G}_m -torsor $N_{D/Y} \setminus 0$, which is just $\mathbb{G}_m(D)$. \square

5.3. Comparison of geometric objects. Recall from Definition 3.2.2 the map $\Phi^{(X,D)}$ associated to a pair (X, D) and the map $\Phi^{X/B}$ associated to a degeneration $(X/B, B_0)$.

Proposition 5.3.1. *Assume D is smooth, proper and connected. Then the functors $\Phi^{(Y,D)}$ and $\Phi^{X/B}$ induce equivalences on geometric points.*

Proof. Indeed, on both sides the geometric objects are expansions, which are indexed up to isomorphism by non-negative integers in either case. Furthermore, the automorphism group of an expansion of length n over an algebraically closed field k is $\mathbb{G}_m(k)^n$ (as we saw in Lemma 5.2.1) both in the stack of expansions and the stack of naïve expansions. The map $\mathbb{G}_m(k)^n \rightarrow \mathbb{G}_m(k)^n$ is an isomorphism, so the functor is also fully faithful as required. \square

5.4. The trouble with naïve expansions. Throughout the discussion so far, we have used naïve expansions as a building block of our stacks of expansions. The stacks \mathcal{T} , \mathfrak{T} and \mathcal{T}_{\sim} themselves are defined as stacks of naïve expansions of a universal object. Li's stacks will be studied as substacks of the corresponding stacks of naïve expansions. And the proofs of the main results use them in several steps.

So why not define the stacks of expansions using naïve expansions and be done with it? There are several issues that arise with $\mathcal{T}_{\text{naïve}}(Y, D)$ and $\mathfrak{T}_{\text{naïve}}(X/B, B_0)$. We describe in the case of the stack of pairs, the case of degenerations being similar.

5.4.1. Deforming the normal bundle. First, there are problems with deformation spaces which prevent these stack from being algebraic in general. Assume Y is smooth and projective, D is smooth an irreducible, but assume $H^1(D, \mathcal{O}_D) \neq 0$. Then the normal bundle $N_{D/Y}$ has a non-constant deformation to another line bundle L . It follows that the naïve expansion $Y' = Y \sqcup_D P$ admits a non-constant deformation to $Y'' = Y \sqcup_D \mathbb{P}_D(\mathcal{O} \oplus L)$, and the latter is not an expansion.

Assuming $H^1(D, \mathcal{O}_D) = 0$ would clearly be too restrictive. Jun Li overcame this issue by defining a stack where local models are fixed, excluding such deformations by definition. Alternatively, one can circumvent the issue by endowing naïve expansions with logarithmic structures, as is done by Kim [Kim10]. We would be content if this covered all the situations we care about, but this is not the case.

5.4.2. *Disconnected divisor.* Consider now the case where Y is smooth and projective, D is smooth, $H^1(D, \mathcal{O}_D) = 0$ but D decomposes as $D_1 \sqcup D_2$. Then the expansion $Y' = Y \sqcup_D P_D$ admits a non-constant deformation to $Y'' = Y \sqcup_{D_1} P_{D_1}$, so again the stack of naïve expansions is not algebraic. Li's approach still gives us a good stack of expansions; the most natural logarithmic approach to expansions diverges: it gives the stack $\mathcal{T}_{\text{naïve}}(Y, D_1) \times \mathcal{T}_{\text{naïve}}(Y, D_2)$. Of course, one can work with Li's solution, but things can still get a bit more complicated.

5.4.3. *Non-proper divisor.* Now drop the assumption that Y and D are proper, so D could be affine of positive dimension. Then the automorphism group of a naïve expansion is no longer finite dimensional, and the diagonal of the stack of expansions is not locally of finite presentation. This case is not resolved by either Li's or Kim's approach. The most common situation where this occurs in the literature is when Y is the total space of a vector bundle over a projective variety Y_0 and D is the pullback of a divisor D_0 on Y_0 .

Of course, such cases can be treated directly, for instance by pulling back expansions of (Y_0, D_0) , which amounts to an ad-hoc application of the principle used in this paper. Our approach using the universal pair $(\mathcal{A}, \mathcal{D})$ allows one to expand any pair without restriction and without need for ad-hoc constructions.

6. COMPARISON WITH LI'S APPROACH

6.1. **Expanded degenerations.** Throughout this section we assume that B is a smooth complex curve, $B_0 = \{b_0\}$ a point, X is smooth, X/B is projective and D is smooth and irreducible.

Corollary 6.1.1 (of Lemma 5.2.1; cf. [Li01, Corollary 1.4]). *If X is proper and D is connected then the automorphism group of an expansion of (X, D) of length ℓ is $(\mathbb{C}^*)^\ell$.*

A family of expanded degenerations over a base scheme S is a family of schemes over S that is isomorphic, *étale*-locally in S , to one of a collection of standard models. We recall the definition of the standard models below.

Suppose that $\pi : X \rightarrow B$ is a flat family whose general fiber is smooth and whose fiber over 0 is the union of two smooth schemes

along a smooth divisor. J. Li constructs a collection of standard models

$$(6) \quad \begin{array}{ccc} X[\ell] & \longrightarrow & X \\ \downarrow & & \downarrow \\ B[\ell] & \longrightarrow & B \end{array}$$

for expansions of X_0 .

Definition 6.1.2 ([Li01, Definition 1.9]). Let $\mathfrak{F}_{\text{Li}}(X/B, b_0)$ be the fibered category whose S -points are families $Z \rightarrow S$ such that, étale-locally in S , there exists a map $S \rightarrow \mathbb{A}^{\ell+1}$ and an isomorphism $Z \cong S \times_{\mathbb{A}^{\ell+1}} X[\ell]$. Morphisms between such families are morphisms of schemes that commute with the projections to $S \times X$.

It will follow from the equivalence proved in Theorem 1.3.2 that up to canonical equivalence, the above definition of $\mathfrak{F}_{\text{Li}}(X/B, b_0)$ does not depend on X . We summarize J. Li's construction of the standard models below, taking a slightly different perspective; see [Li01, Section 1.1] for more details.

For Li's construction, it is enough to work in a small neighborhood of $b_0 \in B$. We can therefore assume that there is an étale map $B \rightarrow \mathbb{A}^1$ whose fiber over $0 \in \mathbb{A}^1$ is $b_0 \in B$. We will construct the standard models $X[\ell] \rightarrow \mathbb{A}^1[\ell]$ of $X \rightarrow \mathbb{A}^1$, and then we can define $B[\ell] = \mathbb{A}^1[\ell] \times_{\mathbb{A}^1} B$, through which the map $X[\ell] \rightarrow \mathbb{A}^1[\ell]$ must factor.

Remark 6.1.3. It is possible to avoid having to pass to an étale neighborhood of b_0 by taking the base of the degeneration to be $\mathcal{A} = [\mathbb{A}^1/\mathbb{G}_m]$ instead of B . The Cartier divisor b_0 gives a map $B \rightarrow \mathcal{A}$, and expansions of the base B are obtained by base change from expansions of \mathcal{A} : we have $B[\ell] = B \times_{\mathcal{A}} \mathcal{A}[\ell]$ and $\mathcal{A}[\ell] = \mathcal{A}^\ell \rightarrow \mathcal{A}$ is the multiplication map.

We leave it to the reader to verify that the discussion in this section carries over word for word replacing the symbol \mathbb{A} by \mathcal{A} (and \mathbb{A}^ℓ by \mathcal{A}^ℓ).

We will have $\mathbb{A}^1[\ell] = \mathbb{A}^{\ell+1}$ and the projection $\mathbb{A}^1[\ell] = \mathbb{A}^{\ell+1} \rightarrow \mathbb{A}^1$ sends (t_0, \dots, t_ℓ) to $\prod t_i$. The construction of $X[\ell]$ is inductive. Assume that we have a construction of $\pi_1^X : X[1] \rightarrow \mathbb{A}^1[1] = \mathbb{A}^2$ for any family $X \rightarrow \mathbb{A}^1$ of the type described in Section 2.2.1. We will see that composing π_1 with the first projection gives a new family $p_1 \circ \pi_1 : X[1] \rightarrow \mathbb{A}^1$ as in 2.2.1. Expanding this family using the same construction, we obtain a new family $X[1][1] \rightarrow \mathbb{A}^2$ and a commutative

diagram

$$\begin{array}{ccc}
 X[1][1] & \longrightarrow & X[1] \\
 \downarrow \pi_1^{X[1]} & & \downarrow \pi_1^X \\
 & & \mathbb{A}^1[1] \\
 & & \downarrow p_1 \\
 \mathbb{A}^1[1] & \xrightarrow{m} & \mathbb{A}^1
 \end{array}$$

where $m : \mathbb{A}^1[1] = \mathbb{A}^2 \rightarrow \mathbb{A}^1$ is the multiplication map and $p_1 : \mathbb{A}^1[1] = \mathbb{A}^2 \rightarrow \mathbb{A}^1$ is the first projection. We can identify $(\mathbb{A}^1[1], m) \times_{\mathbb{A}^1} (\mathbb{A}^1[1], p_1)$ with $\mathbb{A}^3 = \mathbb{A}^1[2]$ and defining $X[2] = X[1][1]$, we obtain from the diagram above a map $X[2] \rightarrow \mathbb{A}^1[2]$. Composing

$$X[2] = X[1][1] \rightarrow \mathbb{A}^1[2] = \mathbb{A}^3 \xrightarrow{p_1} \mathbb{A}^1$$

we obtain a new family as in 2.2.1. Inductively, we obtain $X[\ell + 1] = X[\ell][1] \rightarrow \mathbb{A}^1[\ell] = \mathbb{A}^{\ell+1}$, exactly as above.

It remains to explain the construction of $X[1]$ and verify that the composition $X[1] \rightarrow \mathbb{A}^1[1] \xrightarrow{p_1} \mathbb{A}^1$ is a family of the type described in Section 2.2.1. First, let X' be the base change of $X \rightarrow \mathbb{A}^1$ by the multiplication map $m : \mathbb{A}^2 \rightarrow \mathbb{A}^1$ sending $(t_1, t_2) \mapsto t_1 t_2$. The total space of X' is singular along $D \times (0, 0)$. We construct $X[1]$ as a small resolution of X' .

Let Y_1 and Y_2 be the two irreducible components of X_0 . Each is a divisor in X . Let $(\mathcal{O}_X(Y_i), \mathbf{1}_{Y_i})$ be a line bundle and section such that the vanishing locus of $\mathbf{1}_{Y_i}$ is Y_i . These are determined up to unique isomorphism by Y_i .

Let P be the \mathbb{P}^1 -bundle $\mathbb{P}(\mathcal{O}_{X'} \oplus m^* \mathcal{O}_X(Y_2)) \cong \mathbb{P}(m^* \mathcal{O}_X(Y_1) \oplus \mathcal{O}_{X'})$ where $m : X' \rightarrow X$ (abusively) denotes the projection. If (t_1, t_2) denote the coordinates on \mathbb{A}^2 , then $(t_1, \mathbf{1}_{Y_1})$ is a section of P over the complement of $D \times (0, 0)$ in X' . Let $X[1]$ be the closure of the image of this section in P .

Remark 6.1.4. Étale locally X' admits a smooth morphism to the locus $\{y_1 y_2 = t_1 t_2\}$ inside \mathbb{A}^4 and $X[1]$ resolves the indeterminacy of the rational map $(t_1, y_1) : X' \rightarrow \mathbb{P}^1$.

Composition of the projections $X[1] \rightarrow X'$ and $X' \rightarrow \mathbb{A}^2$ gives us a map $\pi_1 : X[1] \rightarrow \mathbb{A}^2$. Away from $0 \in \mathbb{A}^1$, the fiber of $p_1 \pi_1 : X[1] \rightarrow \mathbb{A}^1$ can be identified with the projection $X \times (\mathbb{A}^1 \setminus 0)^2 \rightarrow (\mathbb{A}^1 \setminus 0)$ on the last component. It is therefore smooth away from 0. Over $0 \in \mathbb{A}^1$, the fiber is the union of two smooth schemes along a smooth divisor (one is Y_2 and the other is the blow-up of $Y_1 \times \mathbb{A}^1$ along $D \times 0$, where D

is the divisor along which Y_1 is joined to Y_2 in X_0). We may therefore iterate the construction to obtain the standard models.

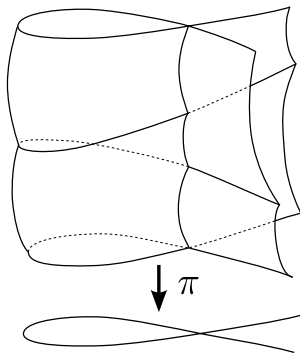
Remark 6.1.5. In [Li01, Definition 1.9], families of expansions are required to be isomorphic to the standard models over an open cover. This open cover must be taken in the analytic topology since, as the following variant of Hironaka's famous example demonstrates, Zariski open covers do not suffice to obtain a stack in the étale topology. This is why we have insisted on gluing in the étale topology above.

Let \tilde{S} be a pair of rational curves joined at two nodes. Label the nodes \tilde{s}_1 and \tilde{s}_2 and the components \tilde{S}_1 and \tilde{S}_2 . This carries a free action of $\mathbb{Z}/2\mathbb{Z}$ which exchanges the components and the nodes. Let S be the quotient, which is a rational curve joined to itself at a single node.

Over \tilde{S} we construct an equivariant family \tilde{Y} of expansions of $X = \mathbb{P}^1 \sqcup_{\text{point}} \mathbb{P}^1$. On the complement of \tilde{s}_1 , we take the family induced from the standard model $X[1]$ by the inclusion of $\tilde{S} - \tilde{s}_1$ as the coordinate axes in \mathbb{A}^2 . Over $\tilde{S} - \tilde{s}_2$, we take the family induced the same inclusion, but with the axes reversed.

Gluing these together, we get a family over \tilde{S} whose fiber over \tilde{s}_1 has 3 components. Passing from \tilde{s}_1 to \tilde{S}_1 smoothes the second node in the fiber over \tilde{s}_1 so that a generic fiber of \tilde{Y} over \tilde{S}_1 has two components. Passing to \tilde{s}_2 degenerates the first of these components to the union of two components, so that the fiber over \tilde{s}_2 has 3 components. Passing now to \tilde{S}_2 smoothes the second node, and passing to \tilde{s}_1 degenerates the first component.

There is a free action of $\mathbb{Z}/2\mathbb{Z}$ on \tilde{Y} making the projection to \tilde{S} equivariant. Let $\pi : Y \rightarrow S$ be the quotient:



This family is not induced in *any* Zariski neighborhood S° of the node from any $X[n]$ via a map $S^\circ \rightarrow \mathbb{A}^\ell$. Indeed, if it were then Y

would be quasi-projective. But if m_1, m_2, m_3 are the degrees of the restriction of an ample line bundle to the fiber over the node, we get

$$\begin{aligned} m_1 + m_2 &= m_1 \\ m_2 + m_3 &= m_3, \end{aligned}$$

so any map to a projective space must collapse the middle component. Thus this family does not appear in the *Zariski* stack.

This point was brought up and clarified in discussions with Michael Thaddeus.

6.2. Expanded pairs. J. Li constructs the moduli space of expanded pairs by reduction to the construction for expanded degenerations: the first expanded pair is a degeneration of the kind studied in the last section.

If (X, D) is a smooth pair, the first expansion $X[1]$ is obtained by blowing up $X \times \mathbb{A}^1$ at the locus $(D, 0)$. This gives a family over \mathbb{A}^1 whose fiber over 0 is the union of X and $\mathbb{P}(N_{D/X} \oplus \mathcal{O}_D)$ along D . The proper transform of $D \times \mathbb{A}^1$ is a divisor in $X[1]$.

Expanding $X[1] \rightarrow \mathbb{A}^1$ as in the last section, we get the standard models $X[\ell] \rightarrow \mathbb{A}^\ell$. Note however that there is a shift of index as compared to the standard models for degenerations. We use the notation $\mathcal{T}_{\text{Li}}(X, D)$ for the stack defined this way.

6.3. Algebraicity and comparison. We note the canonical embeddings which allow us to compare our stacks with Li's stacks.

Definition 6.3.1. Let $\mathcal{T}_{\text{Li}}(X, D) \hookrightarrow \mathcal{T}_{\text{naïve}}(X, D)$ and $\mathcal{T}_{\text{Li}}(X/B, b_0) \hookrightarrow \mathcal{T}_{\text{naïve}}(X/B, b_0)$ be the embeddings given by viewing Li's objects as naïve expansions and arrows as morphisms of naïve expansions.

Proposition 6.3.2. *The stacks $\mathcal{T}_{\text{Li}}(X, D)$ and $\mathfrak{T}_{\text{Li}}(X/B, B_0)$ are algebraic.*

Proof. To treat degenerations and pairs together, we use a common notation as in the proof of Proposition 5.1.1: X is either the total space of a degeneration or a pair, X' is an expansion, and T_{Li} and $T_{\text{naïve}}$ are the relevant stacks of expansions.

The diagonal of T_{Li} is pulled back from the diagonal of $T_{\text{naïve}}$ via the embedding $T_{\text{Li}} \hookrightarrow T_{\text{naïve}}$. The diagonal of $T_{\text{naïve}}$ is representable by Proposition 5.1.1.

On the other hand, the map $\mathbb{A}^n \rightarrow T_{\text{Li}}$ induced from the standard models is formally smooth by definition [Li01, Definition 1.9]: any object of $T_{\text{Li}}(S)$ is required to lift, étale locally in S , to an object of \mathbb{A}^n . Applying this to an infinitesimal extension of S implies the

formal criterion for smoothness. Furthermore, $\mathbb{A}^n \rightarrow T_{\text{Li}}$ is locally of finite presentation, because \mathbb{A}^n is locally of finite presentation and the diagonal of T_{Li} is locally of finite presentation. Since $\mathbb{A}^n \rightarrow T_{\text{Li}}$ is also representable, this implies that it is smooth and since the maps $\mathbb{A}^n \rightarrow T_{\text{Li}}$ cover T_{Li} , this implies that T_{Li} is an algebraic stack. \square

The following proposition completes the proof of Theorem 1.3.2.

Proposition 6.3.3. *Assume D is smooth, projective, and connected. Then the map $\Phi^{(Y,D)}$ of Definition 3.2.2 factors through an equivalence $\mathcal{T} \rightarrow \mathcal{T}_{\text{Li}}(X, D)$ (for a pair) and the map $\Phi^{X/B}$ factors through $\mathfrak{T} \times_{\mathcal{A}} B \rightarrow \mathfrak{T}_{\text{Li}}(X/B, b_0)$ (for a degeneration).*

Proof. First consider the case of pairs. Note that we have commutative diagrams

$$\begin{array}{ccc} \mathbb{A}^n & & \\ \downarrow & \searrow & \\ \mathcal{T} & \xrightarrow{\Phi} & \mathcal{T}_{\text{naïve}}(X, D) \end{array}$$

coming from Li's standard models. To see that the factorization exists, it will be sufficient to show that the maps $\mathbb{A}^n \rightarrow \mathcal{T}$ form a smooth cover. Since $\mathcal{T} \simeq \mathcal{T}_{\text{GV}}$, it is enough to show that $\mathbb{A}^n \rightarrow \mathcal{T}_{\text{GV}}$ is a versal deformation space for the object Q over $0 \in \mathbb{A}^n$. Recall that Q is a chain of $n + 1$ rational curves with two marked points on the first component and one marked point on the last component. The n directions in \mathbb{A}^n correspond to deforming the n nodes of the chain, and this is well known to be a versal deformation space.

Note that the automorphism group of Q is \mathbb{G}_m^n , acting on \mathbb{A}^n by scaling the coordinates. Therefore we obtain an étale morphism $\mathcal{A}^n = [\mathbb{A}^n/\mathbb{G}_m^n] \rightarrow \mathcal{T}$.

The composition $\mathbb{A}^n \rightarrow \mathcal{T} \rightarrow \mathcal{T}_{\text{Li}}(X, D)$ is also smooth, as we saw in the proof of Proposition 6.3.2. By Lemma 5.2.1, the automorphism group of the image of $0 \in \mathbb{A}^n$ in $\mathcal{T}_{\text{Li}}(X, D)$ is \mathbb{G}_m^n , so there is an induced étale map $\mathcal{A}^n = [\mathbb{A}^n/\mathbb{G}_m^n] \rightarrow \mathcal{T}_{\text{Li}}(X, D)$. Therefore \mathcal{A}^n is étale over both \mathcal{T} and $\mathcal{T}_{\text{Li}}(X, D)$. This implies that the map $\mathcal{T} \rightarrow \mathcal{T}_{\text{Li}}(X, D)$ is étale.

Finally, we note that $\mathcal{T} \rightarrow \mathcal{T}_{\text{Li}}(X, D)$ induces an equivalence on geometric points by Proposition 5.3.1.

Next we consider degenerations. It is possible to deduce that $\Phi^{X/B}$ is an isomorphism formally from the isomorphism between the moduli spaces of expanded pairs and degenerations (Proposition 3.3.1 and Section 6.2), so we will only sketch the geometric argument parallel to the one given above for pairs.

We show that $\Phi^{X/B}$ induces an equivalence $\mathfrak{T} \times_{\mathcal{A}} B \rightarrow \mathfrak{T}_{\text{Li}}(X/B, b_0)$ is an equivalence in the case $B = \mathbb{A}^1$, with the general case following by gluing and base change. To show that $\Phi^{X/B}$ factors through $\mathfrak{T}_{\text{Li}}(X/B, b_0)$, we use the isomorphism $\mathfrak{T} \times_{\mathcal{A}} \mathbb{A}^1 \simeq \mathfrak{M}_{0,0}^{\text{ss}}(X/\mathbb{A}^1, 1)$. We obtain this time that $\mathbb{A}^{n+1} \rightarrow \mathfrak{T} \times_{\mathcal{A}} \mathbb{A}^1$ is versal as before. The automorphism group is \mathbb{G}_m^n , giving an étale map $[\mathbb{A}^{n+1}/\mathbb{G}_m^n] \rightarrow \mathfrak{T}_{\text{Li}}(X/B, b_0)$, and $\mathfrak{T} \times_{\mathcal{A}} B \rightarrow \mathfrak{T}_{\text{Li}}(X/B, b_0)$ is therefore étale. Again the map is an equivalence on geometric objects, completing the argument in this case as well. \square

This completes the proof of our Comparison Theorem [1.3.2](#).

7. LABELS AND TWISTS

7.1. Labelling by a set. It is sometimes desirable—in [\[AF11\]](#), for example—to work with variants of the stack \mathcal{T} or \mathfrak{T} which include labels for the nodes of the fibers.

Let F be the non-smooth locus of the projection from the universal expanded degeneration to \mathfrak{T} . Let \mathcal{S} be a set, which we will call the set of labels. Let $\mathfrak{T}^{\mathcal{S}}$ be the sheaf on \mathfrak{T} whose sections over S are the locally constant functions from F_S to \mathcal{S} . This is an étale sheaf on \mathfrak{T} , so its total space, denoted $\mathfrak{T}^{\mathcal{S}}$, is a stack equipped with a formally étale morphism to \mathfrak{T} .

By definition, objects of the stack $\mathfrak{T}^{\mathcal{S}}$ are expansions of the degeneration $\mathcal{A}^2 \rightarrow \mathcal{A}$ together with a continuous labelling of the non-smooth locus in the set \mathcal{S} , and arrows are arrows of expansions preserving the labelling. The universal expansion $\mathcal{X}^{\mathcal{S}}$ is the pullback, via the projection $\mathfrak{T}^{\mathcal{S}} \rightarrow \mathfrak{T}$ of the universal expansion \mathcal{X} of \mathfrak{T} .

We will also make use of $\mathcal{T}^{\mathcal{S}}$, the stack of expanded pairs with a labelling in \mathcal{S} . In this case, let F be the union of the non-smooth locus of the universal expansion of $(\mathcal{A}, \mathcal{D})$ and its distinguished divisor. Let $\mathcal{T}^{\mathcal{S}}$ be the étale sheaf on \mathcal{T} whose sections over S are the locally constant maps from F_S to \mathcal{S} . Since $F \cong F_0 \sqcup \mathcal{D}$, where F_0 is the non-smooth locus of the universal expansion of $(\mathcal{A}, \mathcal{D})$, and since $\mathcal{T} \cong \mathfrak{T}$ by [Proposition 3.3.1](#), we can identify

$$\mathcal{T}^{\mathcal{S}} \cong \mathfrak{T}^{\mathcal{S}} \times \mathcal{S}.$$

The second factor records the labelling of the distinguished divisor. We denote the universal family over $\mathcal{T}^{\mathcal{S}}$ by $\mathcal{X}^{\mathcal{S}}$.

Lemma 7.1.1. *Let T be either \mathfrak{T} or \mathcal{T} and let $T^{\mathcal{S}}$ be $\mathfrak{T}^{\mathcal{S}}$ or $\mathcal{T}^{\mathcal{S}}$, respectively. The projection $T^{\mathcal{S}} \rightarrow T$ is representable by algebraic spaces.*

Proof. The morphism is formally étale by the discussion above; it is locally of finite presentation over T because both F and $\mathcal{S} \times T$ are [Gro66, Proposition 8.13.1].

We don't yet know that the projection is representable, so we have to check that formal sections algebraize uniquely. Suppose that S is the spectrum of a complete noetherian local ring and S_0 is its closed point. Then the connected components of F_{S_0} are the same as the connected components of F_S , so locally constant functions on F_{S_0} algebraize uniquely to locally constant functions on F_S . \square

Therefore $\mathcal{T}^{\mathcal{S}}$ and $\mathcal{F}^{\mathcal{S}}$ are algebraic:

Proposition 7.1.2. *There are algebraic stacks $\mathcal{F}^{\mathcal{S}}$ and $\mathcal{T}^{\mathcal{S}}$, which respectively parameterize expanded degenerations with nodes labeled by \mathcal{S} and expanded pairs with nodes and distinguished divisor labeled by \mathcal{S} .*

7.2. Twisting choice. In this section we work over a field of characteristic 0. We will now relate the stack \mathcal{T}^{tw} to Olsson's log-twisting construction. It is convenient to use the formalism of Borne and Vistoli [BV10] for logarithmic structures.

The stack \mathcal{T} admits a canonical locally free logarithmic structure $(\mathcal{T}, M_{\mathcal{T}})$ coming from the normal crossings divisor over which the universal expansion fails to be smooth; similarly, there is a locally free logarithmic structure $M_{\mathcal{F}}$ on \mathcal{F} . Pulling back, these give locally free logarithmic structures on $\mathcal{T}^{\mathcal{S}}$ and $\mathcal{F}^{\mathcal{S}}$ for any set \mathcal{S} .

Following [BV10], we think about a logarithmic structure as a homomorphism from an étale sheaf of monoids \overline{M} into \mathcal{A} . Put $T = \mathcal{T}$ or $T = \mathcal{F}$ as the case warrants and write $T^{\mathcal{S}}$ for its labeled analogue. Define F as in Section 7.1. We note that, étale locally in a T -scheme S , the connected components of F are in bijection with the generators of \overline{M}_T . Therefore we can use a map $r : F \rightarrow \mathbb{Z}_{>0}$ to construct a new sheaf of monoids $\overline{M}(r)$ that adds an $r(z)$ -th root to the generator corresponding to a connected component z of F whenever z corresponds to a singularity.

Now, consider a map $\mathfrak{r} : \mathcal{S} \rightarrow \mathbb{Z}_{>0}$; we will call it a *twisting choice* because of the way we will use it below. Composing with the tautological map $F \rightarrow \mathcal{S}$ on $T^{\mathcal{S}}$, we obtain a map $F \rightarrow \mathbb{Z}_{>0}$. We may then construct a sheaf of monoids $\overline{M}(\mathfrak{r})$ on $T^{\mathcal{S}}$, as above.

Definition 7.2.1. Fix a twisting choice $\mathcal{S} \rightarrow \mathbb{Z}_{>0}$. Denote by $\mathcal{T}^{\mathfrak{r}}$ the stack of objects of $\mathcal{T}^{\mathcal{S}}$ together with extensions of the map $\overline{M}_{\mathcal{T}} \rightarrow \mathcal{A}$ to a map $\overline{M}_{\mathcal{T}}(\mathfrak{r}) \rightarrow \mathcal{A}$. Similarly, denote by $\mathcal{F}^{\mathfrak{r}}$ the stack of objects of $\mathcal{F}^{\mathcal{S}}$ together with extensions of the morphism $\overline{M}_{\mathcal{F}} \rightarrow \mathcal{A}$ to a map $\overline{M}_{\mathcal{F}}(\mathfrak{r}) \rightarrow \mathcal{A}$.

Proposition 7.2.2. *The maps $\mathcal{T}^{\mathfrak{r}} \rightarrow \mathcal{T}^{\mathfrak{s}}$ and $\mathfrak{T}^{\mathfrak{r}} \rightarrow \mathfrak{T}^{\mathfrak{s}}$ are of Deligne–Mumford type and are isomorphisms on dense open substacks of source and target.*

Proof. Since $\overline{M}_{\mathcal{T}}$ is trivial on the dense open point of \mathcal{T} the map $\mathcal{T}^{\mathfrak{r}} \rightarrow \mathcal{T}^{\mathfrak{s}}$ is an isomorphism there. To see that the map is of Deligne–Mumford type, we can work étale locally on an S -point of $\mathcal{T}^{\mathfrak{s}}$ and assume that the logarithmic structure is free, at which point $\mathcal{T}^{\mathfrak{r}} \times_{\mathcal{T}^{\mathfrak{s}}} S$ becomes a root stack, which is Deligne–Mumford because we work in characteristic 0. The proof for $\mathfrak{T}^{\mathfrak{r}}$ is identical. \square

The pre-image of the divisor that was used to define the logarithmic structure on $\mathfrak{T}^{\mathfrak{s}}$ is a normal crossings divisor in the universal expansion, which we will denote $\mathcal{X}^{\mathfrak{s}}$. It therefore induces a locally free logarithmic structure on $\mathcal{X}^{\mathfrak{s}}$. Étale locally in \mathfrak{T} (i.e., étale locally in S for each S -point of \mathfrak{T}) each generator of $\overline{M}_{\mathfrak{T}}$ maps, away from F , to a generator of $\overline{M}_{\mathcal{X}}$, and to a sum of two generators near F . Define $\overline{M}_{\mathcal{X}}(\mathfrak{r})$ to be the sheaf of monoids obtained by adjoining an $\mathfrak{r}(z)$ -th root to each of these generators.

For the universal family $\mathcal{X}^{\mathfrak{s}}$ over $\mathcal{T}^{\mathfrak{s}}$ we apply the same procedure with the logarithmic structure associated to the normal crossings divisor associated to both singular locus and marking.

Definition 7.2.3. Denote by $\mathcal{X}^{\mathfrak{r}}$ over \mathcal{X} the moduli stack of extensions of the map $\overline{M}_{\mathcal{X}} \rightarrow \mathcal{A}$ to $\overline{M}_{\mathcal{X}}(\mathfrak{r}) \rightarrow \mathcal{A}$; and similarly for $\mathcal{X}^{\mathfrak{r}}$ over \mathcal{X} . We call these the stacks of \mathfrak{r} -twisted expansions.

Lemma 7.2.4. (1) *The fibers of $\mathcal{X}^{\mathfrak{r}}$ over $\mathfrak{T}^{\mathfrak{r}}$ are twisted expansions of \mathcal{A}^2 over \mathcal{A} . The fibers of $\mathcal{X}^{\mathfrak{r}}$ over $\mathcal{T}^{\mathfrak{r}}$ are twisted expansions of \mathcal{A} .*

(2) *The resulting morphism $\mathfrak{T}^{\mathfrak{r}} \rightarrow \mathfrak{T}^{\text{tw}} \times_{\mathfrak{T}} \mathfrak{T}^{\mathfrak{s}}$ or $\mathcal{T}^{\mathfrak{r}} \rightarrow \mathcal{T}^{\text{tw}} \times_{\mathcal{T}} \mathcal{T}^{\mathfrak{s}}$ is an isomorphism onto the open substack consisting of pairs $(X^{\text{tw}'}, \phi : F \rightarrow \mathcal{S})$, with $X^{\text{tw}'}$ a twisted expansion and $\phi : F \rightarrow \mathcal{S}$ a labelling as before, such that the order of the stabilizer of a point z lying over $\bar{z} \in F$ is $\mathfrak{r}(\phi(\bar{z}))$.*

(3) *When $\mathcal{S} = \mathbb{Z}_{>0}$ and $\mathfrak{r} : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}_{>0}$ is the identity, we have $\mathfrak{T}^{\mathfrak{r}} = \mathfrak{T}^{\text{tw}}$ and $\mathcal{T}^{\mathfrak{r}} = \mathcal{T}^{\text{tw}}$.*

Proof. We can check the statements after pulling back $\mathcal{X}^{\mathfrak{r}} \rightarrow \mathcal{A}$ via the map $\mathbb{A}^1 \rightarrow \mathcal{A}$, and similarly pulling back $\mathcal{X}^{\mathfrak{r}} \rightarrow \mathcal{A}^2$ via the map $X \rightarrow \mathcal{A}^2$. Also the third statement follows from the second.

According to [Ols07], [AOV11b] there is an equivalence of categories between log twisted curves and twisted curves. Thus the stack obtained from a nodal curve C/S , with canonical log structure $\overline{M}_C \rightarrow \mathcal{A}$ over $\overline{M}_S \rightarrow \mathcal{A}$ by imposing a simple extension of the log structure $\overline{M}_C \subset$

$\overline{M}'_C \rightarrow \mathcal{A}$ compatible with $\overline{M}_S \subset \overline{M}'_S \rightarrow \mathcal{A}$, is a twisted curve, and every twisted curve is canonically obtained in this manner. The order of the stabilizer of a point z corresponding to the i -th component of the locally free log structure is precisely the order of the i -th component of the quotient $\overline{M}'_C/\overline{M}_C$. In our situation the simple extension is $\overline{M} \subset \overline{M}(\mathfrak{r}) \rightarrow \mathcal{A}$. \square

7.3. Partial untwisting. Let $\mathfrak{r}, \mathfrak{r}' : \mathcal{S} \rightarrow \mathbf{Z}_{>0}$ be two twisting choices. We write that $\mathfrak{r} \prec \mathfrak{r}'$, and say that \mathfrak{r} *divides* \mathfrak{r}' , if $\mathfrak{r}(s)$ divides $\mathfrak{r}'(s)$ for every $s \in \mathcal{S}$. We saw in Definitions 7.2.1 and 7.2.1 that $\mathcal{T}^\mathfrak{r}$ can be constructed as a moduli space of enlargements of the log. structure on $\mathcal{T}^\mathcal{S}$ to one with characteristic monoid $M_{\mathcal{T}}(\mathfrak{r})$ and that its universal object $\mathcal{X}^\mathfrak{r}$ is the moduli space of extensions of $M_{\mathcal{X}}(\mathfrak{r})$ admits a similar description. If $\mathfrak{r} \prec \mathfrak{r}'$ then there are canonical inclusions $M_{\mathcal{T}}(\mathfrak{r}) \subset M_{\mathcal{T}}(\mathfrak{r}')$ and $M_{\mathcal{X}}(\mathfrak{r}) \subset M_{\mathcal{X}}(\mathfrak{r}')$ inducing a commutative diagram

$$\begin{array}{ccc} \mathcal{X}^{\mathfrak{r}'} & \longrightarrow & \mathcal{X}^{\mathfrak{r}} \\ \downarrow & & \downarrow \\ \mathcal{T}^{\mathfrak{r}'} & \longrightarrow & \mathcal{T}^{\mathfrak{r}}. \end{array}$$

Moreover, these maps are compatible with the projections to $\mathcal{X}^\mathcal{S}$ over $\mathcal{T}^\mathcal{S}$, so we obtain

Proposition 7.3.1. *If $\mathfrak{r}, \mathfrak{r}' : \mathcal{S} \rightarrow \mathbf{Z}_{>0}$ are twisting choices such that $\mathfrak{r} \prec \mathfrak{r}'$ then there is a map $\mathcal{T}^{\mathfrak{r}'} \rightarrow \mathcal{T}^{\mathfrak{r}}$, compatible with the projections to $\mathcal{T}^\mathcal{S}$, that is an isomorphism on dense open substacks of source and target. If $\mathfrak{r} \prec \mathfrak{r}' \prec \mathfrak{r}''$, the resulting triangle is commutative in a canonical way.*

7.4. Split expansions. It is particularly important in degeneration formulas to understand the fiber of $\mathfrak{Z} \rightarrow \mathcal{A}$ over the origin $0 \in \mathbb{A}^1 \rightarrow \mathcal{A}$, and the corresponding families of expanded degenerations. Let $\mathcal{T}_0 = 0 \times_{\mathcal{A}} \mathfrak{Z}$ be this locus, and let $\mathcal{T}_0^{\text{tw}} = 0 \times_{\mathcal{A}} \mathfrak{Z}^{\text{tw}}$ where $\mathfrak{Z}^{\text{tw}} \rightarrow \mathcal{A}$ is the composition of the untwisting map $\mathfrak{Z}^{\text{tw}} \rightarrow \mathfrak{Z}$ and the projection $\mathfrak{Z} \rightarrow \mathcal{A}$.

Define $\mathcal{T}_{\text{spl}} = \mathcal{T} \times \mathcal{T}$. If $(\mathcal{Y}_1, \mathcal{E}_1)$ and $(\mathcal{Y}_2, \mathcal{E}_2)$ denote two expanded pairs over S then we can attach \mathcal{Y}_1 to \mathcal{Y}_2 along the isomorphisms $\mathcal{E}_1 \xrightarrow{\sim} \mathcal{D} \xleftarrow{\sim} \mathcal{E}_2$ (cf. [AGV08, Appendix A]). This gives a map $\mathcal{T}_{\text{spl}} \rightarrow \mathcal{T}_0$.

We refer to [Cos06] for the notion of maps of pure degree.

Proposition 7.4.1. *$\mathcal{T}_{\text{spl}} \rightarrow \mathcal{T}_0$ has pure degree 1.*

Proof. The morphism is an isomorphism above the dense open substack of \mathcal{T}_0 where the fibers of the singular locus of the universal expansion have one irreducible component. \square

The proposition above is sufficient to understand splitting in the untwisted case. The twisted case is less straightforward. Instead of a direct map to relate $\mathcal{T}_{\text{spl}}^{\text{tw}}$ to $\mathcal{T}_0^{\text{tw}}$, we have a correspondence

$$\begin{array}{ccc} & \mathcal{T}_{\text{spl}}^{\text{tw}} & \\ & \swarrow \quad \searrow & \\ \mathcal{T}^{\text{tw}} \times_{\mathbf{Z}_{>0}} \mathcal{T}^{\text{tw}} & & \mathcal{T}_{\text{spl}} \times_{\mathcal{T}_0} \mathcal{T}_0^{\text{tw}} \xrightarrow{\pi} \mathcal{T}_0^{\text{tw}}. \end{array}$$

(All of the stacks in the diagram will be defined below.) The map π is obtained from base change from $\mathcal{T}_{\text{spl}} \rightarrow \mathcal{T}_0$ so is of pure degree 1. We will see below that both of the other two maps are of the same pure degree $\frac{1}{\mathbf{r}}$, which permits one, using Costello's theorem [Cos06, Theorem 5.0.1], to translate directly between one stack and the other in questions of virtual enumerative geometry [AF11, Section 5].

Let $\mathcal{T}_{\text{spl}}^{\text{tw}}$ be the stack whose S -points are pairs of twisted expanded pairs $(\mathcal{A}'_1, \mathcal{D}'_1)$ and $(\mathcal{D}'_2, \mathcal{D}'_2)$, together with a band-inverting isomorphism of \mathbb{G}_m -gerbes $\mathcal{D}'_1 \rightarrow \mathcal{D}'_2$ and a commutative diagram

$$\begin{array}{ccc} \mathcal{D}'_1 & \longrightarrow & \mathcal{D}'_2 \\ & \searrow & \swarrow \\ & \mathcal{D} & \end{array}$$

where the maps $\mathcal{D}'_i \rightarrow \mathcal{D}$ are those induced by untwisting. Note that there can be no such isomorphism if the twisting of the distinguished divisors \mathcal{D}'_1 and \mathcal{D}'_2 are different. Provided that the orders of twisting are the same, the desired isomorphism always exists locally in the base. The 2-automorphism group of such an isomorphism is canonically isomorphic to μ_r . This proves the first half of the following proposition.

Proposition 7.4.2. *Let \mathbf{r} be the locally constant function recording the order of twisting at the splitting divisor.*

- (1) *The forgetful map $\mathcal{T}_{\text{spl}}^{\text{tw}} \rightarrow \mathcal{T}^{\text{tw}} \times_{\mathbf{Z}_{>0}} \mathcal{T}^{\text{tw}}$ makes $\mathcal{T}_{\text{spl}}^{\text{tw}}$ into a gerbe banded by μ_r over $\mathcal{T}^{\text{tw}} \times_{\mathbf{Z}_{>0}} \mathcal{T}^{\text{tw}}$.*
- (2) *The map $\mathcal{T}_{\text{spl}}^{\text{tw}} \rightarrow \mathcal{T}_0^{\text{tw}} \times_{\mathcal{T}_0} \mathcal{T}_{\text{spl}}$ has pure degree $\frac{1}{\mathbf{r}}$.*

For the second claim, it is enough to consider a dense open substack of $\mathcal{T}_0^{\text{tw}} \times_{\mathcal{T}_0} \mathcal{T}_{\text{spl}}$. Let $\mathcal{U} \subset \mathcal{T}$ be the open substack of \mathfrak{X} over which the corresponding expansion of $\mathcal{A}^2 \rightarrow \mathcal{A}$ has at most one node in the fibers; define \mathcal{U}^{tw} to be the pullback of \mathcal{U} under the untwisting map, \mathcal{U}_0 its intersection with \mathcal{T}_0 , etc. Note that $\mathcal{U}_{\text{spl}} = \mathcal{U}_0$, so $\mathcal{U}_0^{\text{tw}} \times_{\mathcal{U}_0} \mathcal{U}_{\text{spl}} = \mathcal{U}_0^{\text{tw}}$.

Now, $\mathcal{U} \cong \mathcal{A}$ and \mathcal{U}^{tw} is isomorphic to the union of infinitely many copies of \mathcal{A} , indexed by $\mathbb{Z}_{>0}$, and joined along their open points. If

the open substack labeled by r is identified with \mathcal{A} then the untwisting map $\mathcal{U}^{\text{tw}} \rightarrow \mathcal{U}$ is identified with the r -th power map. Thus

$$\mathcal{U}_0^{\text{tw}} \cong \coprod_{r \in \mathbf{N}} (\mathcal{A}, [r]) \times_{\mathcal{A}} 0 \cong \coprod_{r \in \mathbf{N}} \left[(\text{Spec } \mathbf{C}[t]/t^r) / \mu_r \right]$$

The notation in the second term is meant to indicate the fiber product of $0 \rightarrow \mathcal{A}$ and the r -th power map $[r] : \mathcal{A} \rightarrow \mathcal{A}$. On the other hand,

$$\mathcal{U}_{\text{spl}}^{\text{tw}} \cong \coprod_{r \in \mathbf{N}} B\mu_r \cong \coprod_{r \in \mathbf{N}} \left[(\text{Spec } \mathbf{C}[t]/t) / \mu_r \right]$$

That is, on the r -th component, $\mathcal{U}_0^{\text{tw}}$ is given by the equation $t^r = 0$ and $\mathcal{U}_{\text{spl}}^{\text{tw}}$ is given by $t = 0$. This proves the proposition. \square

8. OTHER INCARNATIONS OF THE STACKS OF EXPANSIONS

The stack \mathcal{T} , its sister \mathfrak{T} and their twisted cousins were described in terms of the stack \mathcal{A} . Since the stack \mathcal{A} itself has modular interpretations, we can in turn interpret the various stacks of expansions in corresponding modular terms.

We have repeatedly used the fact that a map from S to \mathcal{A} corresponds to a line bundle on S with a global section. In Section 8.2 we use this fact to interpret \mathcal{T} et al. as a stacks of *configurations of line bundles*. This perspective was inspired by Borne's and Vistoli's approach to logarithmic structures [BV10].

Any line bundle with section induces a logarithmic structure, and indeed, \mathcal{A} is the moduli space of rank 1 Deligne–Faltings log. structures (which is an open substack of Olsson's stack LOG of all logarithmic structures). Taking this point of view, we show in Section 8.1 that \mathcal{T} can be viewed as the moduli space of *aligned logarithmic structures*.

Finally, the stacks of configurations of line bundles have a readily available étale covering by stacks of the form \mathcal{A}^n . In Section 8.3 we sketch how this point of view gives a concrete construction of \mathcal{T} in terms of a categorical colimit of stacks.

8.1. Aligned logarithmic structures. It is convenient to start with the logarithmic picture.

Definition 8.1.1. An *aligned logarithmic structure* on a scheme S is a locally free logarithmic structure $M \rightarrow \mathcal{O}_S$, together with a subset $\overline{M}_0 \subset \overline{M}$ such that on each fiber the generators of \overline{M} can be labeled $\{e_1, \dots, e_n\}$ so that the corresponding fiber of \overline{M}_0 consists of the elements

$$0, e_1, e_1 + e_2, e_1 + e_2 + e_3, \dots, e_1 + \dots + e_n.$$

Let \mathcal{T}_{\log} be the moduli stack of aligned logarithmic structures.

We similarly define a twisted version: let $\mathcal{T}_{\log}^{\text{tw}}$ be the moduli stack of aligned logarithmic structures along with a simple extension $M \hookrightarrow N$.

We construct an aligned logarithmic structure on \mathcal{T} . Let $\mathcal{X} \rightarrow \mathcal{T}$ be the universal expansion of the pair $(\mathbb{P}^1, 0)$. Then \mathcal{X} is a family of pre-stable rational curves, hence gives rise to a logarithmic structure M on \mathcal{T} . Let \overline{M} be the characteristic monoid of M . The generators in a fiber of \overline{M} over \mathcal{T} are in bijection with the nodes in the corresponding fiber of \mathcal{X} . In particular, they are totally ordered by their positions on the chain of rational curves.

Let \overline{M}_0 be the sheaf on \mathcal{T} whose sections are sheaves of subsets of M satisfying the following two conditions:

- (i) if σ is a section of \overline{M}_0 and $f \in \sigma$ then f restricts to a generator in each fiber of \overline{M} , and
- (ii) if f is an element of a section σ of \overline{M}_0 and e is a section of \overline{M} that restricts to a generator in the fibers with $e < f$ in every fiber, then $e \in \sigma$.

Thus if a fiber of \overline{M} is generated by $e_1 < \dots < e_n$ then the corresponding fiber of \overline{M}_0 is the collection of subsets of \overline{M} ,

$$\emptyset, \{e_1\}, \{e_1, e_2\}, \dots, \{e_1, \dots, e_n\}.$$

There is an embedding $\overline{M}_0 \rightarrow \overline{M}$ sending a subset σ of \overline{M} to $\sum_{f \in \sigma} f$. By looking at the fibers, one can easily verify that this map is injective and $\overline{M}_0 \subset \overline{M}$ is an aligned logarithmic structure.

Proposition 8.1.2. *The map $\mathcal{T} \rightarrow \mathcal{T}_{\log}$ described by the construction above is an isomorphism. Similarly $\mathcal{T}^{\text{tw}} \rightarrow \mathcal{T}_{\log}^{\text{tw}} \times \mathbb{Z}_{>0}$, where the second factor records the twisting along D , is an isomorphism.*

Sketch of proof. It is evident that the functor induces a bijection on isomorphism classes of geometric objects: for \mathcal{T} these are determined by the number of nodes and for \mathcal{T}_{\log} by the cardinality of \overline{M}_0 . Considering automorphisms, the functor in fact gives an equivalence on geometric points. Indeed the automorphism group of an accordion of length n is \mathbb{G}_m^n , and the same holds for a free log structure with a fixed basis of the characteristic monoid. It is also easy to see that the infinitesimal deformation spaces agree so the morphism is étale, giving the result. The twisted case follows from Olsson's theory of log-twisted curves. \square

One can similarly define \mathfrak{T}_{\log} and $\mathfrak{T}_{\log}^{\text{tw}}$ with isomorphisms $\mathfrak{T} \rightarrow \mathfrak{T}_{\log}$ and $\mathfrak{T}^{\text{tw}} \rightarrow \mathfrak{T}_{\log}^{\text{tw}}$.

8.2. Configurations of line bundles. It is possible to remove the sheaf of monoids in the description sketched above and retain only the sheaf \overline{M}_0 . We say that a sheaf E on the étale site of a scheme is a *sheaf of totally ordered finite sets* if E is a sheaf of partially ordered sets and

- (i) E is constructible, and
- (ii) any two sections of E are locally comparable.

There is a stack associated to E whose objects are the sections of E . If x and y are two sections of E , there is a unique morphism from x to y if and only if $x \geq y$.

Note that the sheaves \overline{M}_0 that appeared in the last section are sheaves of totally ordered finite sets.

Let \mathcal{L} be the category of line bundles; its fibers $\mathcal{L}(S)$ are denoted $\mathfrak{Pic} S$ in [BV10]. An S -point of \mathcal{L} is a line bundle on S , and arrows are arbitrary arrows of line bundles, which may vanish. Note that \mathcal{L} differs from $\mathcal{B}\mathbb{G}_m$ and is not a stack in groupoids: a morphism between line bundles is induced from a morphism of their underlying \mathbb{G}_m -torsors if and only if it is an isomorphism.

Definition 8.2.1. Let \mathcal{T}_{to} be the stack whose objects are pairs (E, L) where E is a sheaf of totally ordered finite sets, and $L : E \rightarrow \mathcal{L}$ is a *diagram of line bundles* indexed by E , meaning a morphism of stacks such that

- (i) if $x \geq y$ are sections of E and $L(x) \rightarrow L(y)$ is an isomorphism then $x = y$, and
- (ii) if o is the unique section of E that is minimal in all of the fibers then $L(o) = \mathcal{O}$.

Morphisms in \mathcal{T}_{to} are required to be the identity on $L(o)$ when o is the minimal section of E . We refer to \mathcal{T}_{to} as the stack of *totally ordered configurations of line bundles*.

Somewhat heuristically, we can describe an object of \mathcal{T}_{to} as a diagram of line bundles $\mathcal{L}_n \rightarrow \mathcal{L}_{n-1} \rightarrow \cdots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{O}$, with the agreement that, when $\mathcal{L}_i \rightarrow \mathcal{L}_{i-1}$ is an isomorphism, the diagram is equivalent to the one with $\mathcal{L}'_j = \mathcal{L}_j$ if $j < i$ and $\mathcal{L}'_j = \mathcal{L}_{j+1}$ if $j \geq i$. The map $\mathcal{L}'_i \rightarrow \mathcal{L}'_{i-1}$ is the composition $\mathcal{L}_{i+1} \rightarrow \mathcal{L}_i \rightarrow \mathcal{L}_{i-1}$.

If M is a logarithmic structure and \overline{M}_0 is an alignment of M then for each section x of \overline{M} , the fiber of M over x is a \mathbb{G}_m -torsor $P(x)$, equipped with a \mathbb{G}_m -equivariant morphism to $\mathcal{O} = P(0)$. For each $x \in M$, let $L(x)$ be the line bundle associated to the \mathbb{G}_m -torsor $P(x)$. If $x \geq y$ then there is some $z \in \overline{M}$ such that $x = y + z$. We get a map $L(z) \rightarrow \mathcal{O}$ and this induces a map

$$L(x) \simeq L(y) \otimes L(z) \rightarrow L(y) \otimes \mathcal{O} \simeq L(y).$$

These maps fit together to give a morphism of stacks $L : \overline{M}_0 \rightarrow \mathcal{L}$. This construction defines a morphism $\mathcal{T}_{\log} \rightarrow \mathcal{T}_{\text{to}}$.

Proposition 8.2.2. *The map $\mathcal{T}_{\log} \rightarrow \mathcal{T}_{\text{to}}$ described above is an equivalence.*

This is an immediate application of Borne’s and Vstoli’s interpretation of logarithmic structures in terms of line bundles, given in [BV10].

The universal expanded degeneration and expanded pair over \mathcal{T}_{to} also admit explicit descriptions in terms of configurations of line bundles. The universal expanded degeneration is the moduli space \mathcal{X} of triples (E, L, L') where E is a sheaf of totally ordered finite sets and L and L' are diagrams of line bundles indexed by E satisfying the following condition:

- (*) if $x < y < z$ are sections of E then the maps $L(y) \rightarrow L(x)$ and $L'(z) \rightarrow L'(y)$ do not vanish simultaneously.

The map $\mathcal{X} \rightarrow \mathcal{T}_{\text{to}}$ sends (E, L, L') to $(E, L \otimes L')$ where $(L \otimes L')(x) = L(x) \otimes L'(x)$ for any section x of E . The map $\mathcal{X} \rightarrow \mathcal{A}^2$ in the commutative diagram (4) sends (E, L, L') to the pair $(L(e) \rightarrow L(o), L'(e) \rightarrow L'(o))$ (note that $L(o) = L'(o) = \mathcal{O}$) where e is the final section of E and o is the initial section.

The universal expanded pair is defined in a very similar way. First, we introduce the following notation: if E is a sheaf of totally ordered finite sets, let E' be the sheaf of totally ordered finite sets obtained by adjoining a maximal object to E . Define \mathcal{Y} to be the category of triples (E, L, L') where L is a diagram of line bundles indexed by E and L' is a diagram of line bundles indexed by E' . We impose the same condition (*) on (E, L, L') , with the modification that z is allowed to be a section of E' . The map $\mathcal{Y} \rightarrow \mathcal{T}_{\text{to}}$ sends (E, L, L') to $(E, L \otimes L'|_E)$; the contraction $\mathcal{Y} \rightarrow \mathcal{A}$ sends (E, L, L') to $L'(e') \rightarrow L'(o) = \mathcal{O}$ where e' is the maximal section of E' and o is the initial section.

Unfortunately, we do not know of an efficient way of describing twisted expansions using configurations of line bundles.

8.3. Colimits. On \mathcal{A}^n there is a natural sequence of homomorphisms of line bundles. Let $(L_1, s_1), \dots, (L_n, s_n)$ be the “coordinates” on \mathcal{A}^n , viewing \mathcal{A} as the moduli space of line bundles with sections. Then we have a sequence of homomorphisms,

$$L_1^\vee \otimes \cdots \otimes L_n^\vee \xrightarrow{s_n} L_1^\vee \otimes \cdots \otimes L_{n-1}^\vee \xrightarrow{s_{n-1}} \cdots \xrightarrow{s_3} L_1^\vee \otimes L_2^\vee \xrightarrow{s_2} L_1^\vee \xrightarrow{s_1} \mathcal{O}.$$

Let $[n]$ denote the totally ordered set $\{0 \leq 1 \leq \cdots \leq n\}$ and let $[n]_{\mathcal{A}^n}$ be the constant sheaf of totally ordered sets on \mathcal{A}^n associated to $[n]$. The sequence above defines a map $L : [n]_{\mathcal{A}^n} \rightarrow \mathcal{L}$. Define E to be the

quotient of $[n]_{\mathcal{A}^n}$ by the relation $i \sim j$ if $L(i) \rightarrow L(j)$ is an isomorphism. This gives a map $\mathcal{A}^n \rightarrow \mathcal{T}_{\text{to}}$.

If $u : [n] \rightarrow [m]$ is a morphism of totally ordered sets it induces a morphism $\mathcal{A}^u : \mathcal{A}^n \rightarrow \mathcal{A}^m$. If (L_i, s_i) are the coordinates on \mathcal{A}^m then $\mathcal{A}^{u^*}(L_i, s_i) = \bigotimes_{u(j)=i} (L_j, s_j)$. If u is an injection then \mathcal{A}^u is an open embedding and it is not hard to check that the diagram

$$\begin{array}{ccc} \mathcal{A}^n & \xrightarrow{\quad} & \mathcal{A}^m \\ & \searrow & \swarrow \\ & \mathcal{T}_{\text{to}} & \end{array}$$

is commutative.

Proposition 8.3.1. *The maps $\mathcal{A}^n \rightarrow \mathcal{T}_{\text{to}}$ are étale and induce an equivalence*

$$\varinjlim_n \mathcal{A}^n \xrightarrow{\sim} \mathcal{T}_{\text{to}}$$

where the colimit is taken in the 2-category of stacks.

Proof. Since the \mathcal{A}^n are all representable and étale over \mathcal{T}_{to} , which is already known to be algebraic, it is enough to see that the map induces an equivalence on geometric points. This is clear. \square

APPENDIX A. BALANCED \mathbb{G}_m ACTION ON TWISTED RATIONAL CURVES

Let C be a chain of twisted rational curves corresponding to a geometric point s of $\mathfrak{M}_{0,2}^{\text{tw ss}}$, so each of the end components has a marked point, which itself may be twisted. If C is not twisted, there is a canonical balanced action of \mathbb{G}_m on C by scaling all components of the chain simultaneously. We show below that there is also a balanced action on a chain of twisted curves, which extends uniquely to families. In fact, there are two canonical actions, due to the inversion automorphism of \mathbb{G}_m , so we choose the convention that the first marked point corresponds to 0 and the second marked point to ∞ .

Action of a group on a stack. To define the action of a group on a stack can be more subtle than to define an action on a scheme. We avoid these complications by building the quotient stack into the definition:

Definition A.1. Let G be an algebraic group-scheme, BG its classifying stack, and $E \rightarrow BG$ the universal torsor. An action of G on a stack X is the data of a stack \overline{X} , a morphism $\overline{X} \rightarrow BG$, and an isomorphism $X \simeq \overline{X} \times_{BG} E$.

Balanced action on fibers. We first define the balanced action on fibers.

Proposition A.2. *Let C be a chain of twisted rational curves over a field with a single, possibly twisted, marking on each end of the chain. Then there is a canonical action of \mathbb{G}_m on C which induces the canonical balanced action on the coarse moduli space.*

Proof. We define the action of \mathbb{G}_m on $X = [\mathbb{A}^1/\mu_r]$ and then glue. Let $\overline{X} = \mathcal{A}$ where the map $\overline{X} \rightarrow \mathcal{B}\mathbb{G}_m$ sends a line bundle and section (L, s) to $L^{\otimes r}$. Then $\overline{X} \times_{\mathcal{B}\mathbb{G}_m} E$ is the moduli space of r -th roots of a trivialized line bundle and section—the r -th root stack of \mathbb{A}^1 at the origin—which is precisely X .

We now obtain the action of \mathbb{G}_m on twisted curves by gluing together the actions on root stacks of \mathbb{A}^1 . Note that gluing along the open orbit of \mathbb{A}^1 is trivial, since the quotient is a point. At a node, the quotient of

$$X = [\mathbb{A}^1/\mu_r] \sqcup_{[0/\mu_r]} [\mathbb{A}^1/\mu_r],$$

glued along the automorphism of $B\mu_r$ induced from the inversion map on μ_r , is $\mathcal{X} = \mathcal{A} \sqcup_{\mathcal{B}\mathbb{G}_m} \mathcal{A}$, glued along the *inversion* isomorphism $\mathcal{B}\mathbb{G}_m \simeq \mathcal{B}\mathbb{G}_m$. Indeed, \mathcal{X} may be identified as the moduli space of (L, x, y) where L is a line bundle, x is a section of L , and y is a section of L^\vee such that $xy = 0$. The structure map $\mathcal{X} \rightarrow \mathcal{B}\mathbb{G}_m$ sends (L, x, y) to $L^{\otimes r}$. Taking the fiber product with $E \rightarrow \mathcal{B}\mathbb{G}_m$ gives the stack constructed from the same data, along with a trivialization of $L^{\otimes r}$. This stack is canonically isomorphic to X . \square

Extension to families.

Proposition A.3. *The balanced action defined above extends uniquely to every family of curves in $\mathfrak{M}_{0,2}^{\text{tw,ss}}$.*

Proof. STEP 1: INFINITESIMAL EXTENSION. Suppose that S is the spectrum of an artinian local ring, and S' is a square-zero extension whose ideal is isomorphic to the residue field. Let C' be an S' -point of $\mathfrak{M}_{0,2}^{\text{tw,ss}}$ and let C be its fiber over S . Assume that we have already constructed a \mathbb{G}_m action on C over S whose restriction to the residue field is the balanced action described above, and that this extension is unique. We show that it can also be extended to an action on C' over S' .

Denote the projection by $\pi : C \rightarrow S$ and let $\pi_0 : C_0 \rightarrow S_0$ be the fiber of π over the residue field. The isomorphism classes of extensions

of C to S' form a torsor under

$$\mathrm{Ext}^1(\mathbb{L}_{C/S}, \pi^* \mathcal{O}_{S_0}) = \mathrm{Ext}^1(\mathbb{L}_{C_0/S_0}, \mathcal{O}_{C_0}) = \sum_x T_x C_1 \otimes T_x C_2$$

where the sum is taken over the nodes x of C_0 , and C_1, C_2 are the components of C_0 joined at x .

The isomorphism classes of equivariant extensions form a torsor under

$$\begin{aligned} \mathrm{Ext}^1(\mathbb{L}_{[C/\mathbb{G}_m]/S \times \mathcal{B}\mathbb{G}_m}, \bar{\pi}^* \mathcal{O}_{S_0}) &= \mathrm{Ext}^1(\mathbb{L}_{[C_0/\mathbb{G}_m]/S \times \mathcal{B}\mathbb{G}_m}, \mathcal{O}_{\bar{C}_0}) \\ &= \mathrm{Ext}^1(\mathbb{L}_{C_0/S_0}, \mathcal{O}_{C_0})^{\mathbb{G}_m}, \end{aligned}$$

the group of \mathbb{G}_m -invariants in $\mathrm{Ext}^1(\mathbb{L}_{C_0/S_0}, \mathcal{O}_{C_0})$. But \mathbb{G}_m acts trivially on $T_x C_1 \otimes T_x C_2$ since the action is balanced, so isomorphism classes of deformations and of equivariant deformations form pseudo-torsors under the same group. The same argument applied to Ext^2 shows that equivariant obstructions coincide with obstructions, which vanish, so these pseudo-torsors are torsors. Therefore the map from equivariant deformations to non-equivariant deformations is a bijection on isomorphism classes.

To see that it is actually an equivalence, we must consider the automorphisms of an equivariant deformation, which can be identified with the \mathbb{G}_m -invariants in the group of all automorphisms:

$$\mathrm{Hom}(\Omega_{C_0/S_0}(p_1 + p_2), \mathcal{O}_{C_0})^{\mathbb{G}_m} \subset \mathrm{Hom}(\Omega_{C_0/S_0}(p_1 + p_2), \mathcal{O}_{C_0}).$$

Since a vector field on a chain of rational curves must vanish at the nodes, an element of $\mathrm{Hom}(\Omega_{C_0/S_0}(p_1 + p_2), \mathcal{O}_{C_0})$ corresponds to one vector field on each of the irreducible components of C_0 , vanishing at the two special points of that component. There is a one-dimensional space of vector fields on \mathbb{P}^1 that vanish at 0 and ∞ . These vector fields lift uniquely to any twisting of \mathbb{P}^1 and are equivariant with respect to the \mathbb{G}_m -action fixing 0 and ∞ . Therefore the inclusion displayed above is an isomorphism, and every automorphism of C' is equivariant.

We may deduce by induction that whenever S is an infinitesimal thickening of a point and C is an S -point of $\mathfrak{M}_{0,2}^{\mathrm{tw}, \mathrm{ss}}$ there is a unique action of \mathbb{G}_m on C over S extending the balanced action on the fiber.

STEP 2: ALGEBRAIZATION. Suppose that C is an S -point of $\mathfrak{M}_{0,2}^{\mathrm{tw}, \mathrm{ss}}$ where $S = \mathrm{Spec} R$ and R is a complete noetherian local ring. By the above steps, we have a balanced action of \mathbb{G}_m on $C_n = C \times_S S_n$ where $S_n = \mathrm{Spec} R/\mathfrak{m}^n$ and \mathfrak{m} is the maximal ideal of R . We wish to extend the action to C .

It is equivalent to show that for any pair of maps $T \rightarrow S$ and $t : T \rightarrow \mathbb{G}_m$ the formal automorphism of C_T over T given by t can be algebraized

uniquely. Since we are proving that the algebraization is unique, we can work locally and assume T is affine. Since $\mathfrak{M}_{0,2}^{\text{tw,ss}}$ is locally of finite presentation, we can also assume that T is locally of finite presentation over S . Then T is noetherian and adic, so by [Gro61, Théorème 5.4.1] a formal morphism $C_T \rightarrow C_T$ can be algebraized uniquely. This gives the algebraization.

STEP 3: APPROXIMATION. To construct the action on an arbitrary family $C \rightarrow S$, it is sufficient to construct it étale locally, and since $\mathfrak{M}_{0,2}^{\text{tw,ss}}$ is locally of finite presentation, it is even enough to assume that S is the spectrum of a henselian local ring R . Using local finite presentation once again, we can even assume that R is the henselization at a prime ideal of a ring that is of finite type over \mathbb{Z} .

Let $T \rightarrow S$ and $t : T \rightarrow \mathbb{G}_m$ be any maps, as before. It is enough to show that we can construct the action of t on C_T , and it is enough to do this locally. Using the local finite presentation of $\mathfrak{M}_{0,2}^{\text{tw,ss}}$, it is enough to assume that T is the spectrum of the henselization of an algebra of finite type over R . We can therefore apply Artin's approximation and find that the automorphism defined by t over \hat{T} can be approximated by an automorphism over T .

We can therefore approximate the action of \mathbb{G}_m on C_S arbitrarily well in an étale neighborhood of each geometric point of \mathbb{G}_m . Since \mathbb{G}_m is quasi-compact, we can approximate the action of \mathbb{G}_m on C arbitrarily well on an étale cover of \mathbb{G}_m . Moreover, we saw before that the infinitesimal deformation theory of this action is trivial so the approximation to the action over $C_{\hat{S}}$ just constructed agrees to all orders with the action on $C_{\hat{S}}$. Therefore the approximation is unique and descends to \mathbb{G}_m from the étale cover on which it was constructed above. This completes the proof. \square

Functoriality.

Proposition A.4. *Any morphism of 2-marked twisted rational curves is \mathbb{G}_m -equivariant with respect to the canonical \mathbb{G}_m -action.*

Proof. Suppose $f : C_1 \rightarrow C_2$ is a morphism of 2-marked rational curves over S and λ is a point of \mathbb{G}_m . The locus of $x \in C_1$ where $\lambda.f(x) = f(\lambda.x)$ is closed since C_2 is separated over S . Its complement is therefore open in C_1 and has an open image in S since C_1 is flat over S . Therefore the locus in S where $f \circ \lambda = \lambda \circ f$ is closed. But this locus clearly contains all of the geometric points of S , so it must be all of S . \square

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