Factorization of birational maps for qe schemes in characteristic 0

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Factorization of birational maps: varieties

Theorem (Włodarczyk, ℵ-Karu-Matsuki-Włodarczyk (2002))

Let $\phi : X_1 \to X_2$ be the blowing up of a coherent ideal sheaf I on a variety X_2 over a field of characteristic 0 and let $U \subset X_2$ be the complement of the support of I. Assume X_1, X_2 are regular. Then ϕ can be factored, functorially for smooth surjective morphisms on X_2, into a sequence of blowings up and down of smooth centers disjoint from U:

$$X_1 = V_0 \overset{\varphi_1}{\dashrightarrow} V_1 \overset{\varphi_2}{\dashrightarrow} \ldots \overset{\varphi_{\ell-1}}{\dashrightarrow} V_{\ell-1} \overset{\varphi_\ell}{\dashrightarrow} V_\ell = X_2 \, .$$

Factorization of birational maps: qe schemes

Theorem (ℵ-Temkin)

Let $\phi : X_1 \to X_2$ be the blowing up of a coherent ideal sheaf I on a qe scheme X_2 over a field of characteristic 0 and let $U \subset X_2$ be the complement of the support of I. Assume X_1, X_2 are regular. Then ϕ can be factored, functorially for regular surjective morphisms on X_2, into a sequence of blowings up and down of regular centers disjoint from U:

$$X_1 = V_0 \overset{\varphi_1}{\dashrightarrow} V_1 \overset{\varphi_2}{\dashrightarrow} \dots \overset{\varphi_{\ell-1}}{\dashrightarrow} V_{\ell-1} \overset{\varphi_\ell}{\dashrightarrow} V_\ell = X_2\ .$$

Regular morphisms and qe schemes

Definition

A morphism of schemes $f : Y \to X$ is said to be **regular** if it is (1) flat and (2) all geometric fibers of $f : X \to Y$ are regular.

Definition

A locally noetherian scheme X is a **qe scheme** if:

- for any scheme Y of finite type over X, the regular locus Y_{reg} is open; and
- For any point $x \in X$, the completion morphism $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to \operatorname{Spec} \mathcal{O}_{X,x}$ is a regular morphism.

Qe schemes are the natural world for resolution of singularities.

Places where qe rings appear

- A of finite type over a field or $\mathbb{Z}$, and localizations.
- $A =$ the formal completion of the above.
- $A = O(X)$, where X is an affinoid germ[1] of a complex analytic space.
- $A = O(X)$, where X is an affinoid Berkovich k-analytic space,
- $A = O(X)$, where X is an affinoid rigid space over k.

In all these geometries we deduce factorization in characteristic 0 from factorization over $\operatorname{Spec} A$, which requires GAGA.

[1] intersection with a small closed polydisc

Factorization step 1: birational cobordism

We follow Włodarczyk's original ideas

- Given $X_1 = Bl_I(X_2)$.
- On $B_0 = \mathbb{P}^1 \times X_2$ consider $I' = I + I_0$, where I_0 is the defining ideal of $\{0\} \times X_2$.
- Set $B_1 = Bl_{I'} B_0$. It contains X_1 as the proper transform of $\{0\} \times X_2$, as well as X_2
- Apply canonical resolution of singularities to B_1, resulting in a regular scheme B, projective over X_2, with $\mathbb{G}_m$ action.

So far, this works for schemes.

Factorization step 2: GIT

- Equivariant embedding $B \subset \mathbb{P}_{X_2}(E_{a_1} \oplus \cdots \oplus E_{a_k})$, with $a_i < a_{i+1}$.
- $B /\!\!/_{a_1} \mathbb{G}_m = X_1, \qquad B /\!\!/_{a_k} \mathbb{G}_m = X_2.$
- Denoting

$$
\begin{aligned}
W_{a_i} &= B /\!\!/_{a_i} \mathbb{G}_m, \\
W_{a_i+} &= B /\!\!/_{a_i+\epsilon} \mathbb{G}_m, \\
W_{a_i-} &= B /\!\!/_{a_i-\epsilon} \mathbb{G}_m
\end{aligned}
$$

we have

$$W_{a_i+} = W_{(a_{i+1})-}$$

and a sequence

$$
\begin{array}{ccccccc}
X_1 = W_{a_2-} & & & W_{a_3-} & & & W_{a_l-} = X_2 \\
& \searrow^{\varphi_{2-}} & & {}^{\varphi_{2+}}\swarrow \quad \searrow^{\varphi_{3-}} & & {}^{\varphi_{(l-1)+}}\swarrow & \\
& & W_{a_2} & & \quad \cdots & &
\end{array}
$$

Factorization step 3: local description

- $B^{ss}_{a_i} \to W_{a_i}$ is affine.
- If $b \in B^{ss}_{a_i}$ is a fixed point, can diagonalize $T_{B^{ss}_{a_i},b}$.
- Tangent eigenvectors lift to eigenfunctions.
- Locally on W_{a_i} get equivariant $B^{ss}_{a_i} \to \mathbb{A}^{\dim +1}$.
- This "chart" is *regular* and *inert.*

Factorization step 4: Luna's fundamental lemma

Definition (Special orbits)

An orbit $\mathbb{G}_m \cdot x \subset B^{ss}_{a_i}$ is *special* if it is closed in the fiber of $B^{ss}_{a_i} \to W_{a_i}$.

Definition (Inert morphisms)

The $\mathbb{G}_m$-equivariant $B^{ss}_{a_i} \to \mathbb{A}^{\dim +1}$ is *inert* if (1) it takes special orbits to special orbits and (2) it preserves inertia groups.

Theorem (Luna's fundamental lemma, [Luna,Alper,ℵ-Temkin])

The regular and inert $\mathbb{G}_m$-equivariant $B^{ss}_{a_i} \to \mathbb{A}^{\dim +1}$ is strongly equivariant, *namely*

$$B^{ss}_{a_i} = \mathbb{A}^{\dim +1} \times_{\mathbb{A}^{\dim +1} /\!\!/ \mathbb{G}_m} W_{a_i}.$$

Factorization step 4: torification

- This is compatible with $W_{a_i \pm} \to W_{a_i}$.
- Locally on W_i the transformations $W_{a_i \pm} \to W_{a_i}$ have toric charts.
- The process of *torification* allows us to assume they are *toroidal* transformations.
- Toroidal factorization is known [Włodarczyk, Morelli, ℵ-Matsuki-Rashid].

Factorization in terms of ideals

Recall that $X_1 = Bl_I(X_2)$.
Our factorization

$$X_1 = V_0 \overset{\varphi_1}{\dashrightarrow} V_1 \overset{\varphi_2}{\dashrightarrow} \dots \overset{\varphi_{\ell-1}}{\dashrightarrow} V_{\ell-1} \overset{\varphi_\ell}{\dashrightarrow} V_\ell = X_2$$

gives a sequence of ideals J_i such that $V_k = Bl_{J_i}(X_2)$, and smooth Z_i such that $\varphi_i^{\pm 1}$ is the blowing up of Z_i.

Transporting to other categories 1: covering

- Assume $X_1 \to X_2$ is a blowing up of an ideal in one of our categories: local, formal, complex, Berkovich, rigid.
- Say we have a finite cover $X_2' = \sqcup U_\alpha \to X_2$ by patches such that $O(U_\alpha)$ is a qe ring which determines U_α: ideals correspond to closed sub-objects, coherent sheaves are acyclic, correspond to modules, etc.
- In complex analysis, $U_\alpha = X_2 \cap \overline{D}$ with $\overline{D}$ a closed polydisc with the restricted sheaf [Frisch, Bambozzi].
- Write $\mathcal{X}_2' = \sqcup \operatorname{Spec} O(U_\alpha)$, write $\mathcal{I}'$ corresponding to I, and $\mathcal{X}_1' = Bl_{\mathcal{I}'}\mathcal{X}_2'$, which is regular.
- Factorization of $\mathcal{X}_1' \to \mathcal{X}_2'$ gives ideals $\mathcal{J}_i'$ and smooth $\mathcal{Z}_i \subset \mathcal{V}_k' := Bl_{\mathcal{J}_i'}(\mathcal{X}_2')$.

Transporting to other categories 2: GAGA

Theorem (Serre's Théorème 3)

In each of these categories, the pullback functor

$$h^* : Coh(\mathbb{P}^r_{\mathcal{X}_2'}) \to Coh(\mathbb{P}^r_{X_2'})$$

is an equivalence which induces isomorphisms on cohomology groups and preserves regularity.

Transporting to other categories 2: patching

- $$X_2 \xleftarrow{\text{cover}} X_2' \xrightarrow{h} \mathcal{X}_2'$$
- We had ideals $\mathcal{J}_i' \subset \mathcal{O}_{\mathcal{X}_2'}$ and smooth $\mathcal{Z}_i \subset \mathcal{V}_k' := Bl_{\mathcal{J}_i'}(\mathcal{X}_2')$ factoring $\mathcal{X}_1' \to \mathcal{X}_2'$.
- We get ideals $J_i' \subset \mathcal{O}_{X_2'}$ and smooth $Z_i' \subset V_k' := Bl_{J_i'}(X_2')$ factoring $X_1' \to X_2'$.
- Functoriality and Temkin's trick ensure that these agree on $X_2' \times_{X_2} X_2'$.
- Descent gives ideals $J_i \subset \mathcal{O}_{X_2}$ and smooth $Z_i \subset V_k := Bl_{J_i}(X_2)$ factoring $X_1 \to X_2$, as needed.

An analytic GAGA: statement

Theorem (Serre's Théorème 3)

Let D be a closed polydisk, $A = \mathcal{O}(D)$. The pullback functor

$$h^* : Coh(\mathbb{P}^r_A) \to Coh(\mathbb{P}^r_D)$$

is an equivalence which induces isomorphisms on cohomology groups.

An analytic GAGA: lemmas

Lemma (Dimension Lemma)

We have $H^i(\mathbb{P}^r_A, \mathcal{F}) = H^i(\mathbb{P}^r_D, h^*\mathcal{F}) = 0$ *for* $i > r$ *and all* $\mathcal{F}$*.*

Proof.

Use Çech covers of $\mathbb{P}^r_D$ by closed standard polydisks! ♠

Lemma (Structure Sheaf Lemma)

We have $H^i(\mathbb{P}^r_A, \mathcal{O}) = H^i(\mathbb{P}^r_D, \mathcal{O})$ *for all* i*.*

Proof.

Use proper base change for

$$\begin{array}{ccc} \mathbb{P}^r_D & \longrightarrow & \mathbb{P}^r_{\mathbb{CP}^n} \\ \pi \downarrow & & \downarrow \varpi \\ D & \longrightarrow & \mathbb{CP}^n. \end{array}$$

♠

An analytic GAGA: Serre's proof - cohomology

Lemma (Twisting Sheaf Lemma)

We have $H^i(\mathbb{P}^r_A, \mathcal{O}(n)) = H^i(\mathbb{P}^r_D, \mathcal{O}(n))$ *for all* i, r, n*.*

Proof.

Induction on r and $0 \to \mathcal{O}_{\mathbb{P}^r_D}(n-1) \to \mathcal{O}_{\mathbb{P}^r_D}(n) \to \mathcal{O}_{\mathbb{P}^{r-1}_D}(n) \to 0$

$$\begin{array}{ccccccc} H^{i-1}(\mathbb{P}^{r-1}_A, \mathcal{O}(n)) & \longrightarrow & H^i(\mathbb{P}^r_A, \mathcal{O}(n-1)) & \longrightarrow & H^i(\mathbb{P}^r_A, \mathcal{O}(n)) & \longrightarrow & H^i(\mathbb{P}^{r-1}_A, \mathcal{O}(n)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{i-1}(\mathbb{P}^{r-1}_D, \mathcal{O}(n)) & \longrightarrow & H^i(\mathbb{P}^r_D, \mathcal{O}(n-1)) & \longrightarrow & H^i(\mathbb{P}^r_D, \mathcal{O}(n)) & \longrightarrow & H^i(\mathbb{P}^{r-1}_D, \mathcal{O}(n)). \end{array}$$

So the result for n is equivalent to the result for $n-1$.

By the Structure Sheaf Lemma it holds for $n = 0$ so it holds for all n. ♠

An analytic GAGA: Serre's proof - cohomology

Proposition (Serre's Théorème 1)

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^r_A$. The homomorphism $h^* : H^i(\mathbb{P}^r_A, \mathcal{F}) \to H^i(\mathbb{P}^r_D, h^*\mathcal{F})$ is an isomorphism for all i.

Proof.

Descending induction on i for all coherent $\mathbb{P}^r_A$ modules, the case $i > r$ given by the Dimension Lemma.

Choose a resolution $0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$ with $\mathcal{E}$ a sum of twisting sheaves. Flatness of h implies $0 \to h^*\mathcal{G} \to h^*\mathcal{E} \to h^*\mathcal{F} \to 0$ exact.

$$\begin{array}{ccccccccc}
H^i(\mathbb{P}^r_A, \mathcal{G}) & \longrightarrow & H^i(\mathbb{P}^r_A, \mathcal{E}) & \longrightarrow & H^i(\mathbb{P}^r_A, \mathcal{F}) & \longrightarrow & H^{i+1}(\mathbb{P}^r_A, \mathcal{G}) & \longrightarrow & H^{i+1}(\mathbb{P}^r_A, \mathcal{E}) \\
\downarrow & & \downarrow = & & \downarrow & & \downarrow = & & \downarrow = \\
H^i(\mathbb{P}^r_D, \mathcal{G}) & \longrightarrow & H^i(\mathbb{P}^r_D, \mathcal{E}) & \longrightarrow & H^i(\mathbb{P}^r_D, \mathcal{F}) & \longrightarrow & H^{i+1}(\mathbb{P}^r_D, \mathcal{G}) & \longrightarrow & H^{i+1}(\mathbb{P}^r_D, \mathcal{E})
\end{array}$$

so the arrow $H^i(\mathbb{P}^r_A, \mathcal{F}) \to H^i(\mathbb{P}^r_D, h^*\mathcal{F})$ surjective, and so also for $\mathcal{G}$, and finish by the 5 lemma. ♠

An analytic GAGA: Serre's proof - Homomorphisms

Proposition (Serre's Théorème 2)

For any coherent $\mathbb{P}^r_A$-modules $\mathcal{F}, \mathcal{G}$ the natural homomorphism

$$\underline{\mathrm{Hom}}_{\mathbb{P}^r_A}(\mathcal{F}, \mathcal{G}) \to \underline{\mathrm{Hom}}_{\mathbb{P}^r_D}(h^*\mathcal{F}, h^*\mathcal{G})$$

is an isomorphism. In particular the functor h^* is fully faithful.

Proof.

By Serre's Théorème 1, suffices to show that $h^*\mathcal{H}om_{\mathbb{P}^r_A}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathbb{P}^r_D}(h^*\mathcal{F}, h^*\mathcal{G})$ is an isomorphism.

$$\begin{aligned}\left(h^*\mathcal{H}om_{\mathbb{P}^r_A}(\mathcal{F}, \mathcal{G})\right)_x &= Hom_{\mathcal{O}_{x'}}(\mathcal{F}_{x'}, \mathcal{G}_{x'}) \otimes_{\mathcal{O}_{x'}} \mathcal{O}_x \\ &= Hom_{\mathcal{O}_x}(\mathcal{F}_{x'} \otimes_{\mathcal{O}_{x'}} \mathcal{O}_x, \mathcal{G}_{x'} \otimes_{\mathcal{O}_{x'}} \mathcal{O}_x) \\ &= \mathcal{H}om_{\mathbb{P}^r_X}(h^*\mathcal{F}, h^*\mathcal{G})_x.\end{aligned}$$

by flatness. ♠

An analytic GAGA: Serre's proof - generation of twisted sheaves

Proposition (Cartan's Théorème A)

For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^r_D$ there is n_0 so that $\mathcal{F}(n)$ is globally generated whenever $n > n_0$.

Proof.

Induction on r.
Suffices to generate stalk at x. Choose $H \ni x$, and get an exact sequence $0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0$. This gives $\mathcal{F}(-1) \stackrel{\varphi_1}{\to} \mathcal{F} \stackrel{\varphi_0}{\to} \mathcal{F}_H \to 0$ which breaks into

$$0 \to \mathcal{G} \to \mathcal{F}(-1) \to \mathcal{P} \to 0 \qquad \text{and} \qquad 0 \to \mathcal{P} \to \mathcal{F} \to \mathcal{F}_H \to 0,$$

where $\mathcal{G}$ and $\mathcal{F}_H$ are coherent sheaves on H,

Serre's proof - generation of twisted sheaves (continued)

Proof of Cartan's Théorème A, continued.

$$0 \to \mathcal{G} \to \mathcal{F}(-1) \to \mathcal{P} \to 0 \qquad \text{and} \qquad 0 \to \mathcal{P} \to \mathcal{F} \to \mathcal{F}_H \to 0,$$

so right terms in

$$H^1(\mathbb{P}^r_D, \mathcal{F}(n-1)) \to H^1(\mathbb{P}^r_D, \mathcal{P}(n)) \to H^2(H, \mathcal{G}(n))$$

and

$$H^1(\mathbb{P}^r_D, \mathcal{P}(n)) \to H^1(\mathbb{P}^r_D, \mathcal{F}(n)) \to H^1(H, \mathcal{F}_H(n))$$

vanish for large n. So $h^1(\mathbb{P}^r_D, \mathcal{F}(n))$ is descending, and when it stabilizes $H^1(\mathbb{P}^r_D, \mathcal{P}(n)) \to H^1(\mathbb{P}^r_D, \mathcal{F}(n))$ is bijective so $H^0(\mathbb{P}^r_X, \mathcal{F}(n)) \to H^0(H, \mathcal{F}_H(n))$ is surjective.
Sections in $H^0(H, \mathcal{F}_H(n))$ generate $\mathcal{F}_H(n)$ by dimension induction, and by Nakayama the result at $x \in H$ follows.

♠

An analytic GAGA: Serre's proof - the equivalence

Peoof of Serre's Théorème 3.

Choose a resolution $\mathcal{O}(-n_1)^{k_1} \stackrel{\psi}{\to} \mathcal{O}(-n_0)^{k_0} \to \mathcal{F} \to 0$.
By Serre's Théorème 2 the homomorphism ψ is the analytification of an algebraic sheaf homomorphism ψ', so the cokernel $\mathcal{F}$ of ψ is also the analytification of the cokernel of ψ'. ♠