Vojta's conjecture and level structures on abelian varieties

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Torsion on elliptic curves

Following [Mazur 1977]...

Theorem (Merel, 1996)

Fix $d \in \mathbb{Z}_{>0}$. There is an integer $c = c(d)$ such that:
For all number fields k with $[k : \mathbb{Q}] = d$ and all elliptic curves E/k,

$$\#E(k)_{\text{tors}} < c.$$

Mazur: $d = 1$.
What about higher dimension?

(Jump to theorem)

Torsion on abelian varieties

Theorem (Cadoret, Tamagawa 2012)

Let k be a field, finitely generated over $\mathbb{Q}$; let p be a prime.

Let $A \to S$ be an abelian scheme over a k-curve S.

There is an integer $c = c(A, S, k, p)$ such that

$$\#A_s(k)[p^\infty] \leq c$$

for all $s \in S(k)$.

What about all torsion?
What about all abelian varieties of fixed dimension together?

Main Theorem

Let A be a g-dimensional abelian variety over a number field k.

A full-level m structure on A is an isomorphism of k-group schemes

$$A[m] \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^g \times (\mu_m)^g$$

Theorem (ℵ, V.-A., M. P. 2017)

Assume Vojta's conjecture.
Fix $g \in \mathbb{Z}_{>0}$ and a number field k.
There is an integer $m_0 = m_0(k,g)$ such that:
For any $m > m_0$ there is no principally polarized abelian variety A/k of dimension g with full-level m structure.

Why not torsion?
What's with Vojta?

Mazur's theorem revisited

- Consider the curves $\pi_m : X_1(m) \to X(1)$.
- $X_1(m)$ parametrizes elliptic curves with m-torsion.
- Observation: $g(X_1(m)) \xrightarrow[m\to\infty]{} \infty$ (quadratically)
- Faltings (1983) $\Longrightarrow X_1(m)(\mathbb{Q})$ finite for large m.
- Manin (1969!):[1] $\Longrightarrow X_1(p^k)(\mathbb{Q})$ finite for some k,
- and by Mordell–Weil $X_1(p^k)(\mathbb{Q}) = \emptyset$ for large k.

But there are infinitely many primes $> m_0$!

(Jump to Flexor–Oesterlé)

[1] Demjanjenko

Aside: Cadoret-Tamagawa

- Cadoret-Tamagawa consider similarly $S_1(m) \to S$, with components $S_1(m)^j$.
- They show $g(S_1^j(p^k)) \longrightarrow \infty, \ldots$
- unless they correspond to torsion on an isotrivial factor of A/S.
- Again this suffices by Faltings and Mordell–Weil for their p^k theorem.

Is there an analogue for higher dimensional base?

Mazur's theorem revisited: Flexor–Oesterlé, Silverberg

Proposition (Flexor–Oesterlé 1988, Silverberg 1992)

There is an integer $M = M(g)$ so that: Suppose $A(\mathbb{Q})[p] \neq \{0\}$, suppose q is a prime, and suppose $p > (1 + \sqrt{q}^M)^{2g}$. Then the reduction of A at q is "not even potentially good".

- p torsion reduced injectively moduo q.
- The reduction is not good because of Lang-Weil: there are just too many points!
- For potentially good reduction, there is good reduction after an extension of degree $< M$, so that follows too.

Remark:

- Flexor and Oesterlé proceed to show that ABC implies uniform boundedness for elliptic curves.
- This is what we follow: Vojta gives a higher dimensional ABC.
- Mazur proceeds in another way

Mazur's theorem revisited after Merel: Kolyvagin-Logachev, Bump–Friedberg–Hoffstein, Kamienny

The following suffices for Mazur's theorem:

Theorem

For all large p, $X_1(p)(\mathbb{Q})$ consists of cusps.

- [Merel] There are many weight-2 cusp forms f on $\Gamma_0(p)$ with analytic rank $\mathrm{ord}_{s=1} L(f,s) = 0$.
- [KL, BFH 1990] The corresponding factor $J_0(p)_f$ has rank 0.
- [Mazur, Kamienny 1982] The composite map $X_1(p) \to J_0(p)_f$ sending cusp to 0 is immersive at the cusp, even modulo small q.
- But reduction of torsion of $J_0(p)_f$ modulo q is injective.
- Combining with Flexor–Oesterlé we get the result.

Is there a replacement for $g > 1$???????

Main Theorem

Let A be a g-dimensional abelian variety over a number field k.

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Theorem (ℵ, V.-A., M. P. 2017)

Assume Vojta's conjecture.

Fix $g \in \mathbb{Z}_{>0}$ and a number field k.

There is an integer $m_0 = m_0(k,g)$ such that:

For any prime $p > m_0$ there is no (pp) abelian variety A/k of dimension g with full-level p structure.

Strategy

- $\widetilde{\mathscr{A}_g} \to \operatorname{Spec}\mathbb{Z}$:= moduli stack of ppav's of dimension g.
- $\widetilde{\mathscr{A}_g}(k)_{[m]}$:= k-rational points of $\widetilde{\mathscr{A}_g}$ corresponding to ppav's A/k admitting a full-level m structure.
- $\widetilde{\mathscr{A}_g}(k)_{[m]} = \pi_m(\widetilde{\mathscr{A}_g}^{[m]}(k))$,
  where $\widetilde{\mathscr{A}_g}^{[m]}$ is the space of ppav with full level.
- $W_i := \overline{\bigcup_{p \geq i} \widetilde{\mathscr{A}_g}(k)_{[p]}}$
- W_i is closed in $\widetilde{\mathscr{A}_g}$ and $W_i \supseteq W_{i+1}$.
- $\widetilde{\mathscr{A}_g}$ is Noetherian, so $W_n = W_{n+1} = \cdots$ for some $n > 0$.
- Vojta for stacks $\Rightarrow$ W_n has dimension ≤ 0.

(Jump to Vojta)

Dimension 0 case (with Flexor–Oesterlé)

- Suppose that $W_n = \overline{\bigcup_{p \geq n} \widetilde{\mathscr{A}_g}(k)_{[p]}}$ has dimension 0.
- representing finitely many geometric isomorphism classes of ppav's.
- Fix a point in W_n that comes from some A/k.
- Pick a prime $\mathfrak{q} \in \operatorname{Spec} \mathscr{O}_k$ of potentially good reduction for A.
- Twists of A with full-level p structure ($p > 2$; $\mathfrak{q} \nmid p$) have good reduction at $\mathfrak{q}$.
- p-torsion injects modulo $\mathfrak{q} \implies p \leq (1 + N\mathfrak{q}^{1/2})^2$. $\square$

There are other approaches!

Towards Vojta's conjecture

- k a number field; S a finite set of places containing infinite places.
- $(\mathscr{X}, \mathscr{D})$ a pair with:
  - $\mathscr{X} \to \operatorname{Spec} \mathscr{O}_{k,S}$ a smooth proper morphism of schemes;
  - $\mathscr{D}$ a fiber-wise normal crossings divisor on $\mathscr{X}$.

  $(X, D) :=$ the generic fiber of $(\mathscr{X}, \mathscr{D})$; $\mathscr{D} = \sum_i \mathscr{D}_i$.
- We view $x \in \mathscr{X}(\bar{k})$ as a point of $\mathscr{X}(\mathscr{O}_{k(x)})$,
  or a scheme $\mathscr{T}_x := \operatorname{Spec} \mathscr{O}_{k(x)} \to \mathscr{X}$.

Towards Vojta: counting functions and discriminants

Definition

For $x \in \mathscr{X}(\bar{k})$ with residue field $k(x)$ define the truncated counting function

$$N_k^{(1)}(D,x) = \frac{1}{[k(x):k]} \sum_{\substack{\mathfrak{q} \in \operatorname{Spec} \mathscr{O}_{k,S} \\ (\mathscr{D}|_{\mathscr{T}_x})_{\mathfrak{q}} \neq \emptyset}} \log \underbrace{|\kappa(\mathfrak{q})|}_{\substack{\text{size of} \\ \text{residue field}}} .$$

and the relative logarithmic discriminant

$$\begin{aligned} d_k(k(x)) &= \frac{1}{[k(x):k]} \log|\operatorname{Disc} \mathscr{O}_{k(x)}| - \log|\operatorname{Disc} \mathscr{O}_k| \\ &= \frac{1}{[k(x):k]} \deg \Omega_{\mathscr{O}_{k(x)}/\mathscr{O}_k}. \end{aligned}$$

Vojta's conjecture

Conjecture (Vojta c. 1984; 1998)

X a smooth projective variety over a number field k.

D a normal crossings divisor on X; H a big line bundle on X.

Fix a positive integer r and $\delta > 0$.

There is a proper Zariski closed $Z \subset X$ containing D such that

$$N_X^{(1)}(D,x) + d_k(k(x)) \geq h_{K_X+D}(x) - \delta h_H(x) - O_r(1)$$

for all $x \in X(\bar{k}) \smallsetminus Z(\bar{k})$ with $[k(x):k] \leq r$.

$d_k(k(x))$ measure failure of being in $\mathscr{X}(k)$
$N_X^{(1)}(D,x)$ measure failure of being in $\mathscr{X}^0(\mathscr{O}_k) = (\mathscr{X} \setminus \mathscr{D})(\mathscr{O}_k)$

Vojta's conjecture: special cases

- $D = \emptyset$; $H = K_X$; $r = 1$; X of general type:
  Lang's conjecture: $X(k)$ not Zariski dense.
- $H = K_X(D)$; $r = 1$; S a finite set of places ; (X, D) of log general type:
  Lang–Vojta conjecture: $\mathscr{X}^0(\mathscr{O}_{k,S})$ not Zariski dense.
- $X = \mathbb{P}^1; r = 1; D = \{0, 1, \infty\}$: Masser–Oesterlé's ABC conjecture.

Extending Vojta to DM stacks

- Recall: Vojta $\Rightarrow$ Lang.
- Example: $X = \mathbb{P}^2(\sqrt{C})$, where C a smooth curve of degree > 6.
- Then $K_X \sim \mathscr{O}(d/2 - 3)$ is big, so X of general type, but $X(k)$ is dense.
- The point is that a rational point might still fail to be integral: it may have "potentially good reduction" but not "good reduction"!
- The correct form of Lang's conjecture is: if X is of general type then $\mathscr{X}(\mathscr{O}_{k,S})$ is not Zariski-dense.

What about a quantitative version? We need to account that even rational points may be ramified.

- Heights and intersection numbers are defined as usual.
- We must define the discriminant of a point $x \in X(k)$.

Discriminant of a rational point

- $\mathscr{X} \to \mathrm{Spec}\,\mathscr{O}_{k,S}$ smooth proper, $\mathscr{X}$ a DM stack.
- For $x \in \mathscr{X}(\bar{k})$ with residue field $k(x)$, take Zariski closure and normalization of its image.
- Get a morphism $\mathscr{T}_x \to \mathscr{X}$, with $\mathscr{T}_x$ a normal stack with coarse moduli scheme $\mathrm{Spec}\,\mathscr{O}_{k(x),S}$.
- The relative logarithmic discriminant is

$$d_k(\mathscr{T}_x) = \frac{1}{\deg \mathscr{T}_x/\mathscr{O}_k} \deg \Omega_{\mathscr{T}_x/\mathscr{O}_k}.$$

Vojta's conjecture for stacks

Conjecture

k number field; S a finite set of places (including infinite ones).

$\mathscr{X} \to \operatorname{Spec}\mathscr{O}_{k,S}$ a smooth proper DM stack.

$X = \mathscr{X}_k$ generic fiber (assume irreducible)

$\underline{X}$ coarse moduli of X; assume projective with big line bundle H.

$\mathscr{D} \subseteq \mathscr{X}$ NC divisor with generic fiber D.

Fix a positive integer r and $\delta > 0$.

There is a proper Zariski closed $Z \subset X$ containing D such that

$$N_X^{(1)}(D,x) + d_k(\mathscr{T}_x) \geq h_{K_X+D}(x) - \delta h_H(x) - O(1)$$

for all $x \in X(\bar{k}) \smallsetminus Z(\bar{k})$ with $[k(x):k] \leq r$.

Vojta is flexible

Proposition ($\aleph$, V.-A. 2017)

Vojta for DM stacks follows from Vojta for schemes.

Key: Vojta showed that Vojta's conjecture is compatible with taking branched covers.

Proposition (Kresch–Vistoli)

- *There is a finite flat surjective morphism* $\pi\colon Y \to \mathscr{X}$
- *with* Y *a smooth projective irreducible scheme*
- *and* $D_Y := \pi^*\mathscr{D}$ *a NC divisor.*

Vojta for $Y \implies$ Vojta for X.

Completing the proof of the Main Theorem

Recall:

- $$W_i := \overline{\bigcup_{p \geq i} \widetilde{\mathscr{A}}_g(k)_{[p]}}$$
- and $W_n = W_{n+1} = \cdots$ for some $n > 0$.
- Want to show: $\dim W_n \leq 0$. Proceed by contradiction.
- Let X is an irreducible positive dimensional component of W_n.
- $X' \to X$ a resolution of singularities.
- $X' \subseteq \overline{X}'$ smooth compactification with $D := \overline{X}' \smallsetminus X$ NC divisor.
- Pick model $(\mathscr{X}, \mathscr{D})$ of $(\overline{X}', D)$ over $\operatorname{Spec} \mathscr{O}_{k,S}$ (Olsson)

Birational geometry

- [Zuo 2000] $K_{\overline{X}'} + D$ is big.
- Remark [Brunebarbe 2017]:
  As soon as $m > 12g$, every subvariety of $\mathscr{A}_g^{[m]}$ is of general type.
  Uses the fact that $\mathscr{A}_g^{[m]} \to \mathscr{A}_g$ is highly ramified along the boundary.
  Implies a Manin-type result for full $[p^r]$-levels.

- Can one prove a result for torsion rather than full level?

- Taking $H = K_{\overline{X}'} + D$ get by Northcott an observation on the right hand side of Vojta's conjecture

$$N_X^{(1)}(D,x) + d_k(\mathscr{T}_x) \geq \underbrace{h_{K_{\overline{X}'}+D}(x) - \delta h_H(x)}_{\text{large for small } \delta \text{ away from some } Z} - O(1)$$

Key Lemma

$X(k)_{[p]} =$ k-rational points of X corresponding to ppav's A/k admitting a full-level p structure.

Lemma

Fix $\epsilon_1, \epsilon_2 > 0$. For all $p \gg 0$ and $x \in X(k)_{[p]}$, we have

(1)

$$N_X^{(1)}(D, x) \leq \epsilon_1 h_D(x) + O(1)$$

and

(1)

$$d_k(\mathscr{T}_x) \leq \epsilon_2 h_D(x) + O(1).$$

Note: $h_D \ll h_H$ outside some Z.
Vojta gives, outside some Z,
$h_H(x) \ll N_X^{(1)}(D, x) + d_k(\mathscr{T}_x) \ll \epsilon h_H(x)$,
giving finiteness outside this Z by Northcott.

(1) $N^{(1)}(D,x) \ll \epsilon_1 h_D(x)$

- x is the image of a rational point on $\mathscr{A}_g^{[p]}$
- $\pi_p : \mathscr{A}_g^{[p]} \to \mathscr{A}_g$ is highly ramified along D (Mumford / Madapusi Pera).
- So whenever $(D|_{\mathscr{T}_x})_{\mathfrak{q}} \neq \emptyset$ its multiplicity is $\gg p$.
- so $N^{(1)}(D,x) \ll h_D(x) \underbrace{\frac{1}{p}}_{\sim \epsilon_1}$.

(2) $d_k(\mathscr{T}_x) \le \epsilon_2 h_D(x)$

- x corresponds to an abelian variety with many p-torsion points.
- Flexor–Oesterlé at any small prime $\Rightarrow h_D(x) \gg p^s$.

- x has semistable reduction outside $p \Rightarrow d_k(\mathscr{T}_x) \ll \log p$
- so $d_k(\mathscr{T}_x) \ll h_D(x) \underbrace{\frac{\log p}{p^s}}_{\sim\epsilon_2}$.