BOOK REVIEW: S.D. CUTKOSKY: RESOLUTION OF SINGULARITIES J. KOLLÁR: LECTURES ON RESOLUTION OF SINGULARITIES

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Resolution of Singularities
by Steven Dale Cutkosky
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Lectures on resolution of singularities
by János Kollár
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1. Hironaka's theorem - then and now

Resoution of singularities in characteristic 0 - what a change of fortunes!

Hironaka's proof [19] appeared 46 years ago. Yet in recent years there is a sense of excitement about the subject: young geometers are giddy about learning it; lectures about it are given in regular courses, summer schools, and conferences; and more people are seriously trying their hands at the notorious problem of resolution of singularities in positive characteristics. What is all the fuss about?

Of course the answer has several aspects. Efforts at good exposition in the last two decades, and the resulting widening of the group of interested researchers is important. Some publicity certainly helped. These two books under review are both an outcome of the process and a new driving force for the excitement. Also essential is new mathematical substance.

The main point, however, is different. Here is a wonderful result of fundamental importance in algebraic and complex geometry. Its uses are far and wide - an impressive list was included already in Grothendieck's address at the Nice congress [17] where Hironaka received the Fields Medal. It has

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every right to be included in a basic graduate course on the subject - after all, our students get to use it right away. Yet for many years it was used as a black box. Grothendieck admitted in his address that he had not fully understood the proof, and the embarrassment is palpable. Much later still, Kollár and Mori [22] referred to the possibility of using weaker results, which did not make them too happy.

The situation now is dramatically changed. Here are two books you can use to introduce your student to the subject. Self-contained proofs are given in Chapter 6 of Cutkosky's book [11] and in Chapter 3 of Kollár's book [21]. I used Kollár's treatment as one topic in a basic graduate course, and it is truly remarkable. After two weeks of introductory material, the students took over lecturing. The proof breaks nicely into manageable pieces, each taking about one hour of lecture. The whole thing took about six weeks, and left ample time for other topics. Others who lectured on their own covered this faster. This is finally the way things should be! If you are teaching algebraic geometry and considering topics for the second term, do consider this subject - you will enjoy it whether or not you are an algebraic geometer. You can choose what fits your style: Kollár's book is in the style of lecture notes, whereas Cutkosky's is more of a textbook. In short, there is no excuse now for using resolution of singularities as a black box - it is well understood and fully transparent. Certainly if you are an algebraic geometer you had better learn this stuff before the next time you meet either author.

Let us now introduce the problem:

Consider an algebraic variety X over a field k. The basic problem of resolution of singularities is the following:

Problem 1.0.1. *Find a proper, birational regular map $X' \to X$ where X' is a nonsingular algebraic variety.*

Such $X' \to X$ is a resolution of singularities of X - in essence one parametrizes the points of X by a nonsingular variety X', in such a way that nothing is missing (the map is proper and dominant, therefore surjective) and yet not too much is changed (the map is birational).

Hironaka solved this problems for varieties over fields of characteristic 0 in a strong sense:

Theorem 1.0.2. *Assume the base field is of characteristic 0. Then there exists a proper birational regular map $X' \to X$ where*

- *X' is a nonsingular algebraic variety,*
- *$X' \to X$ induces an isomorphism over the smooth locus of X, and*
- *$X' \to X$ is obtained by a sequence of* blowings up *of nonsingular subvarieties lying over the singular locus of X.*

The procedure of blowing up is a standard, explicit way of modifying a variety, see Hartshorne [18, I.4 p. 28 and II.7 p. 160]. It is a surgery operation in which one removes a subvariety $Y \subset X$ and replaces its points by the set of normal directions of Y in X. When X and Y are smooth, Y is replaced by $\mathbb{P}(N_{Y/X})$, the projectivization of the normal bundle. In particular when Y is a point and X smooth, Y is replaced by $\mathbb{P}^{\dim X - 1}$.

In his Nice address, Grothendieck insisted that the fact that blowing up is used is important for only one reason: it implies that $X' \to X$ is a projective morphism; so in particular if X is projective then so is X'. In hindsight we know that the use of blowing up is much more important - it allows one to track the change of geometry in the process.

Hironaka's original proof is based on defining an invariant of singularities, whose maximum locus is a smooth subvariety which one can blow up; then one needs to show that the invariant drops, in a set of values satisfying the descending chain condition, hence the process must stop. Constructing the invariant and proving its requisite properties is an extremely complicated process, involving multiple inductions using sequences of operations like blowing up and products, which might not be evidently related to the problem.

It is thus not surprising Grothendieck did not manage to disentangle the proof. His Nice address did not serve to encourage many to enter the subject, which might explain why it took so long to find a simpler one. Grothendieck did contribute [16, §7] by defining a class of schemes ideally suited for considering resolution of singularities. The definition of this class - so called "quasi excellent schemes" - is rather involved. (I cannot explain the choice of such a tortured name.)

Ironically, the current proof, as given in Chapter 3 of [21], is very much the type Grothendieck would have liked, and does not involve any heavy tools not known at the time. No complicated invariants are used, no big inductions required, everything is natural and based on careful common sense arguments. It would fit nicely in Grothendieck's E.G.A. (of which [16] is but one volume) without making a dent.

On the other hand, it must be said that all the main tools are already in Hironaka's work. If you understand current proofs, you can go back to Hironaka's work, and it no longer seems so daunting.

2. The current proofs: main ideas

2.1. **Embedded resolution.** All treatments of Hironaka's theorem start by addressing the *embedded* case: the variety X in question is assumed embedded in a smooth variety M, and one wishes to resolve singularities by repeatedly blowing up smooth subvarieties of M, replacing M by the blown up variety and replacing X by the proper transform. Further, one requires that not just the proper transform, but the whole inverse image of X become nice at the end - all components should be smooth and have normal crossings.

Unfortunately there are varieties which do not embed in smooth varieties; and for those which do, the embedding is not unique. The approach used by Hironaka and most of his followers requires constructing an invariant of singularities which does not depend on the embedding, and using it to determine the center of blowing up. So if you have any variety, locally it embeds in an affine space, and the resolution of such local patches glues together because the invariant says so. This is known as Hironaka's trick. As we will see, there is now an alternative approach.

2.2. **Resolving singularities of ideals.** The next idea is to replace geometry by algebra. Resolving singularities of an embedded variety $X \subset M$ can be essentially reduced to resolving singularities of the associated ideal sheaf I_X of regular functions vanishing along X. This means that after a sequence of blowings up of "allowable" smooth centers, one gets a modification $M' \to M$ where the lifted ideal sheaf $I_X\mathcal{O}_M$ is locally generated by one element of the form $\prod x_i^{m_i}$, where x_i are local parameters. This is sometimes called "principalization of an ideal sheaf". The actual reduction of resolution to principalization requires just a little more argumentation, since one must stop before blowing up a component of X itself, but this is not difficult to arrange.

2.3. **Order reduction.** At this point both Kollár and Cutkosky use a compromise. One considers marked ideals - namely pairs (I, d) of an ideal sheaf with a positive integer. If one proves that one can blow up and transform any (I, d) to (I', d) where I' is an ideal sheaf whose maximal order of vanishing is less than d, then one can principalize any ideal sheaf. However for resolution of singularities this results in a theorem slightly weaker than Hironaka's: the centers of blowing up on M and its transforms are smooth, but their restriction to the transforms of X are not smooth. This is good enough for most purposes, the salient exceptions being questions of equisingularity. More on this below.

With this compromise, resolution of singularities is finally reduced to order reduction of a marked ideal (I, d) using repeated blowing up of nonsingular centers.

2.4. **Maximal contact.** It is natural to search for a way to induct on the dimension of M. The key idea, which works only in characteristic 0, is that of smooth hypersurfaces of maximal contact: order reduction of a marked ideal (I, d) on M is equivalent to order reduction of an associated marked ideal $(C(I), d')$ - called a coefficient ideal - on a hypersurface of maximal contact H. The key example of a hypersurface of maximal contact and a coefficient ideal is explained in [21, 3.14]. In essence, $(C(I), d')$ is obtained by taking the coefficients of elements of I with respect to the defining equation of H. In practice one uses derivatives. Kollár goes one step further: he defines a notion of D-balanced ideals, and shows that one can replace any (I, d) by a

D balanced ideal, which has the property that its restriction to H already serves as a coefficient ideal.

Here we arrive at a major difficulty of Hironaka's method: in characteristic 0, hypersurfaces of maximal contact exist locally, but rarely globalize. A related issue is that local hypersurfaces of maximal contact are not unique. If one is to use local hypersurfaces of maximal contact to resolve singularities, what is there to guarantee that the resolutions will glue together?

This is the point where Kollár's treatment diverges from that of Cutkosky. The books are only three years apart, but in the intervening time Włodarczyk wrote [29], which introduces, in my opinion, a major simplification.

2.5. **The gluing problem.** Cutkosky follows an approach where one uses auxiliary operations to construct an invariant (this is called Hironaka's trick), and one needs to show that the construction is independent of any choices of hypersurfaces of maximal contact.

Włodarczyk noticed that any marked ideal (I, d) can safely be replaced by a marked ideal $(W(I), e)$ - the so called "homogenized ideal" (Włodarczyk) or "MC-invariant ideal" (Kollár) - which has the following two properties:

(1) Order reduction for (I, d) is equivalent to order reduction for the marked ideal $(W(I), e)$, and
(2) for any two smooth hypersurfaces of maximal contact H, H' for $(W(I), e)$ at a point p on M, there is a formal automorphism of M at p which carries H to H' and keeps $W(I)$ invariant.

Therefore *if one has a procedure for order reduction which is invariant under all automorphisms, the result for* $(W(I), e)$ *does not depend on the hypersurface of of maximal contact.*

Włodarczyk still uses singularity invariants in his proof as a way to keep track of improving singularities. Kollár avoids such invariants, and in the process is able to disentangle the proof entirely. There is now a nice third variant by Mustaţă [25], which combines ideas of both.

2.6. **How did we get here?**

2.6.1. *A quiet period.* For some time after Hironaka published his proof, progress on further understanding the result was not enormous. There was work of Grothendieck [17] on excellent schemes and the general setup for resolution of singularities; work of Giraud [15] on maximal contact; work of Hironaka on idealistic exponents as a formalism underlying the proof. Lipman wrote the wonderful classic [23] as an effort to introduce the subject. But he did not touch the proof of the theorem itself. There was of course much other work - but most of the effort was invested in trying to prove new results - either in positive and mixed characteristics or strengthening and generalizing the result in characteristic 0. It must be said that throughout this period Hironaka made significant efforts to teach about the subject around the world, and had students study the subject at Harvard, in Spain and in other places.

2.6.2. *New heros.* Things took a turn at the end of the 1980's, and from two quite different directions. In a sequence of papers beginning with [4], Edward Bierstone and Pierre Milman attempted to understand aspects of resolution of singularities from the bottom up. Taking their motivation from singularities of complex analytic functions, they started by giving elementary arguments for results significantly weaker than Hironaka's, and built up. Their work culminated in [5], a complete proof of Hironaka's theorem basically in its strongest form. During the same period, Orlando Villamayor took the opposite journey: in the very technical paper [28] he showed that Hironaka's resolution algorithm is constructive. Ever since he has been involved in trying to disentangle the result to something understandable, notably the manuscript [14] with Encinas.

2.6.3. *Cutkosky, Włodarczyk and the tenure system.* So we arrive around 2000, when resolution of singularities in characteristic 0 is well understood, but only to a select (but growing) few. There was a growing number of papers on the subject, but it remained largely unpenetrated by the masses. It was time for somebody to write a book. This is [11].

As Cutkosky was teaching his course, writing notes and putting together the book, a new turn was brewing, which in my view was revolutionary. Jarosław Włodarczyk, like many others in the subject, had - and still has - his sights on the positive characteristic case. But luckily for the rest of us, Włodarczyk needed to secure tenure. Thus was born the paper [29]. Apparently this was a small piece in his plan to attack the positive characteristic case. Here Włodarczyk introduced the notion of a homogenized ideal (or MC-invariant ideal in Kollár's version), which does away with Hironaka's trick. Kollár goes one step further - he does away with all complicated invariants. The price is the addition of one short section on the invariance of the resolution under automorphisms. Rather natural and not at all difficult, I contend this section is not much of a price at all.

2.7. **Is there more on Hironaka's theorem?** I have two more points.

2.7.1. *The really really general case.* Remember Grothendieck's general notion of quasi-excellent schemes? The treatments of resolution of singularities in characteristic 0 do not apply directly for these. There is much recent progress by Michael Temkin [27] on this problem, and one can hope for a complete solution in the near future. I want to stress that this is not just yet another crazy abstraction - for instance this generality allows one to unify and combine results in algebraic geometry, complex analytic geometry, p-adic analytic geometry etc.

2.7.2. *A remaining challenge: Hilbert–Samuel to order reduction.* I have mentioned that both Cutkosky and Kollár make a compromise: the centers of blowing up on a smooth ambient variety are non-singular, but their restriction to the proper transforms of X are quite singular. Hironaka's centers are smooth, and in fact the proper transform of X satisfies an equisingularity property (normal flatness) along the centers. For applications in singularity theory this is important.

There is a way to recover Hironaka's theorem in its full force, but paying a significant price in clarity. One can replace the integer d by the Hilbert–Samuel function and use it throughout. Alternatively one can reduce the situation back to marked ideals, replacing the ideal I_X by one whose order of vanishing "presents" the Hilbert–Samuel function of the original ideal - see [5, Chapter III]. The origins of this method go back to Hironaka. This reduction is quite intricate. I propose a challenge to the experts, to produce a treatment of such reduction which is easy enough to fit the treatments of either Kollár or Cutkosky. For the time being the compromise remains the best choice.

3. Other topics of resolution of singularities

From what I have written so far, the reader might get the idea that there is nothing in the two books under review other than Hironaka's theorem. This is not at all the case. And there is more to say about resolution of singularities which did not make it into the books.

3.1. **Curves, surfaces and threefolds.** Both books start from some foundational material, and continue to cover a selection of methods on resolution of singularities of curves. These include: Newton's method for plane curves using Puiseux exponents; normalization; Albanese's method of repeated projection from a point; and embedded resolution of singularities. Cutkosky has a chapter covering in detail a method proposed by Hironaka [7] for resolving surfaces in positive characteristics. Kollár's treats several other methods for resolving surfaces: Jung's method, Albanese's method (which together with Jung's method gives resolution in characteristic > 2), and embedded resolution in characteristic 0.

There are of course important classic topics which did not make it into the books. Abhyankar's difficult work on the subject (see [1] and references therein) is not covered. Neither is Lipman's much more approachable work [24] on surfaces in mixed characteristics. After publication of his book, Cutkosky reworked Abhyankar's proof of resolution of threefolds over algebraically closed fields of characteristic > 5 in [13] - this could serve as a nice additional topic in a course. Cutkosky's book does treat Zariski's valuative method in some detail.

3.2. **De Jong's Alterations.** A major topic not covered here is de Jong's Alterations result [20], in which a resolution of singularities $X' \to X$ is replaced by an alteration - namely a proper, generically finite regular map $Y \to X$. This works in positive and mixed characteristics, and has had numerous applications in arithmetic and geometry. De Jong's result also led to two methods of resolution of singularities in characteristic 0 in arbitrary dimension. One is by Bogomolov and Pantev [6], see also Paranjape [26], which is in some sense a generalization of Jung's method to arbitrary dimension, and I believe is still the quickest way to prove resolution of singularities in characteristic 0, albeit significantly weaker than Hironaka's. The other, by de Jong and the reviewer [2] led to results on toroidalization and semistable reduction of families of varieties [3]. On this there are further quite difficult results by Cutkosky (see overview in [12]), which connect to largely open problems of resolution of singularities of foliations and differential equations, but this would lead us too far afield.

3.3. **So what about positive characteristics?** There are two current streams of work on positive characteristic: attempts to prove resolution in arbitrary dimension, and attempts to make progress in low dimensions.

On the arbitrary dimension work there is not much I am qualified to say: it is no secret that a number of people, including Hironaka, Spivakovsky, Villamayor, Teissier, and more recently Włodarczyk and Kawanoue-Matsuki, have been working hard on the problem. My understanding is that new invariants of singularities have been produced, interesting geometry and algebra discovered, but so far the problem has not found a solution.

There is however definite major progress in low dimensions. Vincent Cossart and Oliver Piltant have been able to resolve singularities of threefolds in any characteristic [9, 10] and seem to have made progress on the arithmetic case. Cossart, Uwe Jannsen and Shuji Saito [8] have rewritten Hironaka's resolution of surfaces in positive and mixed characteristic in its most general and strongest form to date. This paper also provides a thorough history of the subject.

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