DECOMPOSITION OF DEGENERATE GROMOV-WITTEN INVARIANTS

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Abstract. We prove the decomposition formula for stable logarithmic maps.

Contents

1. Introduction

1.1. **Statement of results.** One of the main goals of logarithmic Gromov–Witten theory is to develop new formulas relating the Gromov–Witten invariants of a smooth variety X_η to invariants of a degenerate variety X_0.

Consider a logarithmically smooth and projective morphism $X \to B$, where B is a logarithmically smooth curve having a single point $b_0 \in B$ where the logarithmic structure is nontrivial. In the language of [KKMSD73, AK00], this is the same as saying that the underlying schemes $\underline{X}$ and $\underline{B}$ are provided with a toroidal structure such that $\underline{X} \to \underline{B}$ is a toroidal morphism, and $\underline{\{b_0\}} \subset \underline{B}$ is the toroidal divisor. One defines as in [GS13], see also [Che10, AC11], an algebraic stack $\mathscr{M}(X/B, \beta)$ parametrizing *stable logarithmic maps $f : C \to X$ with discrete data $\beta = (g, A, u_{p_1}, \ldots, u_{p_k})$* from logarithmically smooth curves to X. Here

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- g is the genus of C,
- A is the homology class $\underline{f}_*[\underline{C}]$, which we assume is supported on fibers of $\underline{X} \to \underline{B}$ and
- $u_{p_1}, \ldots, u_{p_k}$ are the *logarithmic types* or *contact orders* of the marked points with the logarithmic strata of X.

Writing $\underline{\beta} = (g, k, A)$ for the non-logarithmic discrete data, there is a natural morphism $\mathscr{M}(X/B, \beta) \to \mathscr{M}(\underline{X}/\underline{B}, \underline{\beta})$ "forgetting the logarithmic structures", which is proper and representable [ACMW14, Theorem 1.1.1]. The map $\mathscr{M}(X/B, \beta) \to \mathscr{M}(\underline{X}/\underline{B}, \underline{\beta})$ is in fact finite, see [Wis16, Corollary 1.2]. There is also a natural morphism $\mathscr{M}(X/B, \beta) \to B$, and we denote its fiber over $b \in B$ by $\mathscr{M}_\beta(X_b/b)$.

Since $X \to B$ is logarithmically smooth there is a perfect relative logarithmic obstruction theory $\mathbf{E}^\bullet \to \mathbf{L}_{\mathscr{M}(X/B,\beta)\,/\,\mathrm{Log}_B}$ giving rise to a virtual fundamental class $[\mathscr{M}(X/B, \beta)]^{\mathrm{virt}}$ and to Gromov–Witten invariants.

An immediate consequence of the formalism is the following (this is indicated after [GS13, Theorem 0.3]):

Th:deformation **Theorem 1.1.1** (Logarithmic deformation invariance). *For any point $\{b\} \stackrel{j_b}{\hookrightarrow} B$ one has*

$$j_b^![\mathscr{M}(X/B, \beta)]^{\mathrm{virt}} = [\mathscr{M}(X_b/b, \beta)]^{\mathrm{virt}}.$$

This implies, in particular, that Gromov–Witten invariants of X_b agree with those of X_{b_0}, and it is important to describe invariants of X_{b_0} in simpler terms.

The main result here is the following:

Th:decomposition **Theorem 1.1.2** (The logarithmic decomposition formula). *Suppose the morphism $X_{b_0} \to b_0$ is logarithmically smooth , and X_{b_0} is simple. Then*

Eq:decomposition
$$[\mathscr{M}(X_0/b_0, \beta)]^{\mathrm{virt}} = \sum_{\tau \in \Omega} m_\tau \cdot \sum_{\mathbf{A} \vdash A} (i_{\tau,\mathbf{A}})_*[\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0, \beta)]^{\mathrm{virt}} \qquad (1.1.1)$$

See Definition 2.1.5 for the notion of simple logarithmic structures. The notation $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0, \beta)$, m_τ and $i_{\tau,\mathbf{A}}$ is briefly explained as follows.

Given discrete data β we define below a finite set $\Omega = \{\tau\}$, which we describe in several equivalent ways.

First, Ω is the set of isomorphism classes τ of *rigid tropical curves* of type β in the polyhedral complex $\Delta(X)$.

A second description comes in the proof: an element $\tau \in \Omega$ is a ray in the tropical moduli space $M^{\mathrm{trop}}(\Sigma(X)/\Sigma(B))$ surjecting to $\Sigma(B) = \mathbb{R}_{\geq 0}$. Each element $\tau \in \Omega$ comes with a multiplicity $m_\tau \in \mathbb{Z}$; in terms of the latter description as a ray, m_τ is the index of the lattice of τ inside the lattice $\mathbb{N}$ of $\Sigma(B)$.

The notation $\mathbf{A}$ stands for a partition of the curve class A into classes $\mathbf{A}(v), v \in V(G)$, where G is the graph underlying τ.

The moduli stack $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)$ is constructed as a stack quotient $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta) = [\mathscr{M}_{\tilde\tau,\mathbf{A}}(X_0/b_0,\beta)/\operatorname{Aut}(\tilde\tau,\mathbf{A})]$, where $\mathscr{M}_{\tilde\tau,\mathbf{A}}(X_0/b_0,\beta)$ parametrizes logarithmic maps marked by a rigidified version $\tilde\tau$ of τ, where the dual graph of each curve admits a contraction to a fixed graph G_τ. Finally, there is a canonical map $i_{\tau,\mathbf{A}} : \mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta) \to \mathscr{M}(X_0/b_0,\beta)$ forgetting the marking by τ.

The following is thus an equivalent formulation of the formula (1.1.1) which is useful for studying the splitting formula and applications:

$$[\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}} = \sum_{(\tilde\tau,\mathbf{A})\,:\,\mathbf{A}\vdash A} \frac{m_{\tilde\tau}}{|\operatorname{Aut}(\tilde\tau,\mathbf{A})|}(i_{\tilde\tau,\mathbf{A}})_*\,[\mathscr{M}_{\tilde\tau,A}(X_0/b_0,\beta)]^{\mathrm{virt}}$$

(see Theorem 4.8.1). [1] ←1

Remark 1.1.3. In general, the sum over $(\tilde\tau,\mathbf{A})$ will be infinite, but because the moduli space is of finite type, all but a finite number of the moduli spaces $\mathscr{M}_{\tilde\tau,A}(X_0/b_0,\beta)$ will be empty.

These theorems form the first two steps towards a general logarithmic degeneration formula. In many cases this is sufficient for meaningful computations, as we show in section 6. These results have precise analogies with results in [Li02], as explained in §6.1. Theorem 1.1.1 is a generalization of [Li02], Lemma 3.10, while Theorem 1.1.2 is a generalization of part of [Li02], Corollary 3.13. There, what is written as $\mathfrak{M}(\mathfrak{Y}_1^{rel}\cup\mathfrak{Y}_2^{rel},\eta)$ plays the role of what is written here as $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)$.

What is missing in our more general situation is any implication that $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)$ can be described in terms of relative invariants of individual irreducible components of X_0. Indeed, this will not be the case, and we give an example in §6.2 in which X_0 has three components meeting normally, with one triple point. We give a log curve contributing to the Gromov-Witten invariant which has a component contracting to the triple point, and this curve cannot be broken up into relative curves on the three irreducible components of X_0.

In fact, a new theory is needed to give a more detailed description of the moduli spaces $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)$ in terms of pieces of simpler curves. In future work, we will define *punctured curves* which replace the relative curves of Jun Li's gluing formula. Crucially, we will explain how punctured curves can be glued together to describe the moduli spaces $\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)$.

The results described here are also analogous to results of Brett Parker proved in his category of exploded manifolds. Theorem 1.1.1 is analogous to [Par11], Theorem 3.7, while Theorem 1.1.2 is analogous to part of [Par11], Theorem 4.6. The aim in proving a full gluing formula is a full logarithmic analogy of that theorem.

[1] (Mark) This should be the formula in the main theorem?

The structure of the paper is as follows. In §2, we review various aspects of logarithmic Gromov-Witten theory, with a special emphasis on the relationship with tropical geometry. While this point of view was present in [GS13], we make it more explicit here, and in particular discuss tropicalization in a sufficient degree of generality as needed here.

§3 proves a first version of the degeneration formula. A toric morphism $X_\Sigma \to \mathbb{A}^1$ from a toric variety X_Σ has fibre over 0 easily described in terms of the corresponding map of fans $\Sigma \to \mathbb{R}_{\geq 0}$. There is a one-to-one correspondence between irreducible components of the fibre and rays of Σ mapping surjectively to $\mathbb{R}_{\geq 0}$, and their multiplicity is determined by the integral structure of this map of rays, see Proposition 3.1.1. The main point is that the equality of Weil divisors (3.1.1) then generalizes to "virtually log smooth" morphisms, in this case the morphism $\mathscr{M}(X/B, \beta) \to B$. Thus one obtains in Proposition 3.4.1 a first version of a decomposition formula, decomposing $\mathscr{M}(X_0/B_0, \beta)$ into "virtual components" $\mathscr{M}_m(X_0/B_0, \beta)$, which can be thought of as reduced unions of those components of $\mathscr{M}(X_0/B_0, \beta)$ appearing with multiplicity m.

§4 then refines this decomposition. In analogy with the purely toric case, we obtain a further decomposition in terms of rays (representing "virtual divisors" of $\mathscr{M}(X/B, \beta)$) in the tropical version of the moduli space of curves, $M^{\mathrm{trop}}(\Sigma(X)/\Sigma(B))$. These rays are interpreted as parameterizing rigid tropical curves, leading to the proof of Theorem 1.1.2.

The second part of the paper turns to some simple applications of the theory. While there are quite a few theoretical papers on log Gromov-Witten invariants, there is still a hole in the literature as far as explicit calculations are concerned. In §5 we explain some simple methods for constructing example of logarithmic curves, building on work of Nishinou and Siebert in [NS06]. This allows us to give some explicit examples of the decomposition formula in the final section of the paper.

1.2. **Acknowledgements.** That there are analogies with Parker's work is not a surprise: we received a great deal of inspiration from his work and had many fruitful discussions with Brett Parker. We also benefited from discussions with Steffen Marcus, Ilya Tyomkin, Martin Ulirsch and Jonathan Wise.

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Sec:convention

1.3. **Convention.** All logarithmic schemes and stacks we consider here are fine and saturated and defined over the complex numbers. We will usually only consider toric monoids, i.e., monoids of the form $P = P_{\mathbb{R}} \cap M$ for $M \cong \mathbb{Z}^n$, $P_{\mathbb{R}} \subseteq M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ a rational polyhedral cone. For P a toric monoid, we write

$$P^\vee = \mathrm{Hom}(P, \mathbb{N}), \quad P^* = \mathrm{Hom}(P, \mathbb{Z}).$$

1.4. **Notation.**
[2] ←[2]

$\Bbbk$	the base field, usually algebraically closed of characteristic zero
$\mathcal{M}$	a fine and saturated logarithmic structure
$\overline{\mathcal{M}}$	the characteristic (or ghost) monoid
$X \to B$	logarithmically smooth family
Log_B	stack of fine and saturated logarithmic structures over B
$\mathcal{D}_m \subset \mathrm{Log}_B$	corresponding to multipliciation by m, see Definition 3.3.1
$b_0 \in B$	special point, with induced structure of logarithmic point
$X_0 \subset X$	fiber $X_0 = X \times_B \{b_0\}$
A	a curve class $A \in H_2(X, \mathbb{Z})$
g	genus of a curve
k	number of marked points
$f : C \to X$	a logarithmic stable map
$\mathbf{p}$	marked points on C, $\mathbf{p} = (p_1, \ldots, p_k)$
u_{p_i}	type or contact order at marked point p_i
q	a node $q \in C$
$\underline{\beta}$	discrete data for a usual stable map: $\underline{\beta} = (g, k, A)$
β	discrete data for a stable logarithmic map: $\beta = (\underline{\beta}, u_{p_1}, \ldots, u_{p_k})$
β'	discrete data for a logarithmic map: $\beta' = (g, u_{p_1}, \ldots, u_{p_k})$
$\mathscr{M}(X/B, \beta)$	moduli stack of stable logarithmic maps of X relative to B
$\mathscr{M}(\underline{X}/\underline{B}, \underline{\beta})$	moduli stack of stable maps of the underlying $\underline{X}$ relative to $\underline{B}$
$\mathscr{M}(X_0/b_0, \beta)$	moduli stack of logarithmic stable maps of the fiber X_0
$\mathcal{X}$	the 1-dimensional Artin stack $\mathcal{A}_X \times_{\mathcal{A}} B$
$\mathcal{X}_0$	the fiber $\mathcal{X} \times_B b_0$ of $\mathcal{X}$
$\mathfrak{M}_B, \mathfrak{M}_{b_0}$	moduli of logarithmic curves over B, respectively b_0
$\mathfrak{M}(\mathcal{X}_0/b_0, \beta')$	moduli of logarithmic maps
$\mathfrak{M}_m(\mathcal{X}_0/b_0, \beta')$	moduli of logarithmic maps with multiplicity m
$\Sigma(X)$	the cone complex of X
$\Delta(X)$	the polyhedral complex of X
G	a graph
$E(G)$	set of edges of G

[2](Qile) Add reference and type-setting for the notations.

$V(G)$	set of vertices of G
$L(G)$	set of legs of G
ℓ	a length function $\ell : E(G) \to \mathbb{R}_{>0}$
Γ	a tropical curve $\Gamma = (G, \ell)$
$\tilde{\tau}$	a tropical curve in $\Delta(X)$
τ	isomorphism class of $\tilde{\tau}$
$\mathbf{A} \vdash A$	a partition of the curve class A
$\mathscr{M}_{\tilde{\tau},\mathbf{A}}(X_0/b_0, \beta)$	moduli of stable logarithmic maps marked by $\tilde{\tau}$
$\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0, \beta)$	moduli of stable logarithmic maps marked by the isomorphism class τ
$\mathscr{M}(X/B, \beta, \sigma)$	moduli of stable logarithmic maps with with point constraints
$p^\dagger$	the standard log point $\mathrm{Spec}(\mathbb{N} \to \Bbbk)$
$Y^\dagger$	$:= Y \times p^\dagger$
$\mathscr{M}(Y^\dagger/b_0, \beta, \sigma)$	moduli of stable logarithmic maps with point constraints
$\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0, \beta, \sigma)$	moduli of marked stable logarithmic maps with point constraints

Part 1. Theory

sec:prelim

2. Preliminaries

2.1. Cone complexes associated to logarithmic stacks.

2.1.1. *The category of cones.* We consider the category of rational polyhedral cones, wich we denote by **Cones**. The objects of **Cones** are pairs $\sigma = (\sigma_\mathbb{R}, N)$ where $N \cong \mathbb{Z}^n$ is a lattice and $\sigma_\mathbb{R} \subseteq N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ is a top-dimensional strictly convex rational polyhedral cone. A morphism of cones $\varphi : \sigma_1 \to \sigma_2$ is a homomorphism $\varphi : N_1 \to N_2$ which takes $\sigma_{1\mathbb{R}}$ into $\sigma_{2\mathbb{R}}$. Such a morphism is a *face morphism* if it identifies $\sigma_{1\mathbb{R}}$ with a face of $\sigma_{2\mathbb{R}}$ and N_1 with a saturated sublattice of N_2. If we need to specify that N is associated to σ we write N_σ instead.

2.1.2. *Generalized cone complexes.* Recall from [ACP12] that a *generalized cone complex* is a topological space with a presentation as the colimit of an arbitrary finite diagram in the category **Cones** with all morphisms being face morphisms. If Σ denotes a generalized cone complex, we write $\sigma \in \Sigma$ if σ is a cone in the diagram yielding Σ, and write $|\Sigma|$ for the underlying topological space. A morphism of generalized cone complexes $f : \Sigma \to \Sigma'$ is a continuous map $f : |\Sigma| \to |\Sigma'|$ such that for each $\sigma_\mathbb{R} \in \Sigma$, the induced map $\sigma \to |\Sigma'|$ factors through a morphism $\sigma \to \sigma' \in \Sigma'$.

Note that two generalized cone complexes can be isomorphic yet not have the same presentation. In particular, [ACP12, Proposition 2.6.2] gives a good choice of presentation, called a *reduced* presentation. This presentation has the property that every face of a cone in the diagram is in the diagram, and every isomorphism in the diagram is a self-map.

2.1.3. *Generalized polyhedral complexes.* We can similarly define a *generalized polyhedral complex*, where in the above set of definitions pairs $(\sigma_{\mathbb{R}}, N)$ live in the category **Poly** of rationally defined polyhedra. This is more general than cones, as any cone σ is in particular a polyhedron (usually unbounded). For example, an affine slice of a fan is a polyhedral complex.

2.1.4. *The tropicalization of a logarithmic scheme.* Now let X be a Zariski fs log scheme of finite type. For the generic point η of a stratum of X, its characteristic monoid $\overline{\mathcal{M}}_{X,\eta}$ defines a dual monoid $(\overline{\mathcal{M}}_{X,\eta})^\vee := \mathrm{Hom}(\overline{\mathcal{M}}_{X,\eta}, \mathbb{N})$ lying in a group $(\overline{\mathcal{M}}_{X,\eta})^* := \mathrm{Hom}(\overline{\mathcal{M}}_{X,\eta}, \mathbb{Z})$, see Section 1.3, hence a dual cone

$$\sigma_\eta := \big((\overline{\mathcal{M}}_{X,\eta})^\vee_{\mathbb{R}},\, (\overline{\mathcal{M}}_{X,\eta})^*\big).$$

If η is a specialization of η', then there is a well-defined generization map $\overline{\mathcal{M}}_{X,\eta} \to \overline{\mathcal{M}}_{X,\eta'}$ since we assumed X is a Zariski logarithmic scheme. Dualizing, we obtain a face morphism $\sigma_{\eta'} \to \sigma_\eta$. This gives a diagram of cones indexed by strata of X with face morphisms, and hence gives a generalized cone complex $\Sigma(X)$. We call this the *tropicalization* of X, following [GS13], App. B. [3]

This construction is functorial: given a morphism of log schemes $f : X \to Y$, one obtains from the map $f^\flat : f^{-1}\overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_X$ an induced map of generalized cone complexes $\Sigma(f) : \Sigma(X) \to \Sigma(Y)$.

Def:simple **Definition 2.1.5.** We say X is *monodromy free* if X is a Zariski log scheme and for every $\sigma \in \Sigma(X)$, the natural map $\sigma \to |\Sigma(X)|$ is injective on the interior of any face of σ, see [GS13, Definition B.2]. We say X is *simple* if the map is injective on every σ.

Following [Uli15], we can then define the generalized cone complex associated with a finite type logarithmic stack X, in particular allowing for logarithmic schemes X in the étale topology. One can always find a cover $X' \to X$ in the smooth topology with X' a union of simple log schemes, and with $X'' = X' \times_X X'$, we define $\Sigma(X)$ to be the colimit of $\Sigma(X'') \rightrightarrows \Sigma(X')$. The resulting generalized cone complex is independent of the choice of cover.

Examples 2.1.6. (1) If X is a toric variety with the canonical toric logarithmic structure, then $\Sigma(X)$ is abstractly the fan defining X. It is missing the embedding of $|\Sigma(X)|$ as a fan in a vector space $N_{\mathbb{R}}$, and should be viewed as a piecewise linear object.

[3] This terminology differs slightly from that of [Uli], where the tropicalization is a canonically defined tropical subset of the compactified cone complex. Hopefully this will not cause confusion.

(2) Let $\Bbbk$ be a field and $X = \operatorname{Spec}(\mathbb{N} \to \Bbbk)$ the standard log point with $\mathcal{M}_X = \Bbbk^\times \times \mathbb{N}$. Then $\Sigma(X)$ consists of the ray $\mathbb{R}_{\geq 0}$.

(3) Let C be a curve with an étale logarithmic structure with the property that $\overline{\mathcal{M}}_C$ has stalk $\mathbb{N}^2$ at any geometric point, but has monodromy of the form $(a, b) \mapsto (b, a)$, so that the pull-back of $\overline{\mathcal{M}}_C$ to a double cover $C' \to C$ is constant but $\overline{\mathcal{M}}_C$ is only locally constant. Then $\Sigma(C)$ can be described as the quotient of $\mathbb{R}^2_{\geq 0}$ by the automorphism $(a, b) \mapsto (b, a)$. If we use the reduced presentation, $\Sigma(C)$ has one cone each of dimensions 0, 1 and 2.

See [GS13, Example B.1] for a further example which is Zariski but not monodromy-free.

2.2. **Artin fans.** Let W be a fine and saturated algebraic log stack. We are quite permissive with algebraic stacks, as delineated in [Ols03, (1.2.4)-(1.2.5)], since we need to work with stacks with non-separated diagonal. An Artin stack logarithmically étale over $\operatorname{Spec} \Bbbk$ is called an *Artin fan*.

The logarithmic structure of W is encoded by a morphism $W \to \mathrm{Log}$, see [Ols03]. For many purposes this needs to be refined, since different strata of W may map to the same point of Log, and we wish to distinguish strata. Following preliminary notes written by two of us (Chen and Gross), the paper [AW13] introduces a canonical Artin fan $\mathcal{A}_W$ associated functorially to a logarithmically smooth fs log scheme W. This was generalized in [ACMW14, Prop. 3.1.1]:

Theorem 2.2.1. *Let X be a logarithmic algebraic stack which is locally connected in the smooth topology. Then there is an initial factorization of the map $X \to \mathrm{Log}$ through a strict étale morphism $\mathcal{A}_X \to \mathrm{Log}$ which is representable by algebraic spaces.*

There is a more explicit description of $\mathcal{A}_X$ in terms of the cone complex $\Sigma(X)$. For any cone $\sigma \subseteq N_{\mathbb{R}}$, let $P = \sigma^\vee \cap M$ be corresponding monoid. We write

APdefinition
$$\mathcal{A}_\sigma = \mathcal{A}_P := [\operatorname{Spec} \Bbbk[P] / \operatorname{Spec} \Bbbk[P^{\mathrm{gp}}]]. \tag{2.2.1}$$

This stack carries the standard toric logarithmic structure coming from the global chart $P \to \Bbbk[P]$. We then have:

Artinfanaslimit

Proposition 2.2.2. *Let X be a logarithmic Deligne-Mumford stack, with cone complex $\Sigma(X)$ a colimit of a diagram of cones $s : I \to$ **Cones**. Then $\mathcal{A}_X$ is the colimit as sheaves over* Log *of the corresponding diagram of sheaves given by $I \ni i \mapsto \mathcal{A}_{s(i)}$.*

Proof. This follows from the construction of $\mathcal{A}_X$ in [ACMW14]. ♠

Remark 2.2.3. Unlike $\Sigma(X)$, the formation of $\mathcal{A}_X$ is not functorial for all logarithmic morphisms $Y \to X$. This is a result of the fact that the morphism $Y \to \mathrm{Log}$ is not the composition $Y \to X \to \mathrm{Log}$, unless $Y \to X$ is strict. Note also that not all Artin fans $\mathcal{A}$ are of the form $\mathcal{A}_X$, since $\mathcal{A} \to \mathrm{Log}$ may fail to be representable.

thmic maps

2.3. Stable logarithmic maps and their moduli.

2.3.1. *Definition.* We fix a log morphism $X \to B$ with the logarithmic structure on X being defined in the Zariski topology. Recall from [GS13], [Che10] and [AC11]:

Definition 2.3.2. A *stable logarithmic map* $(C/W, \mathbf{p}, f)$ is a commutative diagram

gmapdiagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d] \\ W \arrow[r] & B \end{tikzcd} \tag{2.3.1}$$

where

(1) $\pi : C \to W$ is a proper logarithmically smooth and integral morphism of log schemes together with a tuple of sections $\mathbf{p} = (p_1, \ldots, p_k)$ of $\underline{\pi}$ such that every geometric fibre of π is a reduced and connected curve, and if $U \subset \underline{C}$ is the non-critical locus of $\underline{\pi}$ then $\overline{\mathcal{M}}_C|_U \cong \underline{\pi}^*\overline{\mathcal{M}}_W \oplus \bigoplus_{i=1}^k p_{i*}\mathbb{N}_W$.

(2) For every geometric point $\bar{w} \to \underline{W}$, the restriction of $\underline{f}$ to $\underline{C}_{\bar{w}}$ is an ordinary stable map.

2.3.3. *Basic maps.* The crucial concept for defining moduli of stable logarithmic maps is the notion of *basic* stable logarithmic maps. To explain this in tropical terms, we begin by summarizing the discussion of [GS13], §1 where more details are available. The terminology used in [Che10, AC11] is *minimal* stable logarithmic maps.

2.3.4. *Induced maps of monoids.* Suppose given a stable logarithmic map $(C/W, \mathbf{p}, f)$ with $W = \mathrm{Spec}(Q' \to \Bbbk)$, with Q' an arbitrary sharp fs monoid and $\Bbbk$ an algebraically closed field. We will use the convention that a point denoted $p \in C$ is always a marked point, and a point denoted $q \in C$ is always a nodal point. Denoting $\underline{Q}' = \pi^{-1}Q'$, the morphism $\pi^\flat$ of logarithmic structures induces a homomorphism of sheaves of monoids $\psi : \underline{Q}' \to \overline{\mathcal{M}}_C$. Similarly $f^\flat$ induces $\varphi : f^{-1}\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C$.

2.3.5. *Structure of ψ.* The homomorphism ψ is an isomorphism when restricted to the complement of the special (nodal or marked) points of C. The sheaf $\overline{\mathcal{M}}_C$ has stalks $Q' \oplus \mathbb{N}$ and $Q' \oplus_{\mathbb{N}} \mathbb{N}^2$ at marked points and nodal points respectively. The latter fibred sum is determined by a map

Eq:rho_q

$$\begin{aligned} \mathbb{N} &\to Q \\ 1 &\mapsto \rho_q \end{aligned} \tag{2.3.2}$$

and the diagonal map $\mathbb{N} \to \mathbb{N}^2$ (see Def. 1.5 of [GS13]). The map ψ at these special points is given by the inclusion $Q' \to Q' \oplus \mathbb{N}$ and $Q' \to Q' \oplus_{\mathbb{N}} \mathbb{N}^2$ into the first component for marked and nodal points respectively.

Sec:phi

2.3.6. *Structure of φ.* For $\bar{x} \in C$ a geometric point with underlying scheme-theoretic point x, the map φ induces maps $\varphi_{\bar{x}} : P_x \to \overline{\mathcal{M}}_{C,\bar{x}}$ for

$$P_x := \overline{\mathcal{M}}_{X,f(\bar{x})}$$

(well-defined independently of the choice of $\bar{x} \to x$ since the logarithmic structure on X is Zariski). Following Discussion 1.8 of [GS13], we have the following behaviour at three types of points on C:

(i) $x = \eta$ is a generic point, giving a local homomorphism[4] of monoids

$$\varphi_{\bar{\eta}} : P_\eta \to Q'.$$

(ii) $x = p$ a marked point, giving u_p the composition

$$u_p : P_p \xrightarrow{\varphi_{\bar{p}}} Q' \oplus \mathbb{N} \xrightarrow{\mathrm{pr}_2} \mathbb{N}.$$

The element $u_p \in P_p^\vee$ is called the *contact order* at p.

(iii) $x = q$ a node contained in the closures of η_1, η_2. Then there are generization maps $\chi_i : P_q \to P_{\eta_i}$, and there exists a homomorphism

$$u_q : P_q \to \mathbb{Z}$$

such that

veta1veta2diffeq

$$\varphi_{\bar{\eta}_2}\big(\chi_2(m)\big) - \varphi_{\bar{\eta}_1}\big(\chi_1(m)\big) = u_q(m) \cdot \rho_q, \tag{2.3.3}$$

with ρ_q given in Equation (2.3.2), see [GS13], (1.8). This data completely determines the local homomorphism $\varphi_{\bar{q}} : P_q \to Q' \oplus_{\mathbb{N}} \mathbb{N}^2$.

The choice of ordering η_1, η_2 for the branches of C containing a node is called an *orientation* of the node. We note that reversing the orientation of a node q (by interchanging η_1 and η_2) results in reversing the sign of u_q.

2.3.7. *Dual graphs and combinatorial type.* As customary when studying nodal curves and their maps, a graph G will consist of a set of vertices $V(G)$, a set of edges $E(G)$ and a separate set of *legs* or *half-edges* $L(G)$, with appropriate incidence relations between vertices and edges, and between vertices and half-edges. In order to obtain the correct notion of automorphisms, we also implicitly use the convention that every edge $E \in E(G)$ of G is a pair of *orientations of E* or as a pair of *half-edges of E* (disjoint from $L(G)$), so that the automorphism group of a graph with a single loop is $\mathbb{Z}/2\mathbb{Z}$.

[4]A homomorphism of monoids $\varphi : P \to Q$ is *local* if $\varphi^{-1}(Q^\times) = P^\times$.

Let G_C be the dual intersection graph of C. This is the graph which has a vertex v_η for each generic point η of C, an edge E_q joining v_{η_1}, v_{η_2} for each node q contained in the closure of η_1 and η_2, and where E_q is a loop if q is a double point in an irreducible component of C. Note that an orientation on a node gives rise to an orientation on the corresponding edge. Finally G_C has a leg L_p with endpoint v_η for each marked point p contained in the closure of η.

Definition 2.3.8. Let $(C/S, \mathbf{p}, f)$ be a logarithmic stable map. The *combinatorial type* of $(C/S, \mathbf{p}, f)$ consists of the following data:

(1) The dual graph G_C.
(2) The contact data u_p corresponding to marked points of C.
(3) The contact data u_q corresponding to oriented nodes of C.

2.3.9. *The basic monoid.* Given a combinatorial type of a logarithmic map $(C/S, \mathbf{p}, f)$ we define a monoid Q by first defining its dual

$$Q^\vee = \left\{ \big((V_\eta)_\eta, (e_q)_q\big) \in \bigoplus_\eta P_\eta^\vee \oplus \bigoplus_q \mathbb{N} \,\middle|\, \forall q : V_{\eta_2} - V_{\eta_1} = e_q u_q \right\}.$$

Here the sum is over generic points η of C and nodes q of C. We then set

$$Q := \mathrm{Hom}(Q^\vee, \mathbb{N}).$$

It is shown in [GS13], §1.5, that Q is a sharp monoid, necessarily fine and saturated by construction.

Given a stable logarithmic map $(C/W, \mathbf{p}, f)$ over $W = \mathrm{Spec}(Q' \to \Bbbk)$ of the given combinatorial type, we obtain a canonically defined map

candefmap (2.3.4)
$$Q \to Q'$$

which is most easily defined as the transpose of the map

$$(Q')^\vee \to Q^\vee \subseteq \bigoplus_\eta P_\eta^\vee \oplus \bigoplus_q \mathbb{N},$$

given by

$$m \mapsto \big((\varphi_{\bar\eta}^t(m))_\eta, (m(\rho_q))_q\big).$$

where $\varphi_{\bar\eta}$ is defined in Section 2.3.6 and ρ_q is given in Equation 2.3.2.

Definition 2.3.10 (Basic maps)**.** Let $(C/W, \mathbf{p}, f)$ be a stable logarithmic map. If $W = \mathrm{Spec}(Q' \to \Bbbk)$ for some Q' and algebraically closed field $\Bbbk$, we say this stable logarithmic map is *basic* if the above map $Q \to Q'$ is an isomorphism. If W is an abitrary fs log scheme, we say the stable map is *basic* if $(C_{\bar w}/\bar w, \mathbf{p}, f|_{C_{\bar w}})$ is basic for all geometric points $\bar w \in W$.

2.3.11. *Degree data and class.* In what follows, we will need to make a choice of a notion of *degree data* for curves in X; we will write the group of degree data as $H_2(X)$. This could be 1-cycles on X modulo algebraic or numerical equivalence, or it could be $\mathrm{Hom}(\mathrm{Pic}(X),\mathbb{Z})$. If we work over $\mathbb{C}$, we can use ordinary singular homology $H_2(X,\mathbb{Z})$. A key requirement $H_2(X)$ must satisfy, which satisfied by any of the above choices, is that any family of stable maps $\underline{f}:\underline{C}/\underline{W}\to\underline{X}$ should induce a well-defined class $\underline{f}_*[C_{\bar w}]\in H_2(X)$ for $\bar w\in\underline{W}$ a geometric point; further, if $\underline{W}$ is connected, this class should be independent of the choice of $\bar w$.

Definition 2.3.12. A *class* β of stable logarithmic maps to X consists of the following:

(i) The data $\underline{\beta}$ of an underlying ordinary stable map, i.e., the genus g, the number of marked points k, and data $A\in H_2(X)$.
(ii) Integral elements $u_{p_1},\ldots,u_{p_k}\in|\Sigma(X)|$. [5]

We say a stable logarithmic map $(C/W,\mathbf{p},f)$ *is of class* β if two conditions are satisfied. First, the underlying ordinary stable map must be of type $\underline{\beta}=(g,k,A)$. Second, define the closed subset $\underline{Z}_i\subseteq\underline{X}$ to be the union of strata with generic points η such that u_{p_i} lies in the image of $\sigma_\eta\to|\Sigma(X)|$. Then for any i we have $\mathrm{im}(\underline{f}\circ p_i)\subset\underline{Z}_i$ and for any geometric point $\bar w\to\underline{W}$ such that $p_i(\bar w)$ lies in the stratum of X with generic point η, the composed map

$$\overline{\mathcal{M}}_{X,\bar\eta}=\overline{\mathcal{M}}_{X,\underline{f}(p_i(\bar w))}\xrightarrow{\overline{f}^\flat}\overline{\mathcal{M}}_{C,p_i(\bar w)}=\overline{\mathcal{M}}_{W,\underline{w}}\oplus\mathbb{N}\xrightarrow{\mathrm{pr}_2}\mathbb{N},$$

thought of as an element of σ_η, maps to $u_{p_i}\in|\Sigma(X)|$ under the map $\sigma_\eta\to|\Sigma(X)|$. In particular, s_i specifies the contact order u_{p_i} at the marked point $p_i(\bar w)$.

We emphasize that the class β does not specify the contact orders u_q at nodes.

Definition 2.3.13. Let $\mathscr{M}(X/B,\beta)$ denote the stack of basic stable logarithmic maps of class β. This is the category whose objects are basic stable logarithmic maps $(C/W,\mathbf{p},f)$ of class β, and whose morphisms $(C/W,\mathbf{p},f)\to(C'/W',\mathbf{p},f')$ are commutative diagrams

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d] & C' \arrow[r, "f'"] \arrow[d] & X \arrow[d] \\ W \arrow[r] & W' \arrow[r] & B \end{tikzcd}$$

with the left-hand square cartesian, $W\to W'$ strict, and $f=f'\circ g$.

[5]We remark that this definition of contact orders is different than that given in [GS13, Definition 3.1]. Indeed, the definition given there does not work when X is not monodromy free, and [GS13, Remark 3.2] is not correct in that case. However, [GS13, Definition 3.1] may be used in the monodromy free case.

Theorem 2.3.14. *If $X \to B$ is proper, then $\mathscr{M}(X/B,\beta)$ is a proper Deligne-Mumford stack. If furthermore $X \to B$ is logarithmically smooth, then $\mathscr{M}(X/B,\beta)$ carries a perfect obstruction theory, defining a virtual fundamental class $[\mathscr{M}(X/B,\beta)]^{\mathrm{virt}}$ in the rational Chow group of the underlying stack $\underline{\mathscr{M}}(X/B,\beta)$.*

Proof. The stack $\mathscr{M}(X/B,\beta)$ is constructed in [GS13] in general for X a Zariski log scheme and in [Che10], [AC11] with the stronger assumption that there is a monoid P and a sheaf homomorphism $\underline{P} \to \overline{\mathcal{M}}_Y$ which locally lifts to a chart. Properness was proved in the latter two references in those cases, and in [GS13] with a certain hypothesis, combinatorial finiteness, see [GS13], Def. 3.3. Properness was shown in general in [ACMW14].

The existence of a perfect obstruction theory when $X \to B$ is logarithmically smooth was proved in [GS13], §5. ♠

We remark that the stack $\mathscr{M}(X/B,\beta)$ with its logarithmic structure defines a stack in groupoids over the category of logarithmic schemes. As such it parametrizes all stable logarithmic maps, without requiring them to be basic.

elogcurves

2.4. **Stack of prestable logarithmic curves.** In [GS13], §5, one constructs a relative obstruction theory over $\mathfrak{M}_B$, the Artin stack of all prestable logarithmically smooth curves defined over B. We recall briefly how this moduli space is constructed, see [GS13], Appendix A for details.

First, working over a field $\Bbbk$, there is a moduli space $\mathfrak{M}$ of pre-stable basic logarithmic curves over $\mathrm{Spec}\,\Bbbk$, essentially constructed by F. Kato in [Kat00]. Of course $\mathfrak{M}$ is an algebraic log stack over $\mathrm{Spec}\,\Bbbk$.

If B is an arbitrary fs log scheme over $\mathrm{Spec}\,\Bbbk$, a morphism $\underline{W} \to \mathfrak{M} \times_{\mathrm{Spec}\,\Bbbk} \underline{B}$ determines a logarithmic structure on $\underline{W}$ which has, as direct summand, the basic logarithmic structure for the family of curves $\underline{C} \to \underline{W}$ induced by $\underline{W} \to \mathfrak{M}$. Thus $\mathfrak{M} \times_{\mathrm{Spec}\,\Bbbk} \underline{B}$ acquires the structure of an algebraic log stack over the log scheme B.

We then set

$$\mathfrak{M}_B := \mathrm{Log}_{\mathfrak{M}\times B},$$

Olsson's stack parametrizing logarithmic structures over $\mathfrak{M} \times B$. [6]

subsection

2.5. **The tropical interpretation.** The precise form of the basic monoid Q came from a tropical interpretation, which will play an important role here. We review this in our general setting. Given a stable logarithmic map $(C/W,\mathbf{p},f)$, we obtain

[6] In [GS13] one explicitly writes $\mathfrak{M}_B := \mathrm{Log}_B^{\bullet\to\bullet} \times_{\mathrm{Log}_B} (\mathfrak{M} \times_{\mathrm{Spec}\,\Bbbk} B)$.

an associated diagram of cone complexes,

onecomplexdiagram (2.5.1)
$$\begin{tikzcd} \Sigma(C) \arrow[r, "\Sigma(f)"] \arrow[d, "\Sigma(\pi)"'] & \Sigma(X) \arrow[d] \\ \Sigma(W) \arrow[r] & \Sigma(B). \end{tikzcd}$$

This diagram can be viewed as giving a family of tropical curves mapping to $\Sigma(X)$, parameterized by the cone complex $\Sigma(W)$. Indeed, a fibre of $\Sigma(\pi)$ is a graph and the restriction of $\Sigma(f)$ to such a fibre can be viewed as a tropical curve mapping to $\Sigma(X)$. We make this precise.

First, we need to be a bit careful about the diagram giving a presentation of $\Sigma(X)$. To avoid difficulties in notation, we shall assume that in fact X is simple. This is not a restrictive assumption in this paper, as our results will only apply when X is log smooth over the trivial point $\operatorname{Spec} \Bbbk$, and as X is assumed to be Zariski in any event, it follows that X is simple.

We can then use the reduced presentation of $\Sigma(X)$ given by [ACP12, Proposition 2.6.2]: every face of a cone in the diagram presenting $\Sigma(X)$ is in the diagram, and every isomorphism is a self-map. But since X is in particular monodromy free, all isomorphisms are identities, and simplicity then implies that if $\tau, \sigma \in \Sigma(X)$ with the image of τ in $|\Sigma(X)|$ a face of the image of σ, then there is a unique face map $\tau \to \sigma$ in the diagram.

weightedcurvedef **Definition 2.5.1.** Let G be a connected graph. A *genus-weight function* is a function $\mathbf{g} : V(G) \to \mathbb{N}$. The pair $(G, \mathbf{g})$ is called *a genus-weighted graph*, and the value $\mathbf{g}(v)$ is called the genus associated to the vertex v.

tropicalcurvedef **Definition 2.5.2.** A *tropical curve* $\Gamma = (G, \mathbf{g}, \ell)$ *of combinatorial type* $(G, \mathbf{g})$ is an assignment
$$\ell : E(G) \to \mathbb{R}_{>0}$$
of lengths to each edge of G.

There is an evident geometric realization of Γ as a metric space. We denote it $|\Gamma|$.

One imagines a vertex v with weight $\mathbf{g}(v)$ representing $\mathbf{g}(v)$ infinitesimally small loops incident to v. The genus of Γ is defined by the expression

Eq:genus (2.5.2)
$$g(\Gamma) = b_1(|\Gamma|) + \sum_{v \in V(G)} \mathbf{g}(v).$$

It depends only on the genus-weighted graph $(G, \mathbf{g})$.

pcurveinsigmaxdef **Definition 2.5.3.** A *tropical curve in* $\Sigma(X)$ consists of the following data:

(1) A tropical curve $\Gamma = (G, \mathbf{g}, \ell)$ as in Definition 2.5.2.

(2) A map
$$\boldsymbol{\sigma} : V(G) \cup E(G) \cup L(G) \to \Sigma(X)$$
(viewing $\Sigma(X)$ as a set of cones). This data must satisfy the condition that if E is either a leg or edge incident to a vertex v, then there is an inclusion of faces $\boldsymbol{\sigma}(v) \subseteq \boldsymbol{\sigma}(E)$ in the (reduced) presentation of $\Sigma(X)$.

(3) Leg ordering: writing $k = \#L(G)$, we are given a bijection between $L(G)$ and $\{1, \ldots, k\}$.

(4) Edge marking: for each edge $E_q \in E(G)$ with a choice of orientation, an element $u_q \in N^{\mathrm{gp}}_{\boldsymbol{\sigma}(E_q)}$; reversing the orientation of E_q results in replacing u_q by $-u_q$.

(5) Leg marking: for each leg $E_p \in L(G)$ an element $u_p \in N_{\boldsymbol{\sigma}(E_p)} \cap \boldsymbol{\sigma}(E_p)$.

(6) A map $f : |\Gamma| \to |\Sigma(X)|$ satisfying conditions
   (a) For $v \in V(G)$ we have $f(v) \in \mathrm{Int}(\boldsymbol{\sigma}(v))$.
   (b) Let $E_q \in E(G)$ be an edge with endpoints v_1 and v_2, oriented from v_1 to v_2. Then
      (i) $f(\mathrm{Int}(E_q)) \subseteq \mathrm{Int}(\boldsymbol{\sigma}(E_q))$,
      (ii) f maps E_q affine linearly to the line segment in $\boldsymbol{\sigma}(E_q)$ joining $f(v_1)$ and $f(v_2)$, with the points $f(v_1)$ and $f(v_2)$ viewed as elements of $\boldsymbol{\sigma}(E_q)$ via the unique inclusions of faces $\boldsymbol{\sigma}(v_i) \subseteq \boldsymbol{\sigma}(E_q)$.
      (iii) we have [7]
$$f(v_2) - f(v_1) = \ell(E_q)u_q.$$
   (c) For $E_p \in L(G)$ a leg with vertex v, then
$$f(\mathrm{Int}(E_p)) \subseteq \mathrm{Int}(\boldsymbol{\sigma}(E_p)),$$
   and f maps E_p affine linearly to the ray
$$f(v) + \mathbb{R}_{\geq 0}u_p \subseteq \boldsymbol{\sigma}(E_p).$$

The *combinatorial type* $\widetilde{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, u)$ of a decorated tropical curve in $\Sigma(X)$ is the graph G along with data (2)-(5) above. Note that we are suppressing the leg numbering, viewing the set $L(G)$ as identical with $\{1, \ldots, k\}$.

2.5.4. *Tropical curves from logarithmically smooth curves.* Now suppose
$$W = \mathrm{Spec}(Q \to \Bbbk)$$
for some monoid Q. Then the diagram (2.5.1) is intepreted as follows. First,
$$\Sigma(W) = Q^\vee_{\mathbb{R}} := \mathrm{Hom}(Q, \mathbb{R}_{\geq 0}).$$

[7]Note that u_q and ℓ are required data for a tropical map, as they cannot in general recovered from the image.

For any point $m \in Q^\vee_{\mathbb{R}}$, the inverse image $\Sigma(\pi)^{-1}(m)$ is a tropical curve. When m lies in the interior of $Q^\vee_{\mathbb{R}}$, then the combinatorial type of the curve is the dual intersection graph Γ_C of C. Explicitly:

(i) If η is a generic point of C, then $\sigma_\eta = Q^\vee_{\mathbb{R}}$ and $\Sigma(\pi)|_{\sigma_\eta}$ is the identity. Thus each fibre of $\Sigma(\pi)|_{\sigma_\eta}$ is a point v. We put the weight $\mathbf{g}(v) = g(C_\eta)$, the geometric genus of the component with generic point η.

(ii) If q is a node of C, then

$$\sigma_q = \mathrm{Hom}(Q \oplus_{\mathbb{N}} \mathbb{N}^2, \mathbb{R}_{\geq 0}) = Q^\vee_{\mathbb{R}} \times_{\mathbb{R}_{\geq 0}} \mathbb{R}^2_{\geq 0},$$

where the maps $Q^\vee_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ and $\mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ are given by $\rho_q \in Q$ and $(a,b) \mapsto a+b$ respectively. Thus the fibre of $\Sigma(\pi)|_{\sigma_q}$ over $m \in Q^\vee_{\mathbb{R}}$ is an interval. This interval admits an integral affine isomorphism to an interval of affine length $m(\rho_q)$. Since $\rho_q \in Q$, this length is non-zero if $m \in \mathrm{Int}(Q^\vee_{\mathbb{R}})$. If the corresponding edge of $\Gamma = \Sigma(\pi)^{-1}(m)$ is called E_q, then we set $\ell(E_q) = m(\rho_q)$.

(iii) If $p \in C$ is a marked point, then $\sigma_p = Q^\vee_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$, and $\Sigma(\pi)|_{\sigma_p}$ is the projection onto the first component. Thus a fibre of $\Sigma(\pi)_{\sigma_p}$ is a ray we denote by E_p.

This analysis then makes clear the claim that Γ is the dual intersection graph Γ_C of C whenever $m \in \mathrm{Int}(Q^\vee_{\mathbb{R}})$. However the tropical structure of Γ_C, i.e., the lengths of the edges, depends on m.

If m lies in the boundary of $Q^\vee_{\mathbb{R}}$, then $\Sigma(\pi)^{-1}(m)$ is obtained from Γ_C by contracting the bounded edges E_q such that $m(\rho_q) = 0$. For example, $\Sigma(\pi)^{-1}(0)$ consists of a single vertex with an attached unbounded edge for each marked point of C. In general, if $m \in \mathrm{Int}(\tau)$, $m' \in \mathrm{Int}(\sigma)$ with $\tau \subseteq \sigma$ faces of $Q^\vee_{\mathbb{R}}$, then there is a continuous map with connected fibres

contractionmap

$$\xi : \Sigma(\pi)^{-1}(m') \to \Sigma(\pi)^{-1}(m) \tag{2.5.3}$$

which contracts precisely those edges of the first graph whose lengths go to zero over m. This is compatible with the weight $\mathbf{g}$ in the sense that for a vertex $v \in \Sigma(\pi)^{-1}(m)$ we have $\mathbf{g}(v) = g(\xi^{-1}(v))$, where $g(\xi^{-1}(v))$ is calculated using Equation (2.5.2).

-to-tropical-maps

2.5.5. *Tropical curves in $\Sigma(X)$ from stable logarithmic maps.* Continuing with $W = \mathrm{Spec}(Q \to \kappa)$, the map $\Sigma(f)$ encodes the map $\varphi : f^{-1}\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_C$ and defines a family of tropical curves in $\Sigma(X)$ in the sense of Definition 2.5.3. The data (1)-(6) are specified as follows, given $m \in \mathrm{Int}(Q^\vee_{\mathbb{R}})$:

(1) Identify Γ_C with $\Sigma(\pi)^{-1}(m)$, which we have seen is a tropical curve.

(2) An element x of $V(G) \cup E(G) \cup L(G)$ corresponds to a point $\bar{x} \in C$ — either a generic point, a double point, or a marked point. Define

$$\boldsymbol{\sigma}(x) := (P_x)^\vee_{\mathbb{R}} = \mathrm{Hom}(\overline{\mathcal{M}}_{X,f(\bar{x})}, \mathbb{R}_{\geq 0}) \in \Sigma(X).$$

(3) If the marked points are $p_1, \ldots, p_k$, the bijection $L(G) \leftrightarrow \{1, \ldots, k\}$ is $E_{p_i} \leftrightarrow i$.
(4) The edge marking data u_q are the vectors associated to $C \to X$ defined in §2.3; note u_q depends on a choice of orientation of E_q and is replaced by $-u_q$ when the orientation is reversed.
(5) The leg markings u_p are defined in the same section.
(6) For v_η a vertex of Γ_C, we have by definition of $\Sigma(f)$ that with $\varphi_{\bar\eta} : P_\eta \to Q$,

$$\Sigma(f)(v_\eta) = \varphi_{\bar\eta}^t(m) \in \mathrm{Int}(\boldsymbol{\sigma}(v_\eta)),$$

the latter since $P_\eta \to Q$ is a local homomorphism. This shows item (6)(a) of Definition 2.5.3. Property (6)(b) follows from (2.3.3). The remaining properties of a tropical curve in $\Sigma(X)$ are easily checked.

al-tropical

2.5.6. *Basic maps and tropical universal families.* Basicness of the map f can then be recast as follows. If $(C/W, f)$ is basic, then the above family of tropical curves is universal. Indeed, the definition of the dual of the basic monoid $Q^\vee$ precisely encodes the data of a tropical curve in $\Sigma(X)$ with the combinatorial type described above. A tuple $((V_\eta)_\eta, (e_q)_q) \in \mathrm{Int}(Q^\vee_{\mathbb{R}})$ specifies a unique tropical curve $\Sigma(f) : \Gamma_C \to \Sigma(X)$ of the given combinatorial type with $\Sigma(f)(v_\eta) = V_\eta$.

Given another stable logarithmic map $(C'/W', \mathbf{p}, f')$ over $W' = \mathrm{Spec}(Q' \to \kappa)$ coinciding with $(C/W, \mathbf{p}, f)$ at the scheme level, the canonically defined map (2.3.4) can be viewed at the tropical level as a map $(Q')^\vee_{\mathbb{R}} \to Q^\vee_{\mathbb{R}}$. This map takes a point $m \in \mathrm{Int}((Q')^\vee_{\mathbb{R}})$ to the data specifying the map $\Sigma(f')|_{\Sigma(\pi)^{-1}(m)} : \Gamma_C \to \Sigma(X)$.

dded-cones

Remark 2.5.7. Note that if W is not a log point, the diagram (2.5.1) still exists, but the fibres of $\Sigma(\pi)$ may not be the expected ones. In particular, if $\bar w$ is a geometric point of W, there is a functorial diagram

$$\begin{tikzcd} \Sigma(C_{\bar w}) \arrow[r] \arrow[d] & \Sigma(C) \arrow[d] \\ \Sigma(\bar w) \arrow[r] & \Sigma(W) \end{tikzcd}$$

but this diagram need not be Cartesian. This might reflect monodromy in the family W. For example, it is easy to imagine a situation where $C_{\bar w}$ has two irreducible components and two nodes for every geometric point $\bar w$, but the nodal locus of $C \to W$ is irreducible, as there is monodromy interchanging the two nodes. Then a fibre of $\Sigma(C) \to \Sigma(W)$ may consist of two vertices joined by a single edge, while a fibre of $\Sigma(C_{\bar w}) \to \Sigma(\bar w)$ will have two vertices joined by two edges. Similarly, there may be monodromy interchanging irreducible components, hence a fibre of $\Sigma(C) \to \Sigma(W)$ may have fewer vertices than $C_{\bar{w}}$ has irreducible components.[8]

ec:torictovirtual

3. From toric decomposition to virtual decomposition

3.1. **The toric picture.** There is an underlying fact — a simple decomposition formula in toric varieties — which makes the decomposition formula possible. The first ingredient is the following. Let W be a toric variety and $\pi : W \to \mathbb{A}^1$ a toric morphism. We write Σ_W and $\Sigma_{\mathbb{A}^1}$ for the associated fans, noting that these can be abstractly identified with $\Sigma(W)$ and $\Sigma(\mathbb{A}^1)$ as cone complexes, although the latter do not naturally lie inside a vector space. Associated to π we have a morphism of fans $\Sigma_\pi : \Sigma_W \to \Sigma_{\mathbb{A}^1}$. Prime toric divisors in W correspond to rays in Σ_W. Let $\Omega_W := \Sigma_W^{(1)}$ be the set of these rays, and for each ray $\tau \in \Omega_W$ write D_τ for the corresponding toric divisor. Write M_W for the character lattice and N_W for the lattice of 1-parameter subgroups of the torus of W, and similarly write $M_{\mathbb{A}^1}$, $N_{\mathbb{A}^1}$. We write $N_\tau = (N \cap \tau)^{\mathrm{gp}}$ for the lattice of integral points tangent to τ. We denote by $\tau_{\mathbb{A}^1} \in \Sigma(\mathbb{A}^1)$ the unique one-dimensional cone. The toric decomposition formula is the following standard observation:

prop:obvious

Proposition 3.1.1. *(1) For $\tau \in \Omega_W$, we have isomorphisms $\tau \cap N_\tau \simeq \mathbb{N}$ and $\tau_{\mathbb{A}^1} \cap N_{\mathbb{A}^1} \simeq \mathbb{N}$, and the map $\tau \cap N_\tau \to \tau_{\mathbb{A}^1} \cap N_{\mathbb{A}^1}$ between them is given by multiplication by a non-negative integer m_τ.*
(2) The multiplicity of the divisor $\pi^*(\{0\})$ along D_τ is m_τ.
In other words, we have an equality of Weil divisors

Eq:toric

$$\pi^*(\{0\}) = \sum_\tau m_\tau D_\tau. \tag{3.1.1}$$

Proof. (1) follows since τ is a rational ray. The map Σ_π is given by a linear function $m : N \to \mathbb{Z}$, and hence π is given by the regular monomial z^m. It is standard that the order of vanishing of z^m on the divisor D_τ is the value of m on the generator of $\tau \cap N_\tau$. But this value is precisely m_τ, giving the result. ♠

Remark 3.1.2. In fact, the data m_τ only depends on the map $\Sigma(\pi) : \Sigma(W) \to \Sigma(\mathbb{A}^1)$ as abstract cone complexes, as the lattice N_τ is the lattice giving the integral structure on cones $\tau \in \Sigma(W)$ corresponding to the codimension one strata of W.

position-toroidal

3.2. **Decomposition in the toroidal case.** The proposition immediately applies to a logarithmically smooth morphism $W \to B$, where B is a smooth curve with toroidal divisor $\{b_0\}$: associated to $W \to B$ we have a morphism of generalized cone complexes $\Sigma(W) \to \Sigma(B)$. We have again that $\Sigma(B) \simeq \mathbb{R}_{\geq 0}$ with the lattice

[8] We have been informed that forthcoming work Chan, Cavalieri, Ulirsch and Wise redefines moduli of tropical curves as stacks, in which case this issue will be resolved.

$N_B \simeq \mathbb{Z}$. We have a correspondence between rays $\tau \in \Sigma(W)$ and toroidal divisors $D_\tau \subset W$. For a ray τ with integral lattice N_τ, we have $\tau \cap N_\tau \simeq \mathbb{N}$, and the monoid homomorphism $\tau \cap N_\tau \to \tau_B \cap N_B$ is multiplication by an integer m_τ.

on-toroidal **Corollary 3.2.1.**

Eq:toroidal (3.2.1)
$$\pi^*(\{b_0\}) = \sum_\tau m_\tau D_\tau.$$

Proof. Fix a geometric point x on the open stratum of D_τ. We may assume that we have a toroidal chart, namely a commutative diagram

$$\begin{tikzcd}
V_W \arrow[d, "\pi_V"'] & U_W \arrow[l] \arrow[r] \arrow[d] & W \arrow[d, "\pi"] \\
V_B & U_B \arrow[l] \arrow[r] & B
\end{tikzcd}$$

where

1. all the horizontal arrows are étale,
2. $U_W \to W$ is an étale neighborhood of x,
3. $\pi_V : V_W \to V_B$ is a toric morphism of affine toric varieties.

Since B is a curve, $V_B \simeq \mathbb{A}^1$. Write 0 for its origin. Replacing V_W and U_W by open sets we may assume V_W contains a unique toric divisor D_V. Then the multiplicity of $\pi_W^*(\{b_0\})$ along D_τ coincides with the multiplicity of $\pi_V^*(\{0\})$ along D_V. This is m_τ by Equation (3.1.1). ♠

3.3. **Decomposition in the stack of logarithmic structures.** In this paper we apply Equation (3.2.1) in the generality of Artin stacks. We continue to work with (B, b_0) a pointed smooth curve. We have Log_B, Olsson's stack of log schemes over B, the objects of which are log morphisms $X \to B$.[9] There is a morphism $\mathrm{Log}_B \to B$, the forgetful map which forgets the logarithmic structures. Viewing $b_0 \in B$ as the standard log point, we similarly have a morphism $\mathrm{Log}_{b_0} \to b_0$.

Dmdef **Definition 3.3.1.** Define the closed substack $\mathcal{D}_m$ of Log_B as follows. For $m \in \mathbb{N}$, let $m : \mathbb{N} \to \mathbb{N}$ denote the multiplication by m map, inducing $\mathcal{A}_\mathbb{N} \to \mathcal{A}_\mathbb{N} = \mathcal{A}$. This morphism of log stacks induces by projection a morphism of log stacks $\mathcal{A}_\mathbb{N} \times_\mathcal{A} B \to B$, hence a morphism of stacks $m : \mathcal{A}_\mathbb{N} \times_\mathcal{A} B \to \mathrm{Log}_B$. We take $\mathcal{D}_m$ to be the closure of the image of $[0/\mathbb{G}_m] \times_\mathcal{A} B$ under m with the reduced induced stack structure.

We observe:

[9]We emphasize that, since we assume all our logarithmic structures are fine and saturated, this stack Log_B parametrizes only fine and saturated logarithmic structures, and is only an open substack of Olsson full stack of logarithmic structures. Olsson denotes our stack Log_B by $\mathcal{T}or_B$.

ecompositionfacts **Lemma 3.3.2.** *(1)* $\mathrm{Log}_B \to B$ *is logarithmically étale.*

(2) $\mathrm{Log}_B \times_B b_0 \cong \mathrm{Log}_{b_0}$.

(3) For each $m \in \mathbb{N}$, $\mathcal{D}_m \subset \mathrm{Log}_B$ *is a generically reduced prime divisor. When* $m > 0$ *the divisor* $\mathcal{D}_m$ *is contained in* Log_{b_0}.

(4) For $m > 0$ *write* $i_m : \mathcal{D}_m \to \mathrm{Log}_{b_0}$ *for the embedding above. For any finite type open substack* $U \subset \mathrm{Log}_B$*, we have the following identity in the Chow group* $A_*(U)$ *of* U *introduced in* [Kre99]*:*

Eq:Log-p-true

$$[\mathrm{Log}_{b_0} \cap U] = \sum_m m \cdot i_{m*}[\mathcal{D}_m \cap U], \tag{3.3.1}$$

For convenience, we denote the above identity formally as follows:

Eq:Log-p

$$[\mathrm{Log}_{b_0}] = \sum_m m \cdot i_{m*}[\mathcal{D}_m] \tag{3.3.2}$$

without specifying the specific choice of U.

Remark 3.3.3. In [Kre99], the Chow groups are constructed for Artin stacks of finite type which admits finite sum of cycles. Note that the stack Log_B is not of finite type, and the summation in (3.3.2) has infinitely many non-zero terms. The equation (3.3.2) is not an identity of Chow cycles in the sense of [Kre99].

Proof. (1) This is generally true for any scheme B, but to simplify notation and clarify the structure of Log_B, we just show it for the given B. It is sufficient to restrict to an étale cover of Log_B, which is described in [Ols03, Corollary 5.25]. One can cover Log_B by stacks indexed by morphisms of monoids $\mathbb{N} \to P$. We set $\mathcal{A} := \mathcal{A}_{\mathbb{N}} = [\mathbb{A}^1/\mathbb{G}_m]$ (following the notation of (2.2.1)). By [Cad07, Lemma 2.1.1], an object of the category $\mathcal{A}$ over a scheme $\underline{W}$ is a pair (L, s) where L is a line bundle on $\underline{W}$ and s is a section of L. In particular, taking a line bundle L on B corresponding to the divisor b_0, and taking a section s of L vanishing precisely once, at b_0, we obtain a strict morphism $B \to \mathcal{A}$. (In fact, $\mathcal{A}$ is the Artin fan of B and this is the canonical morphism).

Then a morphism $\mathbb{N} \to P$ gives the element $\mathcal{A}_P \times_{\mathcal{A}} B$ of the étale open cover of Log_B. Here the morphism $\mathcal{A}_P \to \mathcal{A}$ is functorially induced by $\mathbb{N} \to P$. The composition of $\mathcal{A}_P \times_{\mathcal{A}} B \to \mathrm{Log}_B$ with the forgetful morphism $\mathrm{Log}_B \to B$ is just the projection to B, the base-change of $\mathcal{A}_P \to \mathcal{A}$, which is étale by [Ols03, Corollary 5.23].

(2) is clear since giving a morphism $\underline{W} \to \mathrm{Log}_B \times_B b_0$ is the same thing as giving a log morphism $W \to B$ which factors through the inclusion $b_0 \hookrightarrow B$.

For (3), first note that the image of $[0/\mathbb{G}_m] \times_{\mathcal{A}} B$ under the projection to B is b_0 except in the case when $m = 0$, in which case the image is $B \smallsetminus \{b_0\}$. Thus $\mathcal{D}_m$ is contained in Log_{b_0} for $m \geq 1$.

The divisor $\mathcal{D}_m$ can be described in terms of the étale cover of Log_B given above. Fix $\varphi : \mathbb{N} \to P$ giving $\mathcal{A}_P \times_{\mathcal{A}} B \to \mathrm{Log}_B$ étale. For each toric divisor D_τ of $\operatorname{Spec} \Bbbk[P]$, there is a monoid homomorphism $P \to \mathbb{N}$ given by order of vanishing, i.e., $p \mapsto \operatorname{ord}_{D_\tau}(z^p)$. The composition $\mathbb{N} \to \mathbb{N}$ is multiplication by the integer m_τ, precisely the order of vanishing of $z^{\varphi(1)}$ along D_τ. Also write φ for the induced maps $\varphi : \operatorname{Spec} \Bbbk[P] \to \mathbb{A}^1$ and $\varphi : \mathcal{A}_P \to \mathcal{A}$. At the generic point ξ of D_τ, the map $\overline{\mathcal{M}}_{\mathbb{A}^1, \varphi(\bar{\xi})} \to \overline{\mathcal{M}}_{\operatorname{Spec} \Bbbk[P], \bar{\xi}}$ coincides with $m_\tau : \mathbb{N} \to \mathbb{N}$ if $m_\tau > 0$; otherwise it is the map $0 \to \mathbb{N}$. Thus we see that the pull-back of $\mathcal{D}_m$ to $\mathcal{A}_P \times_{\mathcal{A}} B$ is $\sum_{D_\tau : m_\tau = m} [D_\tau / \operatorname{Spec} \Bbbk[P^{\mathrm{gp}}]] \times_{\mathcal{A}} B$. In particular, $\mathcal{D}_m$ is generically reduced, and we can write as divisors

decomp-Log

$$\varphi^*(b_0) = \sum_\tau m_\tau [D_\tau / \operatorname{Spec} \Bbbk[P^{\mathrm{gp}}]] \times_{\mathcal{A}} B. \tag{3.3.3}$$

under the induced map $\varphi : \mathcal{A}_P \times_{\mathcal{A}} B \to B$.

Consider a finite type open substack $U \subset \mathrm{Log}_B$. By [Ols03, Corollary 5.25], there is a finite set of monoid homomorphisms $\{\phi_i : \mathbb{N} \to P_i\}$ such that U is contained in the image of $\cup \phi_i : \cup_i \mathcal{A}_{P_i} \times_{\mathcal{A}} B \to \mathrm{Log}_B$. To prove (3.3.1), we may first apply the identity (3.3.3) to the image of $\cup \phi_i$, then restrict it to U. This implies (4). ♠

position-m

3.4. **Decomposition for the moduli space: first step.** We now fix a proper and logarithmically smooth morphism $X \to B$ with B a smooth curve with divisorial logarithmic structure given by $b_0 \in B$. Fix a class β of stable logarithmic map. The moduli space $\mathscr{M}(X/B, \beta)$ is neither a toric variety nor logarithmically smooth over B. Its saving grace is the fact that it has a perfect obstruction theory over $\mathfrak{M}_B$, see Section 2.4. For $b \in B$ an arbitrary closed point, $b \neq b_0$, we have the following diagram with all squares Cartesian, as is easily checked with the same argument as in Lemma 3.3.2, (2):

$$\begin{tikzcd}
\mathscr{M}(X_b/b, \beta) \arrow[r, hook, "j_b"] \arrow[d] & \mathscr{M}(X/B, \beta) \arrow[d] & \mathscr{M}(X_0/b_0, \beta) \arrow[l, hook', "j_{b_0}"'] \arrow[d] \\
\mathfrak{M}_b \arrow[r, hook] \arrow[d] & \mathfrak{M}_B \arrow[d] & \mathfrak{M}_{b_0} \arrow[l, hook'] \arrow[d] \\
\mathrm{Log}_b \arrow[r, hook] \arrow[d] & \mathrm{Log}_B \arrow[d] & \mathrm{Log}_{b_0} \arrow[l, hook'] \arrow[d] \\
b \arrow[r, hook] & B & b_0 \arrow[l, hook']
\end{tikzcd}$$

Consider the complex $\mathbf{E}^\bullet := \left(R\pi_* f^* T_{X/B}\right)^\vee$, where $T_{X/B}$ stands for the logarithmic tangent bundle and the dual is taken in the derived sense: $\mathbf{F}^\vee := R\mathrm{Hom}(\mathbf{F}, \mathcal{O}_W)$.

This is a perfect 2-term complex supported in degrees 0 and -1 admitting a morphism to the cotangent complex $\mathbf{L}_{\mathscr{M}(X/B,\beta)\,/\,\mathfrak{M}_B}$. Since $\mathfrak{M}_B$ is pure-dimensional, this provides a well-defined virtual fundamental class $[\mathscr{M}(X/B,\beta)]^{\mathrm{virt}}$, as shown in [GS13, §5].

The obstruction theory for $\mathscr{M}(X/B,\beta)$ pulls back to the obstruction theory for $\mathscr{M}(X_b/b,\beta)$ and $\mathscr{M}(X_0/b_0,\beta)$, and hence by [BF97, Proposition 7.2], we have:

Proposition 3.4.1.
$$[\mathscr{M}(X_b/b,\beta)]^{\mathrm{virt}} = j_b^![\mathscr{M}(X/B,\beta)]^{\mathrm{virt}}$$
and
$$[\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}} = j_{b_0}^![\mathscr{M}(X/B,\beta)]^{\mathrm{virt}}.$$

This allows us to focus on $\mathscr{M}(X_0/b_0,\beta)$.

Write
$$\mathfrak{M}_m := \mathfrak{M}_{b_0} \times_{\mathrm{Log}_{b_0}} \mathcal{D}_m$$
$$\mathscr{M}_m(X_0/b_0,\beta) := \mathscr{M}(X_0/b_0,\beta) \times_{\mathrm{Log}_{b_0}} \mathcal{D}_m,$$
which amounts to adding a column on the right of the diagram above

$$\begin{tikzcd}
\mathscr{M}(X/B,\beta) \arrow[d] & \mathscr{M}(X_0/b_0,\beta) \arrow[l, hook', "j_{b_0}"'] \arrow[d] & \mathscr{M}_m(X_0/b_0,\beta) \arrow[l, hook'] \arrow[d] \\
\mathfrak{M}_B \arrow[d] & \mathfrak{M}_{b_0} \arrow[l, hook'] \arrow[d] & \mathfrak{M}_m \arrow[l, hook'] \arrow[d] \\
\mathrm{Log}_B \arrow[d] & \mathrm{Log}_{b_0} \arrow[l, hook'] \arrow[d] & \mathcal{D}_m \arrow[l, hook'] \\
B & b_0 \arrow[l, hook'] &
\end{tikzcd}$$

Note $\mathscr{M}_m(X_0/b_0,\beta)$ has a natural perfect obstruction theory over $\mathfrak{M}_m$, pulled back from $\mathscr{M}(X_0/b_0,\beta) \to \mathfrak{M}_{b_0}$. There is a natural map we also denote $i_m : \mathscr{M}_m(X_0/b_0,\beta) \to \mathscr{M}(X_0/b_0,\beta)$. Applying a result of Costello, [Cos06] as refined by Manolache [Man12], gives the following:

Proposition 3.4.2.

q:decomposition-1

$$[\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}} = \sum_m m \cdot i_{m*}[\mathscr{M}_m(X_0/b_0,\beta)]^{\mathrm{virt}} \tag{3.4.1}$$

Proof. First consider the closed substack $m\mathcal{D}_m$ of Log_{b_0} defined as follows on the étale cover of Log_{b_0} described in the proof of Lemma 3.3.2. A choice $\varphi : \mathbb{N} \to P$ yields an element $\mathcal{A}_P \times_{\mathcal{A}} b_0$ of the étale cover of Log_{b_0}. Now with $b_0 \to \operatorname{Spec} \Bbbk[\mathbb{N}]$ taking b_0 to 0, this element of the cover is a quotient of $\operatorname{Spec} \Bbbk[P] \times_{\operatorname{Spec} \Bbbk[\mathbb{N}]} b_0$ by a torus action. The latter scheme is $\operatorname{Spec} \Bbbk[P]/(z^{\varphi(1)})$. The ideal $(z^{\varphi(1)})$ has a primary decomposition in $\Bbbk[P]$, $(z^{\varphi(1)}) = \bigcap_{i \in I} \mathfrak{q}_i$ with $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ a prime of height one. Let $J \subseteq I$ be the index set corresponding to those i such that $\mathfrak{p}_i$ corresponds to an irreducible component of $\mathcal{D}_m$ pulled back to $\operatorname{Spec} \Bbbk[P]$. Then $m\mathcal{D}_m$ is the closed substack which pulls back the closed subscheme of $\operatorname{Spec} \Bbbk[P]$ defined by the ideal $\bigcap_{i \in J} \mathfrak{q}_i$.

By Equation (3.3.2) we have that $\sqcup(m\mathcal{D}_m) \to \mathrm{Log}_{b_0}$ is of pure degree 1 in the sense of [Cos06, Theorem 5.0.1] and [Man12, Prop. 5.29]. Consider the cartesian diagram

$$\begin{tikzcd} \coprod_m \left(\mathscr{M}(X_0/b_0, \beta) \times_{\mathrm{Log}_{b_0}} m\mathcal{D}_m \right) \arrow[r, "\Phi"] \arrow[d] & \mathscr{M}(X_0/b_0, \beta) \arrow[d] \\ \coprod_m (m\mathcal{D}_m) \arrow[r] & \mathrm{Log}_{b_0}. \end{tikzcd}$$

The pullback of the perfect obstruction theory of $\mathscr{M}(X_0/b_0, \beta)$ relative to $\mathfrak{M}_{b_0}$ is a perfect obstruction theory of $\coprod_m \left(\mathscr{M}(X_0/b_0, \beta) \times_{\mathrm{Log}_{b_0}} m\mathcal{D}_m \right)$ relative to $\coprod_m \mathfrak{M}_{b_0} \times_{\mathrm{Log}_{b_0}} (m\mathcal{D}_m)$. It follows from [Cos06, Theorem 5.0.1] or [Man12, Propostion 5.29] that

push-union

$$\Phi_* \left[\coprod \left(\mathscr{M}(X_0/b_0, \beta) \times_{\mathrm{Log}_{b_0}} m\mathcal{D}_m \right) \right]^{\mathrm{virt}} = [\mathscr{M}(X_0/b_0, \beta)]^{\mathrm{virt}}. \tag{3.4.2}$$

For each m consider the cartesian diagram

$$\begin{tikzcd} \mathscr{M}_m(X_0/b_0, \beta) \arrow[r, "i'_m"] \arrow[d] & \mathscr{M}(X_0/b_0, \beta) \times_{\mathrm{Log}_{b_0}} m\mathcal{D}_m \arrow[d] \\ \mathcal{D}_m \arrow[r] & m\mathcal{D}_m. \end{tikzcd}$$

The morphism $\mathcal{D}_m \to m\mathcal{D}_m$ is of pure degree $1/m$, in the sense that $[\mathcal{D}_m] = (1/m)[m\mathcal{D}_m]$. Again by [Man12, Propostion 5.29]

Eq:push-m

$$m \cdot i'_{m*}[\mathscr{M}_m(X_0/b_0, \beta)]^{\mathrm{virt}} = \left[\mathscr{M}(X_0/b_0, \beta) \times_{\mathrm{Log}_{b_0}} m\mathcal{D}_m \right]^{\mathrm{virt}}. \tag{3.4.3}$$

Combining (3.4.2) and (3.4.3) we obtain the result. ♠

As it stands, (3.4.1) is not easy to use: it says that a piece $\mathscr{M}_m(X_0/b_0, \beta)$ of the moduli space appears with multiplicity m. We need to describe in a natural, combinatorial way what $\mathscr{M}_m(X_0/b_0, \beta)$ is. The combinatorics of logarithmic structures provides an avenue to do this. A bridge to that combinatorial picture is provided by an unobstructed variant $\mathfrak{M}_m(\mathcal{X}_0/b_0, \beta)$ of $\mathscr{M}_m(X_0/b_0, \beta)$.

sec:mainth

4. Tropical moduli spaces and the main theorem

We now show how to construct the desired set Ω in Theorem 1.1.2. This will be the set of isomorphism classes of tropical curves mapping to $\Sigma(X)$ which are rigid, in a sense we will define below. The integer m_τ will be read off immediately from such a tropical curve.

First we introduce a stack $\mathcal{X}$ and prestable logarithmic maps in $\mathcal{X}/B$ which serve as a bridge from geometry to combinatorics.

4.1. Prestable logarithmic maps in $\mathcal{X}$.

Define $\mathcal{X} = \mathcal{A}_X \times_{\mathcal{A}_B} B$ and $\mathcal{X}_0 = \mathcal{X} \times_B \{b_0\} = \mathcal{A}_X \times_{\mathcal{A}_B} \{b_0\}$. Associated to any cone $\sigma \in \Sigma(X)$ is a (locally closed) stratum $X_\sigma \subseteq X$, and we write X_σ^{cl} for the closure of this stratum. Similarly we write $\mathcal{X}_\sigma$ and $\mathcal{X}_\sigma^{\mathrm{cl}}$ for the correspondeing strata of $\mathcal{X}$. In particular, the map $X \to \mathcal{X}$ provides a one-to-one correspondence between logarithmic strata $X^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ of X and logarithmic strata $\mathcal{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ of $\mathcal{X}$. We note that $X^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)} = \mathcal{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)} \times_{\mathcal{X}} X$, and similarly for the underlying schemes.

We use the notation $\beta' = (g, u_{p_i})$ for discrete data in $\mathcal{X}/B$ or $\mathcal{X}_0/b_0$: these are the same as discrete data $\beta = (g, A, u_{p_i})$ in X or X_0 with the curve class A removed.

Denote by $\mathfrak{M}(\mathcal{X}/B, \beta')$ the stack of basic prestable logarithmic maps in $\mathcal{X}/B$, with its natural logarithmic structure. By [ACMW14] it is an algebraic stack provided with a morphism $\mathfrak{M}(\mathcal{X}/B, \beta') \to \mathfrak{M}(\underline{\mathcal{X}}/B, \underline{\beta}')$. Restricting to b_0 we obtain a morphism of algebraic stacks $\mathfrak{M}(\mathcal{X}_0/b_0, \beta') \to \mathfrak{M}(\underline{\mathcal{X}}_0, \underline{\beta}')$. Using notation of Section 3.4 we may further restrict and define

$$\mathfrak{M}_m(\mathcal{X}_0/b_0, \beta') = \mathfrak{M}(\mathcal{X}, \beta') \times_{\mathrm{Log}_B} \mathcal{D}_m = \mathfrak{M}(\mathcal{X}, \beta') \times_{\mathfrak{M}_B} \mathfrak{M}_m.$$

Since $X \to \mathcal{X}$ is strict we obtain natural strict morphisms

$$\mathscr{M}(X/B, \beta) \to \mathfrak{M}(\mathcal{X}/B, \beta')$$
$$\mathscr{M}(X_0/b_0, \beta) \to \mathfrak{M}(\mathcal{X}_0/b_0, \beta')$$
$$\mathscr{M}_m(X_0/b_0, \beta) \to \mathfrak{M}_m(\mathcal{X}_0/b_0, \beta')$$

The key fact is

Proposition 4.1.1. *The morphisms*

$$\mathfrak{M}(\mathcal{X}/B, \beta') \to \mathfrak{M}_B$$
$$\mathfrak{M}(\mathcal{X}_0/b_0, \beta') \to \mathfrak{M}_{b_0}$$
$$\mathfrak{M}_m(\mathcal{X}_0/b_0, \beta') \to \mathfrak{M}_m$$

are strict étale.

Proof. The morphisms are strict by definition.

Recall that $\mathcal{A}_X \to \mathcal{A}$ is logarithmically étale. It follows that $\mathcal{X} = \mathcal{A}_X \times_{\mathcal{A}} B$ is logarithmically étale over B and so $\mathcal{X}_0 \to b_0$ is logarithmically étale. The fact that $\mathfrak{M}(\mathcal{X}/B, \beta') \to \mathfrak{M}_B$ is étale is equivalent to $\mathfrak{M}(\mathcal{X}/B, \beta') \to \mathfrak{M}_{g,k} \times B$ being logarithmically étale, which is proven in [AW13, Proposition 3.1.2]. The result for $\mathfrak{M}(\mathcal{X}_0/b_0, \beta')$ and $\mathfrak{M}_m(\mathcal{X}_0/b_0, \beta')$ follows by pulling back. ♠

Since $\mathfrak{M}_B \to B$ is logarithmically smooth it follows that $\mathfrak{M}(\mathcal{X}/B, \beta') \to B$ is a toroidal morphism. Corollary 3.2.1 says that a meaningful decomposition result for $\mathfrak{M}(\mathcal{X}_0/b_0, \beta')$ or $\mathfrak{M}_m(\mathcal{X}_0/b_0, \beta')$ results from a meaningful description of rays in the polyhedral cone complex of $\mathfrak{M}(\mathcal{X}/B, \beta')$. This is where Section 2.5 becomes useful.

Let $W = \mathrm{Spec}(Q \to \Bbbk)$ and let $(C/W, \mathbf{p}, f : C \to \mathcal{X})$ be an object of $\mathfrak{M}(\mathcal{X}_0/b_0, \beta')$, namely a basic prestable logarithmic map to $\mathcal{X}/B$ lying over b_0. We recall that in Section 2.5.5 we introduced a family of tropical curves in $\Sigma(X)$ over the cone $\tau_Q := (Q^\vee_{\mathbb{R}}, Q^*)$, which is universal since $(C/W, \mathbf{p}, f)$ is basic, see Section 2.5.6. We denote by $\widetilde{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, u)$ the combinatorial type underlying this family. We write τ for the isomorphism class of the combinatorial type $\widetilde{\tau}$ under graph isomorphisms fixing $\mathbf{g}, \boldsymbol{\sigma}$ and u.

able-strata **Proposition 4.1.2.** *(1) The basic prestable map*

$$(C/W, \mathbf{p}, f : C \to \mathcal{X}) \in \mathfrak{M}(\mathcal{X}_0/b_0, \beta')\,(\Bbbk)$$

belongs to a codimension-1 stratum of $\mathfrak{M}(\mathcal{X}/B, \beta')$ if and only if $Q \simeq \mathbb{N}$; equivalently the corresponding tropical moduli space is a ray.

(2) Assume $Q \simeq \mathbb{N}$ with generator v, let $g : Q^\vee \to \mathbb{N}$ be the map associated to $\tau_Q \to \Sigma(B)$ and $m_\tau = g(v)$. The map $(C/W, \mathbf{p}, f : C \to \mathcal{X})$ lies in $\mathfrak{M}(\mathcal{X}_0/b_0, \beta')$ if and only if $m_\tau \neq 0$, in which case

$$(C/W, \mathbf{p}, f : C \to \mathcal{X}) \in \mathfrak{M}_{m_\tau}(\mathcal{X}_0/b_0, \beta').$$

Denote by $\mathfrak{M}(\mathcal{X}_0/b_0, \beta')_\tau \subset \mathfrak{M}_{m_\tau}(\mathcal{X}_0/b_0, \beta')$ the divisor corresponding to prestable maps whose corresponding tropical curve has discrete data τ. Denote also

$$\mathscr{M}(X_0/b_0, \beta)_\tau = \mathfrak{M}(\mathcal{X}_0/b_0, \beta')_\tau \times_{\mathfrak{M}_{m_\tau}(\mathcal{X}_0/b_0, \beta')} \mathscr{M}(X_0/b_0, \beta).$$

Corollary 3.2.1 gives a decomposition

$$[\mathfrak{M}(\mathcal{X}_0/b_0, \beta')] = \sum_\tau m_\tau [\mathfrak{M}(X_0/b_0, \beta)_\tau],$$

and an application of Costello's theorem to Equation 3.4.1 gives

position-2 (4.1.1)
$$[\mathscr{M}(X_0/b_0, \beta)]^{\mathrm{virt}} = \sum_\tau m_\tau \cdot i_{m_\tau *}[\mathscr{M}(X_0/b_0, \beta)_\tau]^{\mathrm{virt}}.$$

Below we make this more precise, in a way amenable to immediate computations and appropriate for our work on the degeneration formula.

We start by explicitly delineating the available combinatorial data.

Sec:decorated

4.2. **Decorated tropical curves.** We fix $X \to B$ logarithmically smooth over the one-dimensional base (B, b_0) as usual. As in the discussion in §2.5, we can use the fact that X is Zariski and logarithmically smooth over B to conclude that X is simple, and hence take the reduced presentation for $\Sigma(X)$. The cones of $\Sigma(X)$ are then indexed by the strata of X.

Let $\rho : \Sigma(X) \to \Sigma(B) = \mathbb{R}_{\geq 0}$ be the induced map on generalized cone complexes.[10] We will write $\Delta(X) = \rho^{-1}(1)$, which can be thought of as a generalized polyhedral complex.

Note that just as the cones of $\Sigma(X)$ are in one-to-one correspondence with strata of X, the polyhedra of $\Delta(X)$ are in one-to-one correspondence with the strata of X contained in X_0. If $\sigma \in \Delta(X)$, we write X_σ for the corresponding (locally closed) strata of X_0 and X_σ^{cl} for its closure.

decoratedef

Definition 4.2.1. A *decorated tropical curve* $(\Gamma \to \Delta(X), \mathbf{A})$ *in* $\Delta(X)$ consists of the following data:

(i) A tropical curve $\Gamma \to \Sigma(X)$ as in Definition 2.5.3 factoring through the inclusion $\Delta(X) \hookrightarrow \Sigma(X)$.

(ii) Decoration:
$$\mathbf{A} : V(G) \to \coprod_{\sigma \in \Delta(X)} H_2(X_\sigma^{\mathrm{cl}}),$$
with $\mathbf{A}(v) \in H_2(X^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)})$.

The *decorated graph* (or *combinatorial type*) $(\widetilde{\tau}, \mathbf{A})$ of a decorated tropical curve in $\Delta(X)$ is the combinatorial type $\widetilde{\tau} = (G, \mathbf{g}, \boldsymbol{\sigma}, u)$ of a tropical curve (see Definition 2.5.3) in $\Sigma(X)$ with all u_x mapping to 0 under the map $N_{\boldsymbol{\sigma}(x)} \to \mathbb{Z} = N_{\Sigma(B)}$, along with the data $\mathbf{A}$ in (ii) above.

Recall that given a tropical curve, we have defined its genus as
$$g(\Gamma) = b_1(\Gamma) + \sum_v \mathbf{g}(v).$$
Given a decoration $\mathbf{A}$, we define its curve class as
$$A(\mathbf{A}) = \sum_v \mathbf{A}(v) \in H_2(X_0),$$
where $\mathbf{A}(v)$ is viewed as an element of $H_2(X_0)$ via the push-foward map $H_2(X^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}) \to H_2(X_0)$. When $A = A(\mathbf{A})$ we say that $\mathbf{A}$ is a partition of A and write $\mathbf{A} \vdash A$. Clearly both genus $g(\Gamma)$ and curve class $A(\mathbf{A})$ depend only on the combinatorial type $(\widetilde{\tau}, \mathbf{A})$.

[10] In fact these are cone complexes, since Zariski logarithmically smooth schemes have no monodromy. Find the precise reference for this!

An *isomorphism* ϕ between decorated tropical curves $(f_1 : \Gamma_1 \to \Delta(X), \mathbf{A}_1)$ and $(f_2 : \Gamma_2 \to \Delta(X), \mathbf{A}_2)$ in $\Delta(X)$ is an isomorphism $\phi : \Gamma_1 \to \Gamma_2$ of tropical curves (necessarily preserving the genus decoration and ordering of legs), such that

1. $f_1 = f_2 \circ \phi$,
2. $\mathbf{A}_2(\phi(v)) = \mathbf{A}_1(v)$ for all $v \in V(G)$,
3. $u_{\phi(q)} = u_q$ and $u_{\phi(p)} = u_p$ for all edges and legs.

This defines the *automorphism group* $\mathrm{Aut}(\Gamma \to \Delta(X), \mathbf{A})$ of a decorated tropical curve in $\Delta(X)$. In the same manner we define isomorphisms of decorated graphs and the *automorphism group* $\mathrm{Aut}(\widetilde{\tau}, \mathbf{A})$ of a decorated graph $(\widetilde{\tau}, \mathbf{A})$, and similarly for $\mathrm{Aut}(\widetilde{\tau})$.

4.3. **Contractions and rigid curves.** Fix the combinatorial type $(\widetilde{\tau}, \mathbf{A})$ of a decorated tropical curve in $\Delta(X)$. The space $M^{trop}_{\widetilde{\tau},\mathbf{A}}(\Delta(X))$ of decorated tropical curves $(f : \Gamma \to \Delta(X), \mathbf{A})$ in $\Delta(X)$ with these data is an open, possibly unbounded, polyhedron determined by the positions $f(v)$ of vertices of Γ and the lengths $\ell(E_q)$ of the compact edges for which $u_q = 0$. Note that we consider maps with the fixed graph G and do not identify maps differing by an automorphism in $\mathrm{Aut}(\Gamma \to \Delta(X), \mathbf{A})$. This polyhedron is rational, since $\Delta(X)$ is a complex of rationally defined polytopes and the equations in Definition 2.5.3(6)(b)(iii) have rational coefficients.

The interior of each face of $M^{\mathrm{trop}}_{\widetilde{\tau},\mathbf{A}}(\Delta(X))$ is naturally identified with $M^{\mathrm{trop}}_{\widetilde{\tau}',\mathbf{A}'}(\Delta(X))$, where $(\widetilde{\tau}', \mathbf{A}'), \widetilde{\tau}' = (G', \mathbf{g}', \boldsymbol{\sigma}', u')$ is a *contraction* of the decorated graph $(\widetilde{\tau}, \mathbf{A})$, namely:

1. $\pi : G \to G'$ is a graph contraction, preserving the ordering of legs $L(G) = L(G')$.
2. For $v' \in V(G')$we have $\mathbf{g}'(v') = g(\pi^{-1}(v'))$, the genus of the inverse image graph with genus function $\mathbf{g}|_{\pi^{-1}(v')}$.
3. Whenever $\pi(v) = v'$ we have that $\boldsymbol{\sigma}'(v')$ is a face of $\boldsymbol{\sigma}(v)$.
4. For v' a vertex of G' we have $\mathbf{A}(v') = \sum_{\pi(v)=v'} \mathbf{A}(v)$. Here $\mathbf{A}(v)$ is viewed as a curve class on the stratum $X^{\mathrm{cl}}_{\boldsymbol{\sigma}(v')}$, which contains $X^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ by the previous condition.
5. Whenever an edge $E_q = E_{q'}$ is not contracted under π we have $\boldsymbol{\sigma}'(E_{q'})$ is a face of $\boldsymbol{\sigma}(E_q)$.
6. Whenever an edge $E_q = E_{q'}$ is not contracted under π we have $u_{q'} = u_q$.
7. For every leg $E_p = E_{p'}$ we have $\boldsymbol{\sigma}'(E_{p'})$ is a face of $\boldsymbol{\sigma}(E_p)$.
8. For every leg $E_p = E_{p'}$ we have $u_{p'} = u_p$.

rigiddef **Definition 4.3.1.** A decorated tropical curve in $\Delta(X)$ is *rigid* if it is not contained in a non-trivial family of decorated tropical curves of the same decorated graph. In other words, the relevant polyhedron $M^{\mathrm{trop}}_{\widetilde{\tau},\mathbf{A}}(\Delta(X))$ is a point. Note that this notion depends only on the decorated graph $(\widetilde{\tau}, A)$, or just $\widetilde{\tau}$, so it makes sense to say that

$(\widetilde{\tau}, A)$ is rigid. Thus when $(\widetilde{\tau}, A)$ is rigid it uniquely determines a rigid decorated tropical curve in $\Delta(X)$, and we will sometimes refer to $(\widetilde{\tau}, A)$ as a rigid decorated tropical curve.

As we saw above, the cells of the tropical moduli space are defined by linear inequalities with rational coefficients, therefore a rigid tropical curve must be rationally defined, i.e., $f(v_\eta)$ are rational points of $\Delta(X)$ and $\ell(E_q) \in \mathbb{Q}$. We define the *multiplicity* of a rigid decorated tropical curve $m_{\widetilde{\tau}} \in \mathbb{Z}_{>0}$ to be the smallest positive integer $m_{\widetilde{\tau}}$ such that $m_{\widetilde{\tau}}\ell(E_q) \in \mathbb{Z}$ for all edges E_q and $m_{\widetilde{\tau}} f(v)$ is integral (i.e., lies in $N_{\boldsymbol{\sigma}(v)}$) for each vertex v of Γ.

We will see in Proposition 4.6.1 that $m_{\widetilde{\tau}}$ is compatible with the multiplicity m_τ of Proposition 4.1.2.

4.4. **Decorated logarithmic maps in X_0.** Fix a rigid decorated graph $(\widetilde{\tau}, \mathbf{A})$ in $\Delta(X)$. We explicitly write each edge E_q as a pair of half-edges named $E_{q,1}, E_{q,2}$; a leg is already considered a half-edge, having only one endpoint. We define the stratum function on half-edges by $\boldsymbol{\sigma}(E_{q,i}) := \boldsymbol{\sigma}(E_q)$. For each vertex v of Γ we write H_v for the set of half-edges (of the form $E_{q,i}$ or E_p) incident to v. We write $\beta_v = (\mathbf{g}(v), u|_{H_v}, \mathbf{A}(v))$.

Def:decorated-map

Definition 4.4.1. A *stable logarithmic map in X_0 over a base scheme S of class β marked by $(\widetilde{\tau}, \mathbf{A})$* is the following data:

(1) An object $f : C \to X_0$ of $\mathscr{M}_{m_{\widetilde{\tau}}}(X_0/b_0, \beta) := \mathscr{M}(X_0/b_0, \beta) \times_{\mathrm{Log}_{b_0}} \mathcal{D}_{m_{\widetilde{\tau}}}$ over the scheme S.

(2) For each vertex v a stable map $\underline{f}_v : \underline{C}_v \to \underline{X}^{\mathrm{cl}}_{\sigma(v)}$, an object of $\mathscr{M}(\underline{X}^{\mathrm{cl}}_{\sigma(v)}, \underline{\beta}_v)$ over S (with genus $\mathbf{g}(v)$, curve class $\mathbf{A}(v)$, and markings $s_E : S \to \sqcup \underline{C}_v$ labelled by $E \in H_v$) with the marked point corresponding to the half-edge E landing in the stratum $X^{\mathrm{cl}}_{\boldsymbol{\sigma}(E)}$.

These data must satisfy

(i) the underlying curve must coincide with the gluing

$$\underline{C} = (\sqcup \underline{C}_v) / \langle s_{E_{q,1}} = s_{E_{q,2}} \rangle,$$

and

(ii)

$$\underline{f}|_{\underline{C}_v} = \underline{f}_v.$$

A *morphism* of stable logarithmic map in X_0 over S marked by $(\widetilde{\tau}, \mathbf{A})$ is defined as a fiber square as usual.

We denote by $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0, \beta)$ the category of stable logarithmic maps in X_0 over S marked by $(\widetilde{\tau}, \mathbf{A})$.

We denote

$$\mathscr{M}_{\widetilde{\tau}}(X_0/b_0,\beta) := \coprod_{\mathbf{A}\vdash A} \mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0,\beta).$$

duli-scheme **Proposition 4.4.2.** *(1) The category $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0,\beta)$ of stable logarithmic maps in X_0 over S marked by $(\widetilde{\tau},\mathbf{A})$ is a proper Deligne-Mumford stack.*

(2) The mapping $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0,\beta) \to \mathscr{M}_{m_{\widetilde{\tau}}}(X_0/b_0,\beta)$ sending an object of $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0,\beta)$ to stable logarithmic map $f : C \to X_0$ is a morphism of algebraic stacks invariant under $\mathrm{Aut}(\widetilde{\tau},\mathbf{A})$.

The proof is given below along with the proof of Proposition 4.5.2.

4.5. **Decorated logarithmic maps in $\mathcal{X}_0$.** To define a virtual fundamental class on $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0,\beta)$ we use prestable maps in $\mathcal{X}_0/b_0$. As for β' we use the notation $\beta'_v = (\mathbf{g}(v), u|_{H_v})$ for discrete data of maps in $\mathcal{X}_{\boldsymbol{\sigma}(v)}$: these are the same β_v with the curve classes $\mathbf{A}(v)$ removed.

Definition 4.5.1. A *logarithmic map in $\mathcal{X}_0$ over b_0 marked by $\widetilde{\tau}$* is the following data:

(1) A *prestable* logarithmic map $f : C \to \mathcal{X}_0$ which is an object of $\mathfrak{M}_{m_{\widetilde{\tau}}}(\mathcal{X}_0/b_0,\beta') := \mathfrak{M}(\mathcal{X}_0/b_0,\beta') \times_{\mathrm{Log}_{b_0}} \mathcal{D}_{m_{\widetilde{\tau}}}$.

(2) For each vertex v a prestable map $\underline{f}_v : \underline{C}_v \to \underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ over $\underline{S}$, which is an object of $\mathfrak{M}(\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}, \underline{\beta}'_v)$ with genus $\mathbf{g}(v)$ and markings given by H_v. We impose the additional condition that the marking corresponding to half-edge E lands in the stratum $\mathcal{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(E)}$.

These data must satisfy the same conditions (i) and (ii) as in Definition 4.4.1.

We denote by $\mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta')$ the category of stable logarithmic maps in $\mathcal{X}_0$ over S marked by $\widetilde{\tau}$.

oduli-stack **Proposition 4.5.2.** *(1) The category $\mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta')$ of prestable logarithmic maps in X_0 over S marked by $\widetilde{\tau}$ is an algebraic stack.*

(2) The mapping $\mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta') \to \mathfrak{M}_{m_{\widetilde{\tau}}}(\mathcal{X}_0/b_0,\beta')$ sending an object of $\mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta')$ to a prestable logarithmic map $f : C \to \mathcal{X}_0$ is a morphism of algebraic stacks invariant under $\mathrm{Aut}(\widetilde{\tau})$.

(3) The mapping $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0,\beta) \to \mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0,\beta')$ composing maps of an object of $\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0,\beta)$ with the projection $X_0 \to \mathcal{X}_0$ is a morphism of algebraic stacks.

Proof of Propositions 4.4.2 and 4.5.2. Step 1: stacks parametrizing f and $\underline{f}_v$. By [GS13, AC11, ACMW14] the category $\mathscr{M}(X_0/b_0,\beta)$ is a proper Deligne–Mumford stack, see Section 2.3 above. It comes with a morphism $\mathscr{M}(X_0/b_0,\beta) \to \mathscr{M}(\underline{X}_0,\underline{\beta})$.

Similarly, by [ACMW14], $\mathfrak{M}(\mathcal{X}_0/b_0,\beta')$ is an algebraic stack endowed with a morphism $\mathfrak{M}(\mathcal{X}_0/b_0,\beta')\to\mathfrak{M}(\underline{\mathcal{X}_0},\underline{\beta'})$.

Since $\mathscr{M}(X_0/b_0,\beta)\to\mathfrak{M}(\mathcal{X}_0/b_0,\beta')$ is a morphism of fibered categories, it is a morphism of algebraic stacks. Considering the fibered products we obtain a morphism $\mathscr{M}_{m_{\tilde\tau}}(X_0/b_0,\beta)\to\mathfrak{M}_{m_{\tilde\tau}}(\mathcal{X}_0/b_0,\beta')$.

By [BM96, Theorem 3.14] the stack $\mathscr{M}(\underline{X}^{\mathrm{cl}}_{\boldsymbol\sigma(v)},\underline{\beta}_v)$ is a proper Deligne-Mumford stack. Similarly, by [Wis14, Corollary 1.1.1] the stack $\mathfrak{M}(\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol\sigma(v)},\underline{\beta}'_v)$ is algebraic.

Again we obtain morphisms of algebraic stacks $\mathscr{M}(\underline{X}^{\mathrm{cl}}_{\boldsymbol\sigma(v)},\underline{\beta}_v)\to\mathfrak{M}(\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol\sigma(v)},\underline{\beta}'_v)$.

Step 2: requiring marked points to land in the correct strata. For each half-edge $E\in H_v$ we have an evaluation map $e_E:\mathscr{M}(\underline{X}^{\mathrm{cl}}_{\boldsymbol\sigma(v)},\underline{\beta}_v)\to\underline{X}^{\mathrm{cl}}_{\boldsymbol\sigma(v)}$. Define $\mathscr{M}_v:=\bigcap_E e_E^{-1}\underline{X}^{\mathrm{cl}}_{\boldsymbol\sigma(E)}$. This is the proper Deligne–Mumford stack parametrizing maps where the marked point corresponding to E lands in the stratum $\boldsymbol\sigma(E)$. Replacing $\underline{X}^{\mathrm{cl}}_{\boldsymbol\sigma(v)}$ by $\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol\sigma(v)}$ we obtain evaluations $\epsilon_E:\mathfrak{M}(\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol\sigma(v)},\underline{\beta}'_v)\to\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol\sigma(v)}$ and an algebraic stack $\mathfrak{M}_v:=\bigcap_E\epsilon_E^{-1}\underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol\sigma(E)}$.

Step 3: requiring maps to glue at nodes. Let $\mathscr{M}^{prod}=\prod_v\mathscr{M}_v$. The group $\mathrm{Aut}(\tilde\tau)$ acts on this moduli stack. For each edge E_q of Γ we have two evaluation maps $e_{E_{q,i}}:\mathscr{M}^{prod}\to X^{\mathrm{cl}}_{\boldsymbol\sigma(E_{q_1})}=X^{\mathrm{cl}}_{\boldsymbol\sigma(E_{q_2})}$. Define $\mathscr{M}_q=\mathscr{M}^{prod}\times_{(X^{\mathrm{cl}}_{\boldsymbol\sigma(E_{q_1})})^2}X^{\mathrm{cl}}_{\boldsymbol\sigma(E_{q_1})}$, where the map on the left is $e_{E_{q,1}}\times e_{E_{q,2}}$ and the map on the right is the diagonal. It has a natural morphism $\mathscr{M}_q\to\mathscr{M}^{prod}$. Let $\mathscr{M}^{glue}=\mathscr{M}_{q_1}\times_{\mathscr{M}^{prod}}\cdots\times_{\mathscr{M}^{prod}}\mathscr{M}_{q_{|E(\Gamma)|}}$ be the fibered prduct of all these. It is a proper Deligne–Mumford stack parametrizing maps $\underline{f}_v$ with gluing data for a stable map. It therefore carries a family of glued stable maps $\underline{f}:\underline{C}^{glue}\to\underline{X}_0$ by the universal property of pushouts. Hence we have a morphism $\mathscr{M}^{glue}\to\mathscr{M}_{\underline\beta}(\underline{X}_0)$. The action of $\mathrm{Aut}(\tilde\tau)$ on $\mathscr{M}^{prod}$ clearly lifts to $\mathscr{M}^{glue}$ and the morphism $\mathscr{M}^{glue}\to\mathscr{M}(\underline{X}_0,\underline{\beta})$ is invariant.

Replacing X_0 by $\mathcal{X}_0$ we obtain an algebraic stack $\mathfrak{M}^{glue}$, parametrizing maps $\underline{f}_v$ with gluing data. Hence $\mathfrak{M}^{glue}$ carries a family of glued maps $\underline{f}:\underline{C}^{glue}\to\underline{\mathcal{X}}_0$ providing an invariant morphism $\mathfrak{M}^{glue}\to\mathfrak{M}_{\underline\beta}(\underline{\mathcal{X}}_0)$.

The morphism $\mathscr{M}^{glue}\to\mathfrak{M}^{glue}$ is canonically $\mathrm{Aut}(\tilde\tau)$-equivariant.

Step 4: compatibility of maps. We have

$$\mathscr{M}_{\tilde\tau}(X_0/b_0,\beta)=\mathscr{M}^{glue}\times_{\mathscr{M}(\underline{X}_0,\underline{\beta})}\mathscr{M}_{m_{\tilde\tau}}(X_0/b_0,\beta),$$

hence a proper Deligne-Mumford stack, with morphism to $\mathscr{M}_{m_{\tilde\tau}}(X_0/b_0,\beta)$. The action of $\mathrm{Aut}(\tilde\tau)$ canonically lifts.

Replacing X_0 by $\mathcal{X}_0$ we have an algebraic stack $\mathfrak{M}_{\tilde\tau}(\mathcal{X}_0/b_0)=\mathfrak{M}^{glue}\times_{\mathfrak{M}(\underline{\mathcal{X}}_0,\underline{\beta})}\mathfrak{M}_{m_{\tilde\tau}}(\mathcal{X}_0/b_0,\beta')$ with invariant morphism to $\mathfrak{M}_{m_{\tilde\tau}}(\mathcal{X}_0/b_0,\beta')$. The resulting morphism $\mathscr{M}_{\tilde\tau}(X_0/b_0,\beta)\to\mathfrak{M}_{\tilde\tau}(\mathcal{X}_0/b_0,\beta')$ is canonically equivariant.

♠

4.6. **Costello's diagram.** We denote by τ the isomorphism class of $\widetilde{\tau}$, equivalently the unique isomorphism class of rigid tropical curve with combinatorial type $\widetilde{\tau}$ under graph isomorphism fixing the decorations $\mathbf{g}, \boldsymbol{\sigma}$ and u. We also write $m_\tau := m_{\widetilde{\tau}}$.

We denote

$$\begin{aligned}\mathfrak{M}_\tau(\mathcal{X}_0/b_0, \beta') &:= \left[\mathfrak{M}_{\widetilde{\tau}}(\mathcal{X}_0/b_0, \beta') \Big/ \operatorname{Aut}(\widetilde{\tau})\right],\\ \mathscr{M}_\tau(X_0/b_0, \beta) &:= \left[\mathscr{M}_{\widetilde{\tau}}(X_0/b_0, \beta) \Big/ \operatorname{Aut}(\widetilde{\tau})\right]\end{aligned}$$

and

$$\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0, \beta) := \left[\mathscr{M}_{\widetilde{\tau},\mathbf{A}}(X_0/b_0, \beta) \Big/ \operatorname{Aut}(\widetilde{\tau}, \mathbf{A})\right].$$

It follows that

$$\mathscr{M}_\tau(X_0/b_0, \beta) = \coprod_{\mathbf{A} \vdash A} \mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0, \beta).$$

lo-diagram **Proposition 4.6.1.** *(1) We have a cartesian diagram*

$$\begin{tikzcd} \mathscr{M}_\tau(X_0/b_0, \beta) \arrow[r] \arrow[d] & \mathscr{M}_{m_\tau}(X_0/b_0, \beta) \arrow[d] \\ \mathfrak{M}_\tau(\mathcal{X}_0/b_0, \beta') \arrow[r] & \mathfrak{M}_{m_\tau}(\mathcal{X}_0/b_0, \beta') \end{tikzcd}$$

(2) The morphism

$$\Psi : \coprod_{\tau \,:\, m_\tau = m} \mathfrak{M}_\tau(\mathcal{X}_0/b_0, \beta') \to \mathfrak{M}_m(\mathcal{X}_0/b_0, \beta')$$

is of pure degree 1 in the sense of Costello [Cos06].

Proof. (1) Both the $\operatorname{Aut}(\tilde{\tau})$-invariant composed morphisms

$$\mathscr{M}_{\tilde{\tau}}(X_0/b_0, \beta) \to \mathscr{M}_{m_{\tilde{\tau}}}(X_0/b_0, \beta) \to \mathfrak{M}_{m_{\tilde{\tau}}}(\mathcal{X}_0/b_0, \beta')$$

and

$$\mathscr{M}_{\tilde{\tau}}(X_0/b_0, \beta) \to \mathfrak{M}_{\tilde{\tau}}(\mathcal{X}_0/b_0, \beta') \to \mathfrak{M}_{m_{\tilde{\tau}}}(\mathcal{X}_0/b_0, \beta')$$

send a decorated logarithmic map $C \to X$ to the composite morphism $C \to X \to \mathcal{X}$, hence we obtain an $\operatorname{Aut}(\tilde{\tau})$-equivariant morphism

$$\mathscr{M}_{\tilde{\tau}}(X_0/b_0, \beta) \to \mathfrak{M}_{\tilde{\tau}}(\mathcal{X}_0/b_0, \beta') \times_{\mathfrak{M}_{m_{\tilde{\tau}}}(\mathcal{X}_0/b_0, \beta')} \mathscr{M}_{m_{\tilde{\tau}}}(X_0/b_0, \beta).$$

An element of the fibered product consists of

$$\left(\left(f : C \to \mathcal{X}_0, \underline{f}_v : \underline{C}_v \to \underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)} \right) , \tilde{f} : C \to X_0 \right)$$

where the composite $C \xrightarrow{\tilde f} X_0 \to \mathcal{X}_0$ is f. Since $\underline{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)} = \underline{\mathcal{X}}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)} \times_{\underline{\mathcal{X}}_0} \underline{X}_0$ we obtain morphisms $\underline{\tilde f}_v : \underline{C}_v \to \underline{X}^{\mathrm{cl}}_{\boldsymbol{\sigma}(v)}$ which clearly glue together to the given $\underline{\tilde f} : \underline{C} \to \underline{X}_0$. These morphisms are stable since $\tilde f$ is, giving a morphism

$$\mathfrak{M}_{\widetilde\tau}(\mathcal{X}_0/b_0,\beta') \times_{\mathfrak{M}_{m_{\widetilde\tau}}(\mathcal{X}_0/b_0,\beta')} \mathscr{M}_{m_{\widetilde\tau}}(X_0/b_0,\beta) \to \mathscr{M}_{\widetilde\tau}(X_0/b_0,\beta).$$

The functorial nature of the two morphisms we have constructed shows they are inverse to each other, giving that the diagram is cartesian, as required in part (1).

(2) Consider the open locus $\mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ \subset \mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')$ where

- the curves $\underline{C}_v$ are smooth, and
- the image $\underline{f}_v(\underline{C}_v)$ meets the interior $\underline{\mathcal{X}}_v$ of $\underline{\mathcal{X}}^{\mathrm{cl}}_v$.

It suffices to show:

Claim. The morphism $\coprod \mathfrak{M}_{\tau : m_{\widetilde\tau}=m}(\mathcal{X}_0/b_0,\beta')^\circ \to \mathfrak{M}_{m_{\widetilde\tau}}(\mathcal{X}_0/b_0,\beta')$ induced by Ψ is an open embedding, whose image is the union $\mathfrak{M}_{m_{\widetilde\tau}}(\mathcal{X}_0/b_0,\beta')^\circ$ of open strata of $\mathfrak{M}_{m_{\widetilde\tau}}(\mathcal{X}_0/b_0,\beta')$.

Proposition 4.6.1(2) follows from the claim, since under these conditions, Ψ gives an isomorphism between the open set

$$\coprod_{\tau : m_{\widetilde\tau}=m} \mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ = \Psi^{-1}\Psi\left(\coprod \mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ\right)$$

and its open dense image in $\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')$.

To prove the claim, we first show that

$$\Psi(\coprod_{\tau : m_{\widetilde\tau}=m} \mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ) \quad\subset\quad \mathfrak{M}_m(\mathcal{X}_0/b_0,\beta').$$

Indeed the tropical curve of an object

$$(C/W,\mathbf{p},f) \in \Psi\left(\mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ\right)(\Bbbk)$$

is the rigid tropical curve $\widetilde\tau$, and by Proposition 4.1.2 it lies in a codimension-1 stratum of $\mathfrak{M}(\mathcal{X}/B,\beta')$, namely in $\mathfrak{M}_{m_{\widetilde\tau}}(\mathcal{X}_0/b_0,\beta')^\circ$.

We construct a map in the other direction. For any object $(C/W,\mathbf{p},f) \in \mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')^\circ\,(\Bbbk)$ the isomorphism class τ of its tropical curve $\widetilde\tau$ is determined uniquely by the tropicalization process. Further, since $\mathfrak{M}(\mathcal{X}/B,\beta')$ is log smooth over B, Corollary 3.2.1 gives a one-to-one correspondence between rigid tropical curves and open strata of $\mathfrak{M}(\mathcal{X}_0/b_0,\beta')$. In particular, τ must be an isomorphism class of a *rigid* tropical curve, which does not depend on the choice of $\Bbbk$-point in the stratum. This provides a map $\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')^\circ \to \coprod_\tau \mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ$.

We claim that the multiplicity of $\widetilde\tau$ in this construction coincides with the multiplicity m, namely the index of $Q^\vee \to \mathbb{N}$, of the ray corresponding to the stratum in Proposition 4.1.2(2). Indeed using the notation of that proposition, note that $Q^\vee_{\mathbb{R}}$ parameterizes a family of tropical curves with points of $Q^\vee$ corresponding to those curves whose assigned edge lengths are integral and whose vertices map to integral points of $\Sigma(X)$. Thus a primitive generator of $\tilde\tau$ corresponds precisely to the curve $\Psi(\tilde\tau)$ rescaled by a factor of $m_{\tilde\tau}$. [11]

This provides a morphism $\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')^\circ \to \coprod_{\tau : m_\tau} \mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')^\circ$. It is not difficult to show that this is an inverse of Ψ, as needed. [12] ←11 ←12

♠

4.7. Obstruction theories.

We note that $\mathcal{X}_0$ is logarithmically étale over b_0.

Proposition 4.7.1. *(1) The complex $R\pi_* f^* T_{X_0/\mathcal{X}_0}$ defines a perfect relative obstruction theory both on the morphism $\mathscr{M}_\tau(X_0/b_0,\beta) \to \mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta')$ and on the morphism $\mathscr{M}_m(X_0/b_0,\beta) \to \mathfrak{M}_m(\mathcal{X}_0/b_0,\beta')$. This perfect obstruction theory gives rise to virtual fundamental classes*

$$[\mathscr{M}_\tau(X_0/b_0)]^{\mathrm{virt}} \qquad \text{and} \qquad [\mathscr{M}_m(X_0/b_0)]^{\mathrm{virt}}.$$

(2) The resulting virtual fundamental class on $\mathscr{M}_m(X_0/b_0,\beta)$ coincides with the virtual fundamental class defined relative to $\mathfrak{M}_m$ in (3.4.1).

Proof. (1) Since $\mathcal{X} \to \mathrm{Log}_B$ is étale we have $R\pi_* f^* T_{X/\mathcal{X}} = R\pi_* f^* T_{X/\mathrm{Log}_B}$, which is precisely the complex giving rise to the perfect relative obstruction theory for $\mathscr{M}(X/B,\beta) \to \mathfrak{M}_B$ introduced in [GS13]. Since $\mathfrak{M}(\mathcal{X}/B,\beta) \to \mathfrak{M}_B$ is étale this complex induces a perfect relative obstruction theory for $\mathscr{M}(X/B,\beta) \to \mathfrak{M}(\mathcal{X}/B,\beta)$. It is a general fact, see [BL00, Proposition A.1] or [Wis11, Proposition 6.2], that this induces a relative obstruction theory on pullbacks of $\mathscr{M}(X/B,\beta) \to \mathfrak{M}(\mathcal{X}/B,\beta')$ along $\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta') \to \mathfrak{M}(\mathcal{X}/B,\beta')$ or $\mathfrak{M}_\tau(\mathcal{X}_0/b_0,\beta') \to \mathfrak{M}(\mathcal{X}/B,\beta')$.

(2) follows since $\mathfrak{M}_m(\mathcal{X}_0/b_0,\beta') \to \mathfrak{M}_m$ is strict étale. ♠

4.8. Proof of the main theorem.

By Costello and Manolache we get

$$\sum_{\tau\,:\,m_\tau = m} \Psi_* [\mathscr{M}_\tau(X_0/b_0,\beta)]^{\mathrm{virt}} = [\mathscr{M}_m(X_0/b_0)]^{\mathrm{virt}},$$

therefore

$$\sum_\tau m_\tau \Psi_* [\mathscr{M}_\tau(X_0/b_0,\beta)]^{\mathrm{virt}} \quad = \quad [\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}}$$

[11](Dan) Make this compatibility of multiplicities a separate lemma?

[12](Qile) This may be clear on geometric points, but should we be explicit about automorphisms?

and similarly

$$\sum_{(\tau,A)\,:\,\mathbf{A}\vdash A} m_\tau \Psi_* \left[\mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)\right]^{\mathrm{virt}} \quad = \quad \left[\mathscr{M}(X_0/b_0,\beta)\right]^{\mathrm{virt}}.$$

Writing $i_{\tau,\mathbf{A}}$ for Ψ acting on $\mathscr{M}_\tau(X_0/b_0)$, we obtain Theorem 1.1.2.

Since $\mathscr{M}_{\widetilde{\tau},A}(X_0/b_0,\beta) \to \mathscr{M}_{\tau,A}(X_0/b_0,\beta)$ has degree $|\operatorname{Aut}(\widetilde{\tau},\mathbf{A})|$, we have:

Th:main-variant

Theorem 4.8.1.

$$[\mathscr{M}(X_0/b_0,\beta)]^{\mathrm{virt}} = \sum_{(\widetilde{\tau},\mathbf{A})\,:\,\mathbf{A}\vdash A} \frac{m_{\widetilde{\tau}}}{|\operatorname{Aut}(\widetilde{\tau},\mathbf{A})|} (i_{\widetilde{\tau},\mathbf{A}})_* \left[\mathscr{M}_{\widetilde{\tau},A}(X_0/b_0,\beta)\right]^{\mathrm{virt}}.$$

where $i_{\widetilde{\tau},\mathbf{A}}$ is the composite of $i_{\tau,\mathbf{A}}$ with the quotient morphism.

Part 2. Practice

culationalsection

5. Logarithmic modifications and transversal maps

There is a general strategy which is often useful for constructing stable logarithmic maps. This is the most powerful tool we have at our disposal at the moment; eventually, the hope is that gluing technology will replace this construction. However, we expect it to be generally useful, especially in the examples in the next section.

Suppose we wish to construct a stable logarithmic map to X/B, as usual X logarithmically smooth over one-dimensional B with logarithmic structure induced by $b_0 \in B$. Suppose further we wish the stable logarithmic map to map into the fibre X_0 over b_0. This construction is accomplished by a two step process.

5.1. **Logarithmic modifications.** First, we will choose a logarithmic modification $h : \tilde{X} \to X$, i.e., a morphism which is proper, birational, and log étale. The modification h is chosen to accommodate a situation at hand — in our applications the datum of a rigid tropical curve.

Given a modification h, [AW13] constructed a morphism $\mathscr{M}(h) : \mathscr{M}(\tilde{X}/B) \to \mathscr{M}(X/B)$ of moduli stacks of basic stable logarithmic maps, satisfying

$$\mathscr{M}(h)_*([\mathscr{M}(\tilde{X}/B)]^{\mathrm{virt}}) = [\mathscr{M}(X/B)]^{\mathrm{virt}}.$$

The construction of $\mathscr{M}(h)$ is as follows. Given a stable logarithmic map $\tilde{f} : \tilde{C}/W \to \tilde{X}/B$, one obtains on the level of schemes the stabilization of $h\circ\tilde{f}$, i.e., a factorization of $h \circ \tilde{f}$ given by

$$\underline{\tilde{C}}/\underline{W} \xrightarrow{g} \underline{C}/\underline{W} \longrightarrow \underline{X}$$

such that $\underline{C}/\underline{W} \to \underline{X}$ is a stable map. One gives $\underline{C}$ a logarithmic structure $\mathcal{M}_C := g_*\mathcal{M}_{\tilde{C}}$, and with this logarithmic structure one obtains a factorization of $h\circ\tilde{f}$ through C at the level of log schemes, giving $f : C/W \to X/B$. If $\tilde{f}$ was basic, there is no expectation that f is basic, but by [GS13], Proposition 1.22 or [AC11], Corollary 5.11. there is a unique basic map with the same underlying stable map of schemes such that the above constructed f is obained from pull-back from the basic map. This yields the map $\mathscr{M}(h)$.

5.2. **Transversal maps, logarithmic enhancements, and strata.** Second, if we have a stable map to $\underline{X}_0$ which interacts sufficiently well with the strata, we will compute in Theorems 5.4.1 and 5.5.1 the number of log enhancements of this curve. This generalizes a key argument of Nishinou and Siebert in [NS06]. There are two differences: our degeneration $X \to B$ is only logarithmically smooth and not necessarily toric; and the fiber X_0 is not required to be reduced. The precise meaning of "interacting well with logarithmic strata" is as follows:

sversaldef **Definition 5.2.1.** Let $X \to B$ be a logarithmically smooth morphism over B one-dimensional carrying the divisorial logarithmic structure $b_0 \in B$ as usual. Let $X_0^{[d]}$ denote the union of the (open) codimension d logarithmic strata of X_0. Suppose $\underline{f} : \underline{C}/\operatorname{Spec} \Bbbk \to \underline{X}_0$ is a stable map. We say that $\underline{f}$ is a *transverse map* if the image of $\underline{f}$ is contained in $X_0^{[0]} \cup X_0^{[1]}$, and $\underline{f}^{-1}(X_0^{[1]})$ is a finite set.

We codify what it means to take a stable map and endow it with a logarithmic structure:

Definition 5.2.2. Let $f : X \to B$ be as above and $\underline{f} : \underline{C} \to \underline{X}_0$ a stable map. A *logarithmic enhancement* $f : C \to X$ is a stable logarithmic map whose underlying map is $\underline{f}$. Two logarithmic enhancements f_1, f_2 are *isomorphic enhancements* if there is an isomorphism between f_1 and f_2 which is the identity on the underlying $\underline{f}$. Otherwise we say they are *non-isomorphic* or *distinct enhancements*.

We also introduce some terminology to help describe the codimension one strata of X_0:

dlengthdef **Definition 5.2.3.** Let $X \to B$ be as above and S a codimension one stratum of X_0, and let $\bar{y} \in S$. Since X is logarithmically smooth over B, there is an étale neighbourhood $U_{\bar{y}}$ of $\bar{y}$ in X, an étale neighbourhood B' of $b_0 \in B$, a lattice $N = \mathbb{Z}^2$, a two-dimensional rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ and a non-zero element $\rho \in M = \operatorname{Hom}(N, \mathbb{Z})$ along with a commutative diagram

calpicture (5.2.1)
$$\begin{tikzcd} U_{\bar{y}} \arrow[r] \arrow[d] & X_\sigma \arrow[d, "z^\rho"] \\ B' \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

inducing a smooth map $U_{\bar{y}} \to B' \times_{\mathbb{A}^1} X_\sigma$, where X_σ is the toric variety associated to σ. We define:

(1) The *index* $\mathrm{Ind}(S)$ of the stratum S is the index of the sublattice of N generated by the primitive generators of rays of σ.
(2) The *length* $\ell(S)$ is the affine length of the interval $\rho^{-1}(1)$, viewing ρ as a map $\rho : \sigma \to \mathbb{R}_{\geq 0}$.

5.3. **Necessary conditions for the existence of a logarithmic enhancement.**

transprelogdef **Definition 5.3.1.** Let $X \to B$ be as above, and let $\underline{f} : \underline{C}/\operatorname{Spec}\Bbbk \to \underline{X}_0$ be a transverse map. We say $\underline{f}$ is a *transverse pre-logarithmic map* if the following additional conditions on $\underline{f}$ hold:

Constrained node condition: If $x \in \underline{C}$ with $\underline{f}(x) \in X_0^{[1]}$ a singular point of $(X_0)_{\mathrm{red}}$, then x is a node of $\underline{C}$, contained in two distinct irreducible components $\underline{D}_1, \underline{D}_2$ of $\underline{C}$. Furthermore, $\underline{f}(\underline{D}_i) \subset Y_i$ with Y_1, Y_2 two irreducible components of $(X_0)_{\mathrm{red}}$. Let w_i be the order of tangency of $\underline{f} : \underline{D}_i \to Y_i$ along $Y_1 \cap Y_2 \subset Y_i$ at the point $x \in \underline{D}_i$. Let μ_i be the multiplicity of Y_i in X_0. Let S be the stratum of $X_0^{[1]}$ containing $\underline{f}(x)$. Then
(1) $w_1/\mu_1 = w_2/\mu_2$ and
(2) the number

prelogconditions (5.3.1)
$$w_q := \frac{w_1\mu_2\ell(S)}{\mathrm{Ind}(S)} = \frac{w_2\mu_1\ell(S)}{\mathrm{Ind}(S)}$$

is an integer.

Constrained marking condition: If $x \in \underline{C}$ with $\underline{f}(x) \in X_0^{[1]}$ a smooth point of $(X_0)_{\mathrm{red}}$, then x is a smooth point of $\underline{C}$ contained in an irreducible component $\underline{D}$, with $\underline{f}(\underline{D}) \subset Y$ an irreducible component of $(X_0)_{\mathrm{red}}$. Let w be the order of tangency of $\underline{f} : \underline{D} \to Y$ along $Y \cap X_0^{[1]}$ at x. Let S be the codimension one stratum of X_0 containing $\underline{f}(x)$. Then

$$\mathrm{Ind}(S) \bigm| w.$$

Accordingly, a node $x \in \underline{C}$ is a *constrained node* if $\underline{f}(x) \in X_0^{[1]}$ a singular point of $(X_0)_{\mathrm{red}}$, and otherwise it is a *free node*. Similarly a marked point $x \in \underline{C}$ with $\underline{f}(x) \in X_0^{[1]}$ a smooth point of $(X_0)_{\mathrm{red}}$ is a *constrained marking*, otherwise it is a *free marking.*

transprelogtheorem **Theorem 5.3.2.** *Let $X \to B$ be as above, and let $\underline{f} : \underline{C}/\operatorname{Spec}\Bbbk \to \underline{X}_0$ be a transverse map. Suppose that there is an enhancement of $\underline{f}$ to a basic stable logarithmic map $f : C/W \to X/B$. Then*

(1) $\underline{f}$ is a transverse pre-logarithmic map.

(2) The combinatorial type of f is uniquely determined up to possibly a number of marked points p with $u_p = 0$, and the basic monoid Q is

$$Q = \mathbb{N} \oplus \bigoplus_{q \text{ a free node}} \mathbb{N}.$$

(3) The map $W = \operatorname{Spec}(Q \to \Bbbk) \to B$ induces the map $\overline{\mathcal{M}}_{b_0} = \mathbb{N} \to Q$ given by $1 \mapsto (\mu, 0, \ldots, 0)$, where the multiplicity $\mu \in \mathbb{N}$ *is the smallest positive integer divisible by all multiplicities of irreducible components of X_0 intersecting $\underline{f}(C)$ and such that $\mu\ell(S)/w_q$ is an integer for every double point $q \in \underline{C}$ with $\underline{f}(q) \in X_0^{[1]}$, with notation as in the Constrained node condition.*

Proof. Assume given a log enhancement f of $\underline{f}$ defined over the standard log point $W = \operatorname{Spec} \Bbbk^\dagger$; such an enhancement can be obtained by base-change from an arbitrary enhancement. We write $f^\flat : \underline{f}^{-1}\mathcal{M}_X \to \mathcal{M}_C$ for the induced map. We introduce some general notation for the proof. We fix a two-dimensional lattice $N = \mathbb{Z}^2$ with dual lattice M. If $x \in \underline{C}$ is a point with $\underline{f}(x) \in X_0^{[1]}$, then as in Definition 5.2.3, there is an étale neighbourhood U_x of $\underline{f}(x)$ in X, an étale neighbourhood B' of $b_0 \in B$, a two-dimensional rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$ and a non-zero element $\rho \in M$ yielding (5.2.1). Thus in particular if $m \in \sigma^\vee \cap M$, then the monomial z^m can be viewed as an element of the logarithmic structure on X_σ; we write its pull-back to $\mathcal{M}_{U_x}$ as s_m.

Let n_1, n_2 be the two primitive generators of σ, corresponding to divisors Z_1 and Z_2. Then the order of vanishing of z^ρ on Z_i is $\langle \rho, n_i \rangle$. In particular, if $\underline{f}(x)$ is a smooth point of $(X_0)_{\mathrm{red}}$, then $\langle \rho, n_i \rangle = 0$ for some i, say $i = 2$, and the multiplicity in X_0 of the irreducible component Y of $(X_0)_{\mathrm{red}}$ containing $\underline{f}(x)$ is then $\langle \rho, n_1 \rangle$. If $\underline{f}(x)$ is a singular point of $(X_0)_{\mathrm{red}}$, then $\underline{f}(x)$ is contained in two irreducible components Y_1, Y_2 of $(X_0)_{\mathrm{red}}$, of multiplicities $\mu_i = \langle \rho, n_i \rangle$ for $i = 1, 2$.

We will take primitive generators α_1, α_2 of the rays of $\sigma^\vee$, so that $\langle \alpha_i, n_i \rangle = 0$, $\langle \alpha_1, n_2 \rangle > 0$, $\langle \alpha_2, n_1 \rangle > 0$. Note that

eq: index (5.3.2) $$\langle \alpha_1, n_2 \rangle = \langle \alpha_2, n_1 \rangle = \operatorname{Ind}(S),$$

where S is the codimension one stratum of X_0 containing $\underline{f}(x)$.

Step I. Analysis at smooth points of $\underline{C}$ mapping to $X_0^{[1]}$. Let $p \in \underline{C}$ map into $X_0^{[1]}$. Using the setup above, we replace X with an étale open neighbourhood U_p of $\underline{f}(p)$, and C with an étale neighbourhood of p so that we can assume $f^{-1}(X_0^{[1]}) = \{p\}$. Let $Y_1, Y_2 \subset U_p$ be the inverse image of the divisors Z_1, Z_2 of X_σ. We assume that Y_1 is contained in X_0 if $\underline{f}(p)$ is a smooth point of $(X_0)_{\mathrm{red}}$; otherwise Y_1, Y_2 are both contained in X_0. We can further assume $\underline{f}(C) \subseteq Y_1$. Note that z^{α_1} vanishes on Z_2 but not on Z_1. Thus $\alpha_C(f^\flat s_{\alpha_1}) = f^*\alpha_X(s_{\alpha_1}) = f^*(z^{\alpha_1})$ (using the notation z^{α_1}

also for the pull-back of the monomial z^{α_1} to U_p). Now $f^*(z^{\alpha_1})$ vanishes precisely at p, and hence the image of s_{α_1} in $\overline{\mathcal{M}}_C$ is a section supported only at p. Thus p must be a marked point of the curve C, with $\overline{\mathcal{M}}_C = \underline{\mathbb{N}} \oplus \mathbb{N}_p$.

This immediately shows that $\underline{f}(p)$ cannot be a singular point of $(X_0)_{\mathrm{red}}$. Indeed, the map $\Sigma(U_p) \to \Sigma(B')$ is the map $\sigma \to \mathbb{R}_{\geq 0}$ given by $\rho \in M$. On the other hand, a fibre of $\Sigma(C) \to \Sigma(W)$ (see (2.5.1)) is $\mathbb{R}_{\geq 0}$, and since the stable logarithmic map f is defined over B, the induced map $\mathbb{R}_{\geq 0} \to \Sigma(U_p)$ must be constant when composed with the map to $\Sigma(B')$. But if $\underline{f}(p)$ is a singular point of $(X_0)_{\mathrm{red}}$, then necessarily the fibres of $\sigma \to \mathbb{R}_{\geq 0}$ are compact, and hence $\Sigma(C) \to \Sigma(U_p)$ must be constant. Thus $u_p = 0$, i.e., the composed map $\overline{\mathcal{M}}_{X,\underline{f}(p)} = P_p \to \overline{\mathcal{M}}_{C,p} = \mathbb{N} \oplus \mathbb{N}_p \to \mathbb{N}_p$ is zero, where the second map is projection onto the second factor. But we have already seen in the previous paragraph that this map is non-trivial, a contradiction.

Thus the remaining case is that $\underline{f}(p)$ is a smooth point of $(X_0)_{\mathrm{red}}$. Let w and S be as in the Constrained marking condition. Then as α_1 is a primitive generator of a ray of $\sigma^\vee$, and z^{α_1} does not vanish on Y_1, necessarily $z^{\alpha_1}|_{Y_1}$ vanishes to order 1 on $Y_1 \cap Y_2 = Y_1 \cap X_0^{[1]}$. So if we write $C = \operatorname{Spec} \Bbbk[x]$, we must have $\underline{f}^*(z^{\alpha_1}) = \varphi x^w$ for some invertible function φ. Hence the map $u_p : P_p \to \mathbb{N}_p$ satisfies $u_p(\alpha_1) = w$. Also, $u_p(\rho) = 0$ since f is defined over B. Thus, after choosing a basis for N and the dual basis for M, we can assume $n_1 = (a, \operatorname{Ind}(S))$ with $\gcd(a, \operatorname{Ind}(S)) = 1$, $n_2 = (1, 0)$, so that $\alpha_1 = (\operatorname{Ind}(S), -a)$, $\alpha_2 = (0, 1)$, and then ρ is a multiple of α_2. We must then have

$$u_p = (w/\operatorname{Ind}(S), 0),$$

and in particular $\operatorname{Ind}(S)|w$.

This gives the Constrained marking condition, and we have shown that all points p of $\underline{f}^{-1}(X_0^{[1]})$ mapping to smooth points of $(X_0)_{\mathrm{red}}$ must be marked points of the logarithmic structure on C.

Step II. Analysis at free marked points of C and free nodes. We have just seen that given a log enhancement f of $\underline{f}$, every smooth point of $\underline{C}$ mapping to $X_0^{[1]}$ must be marked. Of course, C might contain some additional marked points. We wish to show $u_p = u_q = 0$ for p, q free marked points or nodes.

Write $\{p\}$ for the set of all marked points. Let Γ be the dual intersection graph of $(\underline{C}, \{p\})$. The log enhancement f of $\underline{f}$ yields, using (2.5.1), a family of tropical curves of the form $h : \Gamma \to \Sigma(X)$. Furthermore, $P_\eta = \mathbb{N}$ for any generic point η of $\underline{C}$ by the transversality assumption, so any allowable map h appearing in this family must take the vertex v_η to the ray of $\Sigma(X)$ corresponding to the unique irreducible component of X_0 containing $\underline{f}(\eta)$. Note that such a ray surjects to $\Sigma(B)$. Furthermore, the composition $\Gamma \xrightarrow{h} \Sigma(X) \to \Sigma(B)$ must be the constant map if f is to be a stable logarithmic map to X/B. This implies in particular that all edges

E_p or E_q associated to marked points or free nodes are contracted by h, and thus $u_p = u_q = 0$ for free marked points and free nodes respectively.

It is then clear that regardless of the u_q for constrained nodes, the tropical map h is determined entirely by the lengths of the edges E_q for q free and the image of the curve in $\Sigma(B)$. This implies that the basic monoid Q is

formofQ

$$Q = \mathbb{N} \oplus \bigoplus_{q \text{ free}} \mathbb{N}, \tag{5.3.3}$$

as claimed.

Step III. Analysis of constrained nodes. Fix a constrained node q. Then $\underline{f}(q)$ maps into a codimension one stratum of X_0. Replacing $\underline{C}$ with an étale neighbourhood of q, we can assume that $\underline{f}^{-1}(X_0^{[1]}) = \{q\}$. If $\underline{f}(C) \subseteq Y_i$ for $i = 1$ or 2, the same argument as in Step I shows that there is a section of $\overline{\mathcal{M}}_C$ with support at q. However, since q is a node, this is a contradiction, and thus C splits into two components C_1, C_2 with $\underline{f}(C_i) \subseteq Y_i$.

Because f is defined over the standard log point, $\overline{\mathcal{M}}_{C,q} = S_{e_q}$ for some positive integer e_q, see e.g. [GS13],§1.3. For our purposes, the monoid S_e is most easily described by introducing a lattice $N' = \mathbb{Z}^2$ with dual lattice M', and taking the cone (after choosing a basis)

$$\sigma_e = \langle n'_1 = (0,1), n'_2 = (e,1)\rangle \subset N'_{\mathbb{R}}.$$

Then

$$S_e = \sigma_e^\vee \cap M'.$$

Let α'_1, α'_2 be the primitive generators of $\sigma_e^\vee$ with $\langle \alpha'_i, n'_i\rangle = 0$, and let $\rho' \in \sigma_e^\vee$ be the element inducing the logarithmic map $C \to W$, i.e., $\rho' = (0,1)$ in the chosen basis. The map $P_q \to S_{e_q}$ is then necessarily determined by a map of lattices $N' \to N$ inducing a bijection $\sigma_{e_q} \to \sigma$. Since the map is defined over B, there must be a commutative diagram of cones

gmadiagram

$$\begin{array}{ccc} \sigma_{e_q} & \longrightarrow & \sigma \\ \downarrow & & \downarrow \\ \mathbb{R}_{\geq 0} & \longrightarrow & \mathbb{R}_{\geq 0} \end{array} \tag{5.3.4}$$

The first vertical map is given by $\rho' \in M'$ and the second by $\rho \in M$. The bottom horizontal map is a rescaling by a factor $\alpha_q \in \mathbb{Q}_{>0}$. Thus the map is given on the level of generators of the cones by

gmasigmamap

$$n'_1 \mapsto \alpha_q n_1/\mu_1, \qquad n'_2 \mapsto \alpha_q n_2/\mu_2 \tag{5.3.5}$$

where the quotient by $\mu_i = \langle \rho, n_i\rangle$ is necessary to guarantee commutativity of the above diagram. Note α_q is related to u_q via the definition of u_q (with a suitable choice of orientation of E_q) by

equqeq
$$e_q u_q = \alpha_q (n_2/\mu_2 - n_1/\mu_1). \tag{5.3.6}$$

From this one sees that the dual map $P_q \to S_{e_q}$ is given by

$$\alpha_1 \mapsto \frac{\alpha_q}{\mu_2 e_q}\langle \alpha_1, n_2\rangle \alpha_1', \quad \alpha_2 \mapsto \frac{\alpha_q}{\mu_1 e_q}\langle \alpha_2, n_1\rangle \alpha_2'.$$

Now z^{α_1} is non-vanishing on Y_1 and is zero along on Y_2, and $z^{\alpha_1}|_{Y_1}$ has a zero of order 1 along $Y_1 \cap Y_2$. Thus $\underline{f}^*(z^{\alpha_1})|_{C_1}$ has a zero of order w_1 at q, where w_1 is the order of tangency of the Constrained node condition. Similarly, $\underline{f}^*(z^{\alpha_2})|_{C_2}$ has a zero of order w_2 at q. Locally near q, C is of the form $\operatorname{Spec} \Bbbk[x,y]/(xy)$ and the log structure has chart $S_{e_q} \to \Bbbk[x,y]/(xy)$ given on the generators of S_{e_q} by

$$\alpha_1' \mapsto \varphi_x x, \quad \alpha_2' \mapsto \varphi_y y, \quad \rho' \mapsto 0,$$

where φ_x, φ_y are invertible functions. Thus in particular,

uvxyeq
$$\begin{aligned} f^*(z^{\alpha_1})|_{C_1} &= \alpha_C(f^\flat(s_{\alpha_1}))|_{C_1} = \varphi_x' x^{\alpha_q\langle \alpha_1, n_2\rangle/(\mu_2 e_q)}, \\ f^*(z^{\alpha_2})|_{C_2} &= \alpha_C(f^\flat(s_{\alpha_2}))|_{C_2} = \varphi_y' y^{\alpha_q\langle \alpha_2, n_1\rangle/(\mu_1 e_q)}, \end{aligned} \tag{5.3.7}$$

(again with φ_x', φ_y' some invertible functions) giving

w1w2equations
$$w_1 = \frac{\alpha_q}{\mu_2 e_q}\langle \alpha_1, n_2\rangle, \quad w_2 = \frac{\alpha_q}{\mu_1 e_q}\langle \alpha_2, n_1\rangle. \tag{5.3.8}$$

This is only possible if $w_1/\mu_1 = w_2/\mu_2 = \alpha_q \operatorname{Ind}(S)/(\mu_1\mu_2 e_q)$, keeping in mind (5.3.2). Multiplying (5.3.6) by $\operatorname{Ind}(S)/e_q$ then gives

$$\operatorname{Ind}(S) u_q = w_1\mu_2(n_2/\mu_2 - n_1/\mu_1) = w_2\mu_1(n_2/\mu_2 - n_1/\mu_1).$$

This implies that u_q is a vector whose affine length is

$$\ell(u_q) = w_1\mu_2 \ell(S)/\operatorname{Ind}(S) = w_q.$$

This must be an integer, giving the Constrained node condition.

Step IV. The map $W \to B$. Recall that the basic monoid Q is dual to the monoid $Q^\vee \subset Q^\vee_{\mathbb{R}}$, the latter being the moduli space of tropical curves $h : \Gamma \to \Sigma(X)$ of the given combinatorial type, and $Q^\vee$ consists of those tropical curves whose edge lengths are integral and whose vertices map to integral points of $\Sigma(X)$.

If η is a generic point of $\underline{C}$, denote by μ_η the multiplicity of the irreducible component of $(X_0)_{\mathrm{red}}$ in X_0 containing $\underline{f}(\eta)$. Thus the induced map $\mathbb{N} \to P_\eta$ coming from the structure map $X \to B$ is multiplication by μ_η. Write $\rho : \Sigma(X) \to \Sigma(B)$ for the tropicalization of $X \to B$. The restriction of ρ to the ray of $\Sigma(X)$ corresponding to the irreducible component of X_0 containing $\underline{f}(\eta)$ is multiplication by μ_η. Thus if given a tropical curve $h : \Gamma \to \Sigma(X)$ and μ the image of $\rho \circ h$ in $\Sigma(B)$, we see that $h(v_\eta)$ is integral if and only if $\mu_\eta | \mu$.

The edges of Γ corresponding to free nodes have arbitrary length independent of μ. But an edge corresponding to a constrained node q mapping to a stratum S must have length

$$\mu\ell(S)/\ell(u_q) = \mu\ell(S)/w_q.$$

This must also be integral for h to represent a point in $Q^\vee$. Thus the map $\Sigma(W) \to \Sigma(B)$ must be given by $(\alpha, (\alpha_q)_q) \mapsto \mu\alpha$ where μ is as given in the statement of the theorem. Dually, we obtain the stated description of the map $W \to B$. ♠

5.4. **Existence and count of enhancements of transverse pre-logarithmic maps: reduced case.**

nstruction1 **Theorem 5.4.1.** *Suppose $X \to B$ is as above, and suppose given a transverse prelog map*

$$\underline{f} : (\underline{C}, p_1, \ldots, p_n)/\operatorname{Spec}\Bbbk \to \underline{X}_0$$

whose image is contained in an open subscheme $\underline{U} \subseteq \underline{X}_0$ which is reduced. Suppose further that the marked points $\{p_i\}$ include all points of $\underline{f}^{-1}(X_0^{[1]})$ mapping to non-singular points of X_0. For a constrained node q of $\underline{C}$, let $w_q = w_1 = w_2$ as in (5.3.1), and let $\mu \in \mathbb{N}$ be the multiplicity as given in Theorem 5.3.2, (3).

Then there are

$$\mu^{-1}\prod_q w_q$$

distinct enhancements of $\underline{f}$ to a basic stable logarithmic map, where the log marked points of the log enhancement are precisely $p_1, \ldots, p_n$.

Proof. This is only a very minor variation of a result proved in [NS06], Prop. 7.1, so we give the argument in brief. First, by replacing X_0 by U, we can assume X_0 is reduced in what follows.

By Theorem 5.3.2, the combinatorial type is completely determined, and the basic monoid is $Q = \mathbb{N} \oplus \bigoplus_{q \text{ a free node}} \mathbb{N}_q$, where the sum is over all free nodes, $\mathbb{N}_q := \mathbb{N}$. In the case that X_0 is reduced, it is easy to check that $\operatorname{Ind}(S) = \ell(S)$. Thus the map $\mathbb{N} \to Q$ induced by $W \to B$ is determined by the smallest integer μ such that $\mu\ell(S_q)/w_q$ is an integer for all constrained nodes q (where S_q is the stratum containing $\underline{f}(q)$).

Fix now a morphism $\tau : W = \operatorname{Spec}(Q \to \Bbbk) \to B$ inducing the above map $\mathbb{N} \to Q$ once and for all. (A different choice of morphism would be related by an automorphism of W.) We wish to describe all basic log enhancements $f : C/W \to X_0$ of $\underline{f}$.

Let $\underline{C}^\circ$ be the complement of the nodes and marked points of $\underline{C}$ (the marked points include the non-singular points of $\underline{C}$ mapping to $X_0^{[1]}$). Let $X_0^\circ = X_0 \setminus X_0^{[1]}$, so that

$X_0^\circ \to B$ is strict. Then we obtain a unique diagram

uniquediagram (5.4.1)
$$\begin{tikzcd} C^\circ \arrow[r, "f"] \arrow[d, "\pi"'] & X_0^\circ \arrow[d] \\ W \arrow[r, "\tau"'] & B \end{tikzcd}$$

with π strict. We need to understand how many ways there are of extending this diagram across the marked points and nodes of $\underline{C}$. The extension of the logarithmic structure C° across the marked points of $\underline{C}$ is unique, and it is easy to check that $f : C^\circ \to X_0$ then extends across the marked points. The nodes of $\underline{C}$ need to be treated with more care.

Following the proof of [NS06], for q a node of $\underline{C}$, we can pick $u, v \in \mathcal{O}_{C,\bar{q}}$ with $uv = 0$ and u, v restricting to local parameters respectively on the two branches of C through q. We can then describe any logarithmically smooth extension of $C^\circ \to W$ across q by giving a chart

$$\begin{aligned} Q \oplus_{\mathbb{N}} \mathbb{N}^2 \to & \mathcal{O}_{C,\bar{q}} \\ \big(n, (a,b)\big) \mapsto & \begin{cases} \zeta^a u^a v^b & n = 0 \\ 0 & n \neq 0 \end{cases} \end{aligned}$$

with $\zeta \in \Bbbk^\times$. Here the fibred sum is determined by $\mathbb{N} \to \mathbb{N}^2$ the diagonal map and $\mathbb{N} \to Q$ taking 1 to $\rho_q \in Q$ the smoothing parameter of the node. If q is free, ρ_q is the generator of the summand $\mathbb{N}_q$ of Q. If q is not free, then ρ_q is $\mu\ell(S_q)/w_q$ in the summand $\mathbb{N}$ of Q. Furthermore, the map $C \to W$ is induced by the inclusion $Q \to Q \oplus_{\mathbb{N}} \mathbb{N}^2$. Any choice of $C \to W$ in a neighbourhood of q takes this form for some $\zeta \in \Bbbk^\times$.

If q is a free node, then for any choice of ζ, the diagram (5.4.1) is easily seen to extend across q uniquely, with the map $f^*\mathcal{M}_X \to \mathcal{M}_C$ factoring through $\pi^*\mathcal{M}_W \subset \mathcal{M}_C$, again using strictness of $X_0^\circ \to B$. Thus, given a tuple $(\zeta_q)_q$ ranging over the free nodes q, we obtain an extension of (5.4.1) across the free nodes. However, any two choices of these parameters yield isomorphic curves. Indeed, let $\lambda : W \to W$ be the automorphism of W given by

$$\begin{aligned} Q \times \Bbbk^\times \to & Q \times \Bbbk^\times \\ \big((n, (n_q)_q), r\big) \mapsto & \Big((n, (n_q)_q), r \prod_q \zeta_q^{-n_q}\Big). \end{aligned}$$

Then $\tau \circ \lambda = \tau$, and the pull-back of the curve given by the tuple $(1)_q$ is the curve given by the tuple $(\zeta_q)_q$. We conclude that there is a unique way of extending the diagram (5.4.1) across the free nodes.

If q is a constrained node then we are in the situation of the third paragraph of the proof of [NS06], Prop. 7.1. Fix t a local parameter for B at b_0 so that $\tau^\flat(t) = \big((\mu,(0)_q),1\big) \in Q\times \Bbbk^\times$. We write $e_\tau := (\mu,(0)_q)\in Q$ and $\bar e_\tau := e_\tau/\mu$ the generator of the summand $\mathbb{N}$ of Q. Via a toric chart there exists $x,y\in \mathcal{M}_{X,\overline{f(q)}}\subset \mathcal{O}_{X,\overline{f(q)}}$ such that $xy = t^{\ell(S_q)}$. Furthermore, with appropriate choice of $u,v\in\mathcal{O}_{C,\bar q}$, we have

$$\underline{f}^*(x) = u^{w_q},\quad \underline{f}^*(y) = v^{w_q}.$$

A choice of extension of $C^\circ\to W$ across q is then determined by some $\zeta_q\in\Bbbk^\times$ as above, and we need to understand for which values of ζ_q the morphism $\underline{f}$ extends as a log morphism, and how many choices of such extension exist.

Viewing x,y,t as elements of $\mathcal{M}_{X,\overline{f(q)}}$, we write them as s_x, s_y and s_t, while elements m of $Q\oplus_{\mathbb{N}}\mathbb{N}^2$ determine via the chart at q germs $s_m\in\mathcal{M}_{C,\bar q}$. In particular, if f extends, we must have $s_u,s_v\in\mathcal{M}_{C,\bar q}$ lifting u and v with $f^\flat(s_x) = s_u^{w_q}$, $f^\flat(s_y) = s_v^{w_q}$. Necessarily $s_u = \zeta_q^{-1}s_{(0,(1,0))}$, $s_v = s_{(0,(0,1))}$. By commutativity of (5.4.1), $f^\flat(s_t) = s_{(e_\tau,(0,0))}$. Furthermore, since $s_xs_y = s_t^{\ell(S_q)}$, we have $(s_us_v)^{w_q} = s_{(e_\tau,(0,0))}^{\ell(S_q)} = s_{(\bar e_\tau,(0,0))}^{\mu\ell(S_q)}$, or $s_us_v = \xi s_{(\bar e_\tau,(0,0))}^{\mu\ell(S_q)/w_q}$, where ξ is a w_q-th root of unity. We then see that $\xi=\zeta_q^{-1}$. Thus we only obtain an extension of f across q if ζ_q is a w_q-th root of unity, and then the above description of $f^\flat$ yields a unique such extension. Hence we have w_q extensions across q.

This gives $\prod_q w_q$ extensions over all constrained nodes. However, if $\lambda: W\to W$ is given by

$$\begin{aligned} Q\times\Bbbk^\times &\to Q\times\Bbbk^\times\\ \big((n,(n_q)_q),r\big) &\mapsto \big((n,(n_q)_q), r\zeta^{-n}\big), \end{aligned}$$

then $\tau\circ\lambda=\tau$ provided that ζ is a μ-th root of unity. On the other hand, pull-back by λ replaces the parameter ζ_q associated with a constrained node q with the parameter $\zeta_q\zeta^{\mu\ell(S_q)/w_q}$. This gives the final enumeration. ♠

5.5. **Existence and count of enhancements: general case.** The general situation, when the image of $\underline{f}$ is not contained in the reduced locus of X_0, is subtler. Indeed, the morphism $X_0\smallsetminus X_0^{[1]}\to B$ is not strict on the non-reduced locus, and hence some of the torsors associated to the logarithmic structure on $X_0\smallsetminus X_0^{[1]}$ may be non-trivial. This produces an obstruction to the existence of a diagram (5.4.1), which is really a global obstruction. It is perhaps most convenient to describe this obstruction as follows.

Let $\bar\mu$ be a positive integer and $\underline{\tilde B}\to\underline{B}$ be the degree $\bar\mu$ cyclic cover branched with ramification index $\bar\mu$ over b_0. Let $\underline{\bar X} = \underline{X}\times_{\underline{B}}\underline{\tilde B}$, and let $\underline{\tilde X}\to\underline{\bar X}$ be the normalization, giving a family $\underline{\tilde X}\to\underline{\tilde B}$. It is a standard computation that the inverse image of a multiplicity μ irreducible component of $\underline{X}_0$ in $\underline{\tilde{X}}$ is a union of irreducible components of $\underline{\tilde{X}}_0$, each with multiplicity $\mu/\gcd(\mu,\bar{\mu})$.

At the level of log schemes, in fact $\bar{X}$ carries a fine but not saturated logarithmic structure via the description $\bar{X} = X \times_B \tilde{B}$ in the category of fine log schemes, while $\tilde{X}$ carries an fs logarithmic structure via the description $\tilde{X} = X \times_B \tilde{B}$ in the category of fs logarithmic structures. Here $\tilde{B}$ carries the divisorial logarithmic structure given by $\tilde{b}_0 \in \tilde{B}$, the unique point mapping to b_0.

Similarly, the central fibres are related as follows. The map $\tilde{B} \to B$ induces a map on standard log points $\tilde{b}_0 \to b_0$ induced by $\mathbb{N} \to \mathbb{N}, 1 \mapsto \bar{\mu}$. Then $\bar{X}_0 = X_0 \times_{b_0} \tilde{b}_0$ in the category of fine log schemes, and $\tilde{X}_0 = X_0 \times_{b_0} \tilde{b}_0$ in the category of fs log schemes.

Using this notation, we then have:

urveconstruction2

Theorem 5.5.1. *Suppose given $X \to B$ as above, and let*

$$\underline{f} : (\underline{C}, p_1, \ldots, p_n)/\operatorname{Spec} \Bbbk \to \underline{X}_0$$

be a transverse pre-logarithmic map. Suppose further that the marked points $\{p_i\}$ include all points of $\underline{f}^{-1}(X_0^{[1]})$ mapping to non-singular points of $(X_0)_{\mathrm{red}}$. For a constrained node q of $\underline{C}$ mapping to a stratum S, let w_q be the integer of (5.3.1). Let $\bar{\mu}$ be the least common multiple of multiplicities of all irreducible components of $\underline{X}_0$ which intersect the image of $\underline{f}$. This gives rise to $\tilde{X} \to X$ as above. Then

(1) there is a log enhancement of $\underline{f}$ if and only if there is a lift

$$\underline{\tilde{f}} : \underline{C}/\operatorname{Spec} \Bbbk \to \underline{\tilde{X}}_0$$

of $\underline{f}$.

(2) If such a lift exists, then there exist

$$\frac{\bar{\mu}}{\mu} \prod_q w_q$$

distinct enhancements of $\underline{f}$ to a basic stable logarithmic map.

Proof. If $\underline{f}$ has a log enhancement $f : C \to X_0$, then we can assume f is basic. If μ is as defined in the statement of Theorem 5.3.2, then $\bar{\mu}|\mu$, and the morphism $W \to b_0$ factors through $\tilde{b}_0 \to b_0$. (Note that there are $\bar{\mu}$ such factorizations.) Thus f induces a morphism $C \to \tilde{X}_0 = X_0 \times_{b_0} \tilde{b}_0$ by the universal property of fibre product. This gives the desired lift. Since the lift is still a stable logarithmic curve, the lift is torically transverse.

Conversely, if a lift $\underline{\tilde{f}}$ of $\underline{f}$ exists, then by definition of $\bar{\mu}$, the image of $\underline{\tilde{f}}$ only intersects reduced components of $\tilde{X}_0$. We can then apply Theorem 5.4.1: $\underline{\tilde{f}}$ has a log enhancement $\tilde{f}$, provided the lift is also transverse pre-log. So let q be a constrained node of $\underline{C}$, with $\underline{\tilde{f}}$ mapping q to a strata $\tilde{S}$ of $\tilde{X}_0$, which in turn maps to a strata S of X_0. We can model the morphism $\tilde{X} \to X$ in a neighbourhood of these strata torically via the map of cones $\sigma_e \to \sigma$, as in (5.3.4), where this time the bottom horizontal map is multiplication by $\bar{\mu}$ and $e = \ell(\tilde{S}) = \bar{\mu}\ell(S)$. Suppose the orders of tangency of $\underline{\tilde{f}} : \underline{C} \to \underline{\tilde{X}}$ at q are $\tilde{w}_1, \tilde{w}_2$ as in (1) of Theorem 5.3.2. Then by (5.3.8), with $\alpha_q = \bar{\mu}$ and $e_q = e$, the corresponding orders of tangency of $\underline{f} : \underline{C} \to \underline{X}$ are

$$w_1 = \tilde{w}_1 \frac{\bar{\mu}\,\mathrm{Ind}(S)}{\mu_2 e} = \tilde{w}_1 \frac{\mathrm{Ind}(S)}{\mu_2 \ell(S)}, \qquad w_2 = \tilde{w}_2 \frac{\mathrm{Ind}(S)}{\mu_1 \ell(S)}.$$

Thus the prelog condition (5.3.1) for $\underline{f}$ is equivalent to $\tilde{w}_1 = \tilde{w}_2$, which is the prelog condition for $\underline{\tilde{f}}$. Thus by Theorem 5.4.1, there is a log enhancement $\tilde{f}$ of $\underline{\tilde{f}}$. If $p : \tilde{X} \to X$ denotes the projection, then $p \circ \tilde{f}$ is a log enhancement of $\underline{f}$.

To enumerate the number of possibilities, we first note that by Theorem 5.4.1, given a lift $\underline{\tilde{f}}$, there are $\frac{\bar{\mu}}{\mu} \prod_q w_q$ possible choices for basic stable log enhancements $\tilde{f} : C/W \to \tilde{X}$. For each such enhancement the composition $p \circ \tilde{f}$ is easily seen to be a basic enhancement of $\underline{f}$. There may many different lifts, but the group of $\bar{\mu}$-th roots of unity acts on $\tilde{X}$ over X, and this will identify any two choices of lift. (Equivalently, different choices of lift correspond to the $\bar{\mu}$ different factorizations $W \to \tilde{b}_0 \to b_0$.) Thus the total number of basic log enhancements of $\underline{f}$ is still $\frac{\bar{\mu}}{\mu} \prod_q w_q$. ♠

6. Examples

6.1. The classical case. Suppose $X \to B$ is a simple normal crossings degeneration with $X_0 = Y_1 \cup Y_2$ a reduced union of two irreducible components, with $Y_1 \cap Y_2 = D$ a smooth divisor in both Y_1 and Y_2. In this case, $\Sigma(X) = (\mathbb{R}_{\geq 0})^2$ and the map $\Sigma(X) \to \Sigma(B)$ is given by $(x, y) \mapsto x + y$, so that $\Delta(X)$ admits an affine-linear isomorphism with the unit interval $[0, 1]$, see Figure 1.

Proposition 6.1.1. *In the above situation, let $f : \Gamma \to \Delta(X)$ be a decorated tropical curve. Then f is rigid if and only if every vertex v of Γ maps to the endpoints of $\Delta(X)$ and every edge of Γ surjects onto $\Delta(X)$.*

Note that necessarily every leg of Γ is contracted, as $\Delta(X)$ is compact.

Proof. First note that if an edge E_q is contracted, then $u_q = 0$ and the length of the edge is arbitrary. By changing the length, one sees f is not rigid, see Figure 2 on the left.

Next, suppose v is a vertex with $f(v)$ lying in the interior of $\Delta(X)$. Identifying the latter with $[0, 1]$, we can view $u_q \in \mathbb{Z}$ for any q. Let $E_{q_1}, \ldots, E_{q_r}$ be the edges of Γ adjacent to v with lengths $\ell_1, \ldots, \ell_r$, oriented to point away from v. We can then write down a family f_t of tropical curves, t a real number close to 0, with $f = f_0$,

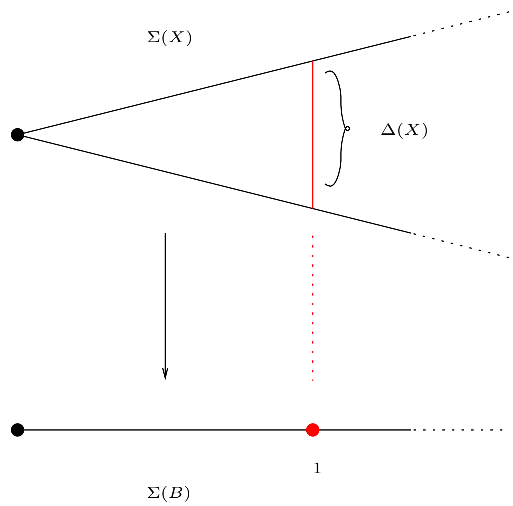

FIGURE 1. The cones $\Sigma(X)$ and $\Sigma(B)$ and the interval $\Delta(X)$ Fig:DeltaX

$f_t(v') = f(v')$ for any vertex $v' \neq v$, and $f_t(v) = f(v) + t$. In doing so, we also need to modify the lengths of the edges E_{q_i}, with $\ell_{i,t} = \ell_i - \frac{t}{u_{q_i}}$. Any unbounded edge attached to v is contracted to $f_t(v)$. So f is not rigid, see Figure 2 on the right. Thus if f is rigid, we see that all vertices of Γ map to endpoints of $\Delta(X)$, and any compact edge is not contracted, hence surjects onto $\Delta(X)$. The converse is clear.

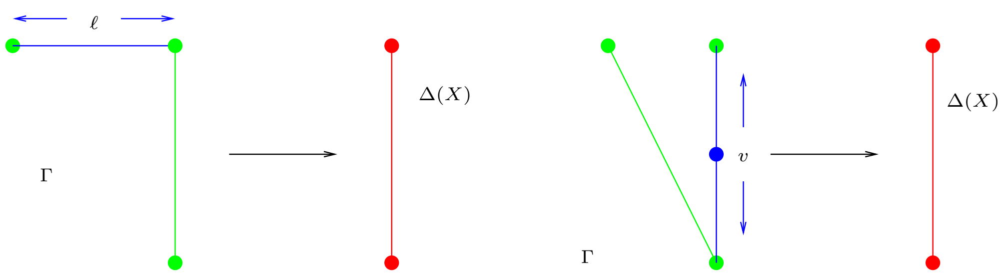

FIGURE 2. A graph with a contracted bounded edge or an interior vertex is not rigid. Fig:not-ri

♠

A choice of decorated rigid tropical curve in this situation is then exactly what Jun Li terms an *admissible triple* in [Li02]. Indeed, by removing $f^{-1}(1/2)$ from Γ, one obtains two graphs (possibly disconnected) Γ_1, Γ_2 with legs and what Jun Li terms *roots* (the half-edges mapping non-trivially to $\Delta(X)$). The weights of a root, in Li's terminology, coincide with the absolute value of the corresponding u_q. The set I in the definition of admissible triple indicates which labels occur for unbounded edges mapping to, say, $0 \in \Delta(X)$. An illustration is given in Figure 3.

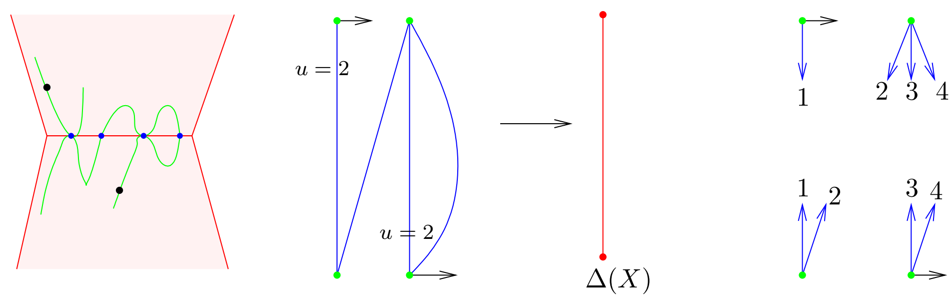

FIGURE 3. A rigid tropical curve is depicted with four edges and two legs, the latter corresponding to marked points with contact order 0. The corresponding admissible triple of Jun Li is depicted on the right, with roots corresponding to half-edges and legs corresponding to the legs of the original graph. The half-edges marked 1 and 3 have $u = 2$. junlicase2

sec:Cubic

6.2. **Rational curves in a pencil of cubics.** It is well-known that if one fixes 8 general points in $\mathbb{P}^2$, the pencil of cubics passing through these 8 points contains precisely 12 nodal rational curves. Blowing up 6 of these 8 points, we get a cubic surface we denote $X_1' \subset \mathbb{P}^3$, and the enumeration of 12 nodal rational cubics translates to the enumeration of 12 nodal plane sections of X_1' passing through the remaining two points p_1, p_2.

We will give here a non-trivial demonstration of the decomposition formula by degenerating the cubic surface to a normal crossings union $H_1 \cup H_2 \cup H_3$ of three blown-up planes.

6.2.1. *Degenerating a cubic to three planes.* Using coordinates $x_0, \ldots, x_3$ on $\mathbb{P}^3$, consider a smooth cubic surface $X_1' \subset \mathbb{P}^3$ with equation

$$f_3(x_0, x_1, x_2, x_3) + x_1x_2x_3 = 0.$$

We then have a family $X' \to B = \mathbb{A}^1$ given by $X' \subseteq \mathbb{A}^1 \times \mathbb{P}^3$ defined by $tf_3 + x_1x_2x_3 = 0$. The fibre X_0' is the union of three planes $H_1' \cup H_2' \cup H_3'$. Pick two sections $p_1, p_2 : B \to X'$ such that $p_i(0) \in H_i'$. This can be achieved by choosing two appropriate points on the base locus $f_3(x_0, x_1, x_2, x_3) = x_1x_2x_3 = 0$.

6.2.2. *Resolving to obtain a normal crossings family.* The total space of X' is not a normal crossings family: it has 9 ordinary double points over $t = 0$, assuming f_3 is chosen generally: these are the points of intersection of the singular lines $H_i' \cap H_j'$ with $f_3 = 0$. One manifestation is the fact that H_i' are Weil divisors which are not

Cartier. By blowing up H_1' followed by H_2', we resolve the ordinary double points. We obtain a family $X \to B$, which is normal crossings, hence logarithmically smooth, in a neighbourhood of $t = 0$, as depicted on the left in Figure 4. Denote by H_i the proper transform of H_i'.

We identify $\Sigma(X)$ with $(\mathbb{R}_{\geq 0})^3$, so that $\Delta(X)$ is identified with the standard simplex $\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1, x_1, x_2, x_3 \geq 0\}$, as depicted on the right in Figure 4.

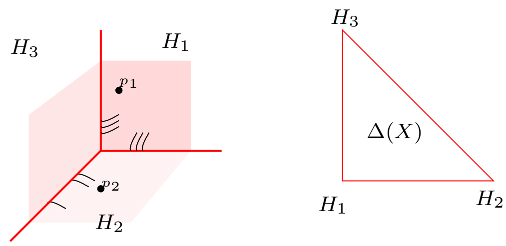

FIGURE 4. The left-hand picture depicts X_0 as a union of three copies of $\mathbb{P}^2$, blown up at 6, 3 or 0 points. The right-hand picture depicts $\Delta(X)$.

cubiccentra

6.2.3. *Limiting curves: triangles.* Since the limit of plane curves on $X_t' = X_t$ should be a plane curve on X_0', limiting curves on X_0 would map to plane sections of X_0' through p_1, p_2. This greatly limits the possible limiting curves — in particular the image in each of H_i' is a line.

General triangles do not occur. It is easy to see that a plane section of X_0' passing through p_1, p_2 whose proper transform in X_0 is a triangle of lines cannot be the image of a stable logarithmic curve of genus zero. This follows from Theorem 5.3.2.

Triangles through double points. On the other hand, consider the total transform of a triangle in X_0' passing through p_1, p_2, and one of the 9 ordinary double points of X'. The resulting curve will be a cycle of 4 rational curves, one of the curves being part of the exceptional set of the blowup of H_1' and H_2'. We can partially normalize this curve at the node contained in the smooth part of X_0, getting a stable logarithmic curve of genus 0. See Figure 5 for one such case.

Tropical picture. We depict to the right the associated rigid tropical curve. Here the lengths of each edge are 1, and the contact data u_q take the values $(-1, 1, 0)$, $(0, -1, 1)$ and $(1, 0, -1)$. This accounts for 9 curves.

Logarithmic enhancement and logarithmic unobstructedness. Note that the above curves are transverse pre-logarithmic curves, and hence by Theorem 5.4.1, each of these curves has precisely one basic logarithmic enhancement. Since the curve is immersed it has no automorphisms. One can use a natural absolute, rather than relative, obstruction theory to define the virtual funamental class, which is governed

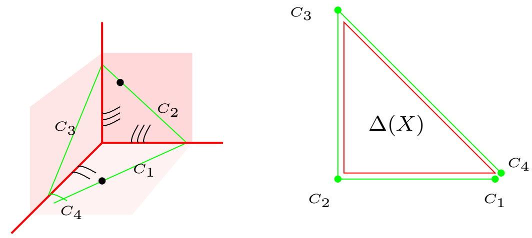

FIGURE 5. Proper transform of a triangle through a double point. The curve is normalized where C_1 and C_4 meet. cubiccase1

by the logarithmic normal bundle. In this case each curve is unobstructed: since it is transverse with contact order 1, the logarithmmic normal bundle coincides with the usual normal bundle. The normal bundle restricts to $O_{\mathbb{P}^1}, O_{\mathbb{P}^1}(1), O_{\mathbb{P}^1}(1)$, and $O_{\mathbb{P}^1}(-1)$ on the respective four components C_1, C_2, C_3 and C_4, hence it is non-special. We note that this does not account for the incidence condition that the marked points land at p_i. This can be arranged, for instance, using (6.3.1) in Section 6.3.2.

It follows that indeed each of these nine curves contributes precisely once to the desired Gromov-Witten invariant.

6.2.4. *Limiting curves: the plane section through the origin.* The far more interesting case is when the plane section of X'_0 passes through the triple point. Then one has a stable map from a union of four projective lines, with the central component contracted to the triple point, see Figure 6 on the left.

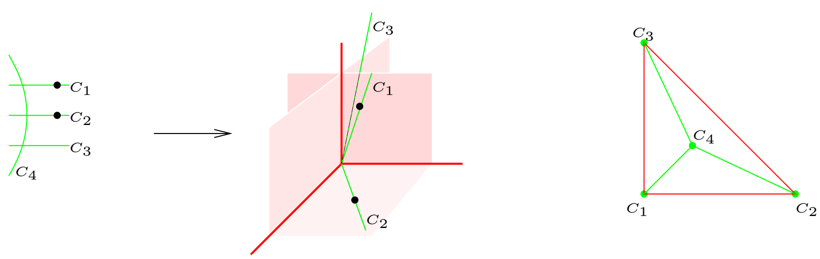

FIGURE 6. A curve mapping to a plane section through the origin, and its tropicalization. cubiccase2

There is in fact a one-parameter family $\underline{W}$ of such stable maps, as the line in H_3 is unconstrained and can be chosen to be any element in a pencil of lines. Only one member of this family lies in a plane, and we will see below that indeed only one member of the family admits a logarithmic enhancement.

Tropical picture. To understand the nature of such a logarithmic curve, we first analyze the corresponding tropical curve. The image of such a curve will be as depicted in Figure 6 on the right, with the central vertex corresponding to the contracted component landing somewhere in the interior of the triangle. However, the tropical balancing condition must hold at this central vertex, by [GS13], Proposition 1.14. From this one determines that the only possibility is that the values of u_q of $(-2,1,1)$, $(1,1,-2)$ and $(1,-2,1)$, all lengths are $1/3$, and the central vertex is $(1/3,1/3,1/3)$. This rigid tropical curve Γ then has multiplicity $m_\Gamma = 3$.

6.2.5. *Logarithmic enhancement using a logarithmic modification.* We now show that only one of the stable maps in the family $\underline{W}$ has a logarithmic enhancement. To do so, we use the techniques of §5, first refining $\Sigma(X)$ to obtain a logarithmic modification of X. The subdivision visible in Figure 6 gives a refinement of $\Sigma(X)$, the central star subdivision of $\Sigma(X)$. This corresponds to the ordinary blow-up $h : \tilde{X} \to X$ at the triple point of X_0. We may then identify logarithmic curves in $\tilde{X}$ and use the induced morphism $\mathscr{M}(\tilde{X}/B) \to \mathscr{M}(X/B)$.

Lifting the map to $\tilde{X}_0$. The central fibre $\tilde{X}_0$ is now as depicted in Figure 7. We then try to build a transverse pre-logarithmic curve in $\tilde{X}$ lifting one of the stable maps of Figure 6. Writing $C = C_1 \cup C_2 \cup C_3 \cup C_4$, with C_4 the central component, we map C_1 and C_2 to the lines L_1 and L_2 as depicted in Figure 7, while C_3 maps to some line L_3 in H_3. On the other hand, by the definition of transverse pre-logarithmic maps, C_4 must map to the exceptional $\mathbb{P}^2 = E$ in such a way that it is triply tangent to ∂E precisely at the points of intersection with L_i, $i = 1,2,3$.

Uniqueness of liftable map. We claim that there is precisely one such map, necessarily with image a curve of degree 3 in $\mathbb{P}^2$, with image as depicted in Figure 7. First, since $L_1 \cap E$, $L_2 \cap E$ are fixed, one can apply the tropical vertex [GPS10] to calculate the number of such maps as 1. One can also deduce this explicitly by considering linear series as follows. The three contact points on $C_4 \simeq \mathbb{P}^1$ can be taken to be $0, 1$ and ∞, and the map $C_4 \to \mathbb{P}^2$ corresponds, up to a choice of basis, to the unique linear system on $\mathbb{P}^1$ spanned by the divisors $3\{0\}$, $3\{1\}$ and $3\{\infty\}$. Since these points map to the coordinate lines, the choice of basis is limited to rescaling the defining sections. The choice of scaling of the defining sections results in fixing the images of 0 and 1, and the image point of ∞ is then uniquely determined.[13]

13→ This determines uniquely the point $L_3 \cap E$, in particular the line L_3 is determined. Thus we see that there is a unique transverse prelogarithmic map $\underline{f} : C \to \underline{\tilde{X}}_0$ such that $\underline{h} \circ \underline{f}$ lies in the family $\underline{W}$ of stable maps to X.

Logarithmic enhancement. We check that this transverse pre-logarithmic curve satisfies the lifting criterion of Theorem 5.5.1: using a base change of degree 3 one replaces the non-reduced $\mathbb{P}^2$ by its triple cover, the cubic $w^3 = xyz$, which is only singular over the intersection points of the axes of $\mathbb{P}^2$. The inverse image of our

[13] (Dan) The middle of figure 7 could be nicer

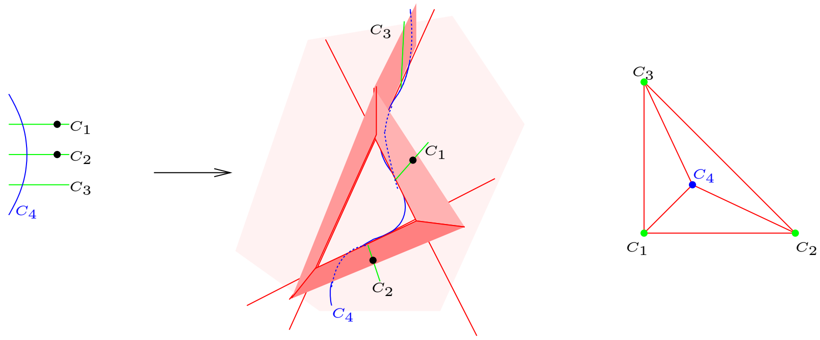

FIGURE 7. cubicblownup

curve consists of three distinct rational curves, each smooth and transverse to the boundary and lying in the smooth locus. Hence the map we denoted $\underline{f}$ lifts (in 3 different ways) to the branched cover, as required by Theorem 5.5.1.

Thus there is precisely $3/3 = 1$ basic log enhancement of this transverse prelogarithmic curve. This gives a basic stable logarithmic map $h \circ f$.

Unobstructedness. Once again we check that $h \circ f$ is unobstructed, if one makes use of an absolute onbstruction theory: the logarithmic normal bundle has degree 0 on each line, hence degree 1 on C_4, and is non-special. Again the map has no automorphisms, which accounts for 1 curve, with multiplicity 3, because $m_\Gamma = 3$. Hence the final accounting is

$$9 + 3 \times 1 \quad = \quad 12,$$

which is the desired result.

6.2.6. *Impossibility of other contributions.* Note our presentation has not been thorough in ruling out other possibilities for stable logarithmic maps, possibly obstructed, contributing to the total. For example, $\underline{W}$ includes curves where L_3 falls into the double point locus of X_0, but a more detailed analysis of the tropical possibilities rules out a possible log enhancement. We leave it to the reader to confirm that we have found all possibilities.

egensection

6.3. **Degeneration of point conditions.** We now consider a situation which is common in applications of tropical geometry; this includes the work of Mikhalkin on tropical curve counting [Mik05]. We fix a pair (Y, D) where Y is a variety over a field $\Bbbk$ and D is a reduced Weil divisor such that the divisorial logarithmic structure on Y is logarithmically smooth over the trivial point $\operatorname{Spec} \Bbbk$. We then consider the trivial family

$$X = Y \times \mathbb{A}^1 \to \mathbb{A}^1 = B,$$

where now X is given the divisorial logarithmic structure with respect to the divisor $(D \times B) \cup (Y \times \{0\})$.

6.3.1. *Evaluation maps and moduli.* Fix a type β of stable logarithmic maps to X over B, getting a moduli space $\mathscr{M}(X/B, \beta)$. We assume now that the curves of type β have n marked points $p_1, \ldots, p_n$ with $u_{p_i} = 0$ — and possibly some additional marked points $x_1, \ldots, x_m$ with non-trivial contact order with D. Given a stable map $(C/W, \mathbf{x}, \mathbf{p}, f)$, a priori for each i we have an evaluation map $\mathrm{ev}_i : (W, p_i^*\mathcal{M}_C) \to X$ obtained by restricting f to the section p_i. Noting that $u_{p_i} = 0$, the map $\mathrm{ev}_i^\flat : (f \circ p_i)^{-1}\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_W \oplus \mathbb{N}$ factors through $\overline{\mathcal{M}}_W$, and thus we have a factorization $\mathrm{ev}_i : (W, p_i^*\mathcal{M}_C) \to W \to X$. In a slight abuse of notation we write ev_i for the morphism $W \to X$ also, and thus obtain a morphism

$$\mathrm{ev} : \mathscr{M}(X/B, \beta) \to X^n := X \times_B \times X \times_B \times \cdots \times_B X.$$

If we choose sections $\sigma_1, \ldots, \sigma_n : B \to X$, we obtain a map

$$\sigma := \prod_{i=1}^{n} \sigma_i : B \to X^n.$$

This allows us to define *the moduli space of curves passing through the given sections,*

$$\mathscr{M}(X/B, \beta, \sigma) := \mathscr{M}(X/B, \beta) \times_{X^n} B,$$

where the two maps are ev and σ.[14]

Sec:VFC-points

6.3.2. *Virtual fundamental class on $\mathscr{M}(X/B, \beta, \sigma)$.* We note that the moduli space $\mathscr{M}(X/B, \beta, \sigma)$ of curves passing through the given sections carries a virtual fundamental class. The perfect obstruction theory is defined by

q:point-condition

$$E^\bullet = \Big(R\pi_*[f^*\Theta_{X/B} \to \bigoplus_{i=1}^{n} (f^*\Theta_{X/B})|_{p_i(W)}]\Big)^\vee, \tag{6.3.1}$$

for the stable map $(\pi : C \to W, \mathbf{x}, \mathbf{p}, f)$. Here the map of sheaves above is just restriction.

6.3.3. *Choice of sections and $\Delta(X)$.* We can now use the techniques of previous sections to produce a virtual decomposition of the fibre over $b_0 = 0$ of $\mathscr{M}(X/B, \beta, \sigma) \to B$. However, to be interesting, we should in general choose the sections to interact with D in a very degenerate way over b_0. In particular, restricting to b_0 (which is now the standard log point), we obtain maps

$$\sigma_i : b_0 \to Y^\dagger,$$

[14] Recall that all fibre products are in the category of fs log schemes.

where $Y^\dagger = Y \times p^\dagger$ is the product with the standard logarithmic point. Note that

$$\Sigma(Y^\dagger) = \Sigma(X) = \Sigma(Y) \times \mathbb{R}_{\geq 0},$$

with $\Sigma(X) \to \Sigma(B)$ the projection to the second factor. So $\Delta(X) = \Sigma(Y)$ and $\Sigma(\sigma_i) : \Sigma(B) \to \Sigma(X)$ is a section of $\Sigma(X) \to \Sigma(B)$ and hence is determined by a point $P_i \in \Delta(X)$, necessarily rationally defined.

6.3.4. *Tropical fibered product.* We wish to understand the fibre product $\mathscr{M}(X/B, \beta, \sigma) := \mathscr{M}(X/B, \beta) \times_{X^n} B$ at a tropical level. We observe

.calproduct **Proposition 6.3.5.** *Let X, Y and S be fs log schemes, with morphisms $f_1 : X \to S$, $f_2 : Y \to S$. Let $Z = X \times_S Y$ in the category of fs log schemes, p_1, p_2 the projections. Suppose $\bar{z} \in Z$ with $\bar{x} = p_1(\bar{z})$, $\bar{y} = p_2(\bar{z})$, and $\bar{s} = f_1(p_1(\bar{z})) = f_2(p_2(\bar{z}))$. Then*

$$\mathrm{Hom}(\overline{\mathcal{M}}_{Z,\bar{z}}, \mathbb{N}) = \mathrm{Hom}(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{N}) \times_{\mathrm{Hom}(\overline{\mathcal{M}}_{S,\bar{s}}, \mathbb{N})} \mathrm{Hom}(\overline{\mathcal{M}}_{Y,\bar{y}}, \mathbb{N})$$

and

$$\mathrm{Hom}(\overline{\mathcal{M}}_{Z,\bar{z}}, \mathbb{R}_{\geq 0}) = \mathrm{Hom}(\overline{\mathcal{M}}_{X,\bar{x}}, \mathbb{R}_{\geq 0}) \times_{\mathrm{Hom}(\overline{\mathcal{M}}_{S,\bar{s}}, \mathbb{R}_{\geq 0})} \mathrm{Hom}(\overline{\mathcal{M}}_{Y,\bar{y}}, \mathbb{R}_{\geq 0}).$$

Proof. The first statement follows immediately from the universal property of fibred product applied to maps $\bar{z}^\dagger \to Z$, where $\bar{z}^\dagger$ denotes the geometric point $\bar{z}$ with standard logarithmic structure. The second statement then follows from the first. ♠

6.3.6. *Tropical moduli space.* We now see a simple interpretation for the tropicalization of $W := \mathscr{M}(X/B, \beta, \sigma)$. If $\bar{w} \in W$ is a geometric point, let Q be the basic monoid associated with $\bar{w}$ as a stable logarithmic map to X. Then by Proposition 6.3.5, we have

$$\mathrm{Hom}(\overline{\mathcal{M}}_{W,\bar{w}}, \mathbb{R}_{\geq 0}) = \mathrm{Hom}(Q, \mathbb{R}_{\geq 0}) \times_{\prod_i \mathrm{Hom}(P_{p_i}, \mathbb{R}_{\geq 0})} \mathbb{R}_{\geq 0}.$$

Here as usual $P_{p_i} = \overline{\mathcal{M}}_{X, f(p_i)}$. The maps defining the fibre product are as follows. The map $\mathrm{Hom}(Q, \mathbb{R}_{\geq 0}) \to \prod_i \mathrm{Hom}(P_{p_i}, \mathbb{R}_{\geq 0})$ can be interpreted as taking a tropical curve $\Gamma \to \Sigma(X)$ to the point of $\mathrm{Hom}(P_{p_i}, \mathbb{R}_{\geq 0})$ which is the image of the contracted edge corresponding to the marked point p_i. The map $\mathbb{R}_{\geq 0} \to \prod_i \mathrm{Hom}(P_{p_i}, \mathbb{R}_{\geq 0})$ is $\prod_i \Sigma(\sigma_i)$ and hence takes 1 to $(P_1, \ldots, P_n)$.

This yields:

Proposition 6.3.7. *Let $m \in \Delta(W)$, and let $\Gamma_C = \Sigma(\pi)^{-1}(m)$. Then $\Sigma(f) : \Gamma_C \to \Delta(X)$ is a tropical curve with the unbounded edges E_{p_i} being mapped to the points P_i. Furthermore, as m varies within its cell of $\Delta(W)$, we obtain the universal family of tropical curves of the same combinatorial type mapping to $\Delta(X)$ and with the edges E_{p_i} being mapped to P_i.*

6.3.8. *Restatement of the decomposition formula.* Denote

$$\mathscr{M}(Y^\dagger/b_0,\beta,\sigma) := \mathscr{M}(X_0/b_0,\beta)\times_{X^n} B$$

and for $\mathbf{A}\vdash A$

$$\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0,\beta,\sigma) := \mathscr{M}_{\tau,\mathbf{A}}(X_0/b_0,\beta)\times_{X^n} B.$$

Theorem 1.1.2 now translates to the following:

omposition-points

Theorem 6.3.9 (The logarithmic decomposition formula for point conditions)**.** *Suppose Y is logarithmically smooth. Then*

$$[\mathscr{M}(Y^\dagger/b_0,\beta,\sigma)]^{\mathrm{virt}} = \sum_{\tau\in\Omega} m_\tau\cdot\sum_{\mathbf{A}\vdash A}(i_{\tau,\mathbf{A}})_*[\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0,\beta,\sigma)]^{\mathrm{virt}}$$

Example 6.3.10. The above discussion allows a reformulation of the approach of [NS06] to Mikhalkin's formula for tropical counts of curves in toric varieties. Take Y to be a toric variety with the toric logarithmic structure, and fix a homology class β of curve and a genus g. By fixing an appropriate number n of points in Y, one can assume that the moduli space of curves of genus g and class β passing through these points has expected dimension 0. Then after choosing suitable degenerating sections $\sigma_1,\ldots,\sigma_n$, one obtains points $P_1,\ldots,P_n\in\Sigma(Y)$, the fan for Y. Then the question of understanding $\mathscr{M}(X/B,\beta,\sigma)$ is reduced to an analysis for each rigid tropical curve in $\Sigma(Y)$ with the correct topology. In particular, the domain curve should have genus g (taking into account the genera assigned to each vertex) and should have $D_\rho\cdot\beta$ unbounded edges parallel to a ray $\rho\in\Sigma(Y)$, where $D_\rho\subseteq Y$ is the corresponding divisor. The argument of [NS06] essentially carries out an explicit analysis of possible logarithmic curves associated with each such rigid curve.

6.4. **An example in $\mathbb{F}_2$.** We now consider a very specific case of the previous subsection. This example deliberately deviates slightly from the toric case mentioned above and exhibits new phenomena.

6.4.1. *A non-toric logarithmic structure on a Hirzebruch surface.* Let Y be the Hirzebruch surface $\mathbb{F}_2$. Viewed as a toric surface, it has 4 toric divisors, which we write as f_0, f_∞, C_0 and C_∞. Here f_0, f_∞ are the fibres of $\mathbb{F}_2\to\mathbb{P}^1$ over 0 and ∞, C_0 is the unique section with self-intersection -2, and C_∞ is a section disjoint from C_0, with C_∞ linearly equivalent to $f_0+f_\infty+C_0$.

We will give Y the (non-toric) divisorial logarithmic structure coming from the divisor $D=f_0+f_\infty+C_\infty$.

6.4.2. *The curves and their marked points.* We will consider rational curves representing the class C_∞ passing through 3 points y_1, y_2, y_3. Of course there should be precisely one such curve.

A general curve of class C_∞ will intersect D in four points, so we will set this up as a logarithmic Gromov-Witten problem by considering genus 0 stable logarithmic maps

$$f : (C, p_1, p_2, p_3, x_1, x_2, x_3, x_4) \to Y,$$

imposing the condition that $f(p_i) = y_i$, and f is constrained to be transversal to f_0, f_∞, C_∞ and C_∞ at x_i for $i = 1, \ldots, 4$ respectively. This transversality determines the vectors u_{x_i}, while we take the contact data $u_{p_i} = 0$.

Since the maps have the points x_3 and x_4 ordered, we expect the final count to amount to 2 rather than 1.

6.4.3. *Choice of degeneration.* We will now see what happens when we degenerate the point conditions as in §6.3, by taking $X = Y \times \mathbb{A}^1$ and considering sections $\sigma_i : \mathbb{A}^1 \to X$, $1 \leq i \leq 3$. We choose these sections to be general subject to the condition that

$$\sigma_1(0) \in f_0, \quad \sigma_2(0) \in f_\infty, \quad \sigma_3(0) \in C_0.$$

Since $C_0 \cap C_\infty = \emptyset$, any curve in the linear system $|C_\infty|$ which passes through this special choice of 3 points must contain C_0, and hence be the curve $f_0 + f_\infty + C_0$.

6.4.4. *The complex* $\Delta(X)$ *and the tropical sections.* Note that $\Delta(X)$ is as depicted in Figure 8, an abstract gluing of two quadrants, not linearly embedded in the plane. The choice of sections σ_i determines points $P_i \in \Sigma(X)$ as explained in §6.3. For example, if, say, the section σ_1 is transversal to $f_0 \times \mathbb{A}^1$, then P_1 is the point at distance 1 from the origin along the ray corresponding to f_0. Since C_0 is not part of the divisor determining the logarithmic structure, P_3 is in fact the origin.

6.4.5. *The tropical curves.* One then considers rigid decorated tropical curves passing through these points.

- The curves must have 7 unbounded edges, E_{p_i}, E_{x_j}.
- The map contracts E_{p_i} to P_i.
- Each E_{x_j} is mapped to an unbounded ray going to infinity in the direction indicating which of the three irreducible components of D the point x_j is mapped to.

6.4.6. *Rigid tropical curves.* It is then easy to see that to be rigid, the tropical curve must have three vertices, v_1, v_2, v_3, with the edge E_{p_i} attached to v_i and v_i necessarily being mapped to P_i.

The location of the E_{x_i} is less clear. One can show using the balancing condition [GS13], Proposition 1.15, that E_{x_1} must be attached to v_1 and E_{x_2} must be attached

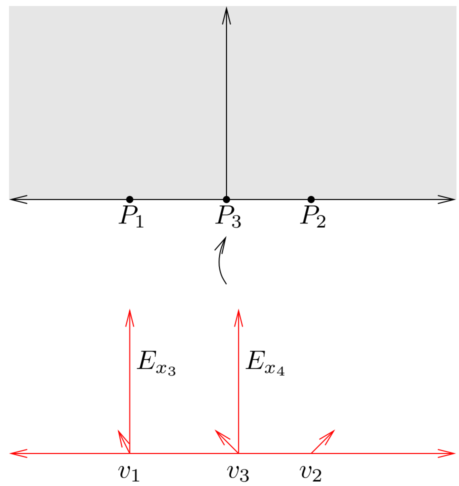

FIGURE 8. The polyhedral complex $\Delta(X) = \Sigma(Y)$. DeltaF2

to v_2. There remains, however, some choice about the location of E_{x_3} and E_{x_4}. Indeed, they may be attached to the vertices v_1, v_2 or v_3 in any manner. Figure 8 shows one such choice.

6.4.7. *Decorated rigid tropical curves.* We must however consider *decorated* rigid tropical curves, and in particular we need to assign curve classes $\beta(v)$ to each vertex v. Let n_i be the number of edges in $\{E_{x_3}, E_{x_4}\}$ attached to the vertex v_i. Since E_{x_3} and E_{x_4} indicate which "virtual" components of the domain curve have marked points mapping to C_∞, it then becomes clear that the class associated to v_1 and v_2 must be $n_1 f$ and $n_2 f$ respectively, while the class associated to v_3 must be $C_0 + n_3 f$, where $n_1 + n_2 + n_3 = 2$.

6.4.8. *The seeming contradiction.* In fact, as we shall see shortly, there are logarithmic curves whose tropicalization yields any one of the curves with $n_1 = n_2 = 1$, and there is no logarithmic curve *over the standard logarithmic point* whose tropicalization is the tropical curve with $n_3 = 2$. Suprisingly at first glance, in fact the only decorated rigid tropical curve which provides a non-trivial contribution to the Gromov-Witten invariant is the one which can not be realised, with $n_3 = 2$. We will also see that the case $n_1 = 2$ or $n_2 = 2$ plays no role.

6.4.9. *Curves with $n_1 = n_2 = 1$ contribute 0.* To explain this seemingly contradictory conclusion, first recall the standard fact that there is a flat family $\underline{\mathcal{X}} \to \underline{\mathbb{A}}^1$ such that $\underline{\mathcal{X}}_0 \cong \mathbb{F}_2$ and $\underline{\mathcal{X}}_t \cong \underline{\mathbb{P}}^1 \times \underline{\mathbb{P}}^1$ for $t \neq 0$. Furthemore, the divisor $f_0 \cup f_\infty \cup C_\infty$ extends to a normal crossings divisor on $\underline{\mathcal{X}}$ with three irreducible components: $\{0\} \times \mathbb{P}^1$, $\{\infty\} \times \mathbb{P}^1$, and a curve of type $(1, 1)$. This endows $\underline{\mathcal{X}}$ with a divisorial logarithmic structure, logarithmically smooth over $\mathbb{A}^1$ with the trivial logarithmic structure. However, no curve of class C_0 or C_0+f in $\mathcal{X}_0$ deforms to $\mathcal{X}_t$ for $t \neq 0$. Hence no curve representing a point in the moduli space $\mathscr{M}_\tau$ for τ one of the decorated rigid tropical curves with $n_3 \leq 1$ deforms. The usual deformation invariance of Gromov-Witten invariants then implies that the contribution to the Gromov-Witten invariant from such a τ is zero.

The fact that this contribution is 0 can also be deduced from our formalism of the gluing formula using punctured maps in [ACGS16].

6.4.10. *Expansion and description of moduli space.* To explore the existence of the relevant logarithmic curves, we again turn to §5. First let us construct a curve whose decorated tropical curve has $n_1 = n_2 = 1$. The image of this curve in $\Delta(X)$ yields a subdivision of $\Delta(X)$ which in turn yields a refinement of $\Sigma(X)$, and hence a log étale morphism $\tilde{X} \to X$. It is easy to see that this is just a weighted blow-up of $f_0 \times \{0\}$ and $f_\infty \times \{0\}$ in $X = Y \times \mathbb{A}^1$; the weights depend on the precise location of P_1 and P_2, but if they are taken to have distance 1 from the origin, the subdivision will correspond to an ordinary blow-up. The central fibre is now as depicted in Figure 9, with the proper transforms of the sections meeting the central fibre at the points p_1, p_2, p_3 as depicted.

The logarithmic curve then has three irreducible components, one mapping to C_0 and the other two mapping to the two exceptional divisors, each isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. These latter two components each map isomorphically to a curve of class $(1,1)$ on the exceptional divisor, and is constrained to pass through p_i and the point where C_0 meets the exceptional divisor. There is in fact a pencil of such curves. We remark that all 7 marked points are visible in Figure 9, but the curves in the exceptional divisors meet the left-most and right-most curves transversally, and not tangent as it appears in the picture. By Theorem 5.4.1, any such stable map then has a log enhancement, and composing with the map $\tilde{X} \to X$ gives a stable logarithmic map over the standard log point whose tropicalization is one of the rigid curves with $n_1 = n_2 = 1$.

One can show that the relevant moduli space in $\tilde{X}_0$ has two components isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, depending on which sides x_3 and x_4 lie. The virtual fundamental class of each component is in fact the top Chern class of the rank-2 trivial bundle, namely 0. This moduli space maps injectively to the moduli space of X_0.

6.4.11. *Curves with $n_1 = n_2 = 0$.* Now consider the case that $n_1 = n_2 = 0$ and $n_3 = 2$. This rigid tropical curve cannot be realised as the tropicalization of a stable logarithmic map over the standard log point. Indeed, to be realised, the curve must have an irreducible component of class $C_0 + 2f = C_\infty$, and we know there is no such curve. However, this tropical curve can in fact be realised as a degeneration of another tropical curve, as depicted in Figure 10.

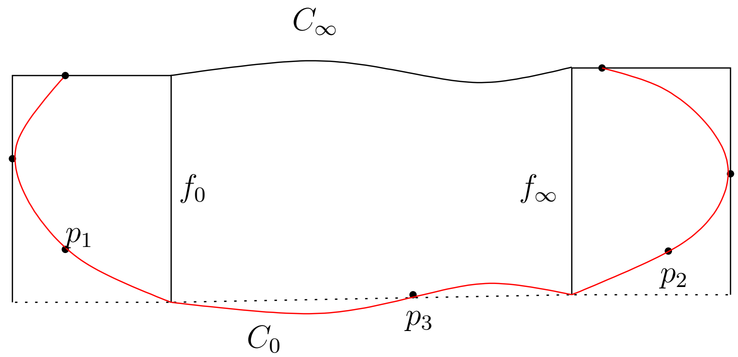

FIGURE 9. F2curvebad

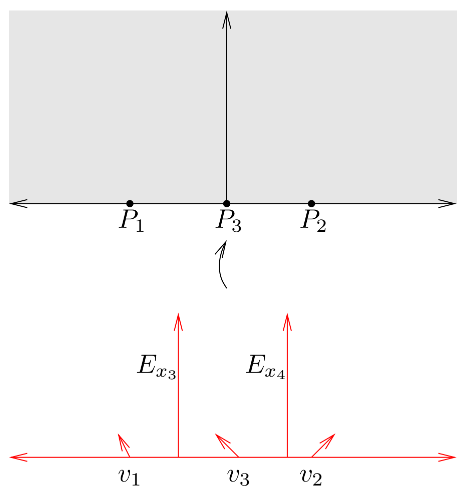

FIGURE 10. F2curvegood

To construct an actual logarithmic curve, we use refinements again. Assume for simplicity of the discussion that P_1 and P_2 have been taken to have distance 2 from the origin. Subdivide $\Delta(X)$ by introducing vertical rays with endpoints P_1 and P_2, and in addition introduce vertical rays which are the images E_{x_3} and E_{x_4}; again for simplicity of the discussion take the endpoint of these rays to be at distance 1 from the origin.

This corresponds to a blow-up $\tilde{X} \to X$ involving four exceptional components, and Figure 11 shows the central fibre of $\tilde{X} \to \mathbb{A}^1$, along with the image of a stable logarithmic map which tropicalizes appropriately. Composing this stable logarithmic map with $\tilde{X} \to X$ then gives a non-basic stable logarithmic map to X over the standard log point. It is not hard to see that the corresponding basic monoid Q

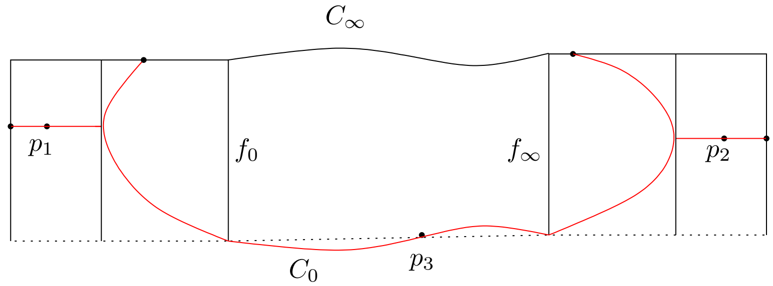

FIGURE 11. F2curvegood2

has rank 3, parameterizing the image of the curve in $\Sigma(B)$ as well as the location of the edges E_{x_3} and E_{x_4}. The degenerate tropical curve where the edges E_{x_3} and E_{x_4} are attached to the vertex v_3 represents a one-dimensional face of $Q^\vee$, so the rigid tropical curve with $n_3 = 2$ does appear in the family $Q^\vee$, but only as a degeneration of a tropical curve which is realisable by an actual stable logarithmic curve over the standard log point.

One can again show that the relevant moduli space in $\tilde{X}_0$ has two components isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This time the virtual fundamental class of each component is the top Chern class of $O(1) \boxplus O(1)$, which has degree 1. In contrast the corresponding moduli space $\mathscr{M}_{n_i=0}(X_0)$ is discrete.

6.4.12. *Curves with* $n_i = 2$. To complete the analysis, we end by noting that the case $n_1 = 2$ or $n_2 = 2$ cannot occur. Consider the case $n_1 = 2$. Any stable logarithmic curve over the standard log point with a tropicalization which degenerates to such a rigid tropical curve must have a decomposition into unions of irreducible components corresponding to the vertices v_1, v_2 and v_3, with the homology class of the image of the stable map restricted to each of these unions of irreducible components being $2[f_0]$, 0 and $[C_0]$ respectively. In particular, this will prevent the possibility of having any irreducible component whose image contains $\sigma_2(0)$. Thus this case does not occur.

6.4.13. *Deformation invariance, toric maps, and fundamental cycles.* Recall that logarithmic Gromov–Witten invariants are deformation invariant, therefore the number 2 of logarithmic stable maps is calculated equally well when p_i map to general points as when they specialize. When the images of p_i are in general position there are precisely two maps - with identical curves but the points x_3, x_4 reordered, which are necessarily unobstructed. The virtual fundamental class at b_0 is represented by the limiting cycle, which corresponds to two logarithmic stable maps on X. We claim that this cycle consists of two logarithmic stable maps with $n_1 = n_2 = 1$. One is bound to ask — how is this possible?

First, let us argue that indeed this is the case. Here the toric picture is helpful since it is unobstructed. Let Y^t be the Hirzebruch surface with its toric logarithmic structure coming from the divisor $f_0 + f_\infty + C_0 + C_\infty$, and let $Y^t \to Y$ be the natural logarithmic map where C_0 is left out. The two general logarithmic stable maps corresponding to p_i generic are disjoint from C_0, hence they lift to Y^t. Using the formalism of [NS06], one can choose the points P_1, P_2, P_3 so that that the limiting curves on Y^t have precisely three components and compose to two maps having $n_1 = n_2 = 1$ on Y.

Next, let us explain why this does not constitute a contradiction within mathematics, and what we must learn from this example.

First, the moduli space $\mathscr{M}(Y^\dagger/b_0, \beta, \sigma)$ has four relevant components, two corresponding to curves with $n_1 = n_2 = 1$ and two corresponding to curves with $n_1 = n_2 = 0$. It has virtual fundamental class represented by the 0-cycle representing the two limiting logarithmic stable maps, which happens to lie on the components with $n_1 = n_2 = 1$. It is also the direct image of the class $[\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0, \beta, \sigma)]^{\mathrm{virt}}$ on the moduli space of curves $\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0, \beta, \sigma)$ mapping to the locus of logarithmic stable maps with $n_1 = n_2 = 1$. But there is nothing here to guarantee that the limiting cycle is the image of any cycle coming from $\mathscr{M}_{\tau,\mathbf{A}}(Y^\dagger/b_0, \beta, \sigma)$, and in fact this is not the case here. (One can obtain the limiting cycle as the image with a different choice of points constraints σ.) Figure 12 describes the situation.

Second, we point out that the virtual decomposition is not compatible with the map $Y^t \to Y$ from the toric logarithmic structure.

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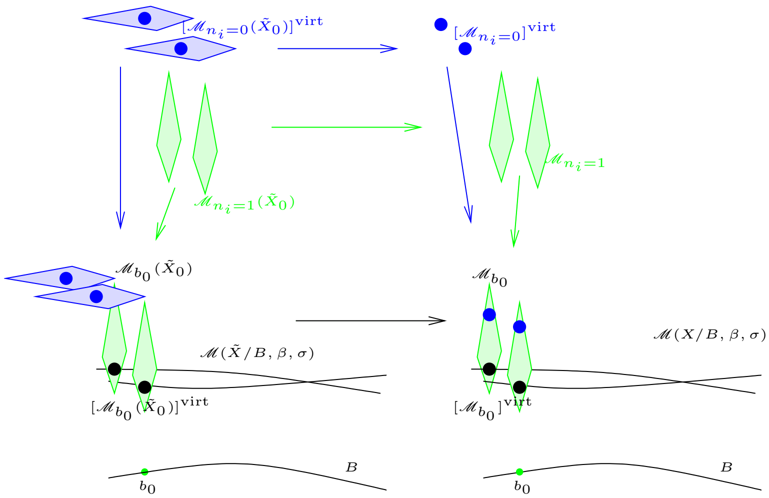

FIGURE 12. On the right: the fiber $\mathscr{M}(Y^\dagger/b_0,\beta,\sigma)$ of the moduli space $\mathscr{M}(X/\mathbb{A}^1,\beta,\sigma)$ and its virtual fundamental class. The class is represented by the limiting cycle marked by $[\mathscr{M}_{b_0}]^{\mathrm{virt}}$, but also by the image of the class $[\mathscr{M}_{n_i=0}]^{\mathrm{virt}}$. On the left: the corresponding picture for the expansion $\tilde{X}$. Fig:dec-6_4

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