LOGARITHMIC STABLE MAPS TO DELIGNE–FALTINGS PAIRS II

DAN ABRAMOVICH AND QILE CHEN

Abstract. We make an observation which enables one to deduce the existence of an algebraic stack of log maps for all Deligne–Faltings log structures (in particular simple normal crossings divisor) from the simplest case with characteristic generated by $\mathbb{N}$ (essentially the smooth divisor case).

Contents

1. Introduction

The idea of logarithmic stable maps was introduced in a legendary lecture by Bernd Siebert in 2001. The program has been on hold for quite a while as Mark Gross and Siebert were pursuing mirror symmetry; they have returned to it only recently. The central object is a stack $\mathcal{K}_\Gamma(Y)$ parametrizing what one calls *logarithmic stable maps* of log-smooth curves into a logarithmic scheme Y with Γ indicating the relevant numerical data, such as genus, marked points, curve class and other indicators related to the logarithmic structure. One needs to show $\mathcal{K}_\Gamma(Y)$ is algebraic and proper. Gross and Siebert approach this by means of probing a logarithmic scheme by the standard log point $\mathrm{Spec}(\mathbb{N} \to \mathbb{C})$. Our understanding is that this method works in great generality.[1] ←1

In [6], the second author considers another combinatorial construction of the stack $\mathcal{K}_\Gamma(Y)$ when the logarithmic structure Y on the underlying scheme $\underline{Y}$ is associated to the choice of a line bundle with a section. The motivating case is that of a pair $(\underline{Y}, \underline{D})$, where $\underline{D}$ is a smooth divisor in the smooth locus of the scheme $\underline{Y}$ underlying Y. This particular situation enables him to approach the degeneration formula of [13, 7, 15] in terms of logarithmic

Date: July 5, 2010.
Abramovich supported in part by NSF grants DMS-0603284 and DMS-0901278.

[1](Dan) Verify with G-S

structures. It should be pointed out that these logarithmic stable maps are not identical to those of Kim [11], though they are closely related.

Our point is that based solely on this special case, one can give a "pure thought" proof of algebricity and properness of the stack $\mathcal{K}_\Gamma(Y)$ whenever Y is a so called *generalized Deligne–Faltings* logarithmic structure. By saying Y is a generalized Deligne–Faltings log structure we mean that there is a fine saturated sharp monoid P and a sheaf homomorphism $P \to \overline{\mathcal{M}}_Y$ which locally lifts to a chart $P \to \mathcal{M}_Y$; the slightly simpler Deligne–Faltings log structure is the case where $P = \mathbb{N}^k$. This in turn covers many of the cases of interest, such as a variety with a simple normal crossings divisor, or a simple normal crossings degeneration of a variety with a simple normal crossings divisor. We generalize it a bit frther in Proposition 3.12. It does not cover the case of a normal crossings divisor which is not simple, but we hope one could cover this case using descent arguments.

The purpose of this note is to set up a general categorical framework which enables us to make this construction. This general setup is of use not only for $\mathcal{K}_\Gamma(Y)$. In particular we have applications, pursued elsewhere [1], to constructing the target of *evaluation maps* of logarithmic Gromov–Witten theory.

All logarithmic schemes in this note are assumed to be fine and saturated logarithmic schemes - abbreviated fs log schemes - unless indicated otherwise.

2. Logarithmic maps: a tale of two categories

2.1. **Stable maps.** Let $\underline{Y}$ be a projective scheme. The stack of stable maps to $\underline{Y}$ is defined as follows: one fixes discrete data $\Gamma = (g, n, \beta)$ where g, n are non-negative integers standing for genus and number of marked points, and β a curve class on $\underline{Y}$. A *pre-stable* map to $\underline{Y}$ over a scheme $\underline{S}$ is a diagram

$$\begin{array}{ccc} \underline{C} & \longrightarrow & \underline{Y} \\ \downarrow & & \\ \underline{S} & & \end{array}$$

where $\underline{C} \to \underline{S}$ is a proper flat family of n-pointed prestable curves, and $\underline{C} \to \underline{Y}$ a morphism. The prestable map is *stable* if on the fibers the groups $\mathrm{Aut}_{\underline{Y}}(\underline{C}_s)$ are finite. Morphisms of prestable curves are defined as cartesian diagrams.

One easily sees that prestable maps form a category fibered in groupoids over the category of schemes. It is an important theorem that it is in fact an algebraic stack $\underline{K}^{pre}(\underline{Y})$, and the substack $\underline{K}_\Gamma(\underline{Y})$ of stable maps of type Γ is proper with projective coarse moduli space [12]. When $\underline{Y}$ is smooth, there is a perfect obstruction theory, giving rise to a virtual fundamental class $[\underline{K}_\Gamma(\underline{Y})]^{\mathrm{virt}}$ underlying the usual algebraic treatment of Gromov–Witten theory [16, 5].

The main result of this note is an analogue of the following evident result: assume $Y = Y_1 \times_{Y_2} Y_3$. Then

$$\underline{K}^{pre}(\underline{Y}) = \underline{K}^{pre}(\underline{Y}_1) \underset{\underline{K}^{pre}(\underline{Y}_2)}{\times} \underline{K}^{pre}(\underline{Y}_3).$$

2.2. **Logarithmic stable maps as a stack over $\mathfrak{LogSch}$.** Let Y be a fs logarithmic scheme with projective underlying scheme $\underline{Y}$. One can repeat the construction above, replacing prestable curves by *log smooth* curves [9], and replacing all morphisms of schemes by morphisms of log schemes: a log prestable map over S is a diagram of *logarithmic* schemes

$$\begin{array}{ccc} C & \longrightarrow & Y \\ \downarrow & & \\ S & & \end{array}$$

where $C \to S$ is a log smooth curve and $C \to Y$ a morphism of log schemes. We define such a map to be stable if the underlying prestable map is stable. Arrows are defined using cartesian diagrams:

$$\begin{array}{ccccc} C' & \longrightarrow & C & \longrightarrow & Y \\ \downarrow & & \downarrow & & \\ S' & \longrightarrow & S & & \end{array}$$

Again it is evidently a category fibered in groupoids, but this time *over the category $\mathfrak{LogSch}$ of fs logarithmic schemes*. It is proven in [6] when the log structure Y is given by a line bundle with section, and in greater generality in the upcoming work of Gross and Siebert, that this category is an fs logarithmic algebraic stack: there is a logarithmic algebraic stack $\mathcal{K}_\Gamma(Y) = (\underline{\mathcal{K}}, \mathcal{M}_{\underline{\mathcal{K}}})$, where the log structure is fs, such that logarithmic stable maps over S are equivalent to log morphisms $S \to \mathcal{K}_\Gamma(Y)$. Denote the universal log smooth curve by $\mathfrak{C} \to \mathcal{K}_\Gamma(Y)$.

It is natural to search for general criteria for algebricity of such logarithmic moduli in analogy to Artin's work [4]. We do not address this general question here.

2.3. **Logarithmic stable maps as a stack over $\mathfrak{Sch}$.** The existence of $\mathcal{K}_\Gamma(Y)$ has immediate strong implications on the structure of log stable maps. Objects of the underlying stack $\underline{\mathcal{K}_\Gamma}(Y)$ over a scheme $\underline{S}$ can be understood as follows: an object is after all an arrow $\underline{S} \to \underline{\mathcal{K}_\Gamma}(Y)$. It automatically gives rise to a cartesian diagram

$$\begin{array}{ccccc} C^{\min} & \longrightarrow & \mathfrak{C} & \longrightarrow & Y \\ \downarrow & & \downarrow & & \\ S^{\min} & \longrightarrow & \mathcal{K}_\Gamma(Y) & & \\ \downarrow & & \downarrow & & \\ \underline{S} & \longrightarrow & \underline{\mathcal{K}}_\Gamma(Y). & & \end{array}$$

in particular an object $S^{\min} \to \mathcal{K}_\Gamma(Y)$, but here *the logarithmic structure* $S^{\min}$ *is pulled back from* $\mathcal{K}_\Gamma(Y)$. Moreover, *every* log stable map factors uniquely through one of this type: Given a log stable map over S we have a morphism $S \to \mathcal{K}_\Gamma(Y)$ by definition, giving rise to an extended diagram

$$\begin{array}{ccccccc} C & \longrightarrow & C^{\min} & \longrightarrow & \mathfrak{C} & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow & & \\ S & \longrightarrow & S^{\min} & \longrightarrow & \mathcal{K}_\Gamma(Y) & & \\ & & \downarrow & & \downarrow & & \\ & & \underline{S} & \longrightarrow & \underline{\mathcal{K}}_\Gamma(Y). & & \end{array}$$

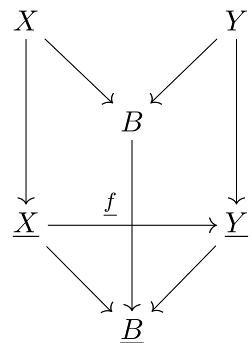
and the top left square is cartesian by F. Kato's theorem [9]. Following B. Kim [11] we call a log map over S *minimal* (not to be confused with log minimal models of the minimal model program) if $S \to \mathcal{K}_\Gamma(Y)$ is strict, namely the log structure on S is the pullback of the log structure on $\mathcal{K}_\Gamma(Y)$. It follows tautologically that the underlying stack $\underline{\mathcal{K}}_\Gamma(Y)$ precisely parametrizes log maps with minimal log structure.

In fact this thought process is reversible: the construction of [6] in the case of a Deligne–Faltings log structure of rank 1 goes by way of constructing a proposed minimal log structure associated to any log map, and verifying that log maps where the log structure is the proposed minimal one are indeed minimal (every object maps uniquely to a minimal one), and form an algebraic stack over $\mathfrak{Sch}$ carrying a logarithmic structure. Our understanding is that Gross and Siebert use this avenue too.[2] 2→ In short, the second categorical interpretation, of $\underline{\mathcal{K}}_\Gamma(Y)$ as a stack over $\mathfrak{Sch}$, takes precedence here.

One is tempted to try to mimic the same construction in general. *This is not the route taken here.* In fact we use the universality of the category $\mathcal{K}_\Gamma(Y)$ over $\mathfrak{LogSch}$ for given Y to deduce its algebricity from cases of simpler Y. It is worthwhile setting this up in general.

[2](Dan) Check with G-S

2.4. **The general setup.** Consider a commutative diagram



where $X \to B$ is an integral morphism of fs log schemes with $\underline{X} \to \underline{B}$ flat and proper, Y is an fs log scheme, and $\underline{X} \to \underline{Y}$ a morphism. We can define a category $\text{Lift}_{\underline{f}}$ fibered in groupoids over $\mathfrak{LogSch}$ whose objects over S are morphisms $S \to B$ of fs log schemes with a lifting $f_S : X_S \to Y_S$ of the underlying morphism $\underline{f}_S : \underline{X}_S \to \underline{Y}_S$. The arrows are again defined by taking cartesian diagrams. We can ask the following question:

problem **Problem 2.5.** Is the category $\text{Lift}_{\underline{f}}$ equivalent to an fs logarithmic algebraic stack $\mathcal{L}ift_{\underline{f}}$? Under what conditions is it proper?

Again this can be interpreted over $\mathfrak{Sch}$: objects of the underlying algebraic stack $\underline{\mathcal{L}ift}_{\underline{f}}$ parametrize lifts over a minimal log structure. This point of view will not be pursued here.

In fact if convenient we can remove the geometry of Y entirely from the picture, mimicking the methods of Olsson [17]: consider $\overline{Y} = \underline{Y} \times_{\underline{B}} B$ and $Z = X \times_{\overline{Y}} Y$. Then Z is a logarithmic scheme over X. The category $\text{Lift}_{\underline{f}}$ is evidently equivalent to the category $\text{Sec}_{X/B}(Z/X)$ whose objects over a log morphism $S \to B$ are sections $X_S \to Z_S$ of $Z_S \to X_S$. Again the statement that $\text{Sec}_{X/B}(Z/X)$ is a log algebraic stack is equivalent to the statement that there is a stack $\underline{\text{Sec}}_{X/B}(Z/X)$ parametrizing sections over minimal log structures.

The main result is the following:

Th:Sec-limits **Theorem 2.6.** *Let $\Delta = (Z_\alpha, \pi_i : Z_{\alpha_i} \to Z_{\beta_i})$ be a finite diagram of fs log schemes over X, with final object X. Assume*

$$Z = \lim_{\leftarrow}(\Delta)$$

in the category of fs log schemes. Then

$$Sec_{X/B}(Z/X) = \lim_{\leftarrow} Sec_{X/B}(Z_\alpha/X)$$

namely it is the limit in the category of fs log schemes.

If the reader finds general categorical limits a bit off-putting, the main cases needed for our applications are (1) fiber products, and (2) equalizers, where one imposes one equation on a log structure coming from two arrows $\mathbb{N} \rightrightarrows P$ on the level of characteristics.

The existence of such limits in the category of fs log schemes is proven in [10]: the case of arbitrary log structures is treated in (1.6), coherent log structures in (2.6), and fine log structures follow from (2.7); the case of fs log structures follows from the analogous adjoint functor with P^{int} replaced by P^{sat}.

Theorem 2.6 applies directly to the category $\mathrm{Lift}_{\underline{f}}$ and Problem 2.5.

We can use the theorem for log stable maps as follows: consider the stack $\underline{\mathcal{K}}_\Gamma(\underline{Y})$. Since the universal curve is prestable, it has a canonical log smooth structure coming from $\mathfrak{M}_{g,n}$ using F. Kato's theorem. We put this in the setup of $\mathrm{Sec}_{X/B}(Z/X)$ by setting $B = \mathcal{K}_\Gamma(\underline{Y})$ with the pull-back log structure coming from $\mathfrak{M}_{g,n}$; $X =$ the universal curve with its canonical log structure; and $Z = X \times_{(\underline{Y} \times_{\underline{B}} B)} Y$ as before. Suppose the log structure on the target Y is a Deligne–Faltings log structure associated to a fine and saturated monoid P. Then P can be written as a colimit

$$P = \lim_{\rightarrow}(\mathbb{N}^k \rightrightarrows \mathbb{N}^m).$$

If we further break this into factors we can write $Y = \lim_{\leftarrow} Y_i$ and $Z = \lim_{\leftarrow} Z_i$, where the Y_i have Deligne–Faltings log structure with characteristic sheaf generated by the constant sheaf $\mathbb{N}$. Theorem 2.6, in conjunction with [6], implies:

Cor:stable-maps

Corollary 2.7. *Assume that the logarithmic structure Y is a Deligne–Faltings log structure. Then the category $\mathcal{K}_\Gamma(Y)$ is a proper logarithmic algebraic stack.*

2.8. **Proof of Theorem 2.6.** An object of $\mathrm{Sec}_{X/B}(Z/X)$ over an arrow $S \to B$ is by definition a section $s : X_S \to Z$, and composing with the canonical maps we get $s_i : X_S \to Z_i$ such that for each arrow $\pi : Z_i \to Z_j$ in Δ we have $\pi_i \circ s_{\alpha_i} = s_{\beta_j}$. This in particular gives us a diagram of objects $s_i \in \mathrm{Sec}_{X/B}(Z_i/X)(S)$ with $\pi_i(s_{\alpha_i}) = s_{\beta_i}$, namely an object of $\lim_{\leftarrow} \mathrm{Sec}_{X/B}(Z_i/X)$. The correspondence on the level of arrows is similar (though maybe more confusing).

The process is completely reversible, hence the equivalence.

3. Further results

3.1. **Obstruction theory.** TO BE REMOVED - INCLUDE IN SEPARATE PAPER

Theorem 3.2. *If Y is log smooth, the stack $\underline{\mathcal{K}}_\Gamma(Y)$ has a virtual fundamental class.*

The proof is identical to [11] - the log relative obstruction theory is perfect. Since the relevant log stack has pure dimension, the virtual fundamental class exists.

3.3. **The zero-dimensional case and evaluations.** TO BE REWRITTEN - MATERIAL IN ANOTHER PAPER

Other cases of Problem 2.5 are of interest. An especially important case is when $X \to B$ is a log marked point: the characteristic of X is obtained as $\overline{\mathcal{M}}_X = \overline{\mathcal{M}}_B \times \mathbb{N}$. Up to a $\mathbb{C}^*$ action, the evaluation maps of $\mathcal{K}_\Gamma(Y)$ land in $\mathrm{Sec}_{X/B}(Z/X)$ where Z is the restriction of $C \times_{\overline{Y}} Y$ to the relevant marking $X \subset C$. A related case is when $X \to B$ is a log node: the characteristic of X is obtained as $\overline{\mathcal{M}}_X = \overline{\mathcal{M}}_B[a,b]/(a+b=e)$ where $e \in \overline{\mathcal{M}}_B$.

A proof of the following result will be given elsewhere:

Proposition 3.4. *Let $X \to B$ be either a log marked point or a log node, and let $Z \to X$ be a morphism with Z an fs log scheme. Then $Sec_{X/B}(Z/X)$ is a log algebraic stack, locally of finite type over S.*

In general this stack is not of finite type - in particular contact orders are encoded here.

A natural way to prove this proposition is by first proving it for X a Deligne–Faltings structure with monoid $\mathbb{N} \to \overline{\mathcal{M}}_X$, use Theorem 2.6 to deduce the case of arbitrary Deligne–Faltings structure, and then apply étale descent to obtain the case of an arbitrary fs structure.

3.5. **Review of the result of** [6]**.** TO BE REWRITTEN WHEN [6] IS CIRCULATED.

We quote the following:

Theorem 3.6. [6] *Let Y carry a log structure associated to a line bundle with section (L,s) on $\underline{Y}$. Then $\mathcal{K}_\Gamma(Y)$ is a proper algebraic stack with projective coarse moduli space.*

We give some ideas about its proof.

3.6.1. *The graph.* Fix a log map $f : C \to Y$ and a point $\xi \in S$. We create a marked graph G, formed out of the dual graph with some extra data coming from f:

(1) a partition of the vertices of G in two types $V(G) = V_0(G) \sqcup V_1(G)$.
(2) Integer weights $c_l \geq 0, l \in E(G)$ on the edges.
(3) Orientation of edges with nonzero weight.

The vertices $V_0(G)$ are precisely those components of C where the section s does not vanish identically. The weights and orientation come by considering the image of the generator of $\mathbb{N}$ in the putative characteristic monoid at a node: it is of the form $e+c_l \log x$ or $e+c_l \log y$, and the orientation determines whether $\log x$ or $\log y$ occurs.

3.6.2. *The monoid.* We associated a monoid $\overline{\mathcal{M}}(G)$ to the marked graph: it is the saturation of the monoid with vertex generators $e_v, v \in V(G)$, edge generators $e_l, l \in E(G)$, modulo vertex equations $e_v = 0, v \in V_0(G)$ and edge equations $e_{v'} = e_v + c_l e_l$ for every edge l with extremities v, v'.

3.6.3. *Minimality.* There is always a unique map $\overline{\mathcal{M}}(G_\xi) \to \overline{\mathcal{M}}_{S,\xi}$ for every point ξ. This follows by looking at the behavior of monoids near nodes. So we define the log structure to be minimal if this map is an isomorphism.

3.6.4. *The stack.* We define $\underline{\mathcal{K}}_\Gamma(Y)$ as the stack of log maps which are minimal.

3.6.5. *Openness.* Minimality is an open condition by considering how graphs and marked graphs contract in small deformations.

3.6.6. *Algebricity.* It follows from Olsson's log stacks [17] that there is a deformation and obstruction theory. So $\underline{\mathcal{K}}_\Gamma(Y)$ is an Artin stack locally of finite presentation.

3.6.7. *Separation.* The stack $\underline{\mathcal{K}}_\Gamma(Y)$ is separated essentially by uniqueness of the minimal structure (there is a bit more to say but that's not hard).

3.6.8. *Properness.* One can prove properness directly, as is done in [6]. For the weak of heart there is another way: it follows directly if we compare it with Kim's version of Jun Li's space.

Proposition 3.7. *Let $\mathcal{K}_\Gamma^{\log}(\underline{Y}, \underline{D})$ be the space of relative log maps according to Kim (with expansions). Then there is a surjection $\mathcal{K}_\Gamma^{\log}(\underline{Y}, \underline{D}) \to \underline{\mathcal{K}}_\Gamma(Y)$.*

The idea is as follows: we can define an *aligned graph* $\underline{G}$ to be a marked graph G along with a strictly orientation preserving surjection $V(G) \to I$ with I a totally ordered set, such that $V_0(G)$ maps to the minimal element of I. We can associate a monoid to an aligned graph, and define a similar stack of aligned log maps $\underline{\mathcal{K}}_\Gamma^{\text{aligned}}(Y)$. There is an obvious surjective forgetful map $\underline{\mathcal{K}}_\Gamma^{\text{aligned}}(Y) \to \underline{\mathcal{K}}_\Gamma(Y)$. The result follows from

Lemma 3.8. *The stack $\underline{\mathcal{K}}_\Gamma^{aligned}(Y)$ is isomorphic to $\mathcal{K}_\Gamma^{\log}(\underline{Y}, \underline{D})$.*

3.9. **Explicit description of minimal log structures for stable maps.** The main result of [6] gives more than just 2.7 for the case $P = \mathbb{N}$: we have an explicit combinatorial description of the minimal log structure associated to a given log map, in terms of weighted oriented graphs. A similar description is possible in general. We present here one of the main cases of interest, namely the case $P = \mathbb{N}^k$.

3.9.1. *The graph.* Fix a log map $f : C \to Y$ and a point $\xi \in S$. We create a k-marked graph G, formed out of the dual graph with the following extra data coming from f:

(1) k partitions of the vertices of G in two types $V(G) = V_0^{(i)}(G) \sqcup V_1^{(i)}(G)$, where $i = 1, \ldots, k$.
(2) k integer weights $c_l^{(i)} \geq 0, i = 1, \ldots, k$ on the edges $l \in E(G)$.
(3) k orientations of edges: whenever an edge l has extremities v, v' and $c_l^{(i)} > 0$, we choose an orientation $v >_i v'$ or $v' >_i v$ of the edge l.

The weights and orientation come by considering the image of the generator of $\mathbb{N}^k$ in the putative characteristic monoid at a node.

3.9.2. *The monoid.* We associated a monoid $\overline{\mathcal{M}}(G)$ to the k-marked graph: it is the saturation of the monoid with vertex generators $e_v^{(i)}, v \in V(G)$, edge generators $e_l, l \in E(G)$, modulo vertex equations $e_v^{(i)} = 0, v \in V_0^{(i)}(G)$ and edge equations $e_{v'}^{(i)} = e_v^{(i)} + c_l^{(i)} e_l$ for every edge l with extremities $v \leq_i v'$. [3] ←3

Proposition 3.10. *There is a canonical morphism $\overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}_S$. A log structure is minimal if and only if this morphism is an isomorphism.*

3.11. **A generalization.** We can weaken the assumption that Y be a Deligne–Faltings structure in corollary 2.7 as follows.

Prop:moreDF **Proposition 3.12.** *Assume that $\underline{Y}$ is projective, Y is fine and saturated and there is a surjective homomorphism of sheaves of monoids $\mathbb{N}^k_Y \to \overline{\mathcal{M}}_Y$. Then the category $\mathcal{K}_\Gamma(Y)$ is a proper logarithmic algebraic stack. Moreover, let Y' be the Deligne–Faltings log structure associated to $\mathbb{N}^k_Y$ through local charts $\overline{\mathcal{M}}_Y \to \mathcal{M}_Y$. Then the morphism $\phi : \mathcal{K}_\Gamma(Y) \to \mathcal{K}_\Gamma(Y')$ associated to $Y \to Y'$ is a closed embedding of fs logarithmic stacks, and the map of log structures $\phi^*\mathcal{M}_{\mathcal{K}_\Gamma(Y')} \to \mathcal{M}_{\mathcal{K}_\Gamma(Y)}$ is surjective.*

Proof. By Corollary 2.7 we have that $\mathcal{K}_\Gamma(Y')$ is a proper logarithmic algebraic stack. It therefore suffices to prove the second statement in the proposition.

We construct $\mathcal{K}_\Gamma(Y)$ locally over $\mathcal{K}_\Gamma(Y')$. We may assume we have a scheme of finite type B', a log stable map $(C'/B', C' \to Y')$. We claim that $B := B' \times_{\mathcal{K}_\Gamma(Y')} \mathcal{K}_\Gamma(Y) \to B'$ is a closed immersion, and the log structure of B' surjects to that of B.

We take $Z = Y \times'_C C \to C'$. Then $\underline{Z}$ is a closed subset of $\underline{C}'$, so there is a universal closed subset of B' where $\underline{Z} = \underline{C}'$. We replace B by this closed subset, with the pullback log structure.

Since the problem is local we may assume that there are sections $\sigma_i : B' \to C'$ landing in the generic locus, and meeting every component of every fiber. We denote $B'_i = B \times_{C'} Z$, where the product is taken via σ_i. Then $B \to B'$ factors through B'_i, and evidently the log structure of B' surjects to that of B'_i. Replacing B' by B'_i one at a time and repeating the previous step, we may assume $Z \to C' \to B'$ has the property that $Z \to C'$ is an isomorphism along the generic locus of each fiber. It follows that $Z \to C'$ is an isomorphism, and then $B = B'$. ♠

References

ACGM [1] D. Abramovich, Q. Chen, D. Gillam and S. Marcus, *The evaluation space of logarithmic stable maps*, in preparation.

[3](Dan) Generalize to arbitrary DF structure; describe the case of a degeneration; write a proof.

AF [2] D. Abramovich and B. Fantechi, *Orbifold techniques in degeneration formulas*, preprint, 2010.

AGV [3] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, Amer. J. Math. **130** (2008), no. 5, 1337–1398. MR MR2450211

Artin [4] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math. **27** (1974), 165–189.

Behrend-Fantechi [5] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

Chen [6] Q. Chen, *Logarithmic stable maps to Deligne–Faltings pairs I*, preprint 2010.

Ionel-Parker [7] Eleny-Nicoleta Ionel and Thomas H. Parker. Relative Gromov-Witten invariants. *Ann. of Math. (2)*, 157(1):45–96, 2003.

Ionel-Parker2 [8] Eleny-Nicoleta Ionel and Thomas H. Parker. The symplectic sum formula for Gromov-Witten invariants. *Ann. of Math. (2)*, 159(3):935–1025, 2004.

FKato [9] Fumiharu Kato. Log smooth deformation and moduli of log smooth curves. *Internat. J. Math.*, 11(2):215–232, 2000.

KKato [10] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

Kim [11] Bumsig Kim. Logarithmic stable maps. `http://arxiv.org/abs/0807.3611`, 2008.

Kontsevich [12] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

Li-Ruan [13] An-Min Li and Yongbin Ruan. Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds. *Invent. Math.*, 145(1):151–218, 2001.

Li [14] Jun Li. Stable morphisms to singular schemes and relative stable morphisms. *J. Differential Geom.*, 57(3):509–578, 2001.

Li-degeneration [15] Jun Li. A degeneration formula of GW-invariants. *J. Differential Geom.*, 60(2):199–293, 2002.

Li-Tian [16] Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. *J. Amer. Math. Soc.*, 11(1):119–174, 1998.

Olsson [17] ———, *Hom-stacks and restriction of scalars*, Duke Math. J. **1 34** (2006), 139-164.

Department of Mathematics, Brown University, Box 1917, Providence, RI 02912, U.S.A.

E-mail address: abrmovic@math.brown.edu, Qile_Chen@Brown.edu