Resolution in Toroidal Orbifolds

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Lecture at AGNeaU-BosCo
March 18, 2017.

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Toroidal orbifolds

Definition (KKMS-D)

$U \subset X$ is toroidal if étale[a] locally it is isomorphic to a toric $T \subset Y$. In other words, for $p \in X$ there is an étale neighborhood $\phi : Z \to X$ and étale map $\psi : Z \to Y$ such that $\phi^{-1}U = \psi^{-1}T$.

[a] or formally, or analytically

Definition

- An orbifold $\mathcal{X}$ is a Deligne–Mumford stack with dense open subscheme, locally $[Y/G]$ with G finite acting faithfully.
- A toroidal orbifold is one which is locally $[Y/G]$, where $U \subset Y$ toroidal and equivariant.

Theorem (KKMS-D, Nizioł)

Any toroidal orbifold has a toroidal resolution of singularities.

Admissible centers

- Anything toroidal is étale locally like $\mathbb{A}^k \times \operatorname{Spec} k[M]$, with the toroidal structure coming from M.
- We use x_i, y, z for non-monomial parameters of $\mathbb{A}^k$ and u, v, w or m_i for elements coming from M.

Definition

- A toroidal submanifold is locally defined by $(x_1, \ldots, x_r)$.
- An admissible center is locally defined by $(x_1, \ldots, x_r, m_1, \ldots, m_s)$.
- A Kummer admissible center is locally defined by $(x_1, \ldots, x_r, m_1^{1/n}, \ldots, m_s^{1/n})$.

Definition

If $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ and $a \in \mathbb{N}$, a Kummer admissible center with ideal $\mathcal{J}$ is $(\mathcal{I}, a)$-admissible if $\mathcal{I} \subset \mathcal{J}^a$. In particular $\mathcal{I}$ vanishes on the center.

Main result: principalization

You can blow up a Kummer center $\mathcal{X}' \to \mathcal{X}$, and $\mathcal{X}'$ is another toroidal orbifold. It is like a weighted blowup on steroids.

Theorem (ℵTW)

Let $\mathcal{X}$ be a toroidal orbifold and $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ a coherent ideal sheaf. Then there is a sequence

$$\mathcal{X}' = \mathcal{X}_n \to \mathcal{X}_{n-1} \to \cdots \to \mathcal{X}_1 \to \mathcal{X}_0 = \mathcal{X}$$

of admissible Kummer blowups[a] such that $\mathcal{I}\mathcal{O}_{\mathcal{X}'}$ is an invertible monomial ideal.

[a] $(\mathcal{I}_j, a_j)$-admissible, where $\mathcal{I}_j = \mathcal{I}_{E_j}^{-a_j}(\mathcal{I}_{j-1}\mathcal{O}_{\mathcal{X}_j})$

- Using destackification [Bergh] / torification [ℵKMW,ℵT] we can replace $\mathcal{X}' \to \mathcal{X}$ by a representable morphism.
- Using toroidal resolution [KKMS-D], we may replace $\mathcal{X}'$ by a smooth one.

From principalization to resolution

- Let $Y \subset X := \mathbb{P}^n$ be a subvariety, $\mathcal{I} = \mathcal{I}_Y$.
- In the principalization sequence for $\mathcal{I}$ write $\mathcal{Y}_i$ for the proper transforms.
- Admissiblity implies: there is a unique time $\mathcal{X}_{i+1} \to \mathcal{X}_i$ where $\mathcal{I}_{\mathcal{Y}_i}$ is blown up.
- Necessarily $\mathcal{Y}_i$ is a toroidal submanifold, so it is a toroidal orbifold itself.
- Destackification replaces $\mathcal{Y}_i \to Y$ by a representable morphism $Y' \to Y$.
- Toroidal resolution gives a resolution $Y'' \to Y' \to Y$.

Cleaning up

An ideal is monomial if it locally corresponds to an ideal of the form $(m_1, \ldots, m_k)$. E.g. $(u, v) \subset k[u, v]$.
Consider the minimal monomial ideal $\mathcal{M} := \mathcal{M}(\mathcal{I})$ containing $\mathcal{I}$. E.g. $\mathcal{I} := (u - v) \subset (u, v)$. Write $\mathcal{D}^\infty$ for the ring of logarithmic differential operators. E.g. $u\frac{\partial}{\partial u}, v\frac{\partial}{\partial v}, 1$.

Theorem (Kollár, ℵTW)

$\mathcal{M}(\mathcal{I}) = \mathcal{D}^\infty(\mathcal{I})$.

E.g. $u\frac{\partial}{\partial u}(u - v) = u, v\frac{\partial}{\partial v}(u - v) = -v$, so $\mathcal{D}^\infty(\mathcal{I}) = (u, v)$

Theorem (Kollár, ℵTW)

Let $X' \to X$ be the normalized blowing up of $\mathcal{M}(\mathcal{I})$, with $\mathcal{M}' = \mathcal{M}\mathcal{O}_{X'}$. Then $\mathcal{I}\mathcal{O}_{X'} = \mathcal{M}' \cdot \mathcal{I}^{cln}$, with $\mathcal{M}(\mathcal{I}^{cln}) = 1$.

E.g. X' = the blowup, locally $\operatorname{Spec} k[u, v/u]$. $\mathcal{M}' = (u)$.
$\mathcal{I}^{cln} = (1 - v/u)$, a clean ideal: $\mathcal{M}(\mathcal{I}^{cln}) = 1$.

examples

(1)
- $\mathcal{I} = (u - v) \subset k[u, v]$
- $\implies \mathcal{M}(\mathcal{I}) = (u - v, u\frac{\partial}{\partial u}(u - v) \ldots) = (u, v)$.
- $X' =$ the blowup, and on the chart Spec $k[u, v/u]$ we have $\mathcal{I}^{cln} = (1 - v/u)$, a clean ideal: $\mathcal{M}(\mathcal{I}^{cln}) = 1$.
- $\mathcal{I}^{cln}$ is admissible and principal.

If we enrich the toroidal structure by $\mathcal{I}^{cln}$, we make it monomial!

(2)
- $\mathcal{I} = (u - v)(u - v - v^2) \subset k[u, v]$
- $\implies \mathcal{M}(\mathcal{I}) = (u^2, uv, v^2)$ (monomials with nonzero coefficients).
- $X' =$ the blowup, $\mathcal{I}^{cln} = (1 - v/u)(1 - v/u - u(v/u)^2)$, a clean ideal.
- Need to do something else!

Order reduction

Definition

Write $\mathcal{D}^{\leq a}$ for the sheaf of logarithmic differential operators of order $\leq a$.
The logarithmic order of a clean ideal $\mathcal{I}$ is the minimum a such that $\mathcal{D}^{\leq a}\mathcal{I} = (1)$.

Theorem (Order reduction)

Let X be a toroidal orbifold, $\mathcal{I}$ a clean ideal with logarithmic order a. Then there is an $(\mathcal{I}, a)$-admissible Kummer sequence such that the transformed ideal $\mathcal{I}_n$ has logarithmic order $< a$.
Furthermore, the procedure assigning to $(X, \mathcal{I})$ the Kummer sequence is functorial for toroidal morphisms.

This implies principalization by induction on a.

Example (3)

- $\mathcal{J} = (u^2, x)$ on $X = \operatorname{Spec} \mathbb{C}[u, x]$, clean.
- Blowing up $\mathcal{J}$: a modification $X' \to X$ with an exceptional divisor E.
- We use E to enrich the toroidal structure.
- Charts:
  - $X'_x := \operatorname{Spec} \mathbb{C}[x, u, v]/(vx - u^2)$, where $v = u^2/x$.
    - Exceptional: $x = 0$, now monomial.
    - $(x, u^2)\mathcal{O}_{X'} = (x)$, invertible monomial ideal.
    - X' a singular toroidal variety.
  - $X'_{u^2} := \operatorname{Spec} \mathbb{C}[u, y]$, where $y = x/u^2$ (nonsingular).
    - Exceptional: $u = 0$, monomial
    - $(x, u^2)\mathcal{O}_{X'} = (u^2)$, invertible monomial ideal.
- So $\mathcal{J}\mathcal{O}_{X'}$ is a monomial ideal on a singular toroidal variety.
- The classical algorithm would have us blow up (x, u) and then an infinitely near point.
- In the case of $\mathcal{J}_{200} = (u^{200}, x)$, the classical algorithm would have us blow up 200 times.

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Analysis of example

- The same blowing up works $\mathcal{I}_1 := \mathcal{J}^2$, as well as its unsaturated variant $\mathcal{I}_2 := (u^4, x^2)$.
- How do we know in all these cases to blow up $\mathcal{J}$?
- Restricting the ideal $\mathcal{J}$ to the hypersurface $\{x = 0\}$ we obtain the monomial ideal (u^2),
- hinting that we should lift this ideal from $\{x = 0\}$ to X, giving (u^2, x).
- What distinguishes x? It defines a toroidal submanifold.
- $x + u^2$ would do just as well.
- Note: $\mathcal{J}$ has *logarithmic order 1*: it contains a regular parameter,

Analysis of example (continued)

- Note: $\mathcal{J}$ has *logarithmic order 1*: it contains a regular parameter,
- namely: restricting $\mathcal{J}$ to the stratum $u = 0$ the resulting ideal (x) defines a toroidal subvariety.
- $\mathcal{I}_1 = \mathcal{J}^2$ or $\mathcal{I}_2$ have logarithmic order $a = 2$:
- on $u = 0$ restrict to (x^2); also $\mathcal{D}^{\leq 2}\mathcal{I}_j = (1)$.
- Following classical methods, pick parameter $x \in \mathcal{D}^{\leq a-1}(\mathcal{I}_j) = \mathcal{D}^{\leq 1}(\mathcal{I}_j)$.
- $\mathcal{J}$ is $(\mathcal{I}_j, 2)$-admissible, in the sense that $\mathcal{I}_j \subseteq \mathcal{J}^2 = \mathcal{J}^a$.
- $\Rightarrow \mathcal{I}_j\mathcal{O}_{X'}$ factors out monomial ideal $(\mathcal{J}\mathcal{O}_{X'})^2$.
- In the example, $\mathcal{I}_j\mathcal{O}_{X'} = (\mathcal{J}\mathcal{O}_{X'})^2$.
- in general *the other factor is automatically clean*

Example (4)

- $\mathcal{I}_3 = (x^2, u)$, logarithmic order $a = 2$.
- $H = \{x = 0\}$ hypersurface of maximal contact, $\mathcal{I}_3\mathcal{O}_H = (u)$
- but (x, u) not admissible, as $\mathcal{I}_3 \not\subseteq (x, u)^2$.
- Kummer ideal sheaf $(x, u^{1/2})$ admissible:
  $\mathcal{I}_3 = (x^2, u) \subseteq (x, u^{1/2})^2 = (x^2, xu^{1/2}, u)$.
- associated blowing up $X' \to X$ with charts:
  - $X'_x := \operatorname{Spec} \mathbb{C}[x, u, v]/(vx^2 = u)$, where $v = u/x^2$ (nonsingular scheme).
    - Exceptional $x = 0$, now monomial.
    - $\mathcal{I}_3 = (x^2, u)$ transformed into (x^2), invertible monomial ideal.
    - Kummer ideal $(x, u^{1/2})$ transformed into monomial ideal (x).
  - The $u^{1/2}$-chart:
    - stack quotient $X'_{u^{1/2}} := \left[\operatorname{Spec} \mathbb{C}[w, y]/\mu_2\right]$,
    - where $y = x/w$ and $\mu_2 = \{\pm 1\}$ acts via $(w, y) \mapsto (-w, -y)$.
    - Exceptional $w = 0$ (monomial).
    - (x^2, u) transformed into invertible monomial ideal $(u) = (w^2)$.
    - $(x, u^{1/2})$ transformed into invertible monomial ideal (w).

Example (4) - continued

- Outside $\{y = 0\}$ this becomes the schematic quotient

$$\mathrm{Spec}\,\mathbb{C}[w, y, y^{-1}]/\mu_2 = \mathrm{Spec}\,\mathbb{C}[y^2, y^{-2}, wy] = \mathrm{Spec}\,\mathbb{C}[v^{-1}, v, x],$$

  an open subscheme in X'_x, allowing gluing of the two charts.
- Note that $X'_{u^{1/2}}$ is again a toroidal orbifold *with respect to the toroidal structure enriched by E*,
- but that the stabilizer of $y = w = 0$ *does not act as a subgroup of the torus*.
- This means that the coarse moduli space is not toroidal in any natural manner, and in order to maintain the toroidal structure the stack structure must remain.
- Classical principalization requires two blowings up, and for (x^{200}, u), it would require 200 blowings up. We need one Kummer blowing up.

Tuning

- Maximal contact elements are local and not unique.
- Włodarczyk introduced the homogenization $\mathcal{H}(\mathcal{I}, a)$.

Theorem

- *Order reduction for $\mathcal{I}$ is equivalent to order reduction for $\mathcal{H}(\mathcal{I})$.*
- *Any two maximal contact elements for $\mathcal{H}(\mathcal{I})$ are related by a local automorphism.*

- So it suffices to prove functorial order reduction.
- Order reduction for $\mathcal{I}_{x=0}$ does not imply order reduction for $\mathcal{I}$.
- Włodarczyk adapted the coefficient ideal $C(\mathcal{I}, a)$ (Villamayor, Bierstone-Milman).

Theorem

Order reduction for $C(\mathcal{I}, a)_{x=0}$ implies order reduction for $\mathcal{I}$.

Next goals

- Extend to qe schemes.
- Extend to good schemes over valuation rings
- Extend to other categories
- Functorial semistable reduction.
- Functorial alteration results.