Punctured logarithmic invariants and gluing

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Work with Qile Chen, Mark Gross and Bernd Siebert

Other work by Parker, Tehrani, Dhruv, Fan-Tseng-Wu-You

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The gluing result

Here is a minimalist[1] statement:

Theorem (ACGS 2020)

The evaluation maps $\widetilde{\mathcal{M}}(X,\tau) \to X^n$ of the moduli stack $\widetilde{\mathcal{M}}(X,\tau)$ of stable marked punctured curves of type τ in a log scheme X are virtually idealized log smooth.

Given an edge of τ with splitting τ' we have a cartesian splitting diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(X/B,\tau) & \longrightarrow & \widetilde{\mathcal{M}}(X,\tau') \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

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There is a lot I need to explain and motivate.

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Rational plane curves

Definition

$$N_d = \#\left\{ \begin{array}{l} C \subset \mathbb{P}^2 \text{ a rational curve,} \\ \deg C = d, \text{ and} \\ p_1, \dots p_{3d-1} \in C \end{array} \right\}.$$

Kontsevich's theorem

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Theorem (Kontsevich)

For $d > 1$ we have

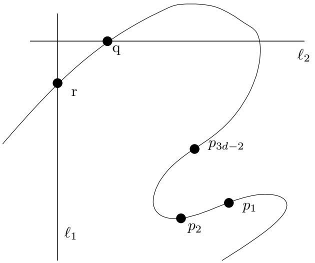
$$N_d = \sum_{\substack{d = d_1 + d_2 \\ d_1, d_2 > 0}} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

The first few numbers are

$$N_1 = 1, \; N_2 = 1, \; N_3 = 12, \; N_4 = 620, \; N_5 = 87304.$$

Kontsevich's theorem: setup

1-parameter family $C \to B$: fix only $p_1 \ldots, p_{3d-2}$,
and two lines ℓ_1, ℓ_2 meeting at a point called p_{3d-3}:

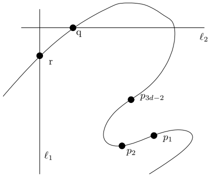


We choose q among $C \cap \ell_1$, and r among $C \cap \ell_2$.

Kontsevich's theorem: preview

The equation

$$N_d = \sum_{\substack{d = d_1 + d_2 \\ d_1, d_2 > 0}} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$



translates to

$$\deg_B(p_1, p_2 | q, r) = \deg_B(p_1, q | p_2, r)$$

coming from the cross ratio map $\lambda : B \to \mathbb{P}^1$.

Kontsevich's theorem: computation

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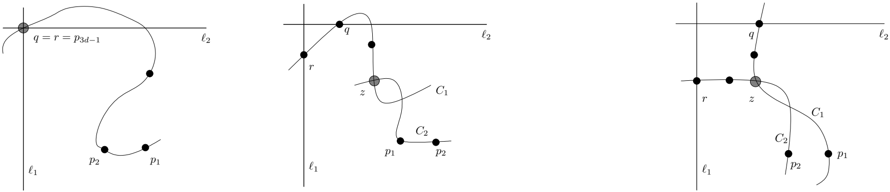

Figure: $N_d \quad + \quad d_1^3 d_2 \binom{3d-4}{3d_1-1} \sum N_{d_1} N_{d_2} \quad = \quad \sum d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N_{d_1} N_{d_2},$

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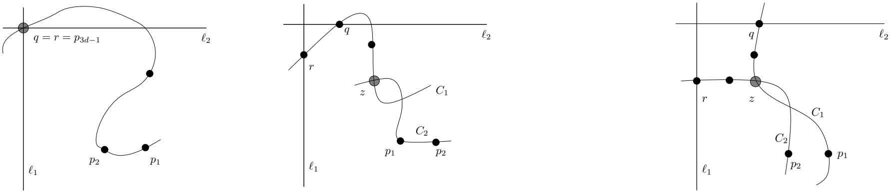

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as needed! ♠

Moduli spaces

- Let $\overline{\mathcal{M}}(X, \tau)$ be the Kontsevich moduli stack of stable maps in X
- with type specified by decorated graph $\tau = (G, h, \beta)$.
- to each vertex v of G we assign a genus $h(v)$ and a curve class $\beta(v)$.
- The legs are marked:

The gluing principle on moduli spaces

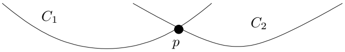
Proposition

The evaluation maps $\overline{\mathcal{M}}(X,\tau) \to X^m$ are virtually smooth.
Given an edge of τ with splitting τ' we have a cartesian splitting diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}(X/B,\tau) & \longrightarrow & \overline{\mathcal{M}}(X,\tau') \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

of stacks with compatible virtual fundamental classes.

The gluing principle on curves



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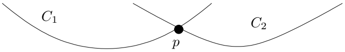

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implying

$$\begin{aligned} Hom(C, X) &= Hom(C_1, X) \underset{Hom(p,X)}{\times} Hom(C_2, X) \\ &= Hom(C_1, X) \underset{X}{\times} Hom(C_2, X) \end{aligned}$$

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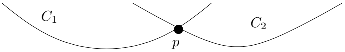

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spreading out to

$$\begin{array}{ccc} \overline{\mathcal{M}}(X/B,\tau) & \longrightarrow & \overline{\mathcal{M}}(X,\tau') \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

From gluing to quantum cohomology

Note that we relied on

$$X = Hom(p, X).$$

One defines quantum cohomology based on the operation

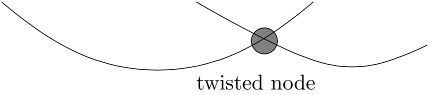
$$\langle \gamma_1 \ldots \gamma_n \rangle_\beta \quad = \quad e_{n+1\,*}\left([M]^{\mathrm{virt}} \cap e_1^*\gamma_1 \cdots e_n^*\gamma_n \right).$$

Associativity is a result of gluing.

Enter orbifolds

- Let $\mathcal{X}$ be a smooth proper DM stack (an algebraic orbifold).
- Let $\overline{\mathcal{M}}(\mathcal{X}, \tau)$ be the moduli stack of stable maps in $\mathcal{X}$
- with type specified by decorated graph τ.
- to each vertex v of G we assign a genus $h(v)$ and a curve class $\beta(v)$.
- The legs are marked by inertia components.
- We insist that every marking is a section of $\mathcal{C}$.

The gluing principle on orbifold curves



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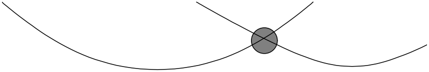

implying

$$Hom(\mathcal{C},\mathcal{X}) = Hom(\mathcal{C}_1,\mathcal{X}) \underset{Hom(\Sigma,\mathcal{X})}{\times} Hom(\mathcal{C}_2,\mathcal{X})$$
$$= Hom(\mathcal{C}_1,\mathcal{X}) \underset{\mathcal{I}\mathcal{X}}{\times} Hom(\mathcal{C}_2,\mathcal{X})$$

The map on the right is the evaluation map e_2. The map on the left is the twisted evaluation map $\check{e}_1$.

Orbifold gluing

Proposition

The evaluation maps $\overline{\mathcal{M}}(\mathcal{X},\tau) \to \mathcal{IX}^m$ are virtually smooth.
Given an edge of τ with splitting τ' we have a cartesian splitting diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}(\mathcal{X},\tau) & \longrightarrow & \overline{\mathcal{M}}(\mathcal{X},\tau') \\ \downarrow & & \downarrow \\ \mathcal{IX} & \longrightarrow & \mathcal{IX} \times \mathcal{IX} \end{array}$$

of stacks with compatible virtual fundamental classes.

From orbifold gluing to orbifold quantum cohomology

Note that we relied on

$$\mathcal{I}\mathcal{X} = Hom(\Sigma, \mathcal{X}).$$

One defines quantum cohomology on $H^*(\mathcal{I}\mathcal{X})$ based on the operation

$$\langle\gamma_1 \dots \gamma_n\rangle_\beta^{\mathcal{X}} \quad = \quad \check{e}_{n+1\,*}\left([M]^{\text{virt}} \cap e_1^*\gamma_1 \cdots e_n^*\gamma_n\right).$$

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Log geometry (K. Kato, Fontaine, Illusie; Ogus)

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- idealized log schemes are étale glued from closed subsets of monomial subschemes of affine toric varieties - the standard-issue idealized log smooth spaces.

Log structures (K. Kato, Fontaine–Illusie)

- a log structure is a monoid homomorphism $\alpha : M \to \mathcal{O}_X$
- such that $\alpha^*\mathcal{O}^\times \to \mathcal{O}^\times$ is an isomorphism.

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- such that $\alpha^*\mathcal{O}^\times \to \mathcal{O}^\times$ is an isomorphism.
- Morphisms are given by natural commutative diagrams. . .
- A key example is the log structure associated to an open $U \subset X$,
- where $M = \mathcal{O}_X \cap \mathcal{O}_U^\times$.

Idealized log structures (Ogus)

- a idealized log structure is a log structure $\alpha : M \to \mathcal{O}_X$
- along with a monoid ideal $K \subset M$,
- such that $\alpha(K) = 0 \in \mathcal{O}$.

Toric and log smooth Log structures (K. Kato)

- When X is a toric variety and U the torus this is a prototypical example of a log smooth structure.
- In this case the monoid is associated to the regular monomials, with $\mathcal{O}^\times$ thrown in.

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- When X is a toric variety and U the torus this is a prototypical example of a log smooth structure.
- In this case the monoid is associated to the regular monomials, with $\mathcal{O}^\times$ thrown in.
- In general X is log smooth if it is étale locally toric.
- A morphism $X \to Y$ is log smooth if it is étale locally a base change of a dominant morphism of toric varieties.

Log curves

- A log curve is a reduced 1-dimensional fiber of a flat log smooth morphism.
- F. Kato showed that these are the same as nodal marked curves, with "the natural" log structure.
- A punctured curve is the idealized version of the above.

Log curves under the microscope

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- A marked point looks like $\mathrm{Spec}(M \to k[x])$ where

$$M = M_S \oplus \mathbb{N} \log x.$$

Punctured curves under the microscope

- A puncturing of a marked curve is a log structure M at a marked point with

$$M_S + \mathbb{N}\log x \quad \subseteq \quad M \quad \subsetneq \quad M_S + \mathbb{Z}\log x.$$

- It is an instance of an idealized log smooth scheme.
- In particular the splitting of a node is a punctrured curve.
- In what follow, I insist that every marking is given with a section.

Splitting

- Consider $X \to \mathbb{A}^1$ the total space of $xy = t$, and
- $C \to S$ given by $\{y = 0\} \to \{t = 0\}$.
- At the origin $M_S + \mathbb{N} \log x \quad \subsetneq \quad M \quad \subsetneq \quad M_S + \mathbb{Z} \log x$.
- It is not a log curve, but rather a punctured curve.

Stable punctured log maps

- Fix X a nice log smooth scheme. It has a cone complex $\Sigma(X)$ with integer lattice.
- A stable punctured log map $C \to X$ is a log morphism with stable underlying morphism of schemes.

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Theorem ([ACGS])

$\mathcal{M}(X,\tau)$, the stack of minimal stable punctured log maps of type τ, is a Deligne–Mumford stack which is finite and representable over $\mathcal{M}(\underline{X},\underline{\tau})$.

Prestable and cut maps (B. Parker)

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- There is a range of choices for the punctured structure.
- For muduli of maps purposes, we use prestable structures:
- It is the minimal puncturing accommodating the map.
- For the purpose of gluing along sections, one can use cut curve structures.
- There is the maximal puncturing accommodating the section.
- The resulting categories are equivalent.

Tropical picture

- X has a cone complex $\Sigma(X)$ with integer lattice.
- $C \to S$ has cone complex $\Sigma(C) \to \Sigma(S)$. The fiber over $u \in \Sigma(S)$ is a tropical curve:
- Components give vertices, nodes give edges, and punctured points give legs.
- Usual marked points give infinite legs.
- Truly punctured points (and cut curves) give finite legs.
- A stable punctured log map gives $\Sigma(C) \to \Sigma(X)$, a family of tropical curves in $\Sigma(X)$.
- The sections mark the legs.
- Minimality is beautifully encoded in this picture. . .

An Analogy: Orbifold vs. Logarithmic cohomology

- If $\mathcal{X}$ is an orbifold, Chen-Ruan defined orbifold and quantum cohomology based on $H^*(\overline{\mathcal{I}}(\mathcal{X}), \mathbb{Q})$.
- $\overline{\mathcal{I}}(\mathcal{X})$, the rigidified inertia stack is the moduli space of orbifold points in $\mathcal{X}$,
- whose components, twisted sectors, correspond to (x, ϕ) where $x \in \mathcal{X}$ and $\phi \in \mathrm{Aut}(x)$.

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- whose components, twisted sectors, correspond to (x, ϕ) where $x \in \mathcal{X}$ and $\phi \in \mathrm{Aut}(x)$.
- Chen Ruan cohomology pairs ϕ with ϕ^{-1}.
- If X is a log scheme, Gross–Hacking–Keel–Siebert. . . define the ring of theta functions,
- based on the moduli space $\mathcal{P}(X)$ of log points in $\mathcal{X}$,
- whose components correspond to (x, u) where $x \in X$ and u a contact order at x, namely $u \in \Sigma(X)(\mathbb{N})$.
- Gluing pairs u and $-u$.

Gluing punctured curves

Lemma

Let C_1°, C_2° be two cut curves with underlying curves $\underline{C}_i$, over a log scheme W with sections $W \to C_i^\circ$ along the puncture.
There is a unique log structure C, log smooth over W on the nodal curve $\underline{C} = \underline{C}_1 \cup^p \underline{C}_2$, with a section at the node, restricting to C_i°. Moreover, C has the coproduct property:

$$Hom(C, X) = Hom(C_1^\circ, X) \times_{Hom(W,X)} Hom(C_2^\circ, X).$$

This has a slightly more involved implication in terms of pre-stable punctured maps.

Gluing punctured curves: moduli

This gives part of the first claim:

Theorem (ACGS 2020)

. . . The following is cartesian:

$$\begin{array}{ccc} \widetilde{\mathcal{M}}(X/B,\tau) & \longrightarrow & \widetilde{\mathcal{M}}(X,\tau') \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X \end{array}$$

The end

Thank you for your attention