Punctured logarithmic maps and punctured invariants

Dan Abramovich, Brown University

Work with Qile Chen, Mark Gross and Bernd Siebert

3CinG - London, Warwick, Cambridge

Other work by Parker, Tehrani, Dhruv, Fan-Tseng-Wu-You

September 18, 2020

Tension

- Virtual fundamental classes in Gromov–Witten theory require working with smooth targets.
- Making full use of deformation invariance in Gromov–Witten theory requires degenerating the target
- Such as $xyz = t$ as $t \to 0$.
- At the very least, étale locally like toric varieties and fibers of toric morphisms
- We need a fairytale world in which these are smooth.

Log geometry

- Observation (Siebert, 2001): Such fairytale world already exists - logarithmic geometry.
- schemes are glued from closed subsets of affine spaces - the standard-issue smooth spaces.
- log schemes are étale glued from closed subsets of affine toric varieties - the standard-issue log smooth spaces.
- (keep this in mind when we go one step further)

Log structures (K. Kato, Fontaine–Illusie)

- a log structure is a monoid homomorphism $\alpha : M \to \mathcal{O}_X$
- such that $\alpha^*\mathcal{O}^\times \to \mathcal{O}^\times$ is an isomorphism.
- Morphisms are given by natural commutative diagrams...
- A key example is the log structure associated to an open $U \subset X$,
- where $M = \mathcal{O}_X \cap \mathcal{O}_U^\times$.

Toric and log smooth Log structures (K. Kato)

- When X is a toric variety and U the torus this is a prototypical example of a log smooth structure.
- In this case the monoid is associated to the regular monomials, with $\mathcal{O}^\times$ thrown in.
- In general X is log smooth if it is étale locally toric.
- A morphism $X \to Y$ is log smooth if it is étale locally a base change of a dominant morphism of toric varieties.

Log curves

- A log curve is a reduced 1-dimensional fiber of a flat log smooth morphism.
- F. Kato showed that these are the same as nodal marked curves, with "the natural" log structure.

Log curves under the microscope

- Say $C \to S$ a log curve, $S = \mathrm{Spec}(M_S \to k)$.
- A general point of C looks like $\mathrm{Spec}(M_S \to k[x])$.
- A node looks like $\mathrm{Spec}(M \to k[x,y]/(xy))$, where

$$M = M_S\langle \log x, \log y\rangle / (\log x + \log y = \log t), t \in M_S.$$

- A marked point looks like $\mathrm{Spec}(M \to k[x])$ where

$$M = M_S \oplus \mathbb{N} \log x.$$

Stable log maps

- Fix X a nice log smooth scheme.
- A stable log map $C \to X$ is a log morphism with stable underlying morphism of schemes.
- Marked points record contact orders with divisors of X.
- These are recorded by integer points $u \in \Sigma(X)(\mathbb{N})$.
- Stable log maps have "standard issue" log structure, called minimal.

Theorem ([GS,C,ACMW])

$\mathcal{M}(X, \tau)$, the stack of minimal stable log maps of type τ, is a Deligne–Mumford stack which is finite and representable over $\mathcal{M}(\underline{X}, \underline{\tau})$.

Tropical picture

- X has a cone complex $\Sigma(X)$ with integer lattice.
- $C \to S$ has cone complex $\Sigma(C) \to \Sigma(S)$. The fiber over $u \in \Sigma(S)$ is a tropical curve:
- Components give vertices, nodes give edges, and marked points give infinite legs.
- A stable log map gives $\Sigma(C) \to \Sigma(X)$, a family of tropical curves in $\Sigma(X)$.
- Minimality is beautifully encoded in this picture...

Logarithmic invariants

- Recall that $\mathcal{M}(\underline{X}, \underline{\tau})$ has a perfect obstruction theory over $\mathfrak{M}_{g,n} \times \underline{X}^n$. This affords invariants by virtual pullback.
- $\mathcal{M}(X, \tau)$ has a POT over $\mathfrak{M}^{ev}(\mathcal{A}_X, \tau)$, where $\mathcal{A}_X$ is the artin fan, a stack-theoretic version of $\Sigma(X)$.
- Here $\mathfrak{M}^{ev}(\mathcal{A}_X, \tau)$ is approximately $\mathfrak{M}(\mathcal{A}_X, \tau) \times_{\mathcal{A}_X^n} X^n$.

Theorem ([GS,C,AC])

$\mathfrak{M}^{ev}(\mathcal{A}_X, \tau)$ is log smooth, and has a fundamental class. This affords invariants by virtual pullback.

An Analogy: Orbifold vs. Logarithmic cohomology

- If $\mathcal{X}$ is an orbifold, Chen-Ruan defined orbifold and quantum cohomology based on $H^*(\overline{\mathcal{I}}_\infty(\mathcal{X}), \mathbb{Q})$.
- $\overline{\mathcal{I}}_\infty(\mathcal{X})$, the rigidified inertia stack is the moduli space of orbifold points in $\mathcal{X}$,
- whose components, twisted sectors, correspond to (x, ϕ) where $x \in \mathcal{X}$ and $\phi \in \mathrm{Aut}(x)$.
- Chen Ruan cohomology pairs ϕ with ϕ^{-1}.
- If X is a log scheme, Gross–Hacking–Keel–Siebert... define the ring of theta functions,
- based on the moduli space $\mathcal{P}(X)$ of log points in $\mathcal{X}$,
- whose components correspond to (x, u) where $x \in X$ and u a contact order at x, namely $u \in \Sigma(X)(\mathbb{N})$.
- what about $-u$?

Splitting?

- Consider $X \to \mathbb{A}^1$ the total space of $xy = t$, and
- $C \to S$ given by $\{y = 0\} \to \{t = 0\}$.
- At the origin $M_S + \mathbb{N} \log x \quad \subsetneq \quad M \quad \subsetneq \quad M_S + \mathbb{Z} \log x$.
- It is not a log curve, but rather a punctured curve.
- Its tropicaliation is a finite leg.

Punctured curves and maps

- A puncturing of a marked curve is a log structure M at a marked point with

$$M_S + \mathbb{N}\log x \quad \subseteq \quad M \quad \subsetneq \quad M_S + \mathbb{Z}\log x.$$

- It is an instance of an idealized log smooth scheme.
- A morphism $f : C \to X$ is prestable if M is generated by $M_S + \mathbb{N}\log x$ and $f^\flat M_X$.

Another example

- Now consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ with log structure given by $D =$ one ruling.
- Let a conic C degenerate to the union of the two rulings $C_0 = D + F$.
- Then D has one marked point and one puncture,
- and (after pre-stabilizing) F has one marked point.

Punctured log maps

- A punctured stable log map $C \to X$ is a prestable log morphism with stable underlying morphism of schemes.
- Punctured points record contact orders with divisors of X.
- These are recorded by integer tangents of the cone complex of X.
- Punctured log maps have "standard issue" minimal log structures.

Theorem ([ACGS])

$\mathcal{M}(X, \tau)$, the stack of minimal punctured stable log maps of type τ, is a Deligne–Mumford stack which is finite and representable over $\mathcal{M}(\underline{X}, \underline{\tau})$.

Punctured invariants

- $\mathcal{M}(X,\tau)$ has a POT over $\mathfrak{M}^{ev}(\mathcal{A}_X,\tau)$, where $\mathcal{A}_X$ is the artin fan.
- $\mathfrak{M}^{ev}(\mathcal{A}_X,\tau)$ is not log smooth, and doesn't have a fundamental class.

Theorem ([ACGS])

$\mathfrak{M}^{ev}(\mathcal{A}_X,\tau)$ *is idealized log smooth. This affords invariants by super-careful virtual pullback.*

.

- The case of $g=0, n=3, u_1, u_2, -u_3 \in \Sigma(X)(\mathbb{N})$ is, fortunately, manageable.

Gluing

- There is a natural finite and representable splitting morphism $\mathcal{M}(X,\tau) \xrightarrow{\delta} \prod \mathcal{M}(X,\tau_i)$.

Theorem (ACGS)

There is a virtual-pullback cartesian diagram

$$\begin{tikzcd}
\mathcal{M}(X,\tau) \arrow[r] \arrow[d] & \prod_{i=1}^r \mathcal{M}(X,\tau_i) \arrow[d] \\
\mathfrak{M}^{ev}(\mathcal{A}_X,\tau) \arrow[r] & \prod_{i=1}^r \mathfrak{M}^{ev}(\mathcal{A}_X,\tau_i)
\end{tikzcd}$$

with horizontal arrows the splitting maps, and the vertical arrows the canonical strict morphisms.

- One needs even more care to relate this to a diagonal map.

The end

Thank you for your attention