Punctured logarithmic maps and punctured invariants

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Other work by Parker, Tehrani, Dhruv, Fan-Tseng-Wu-You

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Virtual fundamental classes in Gromov–Witten theory require working with smooth targets.

Making full use of deformation invariance in Gromov–Witten theory requires degenerating the target

Such as \(xyz = t\) as \(t \to 0\).

At the very least, étale locally like toric varieties and fibers of toric morphisms

We need a fairytale world in which these are smooth.
Log geometry

- Observation (Siebert, 2001): Such fairytale world already exists - logarithmic geometry.
- Schemes are glued from closed subsets of affine spaces - the standard-issue smooth spaces.
- Log schemes are étale glued from closed subsets of affine toric varieties - the standard-issue log smooth spaces.
- (Keep this in mind when we go one step further)
a log structure is a monoid homomorphism $\alpha : M \to \mathcal{O}_X$
such that $\alpha^* \mathcal{O}_X^\times \to \mathcal{O}_X^\times$ is an isomorphism.
Morphisms are given by natural commutative diagrams.
A key example is the log structure associated to an open $U \subset X$, where $M = \mathcal{O}_X \cap \mathcal{O}_U^\times$. 
Toric and log smooth Log structures (K. Kato)

- When $X$ is a toric variety and $U$ the torus this is a prototypical example of a log smooth structure.
- In this case the monoid is associated to the regular monomials, with $\mathcal{O}^\times$ thrown in.
- In general $X$ is log smooth if it is étale locally toric.
- A morphism $X \to Y$ is log smooth if it is étale locally a base change of a dominant morphism of toric varieties.
Log curves

- A **log curve** is a reduced 1-dimensional fiber of a flat **log smooth morphism**.
- F. Kato showed that these are the same as nodal marked curves, with “the natural” log structure.
Log curves under the microscope

- Say $C \rightarrow S$ a log curve, $S = \text{Spec}(M_S \rightarrow k)$.
- A general point of $C$ looks like $\text{Spec}(M_S \rightarrow k[x])$.
- A node looks like $\text{Spec}(M \rightarrow k[x, y]/(xy))$, where
  \[ M = M_S \langle \log x, \log y \rangle / (\log x + \log y = \log t), \quad t \in M_S. \]
- A marked point looks like $\text{Spec}(M \rightarrow k[x])$ where
  \[ M = M_S \oplus \mathbb{N} \log x. \]
Stable log maps

- Fix $X$ a nice log smooth scheme.
- A stable log map $C \to X$ is a log morphism with stable underlying morphism of schemes.
- Marked points record contact orders with divisors of $X$.
- These are recorded by integer points $u \in \Sigma(X)(\mathbb{N})$.
- Stable log maps have “standard issue” log structure, called minimal.

**Theorem ([GS,C,ACMW])**

$\mathcal{M}(X, \tau)$, the stack of minimal stable log maps of type $\tau$, is a Deligne–Mumford stack which is finite and representable over $\mathcal{M}(X, \tau)$. 
Tropical picture

- \( X \) has a cone complex \( \Sigma(X) \) with integer lattice.
- \( C \to S \) has cone complex \( \Sigma(C) \to \Sigma(S) \). The fiber over \( u \in \Sigma(S) \) is a tropical curve:
- Components give vertices, nodes give edges, and marked points give infinite legs.
- A stable log map gives \( \Sigma(C) \to \Sigma(X) \), a family of tropical curves in \( \Sigma(X) \).
- Minimality is beautifully encoded in this picture...
Logarithmic invariants

- Recall that $\mathcal{M}(X, \tau)$ has a perfect obstruction theory over $\mathcal{M}_{g,n} \times \mathcal{X}^n$. This affords invariants by virtual pullback.
- $\mathcal{M}(X, \tau)$ has a POT over $\mathcal{M}^{ev}(A_X, \tau)$, where $A_X$ is the artin fan, a stack-theoretic version of $\Sigma(X)$.
- Here $\mathcal{M}^{ev}(A_X, \tau)$ is approximately $\mathcal{M}(A_X, \tau) \times A_X^n \times \mathcal{X}^n$.

**Theorem ([GS,C,AC])**

$\mathcal{M}^{ev}(A_X, \tau)$ is log smooth, and has a fundamental class. This affords invariants by virtual pullback.
An Analogy: Orbifold vs. Logarithmic cohomology

- If $\mathcal{X}$ is an orbifold, Chen-Ruan defined orbifold and quantum cohomology based on $H^*(\overline{\mathcal{I}}_\infty(\mathcal{X}), \mathbb{Q})$.
- $\overline{\mathcal{I}}_\infty(\mathcal{X})$, the rigidified inertia stack is the moduli space of orbifold points in $\mathcal{X}$, whose components, twisted sectors, correspond to $(x, \phi)$ where $x \in \mathcal{X}$ and $\phi \in \text{Aut}(x)$.
- Chen Ruan cohomology pairs $\phi$ with $\phi^{-1}$.
- If $X$ is a log scheme, Gross–Hacking–Keel–Siebert... define the ring of theta functions,
  - based on the moduli space $\mathcal{P}(X)$ of log points in $\mathcal{X}$,
  - whose components correspond to $(x, u)$ where $x \in X$ and $u$ a contact order at $x$, namely $u \in \Sigma(X)(\mathbb{N})$.
  - what about $-u$?
Splitting?

- Consider $X \to \mathbb{A}^1$ the total space of $xy = t$, and
- $C \to S$ given by $\{y = 0\} \to \{t = 0\}$.
- At the origin $M_S + \mathbb{N} \log x \subsetneq M \subsetneq M_S + \mathbb{Z} \log x$.
- It is not a log curve, but rather a punctured curve.
- Its tropicaliation is a finite leg.
A puncturing of a marked curve is a log structure $M$ at a marked point with

$$M_S + \mathbb{N} \log x \subseteq M \subset M_S + \mathbb{Z} \log x.$$ 

It is an instance of an idealized log smooth scheme.

A morphism $f : C \to X$ is prestable if $M$ is generated by $M_S + \mathbb{N} \log x$ and $f^! M_X$. 
Another example

- Now consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ with log structure given by $D = \text{one ruling}$.
- Let a conic $C$ degenerate to the union of the two rulings $C_0 = D + F$.
- Then $D$ has one marked point and one puncture,
- and (after pre-stabilizing) $F$ has one marked point.
Punctured log maps

- A punctured stable log map $C \to X$ is a prestable log morphism with stable underlying morphism of schemes.
- Punctured points record contact orders with divisors of $X$.
- These are recorded by integer tangents of the cone complex of $X$.
- Punctured log maps have “standard issue” minimal log structures.

Theorem ([ACGS])

$\mathcal{M}(X, \tau)$, the stack of minimal punctured stable log maps of type $\tau$, is a Deligne–Mumford stack which is finite and representable over $\mathcal{M}(X, \tau)$. 
Punctured invariants

- \( \mathcal{M}(X, \tau) \) has a POT over \( \mathcal{M}^{\text{ev}}(A_X, \tau) \), where \( A_X \) is the artin fan.
- \( \mathcal{M}^{\text{ev}}(A_X, \tau) \) is not log smooth, and doesn't have a fundamental class.

**Theorem ([ACGS])**

\( \mathcal{M}^{\text{ev}}(A_X, \tau) \) is idealized log smooth. This affords invariants by super-careful virtual pullback.

- The case of \( g = 0, n = 3, u_1, u_2, -u_3 \in \Sigma(X)(\mathbb{N}) \) is, fortunately, manageable.
There is a natural finite and representable splitting morphism
\[ M(X, \tau) \xrightarrow{\delta} \prod M(X, \tau_i). \]

**Theorem (ACGS)**

There is a virtual-pullback cartesian diagram

\[
\begin{array}{ccc}
M(X, \tau) & \rightarrow & \prod_{i=1}^{r} M(X, \tau_i) \\
\downarrow & & \downarrow \\
M^{ev}(A_X, \tau) & \rightarrow & \prod_{i=1}^{r} M^{ev}(A_X, \tau_i)
\end{array}
\]

with horizontal arrows the splitting maps, and the vertical arrows the canonical strict morphisms.

One needs even more care to relate this to a diagonal map.
The end

Thank you for your attention