

Resolving singularities of varieties and families

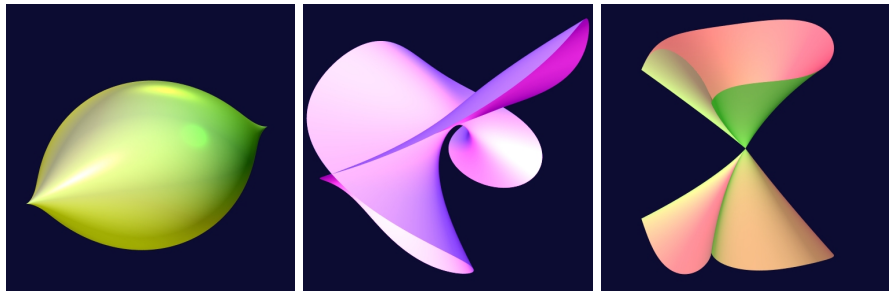
Dan Abramovich
Brown University

Joint work with Michael Temkin and Jarosław Włodarczyk



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On singularities



figures by Herwig Hauser, <https://imaginary.org/gallery/herwig-hauser-classic>

Singularities are beautiful.
Yet we get rid of them.

Resolution of singularities

Definition

A resolution of singularities $X' \rightarrow X$ is a **modification**^a with X' nonsingular inducing an isomorphism over the smooth locus of X .

^aproper birational map

Theorem (Hironaka 1964)

A variety X over a field of characteristic 0 admits a resolution of singularities $X' \rightarrow X$, so that the exceptional locus $E \subset X'$ is a simple normal crossings divisor.^a

^aCodimension 1, smooth components meeting transversally

Always characteristic 0 . . .

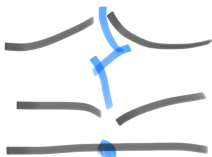


Resolution of families: $\dim B = 1$

Question

When are the singularities of a morphism $X \rightarrow B$ simple?

- If $\dim B = 1$ the simplest one can have by modifying X is $t = \prod x_i^{a_i}$,
 - and if one also allows base change, can have $t = \prod x_i$.
- [Kempf–Knudsen–Mumford–Saint-Donat 1973]



Question

What makes these special?

Log smooth schemes and log smooth morphisms

- A **toric variety** is a normal variety on which $T = (\mathbb{C}^*)^n$ acts algebraically with a dense free orbit.
- Zariski locally defined by equations between monomials.
- A variety X with divisor D is **toroidal** or **log smooth** if étale locally it looks like a toric variety X_σ with its toric divisor $X_\sigma \setminus T$.
- Étale locally it is defined by equations between monomials.
- A morphism $X \rightarrow Y$ is **toroidal** or **log smooth** if étale locally it looks like a torus equivariant morphism of toric varieties.
- The inverse image of a monomial is a monomial.

Resolution of families: higher dimensional base

Question

When are the singularities of a morphism $X \rightarrow B$ simple?

The best one can hope for, after base change, is a **semistable** morphism:

Definition (Karu 2000)

A log smooth morphism, with B smooth, is *semistable* if locally

$$\begin{aligned}t_1 &= x_1 \cdots x_{l_1} \\ &\vdots \\ t_m &= x_{l_{m-1}+1} \cdots x_m\end{aligned}$$

In particular log smooth.

Similar definition by Berkovich, all following de Jong.

The semistable reduction problem

Conjecture [N-Karu]

Let $X \rightarrow B$ be a dominant morphism of varieties.

- (Loose) There is an **alteration** $B_1 \rightarrow B$ and a modification $X_1 \rightarrow (X \times_B B_1)_{\text{main}}$ such that $X_1 \rightarrow B_1$ is semistable.
 - (Tight) If the geometric generic fiber $X_{\bar{\eta}}$ is smooth, such $X_1 \rightarrow B_1$ can be found with $X_{\bar{\eta}}$ unchanged.
-
- One wants the **tight** version in order to compactify smooth families.
 - I'll describe progress towards that.
 - Major early results by [KKMS 1973], [de Jong 1997].
 - Wonderful results in positive and mixed characteristics by de Jong, Gabber, Illusie and Temkin.

Toroidalization and weak semistable reduction

Back to characteristic 0

Theorem (Toroidalization, \aleph -Karu 2000, \aleph -K-Denef 2013)

There is a *modification* $B_1 \rightarrow B$ and a modification $X_1 \rightarrow (X \times_B B_1)_{\text{main}}$ such that $X_1 \rightarrow B_1$ is log smooth and flat.

Theorem (Weak semistable reduction, \aleph -Karu 2000)

There is an *alteration* $B_1 \rightarrow B$ and a modification $X_1 \rightarrow (X \times_B B_1)_{\text{main}}$ such that $X_1 \rightarrow B_1$ is log smooth, flat, with reduced fibers.

- Passing from weak semistable reduction to semistable reduction is a purely combinatorial problem [\aleph -Karu 2000],
- proven by [Karu 2000] for families of surfaces and threefolds, and
- whose restriction to rank-1 valuation rings is proven in a preprint by [Karim Adiprasito - Gaku Liu - Igor Pak - Michael Temkin].

Applications of weak semistable reduction

(with a whole lot of more input)

Theorem (Karu 2000; K-SB 97, Alexeev 94, BCHM 11)

The moduli space of stable smoothable varieties is projective.

Theorem (Viehweg-Zuo 2004)

The moduli space of canonically polarized manifolds is Brody hyperbolic.

Theorem (Fujino 2017)

Nakayama's numerical logarithmic Kodaira dimension is subadditive in families $X \rightarrow B$ with generic fiber F :

$$\kappa_{\sigma}(X, D_X) \geq \kappa_{\sigma}(F, D_F) + \kappa_{\sigma}(B, D_B).$$

Main result

The following result is work-in-progress.

Main result (Functorial toroidalization, \aleph -Temkin-Włodarczyk)

Let $X \rightarrow B$ be a dominant log morphism.

- There are log modifications $B_1 \rightarrow B$ and $X_1 \rightarrow (X \times_B B_1)_{\text{main}}$ such that $X_1 \rightarrow B_1$ is log smooth and flat;
- this is compatible with log base change $B' \rightarrow B$;
- this is functorial, up to base change, with **log** smooth $X'' \rightarrow X$.

This implies the **tight** version of the results of semistable reduction type.

In virtue of functoriality

Theorem (Temkin)

Resolution of singularities holds for excellent schemes, complex spaces, nonarchimedean spaces, p -adic spaces, formal spaces and for stacks.

- This is a consequence of resolution for varieties and schemes, **functorial** for smooth morphisms (submersions). Moreover
- Włodarczyk showed that if one seriously looks for a resolution **functor**, one is led to a resolution theorem.
- Our main result will lead to this generality on families.
- Current application of our main result:

Theorem (Deng 2018)

The moduli space of minimal complex projective manifolds of general type is Kobayashi hyperbolic.

$\dim B = 0$: log resolution via principalization

- To resolve **log** singularities, one embeds X in a **log** smooth $Y \dots$
- \dots which can be done locally.
- One reduces to **principalization** of \mathcal{I}_X (Hironaka, Villamayor, Bierstone–Milman).

Theorem (Principalization ... N-T-W)

Let \mathcal{I} be an ideal on a log smooth Y . There is a functorial logarithmic morphism $Y' \rightarrow Y$, with Y' logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.



Figure: The ideal (u^2, x^2) and the result of blowing up the origin, \mathcal{I}_E^2 . Here u is a monomial but x is not.

Logarithmic order

Principalization is done by **order reduction**, using **logarithmic derivatives**.

- for a monomial u we use $u \frac{\partial}{\partial u}$.
- for other variables x use $\frac{\partial}{\partial x}$.

Definition

Write $\mathcal{D}^{\leq a}$ for the sheaf of **logarithmic** differential operators of order $\leq a$. The **logarithmic order** of an ideal \mathcal{I} is the minimum a such that $\mathcal{D}^{\leq a} \mathcal{I} = (1)$.

Take u, v monomials, x free variable, p the origin.

$$\begin{aligned} \text{logord}_p(u^2, x) &= 1 && \text{(since } \frac{\partial}{\partial x} x = 1) \\ \text{logord}_p(u^2, x^2) &= 2 && \text{logord}_p(v, x^2) = 2 \\ \text{logord}_p(v + u) &= \infty && \text{since } \mathcal{D}^{\leq 1} \mathcal{I} = \mathcal{D}^{\leq 2} \mathcal{I} = \dots = (u, v). \end{aligned}$$

The monomial part of an ideal

Definition

$\mathcal{M}(\mathcal{I})$ is the minimal monomial ideal containing \mathcal{I} .

Proposition (Kollár, X-T-W)

- (1) *In characteristic 0, $\mathcal{M}(\mathcal{I}) = \mathcal{D}^\infty(\mathcal{I})$. In particular $\max_p \log \text{ord}_p(\mathcal{I}) = \infty$ if and only if $\mathcal{M}(\mathcal{I}) \neq 1$.*
- (2) *Let $Y_0 \rightarrow Y$ be the normalized blowup of $\mathcal{M}(\mathcal{I})$. Then $\mathcal{M} := \mathcal{M}(\mathcal{I})\mathcal{O}_{Y_0} = \mathcal{M}(\mathcal{I}\mathcal{O}_{Y_0})$, and it is an invertible monomial ideal, and so $\mathcal{I}\mathcal{O}_{Y_0} = \mathcal{I}_0 \cdot \mathcal{M}$ with $\max_p \log \text{ord}_p(\mathcal{I}_0) < \infty$.*

(1) \Rightarrow (2)

\mathcal{D}_{Y_0} is the pullback of \mathcal{D}_Y , so (2) follows from (1) since the ideals have the same generators.

The monomial part of an ideal - proof

Proof of (1), basic affine case.

- Let $\mathcal{O}_Y = \mathbb{C}[x_1, \dots, x_n, u_1, \dots, u_m]$ and assume $\mathcal{M} = \mathcal{D}(\mathcal{M})$.
- The operators

$$1, u_1 \frac{\partial}{\partial u_1}, \dots, u_l \frac{\partial}{\partial u_l}$$

commute and have distinct systems of eigenvalues on the eigenspaces $u \mathbb{C}[x_1, \dots, x_n]$, for distinct monomials u .

- Therefore $\mathcal{M} = \bigoplus u \mathcal{M}_u$ with ideals $\mathcal{M}_u \subset \mathbb{C}[x_1, \dots, x_n]$ stable under derivatives,
- so each \mathcal{M}_u is either (0) or (1).
- In other words, \mathcal{M} is monomial.



The general case requires more commutative algebra.

$\dim B = 0$: sketch of argument

- In characteristic 0, if $\log \text{ord}_p(\mathcal{I}) = a < \infty$, then $\mathcal{D}^{\leq a-1}\mathcal{I}$ contains an element x with derivative 1, a maximal contact element.
- Carefully applying induction on dimension to an ideal on $\{x = 0\}$ gives order reduction (Encinas–Villamayor, Bierstone–Milman, Włodarczyk):

Proposition (...N-T-W)

Let \mathcal{I} be an ideal on a logarithmically smooth Y with

$$\max_p \log \text{ord}_p(\mathcal{I}) = a.$$

There is a functorial logarithmic morphism $Y_1 \rightarrow Y$, with Y_1 logarithmically smooth, such that $\mathcal{I}\mathcal{O}_{Y'} = \mathcal{M} \cdot \mathcal{I}_1$ with \mathcal{M} an invertible monomial ideal and

$$\max_p \log \text{ord}_p(\mathcal{I}_1) < a.$$

Arbitrary B

(Work in progress)

Main result (N-T-W)

Let $Y \rightarrow B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{I} \subset \mathcal{O}_Y$ an ideal. There is a log morphism $B' \rightarrow B$ and functorial log morphism $Y' \rightarrow Y$, with $Y' \rightarrow B'$ logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

- This is done by **relative order reduction**, using **relative logarithmic derivatives**.

Definition

Write $\mathcal{D}_{Y/B}^{\leq a}$ for the sheaf of **relative logarithmic differential operators** of order $\leq a$. The **relative logarithmic order** of an ideal \mathcal{I} is the minimum a such that $\mathcal{D}_{Y/B}^{\leq a}\mathcal{I} = (1)$.

The new step

- $\mathcal{M} := \mathcal{D}_{Y/B}^\infty \mathcal{I}$ is an ideal which is **monomial along the fibers**.
- $\text{relord}_p(\mathcal{I}) = \infty$ if and only if $\mathcal{M} := \mathcal{D}_{Y/B}^\infty \mathcal{I}$ is a nonunit ideal.

Monomialization Theorem [N-T-W]

Let $Y \rightarrow B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{M} \subset \mathcal{O}_Y$ an ideal with $\mathcal{D}_{Y/B} \mathcal{M} = \mathcal{M}$. There is a log morphism $B' \rightarrow B$ with saturated pullback $Y' \rightarrow B'$, such that $\mathcal{M} \mathcal{O}_{Y'}$ a monomial ideal.

After this one can proceed as in the case “ $\dim B = 0$ ”.

Proof of Monomialization Theorem, special case

Let $Y = \text{Spec } \mathbb{C}[u, v] \rightarrow B = \text{Spec } \mathbb{C}[w]$ with $w = uv$, and $\mathcal{M} = (f)$.

Proof in this special case.

- Every monomial is either $u^\alpha w^k$ or $v^\alpha w^k$.
- Once again the operators $1, u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v}$ commute and have different eigenvalues on u^α, v^α .
- Expanding $f = \sum u^\alpha f_\alpha + \sum v^\beta f_\beta$, the condition $\mathcal{M} = \mathcal{D}_{Y/B} \mathcal{M}$ gives that only one term survives,
- say $f = u^\alpha f_\alpha$, with $f_\alpha \in \mathbb{C}[w]$.
- Blowing up (f_α) on B has the effect of making it monomial, so f becomes monomial.



The general case is surprisingly subtle.

Order reduction: Example 1

- Consider $Y_1 = \text{Spec } \mathbb{C}[u, x]$ and $D = \{u = 0\}$.
- Let $\mathcal{I} = (u^2, x^2)$.
- If one blows up (u, x) the ideal is principalized:



- ▶ on the u -chart $\text{Spec } \mathbb{C}[u, x']$ with $x = x'u$ we have $\mathcal{I}\mathcal{O}_{Y'_1} = (u^2)$,
 - ▶ on the x -chart $\text{Spec } \mathbb{C}[u', x]$ with $u' = xu'$ we have $\mathcal{I}\mathcal{O}_{Y'} = (x^2)$,
 - ▶ which is exceptional hence monomial.
- This is in fact the only functorial **admissible** blowing up.

Order reduction: Example 2

- Consider $Y_2 = \text{Spec } \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$.
- Example 1 is the pullback of this via the log smooth $v = u^2$.
- Functoriality says: we need to blow up an ideal whose pullback is (u, x) .
- This means we need to blow up $(v^{1/2}, x)$.
- What is this? What is its blowup?

Kummer ideals

Definition

- A **Kummer monomial** is a monomial in the Kummer-étale topology of Y (like $v^{1/2}$).
- A **Kummer monomial ideal** is a monomial ideal in the Kummer-étale topology of Y .
- A **Kummer center** is the sum of a Kummer monomial ideal and the ideal of a log smooth subscheme.
- Locally $(x_1, \dots, x_k, u_1^{1/d}, \dots, u_\ell^{1/d})$.

Blowing up Kummer centers

Proposition

Let \mathcal{J} be a Kummer center on a logarithmically smooth Y . There is a universal proper birational $Y' \rightarrow Y$ such that Y' is logarithmically smooth and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.

Example 0

$Y = \text{Spec } \mathbb{C}[v]$, with toroidal structure associated to $D = \{v = 0\}$, and $\mathcal{J} = (v^{1/2})$.

- There is no log scheme Y' satisfying the proposition.
- There is a stack $Y' = Y(\sqrt{D})$, the Cadman–Vistoli root stack, satisfying the proposition!

Example 2 concluded

- Consider $Y_2 = \text{Spec } \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$ and $\mathcal{J} = (v^{1/2}, x)$.
- associated blowing up $Y' \rightarrow Y_2$ with charts:
 - ▶ $Y'_x := \text{Spec } \mathbb{C}[v, x, v'] / (v'x^2 = v)$, where $v' = v/x^2$ (nonsingular scheme).
 - ★ Exceptional $x = 0$, now monomial.
 - ★ $\mathcal{I} = (v, x^2)$ transformed into (x^2) , invertible monomial ideal.
 - ★ Kummer ideal $(v^{1/2}, x)$ transformed into **monomial ideal** (x) .
 - ▶ The $v^{1/2}$ -chart:
 - ★ **stack** quotient $X'_{v^{1/2}} := [\text{Spec } \mathbb{C}[w, y] / \mu_2]$,
 - ★ where $y = x/w$ and $\mu_2 = \{\pm 1\}$ acts via $(w, y) \mapsto (-w, -y)$.
 - ★ Exceptional $w = 0$ (monomial).
 - ★ (v, x^2) transformed into invertible monomial ideal $(v) = (w^2)$.
 - ★ $(v^{1/2}, x)$ transformed into invertible **monomial ideal** (w) .

Proof of proposition

Let \mathcal{J} be a Kummer center on a logarithmically smooth Y . There is a universal proper birational $Y' \rightarrow Y$ such that Y' is a logarithmically smooth **stack** and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.

- Choose a stack \tilde{Y} with coarse moduli space Y such that $\tilde{\mathcal{J}} := \mathcal{J}\mathcal{O}_{\tilde{Y}}$ is an ideal.
- Let $\tilde{Y}' \rightarrow \tilde{Y}$ be the blowup of $\tilde{\mathcal{J}}$, with exceptional E .
- Let $\tilde{Y}' \rightarrow B\mathbb{G}_m$ be the classifying morphism of \mathcal{I}_E .
- Y' is the relative coarse moduli space of $\tilde{Y}' \rightarrow Y \times B\mathbb{G}_m$.
- One shows this is independent of choices. ♠