Resolving singularities of varieties and families

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Joint work with Michael Temkin and Jarosław Włodarczyk

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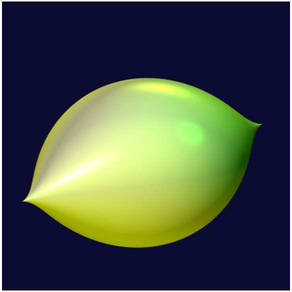
On singularities



figrures by Herwig Hauser, https://imaginary.org/gallery/herwig-hauser-classic

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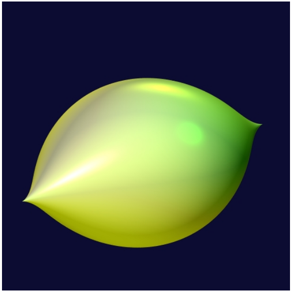
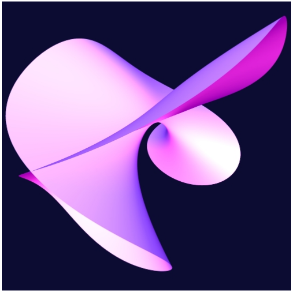

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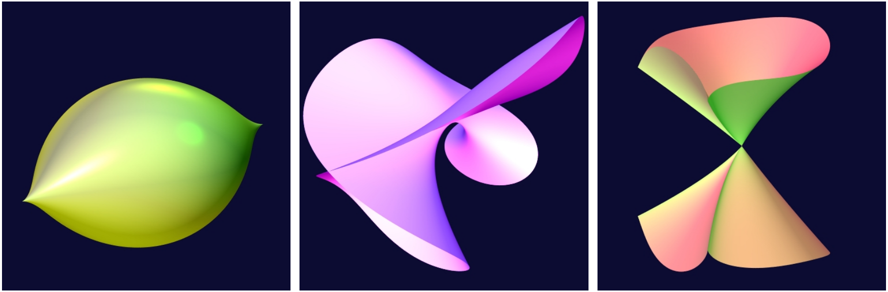

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Singularities are beautiful.

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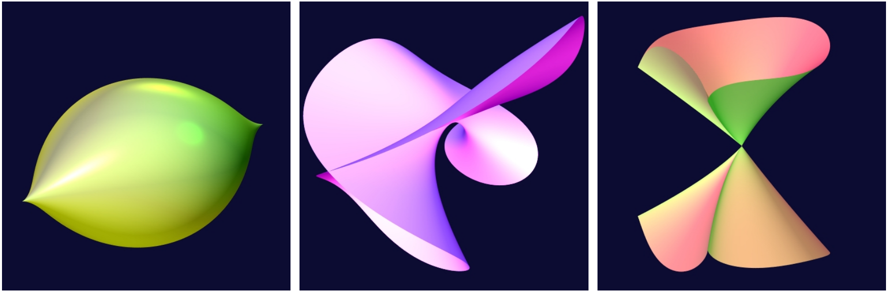

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Singularities are beautiful.
Yet we get rid of them.

Resolution of singularities

Definition

A resolution of singularities $X' \to X$ is a modification[a] with X' nonsingular inducing an isomorphism over the smooth locus of X.

[a] proper birational map

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Theorem (Hironaka 1964)

A variety X over a field of characteristic 0 admits a resolution of singularities $X' \to X$, so that the exceptional locus $E \subset X'$ is a simple normal crossings divisor.[a]

[a] Codimension 1, smooth components meeting transversally

Always characteristic 0 . . .

Resolution of families: $\dim B = 1$

Question

When are the singularities of a morphism $X \to B$ simple?

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- If $\dim B = 1$ the simplest one can have by modifying X is $t = \prod x_i^{a_i}$,
- and if one also allows base change, can have $t = \prod x_i$. [Kempf–Knudsen–Mumford–Saint-Donat 1973]

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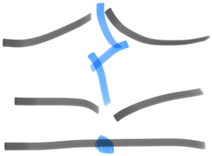

Question

What makes these special?

Log smooth schemes and log smooth morphisms

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- A morphism $X \to Y$ is toroidal or log smooth if étale locally it looks like a torus equivariant morphism of toric varieties.
- The inverse image of a monomial is a monomial.

Resolution of families: higher dimensional base

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Resolution of families: higher dimensional base

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When are the singularities of a morphism $X \to B$ simple?

The best one can hope for, after base change, is a semistable morphism:

Definition (ℵ-Karu 2000)

A log smooth morphism, with B smooth, is *semistable* if locally

$$
\begin{aligned}
t_1 &= x_1 \cdots x_{l_1} \\
\vdots & \quad \vdots \\
t_m &= x_{l_{m-1}+1} \cdots x_m
\end{aligned}
$$

In particular log smooth.

Similar definition by Berkovich, all following de Jong.

The semistable reduction problem

Conjecture [ℵ-Karu]

Let $X \to B$ be a dominant morphism of varieties.

- (Loose) There is an alteration $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{\text{main}}$ such that $X_1 \to B_1$ is semistable.

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- Major early results by [KKMS 1973], [de Jong 1997].
- Wonderful results in positive and mixed characteristics by de Jong, Gabber, Illusie and Temkin.

Toroidalization and weak semistable reduction

Back to characteristic 0

Theorem (Toroidalization, $\aleph$-Karu 2000, $\aleph$-K-Denef 2013)

There is a modification $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{main}$ such that $X_1 \to B_1$ is log smooth and flat.

Theorem (Weak semistable reduction, $\aleph$-Karu 2000)

There is an alteration $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{main}$ such that $X_1 \to B_1$ is log smooth, flat, with reduced fibers.

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- whose restriction to rank-1 valuation rings is proven in a preprint by [Karim Adiprasito - Gaku Liu - Igor Pak - Michael Temkin].

Applications of weak semistable reduction

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The moduli space of canonically polarized manifolds is Brody hyperbolic.

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Theorem (Fujino 2017)

Nakayama's numerical logarithmic Kodaira dimension is subadditive in families $X \to B$ with generic fiber F:

$$\kappa_\sigma(X, D_X) \geq \kappa_\sigma(F, D_F) + \kappa_\sigma(B, D_B).$$

Main result

The following result is work-in-progress.

Main result (Functorial toroidalization, $\aleph$-Temkin-Włodarczyk)

Let $X \to B$ be a dominant log morphism.

- There are log modifications $B_1 \to B$ and $X_1 \to (X \times_B B_1)_{\text{main}}$ such that $X_1 \to B_1$ is log smooth and flat;
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- Włodarczyk showed that if one seriously looks for a resolution functor, one is led to a resolution theorem.
- Our main result will lead to this generality on families.
- Current application of our main result:

Theorem (Deng 2018)

The moduli space of minimal complex projective manifolds of general type is Kobayashi hyperbolic.

$\dim B = 0$: log resolution via principalization

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Theorem (Principalization ... ℵ-T-W)

Let $\mathcal{I}$ be an ideal on a log smooth Y. There is a functorial logarithmic morphism $Y' \to Y$, with Y' logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

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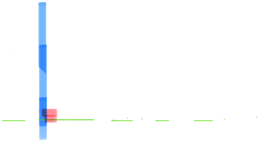

Figure: The ideal (u^2, x^2)
Here u is a monomial but x is not.

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Figure: The ideal (u^2, x^2) and the result of blowing up the origin, $\mathcal{I}_E^2$. Here u is a monomial but x is not.

Logarithmic order

Principalization is done by order reduction, using logarithmic derivatives.

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$\mathrm{logord}_p(v + u) = \infty$ since $\mathcal{D}^{\leq 1}\mathcal{I} = \mathcal{D}^{\leq 2}\mathcal{I} = \cdots = (u, v)$.

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(1) *In cahracteristic 0,* $\mathcal{M}(\mathcal{I}) = \mathcal{D}^\infty(\mathcal{I})$. *In particular* $\max_p \operatorname{logord}_p(\mathcal{I}) = \infty$ *if and only if* $\mathcal{M}(\mathcal{I}) \neq 1$.

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(1)⇒(2)

$\mathcal{D}_{Y_0}$ is the pullback of $\mathcal{D}_Y$, so (2) follows from (1) since the ideals have the same generators.

The monomial part of an ideal - proof

Proof of (1), basic affine case.

- Let $\mathcal{O}_Y = \mathbb{C}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ and assume $\mathcal{M} = \mathcal{D}(\mathcal{M})$.

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- The operators

$$1, u_1 \frac{\partial}{\partial u_1}, \ldots, u_l \frac{\partial}{\partial u_l}$$

commute and have distinct systems of eigenvalues on the eigenspaces $u\,\mathbb{C}[x_1, \ldots, x_n]$, for distinct monomials u.

The monomial part of an ideal - proof

Proof of (1), basic affine case.

- Let $\mathcal{O}_Y = \mathbb{C}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ and assume $\mathcal{M} = \mathcal{D}(\mathcal{M})$.
- The operators
$$1, u_1 \frac{\partial}{\partial u_1}, \ldots, u_l \frac{\partial}{\partial u_l}$$
commute and have distinct systems of eigenvalues on the eigenspaces $u\,\mathbb{C}[x_1, \ldots, x_n]$, for distinct monomials u.
- Therefore $\mathcal{M} = \oplus u\mathcal{M}_u$ with ideals $\mathcal{M}_u \subset \mathbb{C}[x_1, \ldots, x_n]$ stable under derivatives,

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The general case requires more commutative algebra.

$\dim B = 0$: sketch of argument

- In cahracteristic 0, if $\text{logord}_p(\mathcal{I}) = a < \infty$, then $\mathcal{D}^{\leq a-1}\mathcal{I}$ contains an element x with derivative 1, a maximal contact element.

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Proposition (... ℵ-T-W)

Let $\mathcal{I}$ be an ideal on a logarithmically smooth Y with

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There is a functorial logarithmic morphism $Y_1 \to Y$, with Y_1 logarithmically smooth, such that $\mathcal{I}\mathcal{O}_{Y'} = \mathcal{M} \cdot \mathcal{I}_1$ with $\mathcal{M}$ an invertible monomial ideal and

$$\max_p \operatorname{logord}_p(\mathcal{I}_1) < a.$$

Arbitrary B

(Work in progress)

Main result (ℵ-T-W)

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{I} \subset \mathcal{O}_Y$ an ideal. There is a log morphism $B' \to B$ and functorial log morphism $Y' \to Y$, with $Y' \to B'$ logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

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Definition

Write $\mathcal{D}^{\leq a}_{Y/B}$ for the sheaf of relative logarithmic differential operators of order $\leq a$. The relative logarithmic order of an ideal $\mathcal{I}$ is the minimum a such that $\mathcal{D}^{\leq a}_{Y/B}\mathcal{I} = (1)$.

The new step

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Monomialization Theorem [ℵ-T-W]

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{M} \subset \mathcal{O}_Y$ an ideal with $\mathcal{D}_{Y/B}\mathcal{M} = \mathcal{M}$. There is a log morphism $B' \to B$ with saturated pullback $Y' \to B'$, such that $\mathcal{M}\mathcal{O}_{Y'}$ a monomial ideal.

After this one can proceed as in the case "dim $B = 0$".

Proof of Monomialization Theorem, special case

Let $Y = \operatorname{Spec}\mathbb{C}[u,v] \to B = \operatorname{Spec}\mathbb{C}[w]$ with $w = uv$, and $\mathcal{M} = (f)$.

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The general case is surprisingly subtle.

Order reduction: Example 1

- Consider $Y_1 = \operatorname{Spec} \mathbb{C}[u, x]$ and $D = \{u = 0\}$.
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  - on the u-chart $\operatorname{Spec} \mathbb{C}[u, x']$ with $x = x'u$ we have $\mathcal{I}\mathcal{O}_{Y_1'} = (u^2)$,
  - on the x-chart $\operatorname{Spec} \mathbb{C}[u', x]$ with $u' = xu'$ we have $\mathcal{I}\mathcal{O}_{Y'} = (x^2)$,
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- This is in fact the only functorial admissible blowing up.

Order reduction: Example 2

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- This means we need to blow up $(v^{1/2}, x)$.
- What is this? What is its blowup?

Kummer ideals

Definition

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- A Kummer monomial is a monomial in the Kummer-étale topology of Y (like $v^{1/2}$).
- A Kummer monomial ideal is a monomial ideal in the Kummer-étale topology of Y.
- A Kummer center is the sum of a Kummer monomial ideal and the ideal of a log smooth subscheme.
- Locally $(x_1, \ldots, x_k, u_1^{1/d}, \ldots u_\ell^{1/d})$.

Blowing up Kummer centers

Proposition

Let $\mathcal{J}$ be a Kummer center on a logarithmically smooth Y. There is a universal proper birational $Y' \to Y$ such that Y' is logarithmically smooth and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.

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- There is no log scheme Y' satisfying the proposition.
- There is a stack $Y' = Y(\sqrt{D})$, the Cadman–Vistoli root stack, satisfying the proposition!

Example 2 concluded

- Consider $Y_2 = \operatorname{Spec} \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$ and $\mathcal{J} = (v^{1/2}, x)$.

Example 2 concluded

- Consider $Y_2 = \operatorname{Spec} \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$ and $\mathcal{J} = (v^{1/2}, x)$.
- associated blowing up $Y' \to Y_2$ with charts:
  - $Y'_x := \operatorname{Spec} \mathbb{C}[v, x, v']/(v'x^2 = v)$, where $v' = v/x^2$ (nonsingular scheme).
    - ⋆ Exceptional $x = 0$, now monomial.
    - ⋆ $\mathcal{I} = (v, x^2)$ transformed into (x^2), invertible monomial ideal.
    - ⋆ Kummer ideal $(v^{1/2}, x)$ transformed into monomial ideal (x).

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  - The $v^{1/2}$-chart:
    - stack quotient $X'_{v^{1/2}} := \left[\operatorname{Spec} \mathbb{C}[w, y]/\mu_2\right]$,
    - where $y = x/w$ and $\mu_2 = \{\pm 1\}$ acts via $(w, y) \mapsto (-w, -y)$.
    - Exceptional $w = 0$ (monomial).
    - (v, x^2) transformed into invertible monomial ideal $(v) = (w^2)$.
    - $(v^{1/2}, x)$ transformed into invertible monomial ideal (w).

Proof of proposition

Let $\mathcal{J}$ be a Kummer center on a logarithmically smooth Y. There is a universal proper birational $Y' \to Y$ such that Y' is a logarithmically smooth stack and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.

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- Choose a stack $\tilde{Y}$ with coarse moduli space Y such that $\tilde{\mathcal{J}} := \mathcal{J}\mathcal{O}_{\tilde{Y}}$ is an ideal.

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- Y' is the relative coarse moduli space of $\tilde{Y}' \to Y \times B\mathbb{G}_m$.

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- One shows this is independent of choices. ♠