Resolving singularities of varieties and families

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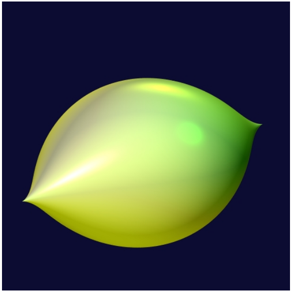
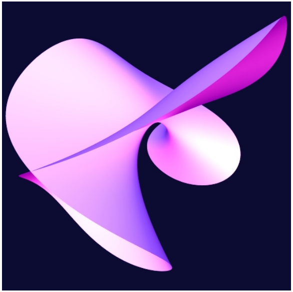
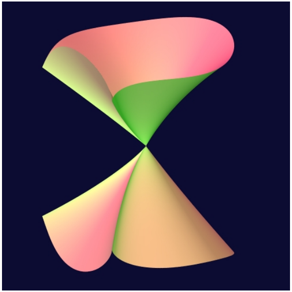
On singularities



Zitrus: $x^2 + z^2 = y^3(1-y)^3$



Kolibri: $x^2 = y^2 z^2 + z^3$



Daisy: $(x^2 - y^3)^2 = (z^2 - y^2)^3$

Real figures by Herwig Hauser, `https://imaginary.org/gallery/herwig-hauser-classic`

Singularities are beautiful.

Why should we “get rid of them”?

Answer 1: to study singularities.

Answer 2: to study the structure of varieties.

Singular and smooth points

Definition

$\{f(x_1, \dots, x_n) = 0\}$ is singular at p if $\frac{\partial f}{\partial x_i}(p) = 0$ for all i.
Otherwise smooth.

In other words, if smooth, $\{f = 0\}$ defines a submanifold of complex codimension 1.

In codimension c, the set $\{f_1 = \cdots = f_k = 0\}$ is smooth when $d(f_1, \dots, f_k)$ has constant rank c.

What is resolution of singularities?

Definition

A resolution of singularities $X' \to X$ is a modification[a] with X' nonsingular inducing an isomorphism over the smooth locus of X.

[a] proper birational map. For instance, blowing up.

Theorem (Hironaka 1964)

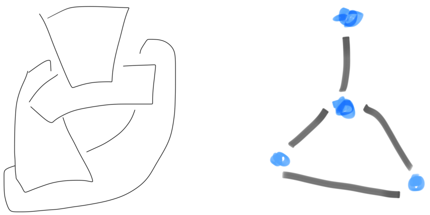
A variety X over a field of characteristic 0 admits a resolution of singularities $X' \to X$, so that the critical locus $E \subset X'$ is a simple normal crossings divisor.[a]

[a] Codim. 1, smooth components meeting transversally - as simple as possible

Always characteristic 0 . . .

Answer 1: Example of invariant - Stepanov's theorem

If $X' \to X$ a resolution with critical $E \subset X'$ a simple normal crossings divisor, define $\Delta(E)$ to be the dual complex of E.



Theorem (Stepanov 2006)

The simple homotopy type of $\Delta(E)$ is independent of the resolution $X' \to X$.

Also work by Danilov, Payne, Thuillier, Harper. . .

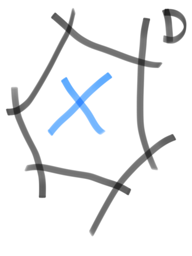
Answer 2: Example of structure result: compactifications

"Working with noncompact spaces is like trying to keep change with holes in your pockets"

Angelo Vistoli

Corollary (Hironaka)

A smooth quasiprojective variety X^0 has a smooth projective compactification X with $D = X \smallsetminus X^0$ a simple normal crossings divisor.

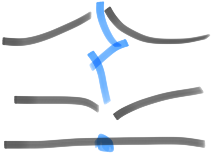


Resolution of families: $\dim B = 1$

Key Question

When are the singularities of a morphism $X \to B$ simple?

- If $\dim B = 1$ the simplest one can have by modifying X is $t = \prod x_i^{a_i}$,
- and if one also allows base change $t = s^k$, can have $s = \prod x_i$. [Kempf–Knudsen–Mumford–Saint-Donat 1973]



Question

What makes these special?

Log smooth schemes and log smooth morphisms

- A toric variety is a normal variety on which $T = (\mathbb{C}^*)^n$ acts algebraically with a dense free orbit.
- Zariski locally defined by equations between monomials.
- A variety X with divisor D is toroidal or log smooth if étale locally it looks like a toric variety X_σ with its toric divisor $X_\sigma \smallsetminus T$.
- Étale locally it is defined by equations between monomials.
- A morphism $X \to Y$ is toroidal or log smooth if étale locally it looks like a torus equivariant morphism of toric varieties.
- The inverse image of a monomial [1] is a monomial.

[1] defining equation of part of D_Y

Resolution of families: higher dimensional base

Question

When are the singularities of a morphism $X \to B$ simple?

The best one can hope for, after base change, is a semistable morphism:

Definition (ℵ-Karu 2000)

A log smooth morphism, with B smooth, is *semistable* if locally

$$\begin{aligned} t_1 &= x_1 \cdots x_{l_1} \\ \vdots \quad & \quad \vdots \\ t_m &= x_{l_{m-1}+1} \cdots x_{l_m} \end{aligned}$$

In particular log smooth.

Similar definition by Berkovich, all inspired by de Jong.

Ultimate goal: the semistable reduction problem

Conjecture [ℵ-Karu]

Let $X \to B$ be a dominant morphism of varieties.

- (Loose) There is a base change[a] $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{\text{main}}$ such that $X_1 \to B_1$ is semistable.
- (Tight) If the geometric generic fiber $X_{\bar\eta}$ is smooth, such $X_1 \to B_1$ can be found with $X_{\bar\eta}$ unchanged.

[a] Alteration: Proper, surjective, generically finite

- One wants the tight version in order to compactify smooth families.
- I'll describe progress towards that.
- Major early results by [KKMS 1973], [de Jong 1997].
- Wonderful results in positive and mixed characteristics by de Jong, Gabber, Illusie and Temkin.

Toroidalization and weak semistable reduction

This is key to what's known:

Theorem (Toroidalization, ℵ-Karu 2000, ℵ-K-Denef 2013)

There is a modification $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{main}$ such that $X_1 \to B_1$ is log smooth and flat.

Theorem (Weak semistable reduction, ℵ-Karu 2000)

There is a base change $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{main}$ such that $X_1 \to B_1$ is log smooth, flat, with reduced fibers.

- Passing from weak semistable reduction to semistable reduction is a purely combinatorial problem [ℵ-Karu 2000],
- proven by [Karu 2000] for families of surfaces and threefolds, and
- whose restriction to rank-1 valuation rings is proven in a preprint by [Karim Adiprasito - Gaku Liu - Igor Pak - Michael Temkin].

Applications of weak semistable reduction

This is already useful for studying families:

Theorem (Karu 2000; K-SB 97, Alexeev 94, BCHM 11)

The moduli space of stable smoothable varieties is projective[a].

[a] in particular bounded and proper

Theorem (Viehweg-Zuo 2004)

The moduli space of canonically polarized manifolds is Brody hyperbolic.

Theorem (Fujino 2017)

Nakayama's numerical logarithmic Kodaira dimension is subadditive in families $X \to B$ with generic fiber F:

$$\kappa_\sigma(X, D_X) \geq \kappa_\sigma(F, D_F) + \kappa_\sigma(B, D_B).$$

Main result

The following result is work-in-progress.

Main result (Functorial toroidalization, $\aleph$-Temkin-Włodarczyk)

Let $X \to B$ be a dominant morphism.

- There are modifications $B_1 \to B$ and $X_1 \to (X \times_B B_1)_{\text{main}}$ such that $X_1 \to B_1$ is log smooth and flat;
- this is compatible with base change $B' \to B$;
- this is functorial, up to base change, with log smooth $X'' \to X$.

This implies the tight version of the results of semistable reduction type.
Application:

Theorem (Deng 2018)

The moduli space of minimal complex projective manifolds of general type is Kobayashi hyperbolic.

dim $B = 0$: log resolution via principalization

- To resolve log singularities, one embeds X in a log smooth Y...
- ...which can be done locally.
- One reduces to principalization of $\mathcal{I}_X$ (Hironaka, Villamayor, Bierstone–Milman).

Theorem (Principalization ...ℵ-T-W)

Let $\mathcal{I}$ be an ideal on a log smooth Y. There is a functorial logarithmic morphism $Y' \to Y$, with Y' logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

Figure: The ideal (u^2, x^2) and the result of blowing up the origin, $\mathcal{I}_E^2$. Here u is a monomial but x is not.

Logarithmic order

Principalization is done by order reduction, using logarithmic derivatives.

- for a monomial u we use $u\frac{\partial}{\partial u}$.
- for other variables x use $\frac{\partial}{\partial x}$.

Definition

Write $\mathcal{D}^{\leq a}$ for the sheaf of logarithmic differential operators of order $\leq a$. The logarithmic order of an ideal $\mathcal{I}$ is the minimum a such that $\mathcal{D}^{\leq a}\mathcal{I} = (1)$.

Take u, v monomials, x free variable, p the origin.

$\text{logord}_p(u^2, x) = 1$ (since $\frac{\partial}{\partial x}x = 1$)

$\text{logord}_p(u^2, x^2) = 2$ $\text{logord}_p(v, x^2) = 2$

$\text{logord}_p(v + u) = \infty$ since $\mathcal{D}^{\leq 1}\mathcal{I} = \mathcal{D}^{\leq 2}\mathcal{I} = \cdots = (u, v)$.

Key new ingredient: The monomial part of an ideal

Definition

$\mathcal{M}(\mathcal{I})$ is the minimal monomial ideal containing $\mathcal{I}$.

Proposition (Kollár, ℵ-T-W)

(1) *In characteristic 0,* $\mathcal{M}(\mathcal{I}) = \mathcal{D}^\infty(\mathcal{I})$. *In particular* $\max_p \operatorname{logord}_p(\mathcal{I}) = \infty$ *if and only if* $\mathcal{M}(\mathcal{I}) \neq 1$.

(2) *Let* $Y_0 \to Y$ *be the normalized blowup of* $\mathcal{M}(\mathcal{I})$. *Then* $\mathcal{M} := \mathcal{M}(\mathcal{I})\mathcal{O}_{Y_0} = \mathcal{M}(\mathcal{I}\mathcal{O}_{Y_0})$, *and it is an invertible monomial ideal, and so* $\mathcal{I}\mathcal{O}_{Y_0} = \mathcal{I}_0 \cdot \mathcal{M}$ *with* $\max_p \operatorname{logord}_p(\mathcal{I}_0) < \infty$.

(1)⇒(2)

$\mathcal{D}_{Y_0}$ is the pullback of $\mathcal{D}_Y$, so (2) follows from (1) since the ideals have the same generators.

The monomial part of an ideal - proof

Proof of (1), basic affine case.

- Let $\mathcal{O}_Y = \mathbb{C}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ and assume $\mathcal{M} = \mathcal{D}(\mathcal{M})$.
- The operators

$$1, u_1 \frac{\partial}{\partial u_1}, \ldots, u_l \frac{\partial}{\partial u_l}$$

  commute and have distinct systems of eigenvalues on the eigenspaces $u\,\mathbb{C}[x_1, \ldots, x_n]$, for distinct monomials u.
- Therefore $\mathcal{M} = \oplus u\mathcal{M}_u$ with ideals $\mathcal{M}_u \subset \mathbb{C}[x_1, \ldots, x_n]$ stable under derivatives,
- so each $\mathcal{M}_u$ is either (0) or (1).
- In other words, $\mathcal{M}$ is monomial.

♠

The general case requires more commutative algebra.

$\dim B = 0$: sketch of argument

- In characteristic 0, if $\operatorname{logord}_p(\mathcal{I}) = a < \infty$, then $\mathcal{D}^{\leq a-1}\mathcal{I}$ contains an element x with derivative 1, a maximal contact element.
- Carefully applying induction on dimension to an ideal on $\{x = 0\}$ gives order reduction (Encinas–Villamayor, Bierstone–Milman, Włodarczyk):

Proposition (…ℵ-T-W)

Let $\mathcal{I}$ be an ideal on a logarithmically smooth Y with

$$\max_p \operatorname{logord}_p(\mathcal{I}) = a.$$

There is a functorial logarithmic morphism $Y_1 \to Y$, with Y_1 logarithmically smooth, such that $\mathcal{I}\mathcal{O}_{Y'} = \mathcal{M} \cdot \mathcal{I}_1$ with $\mathcal{M}$ an invertible monomial ideal and

$$\max_p \operatorname{logord}_p(\mathcal{I}_1) < a.$$

Thank you for your attention!

Adendum 1. Arbitrary B

(Work in progress)

Main result (ℵ-T-W)

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{I} \subset \mathcal{O}_Y$ an ideal. There is a log morphism $B' \to B$ and functorial log morphism $Y' \to Y$, with $Y' \to B'$ logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

- This is done by relative order reduction, using relative logarithmic derivatives.

Definition

Write $\mathcal{D}^{\leq a}_{Y/B}$ for the sheaf of relative logarithmic differential operators of order $\leq a$. The relative logarithmic order of an ideal $\mathcal{I}$ is the minimum a such that $\mathcal{D}^{\leq a}_{Y/B}\mathcal{I} = (1)$.

Adendum 1. The new step

- $\text{relord}_p(\mathcal{I}) = \infty$ if and only if $\mathcal{M} := \mathcal{D}^{\infty}_{Y/B}\mathcal{I}$ is a nonunit ideal which is monomial along the fibers.
- Equivalently $\mathcal{M} = \mathcal{D}_{Y/B}\mathcal{M}$ is not the unit ideal.

Monomialization Theorem [ℵ-T-W]

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{M} \subset \mathcal{O}_Y$ an ideal with $\mathcal{D}_{Y/B}\mathcal{M} = \mathcal{M}$. There is a log morphism $B' \to B$ with saturated pullback $Y' \to B'$, and $\mathcal{M}\mathcal{O}_{Y'}$ a monomial ideal.

After this one can proceed as in the case "$\dim B = 0$".

Adendum 1. Proof of Monomialization, special case

Let $Y = \operatorname{Spec} \mathbb{C}[u, v] \to B = \operatorname{Spec} \mathbb{C}[w]$ with $w = uv$, and $\mathcal{M} = (f)$.

Proof in this special case.

- Every monomial is either $u^\alpha w^k$ or $v^\alpha w^k$.
- Once again the operators $1, u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}$ commute and have different eigenvalues on u^α, v^α.
- Exanding $f = \sum u^\alpha f_\alpha + \sum v^\beta f_\beta$, the condition $\mathcal{M} = \mathcal{D}_{Y/B}\mathcal{M}$ gives that only one term survives,
- say $f = u^\alpha f_\alpha$, with $f_\alpha \in \mathbb{C}[w]$.
- Blowing up (f_α) on B has the effect of making it monomial, so f becomes monomial.

♠

The general case is surprisingly subtle.

Adendum 1. In virtue of functoriality

Theorem (Temkin)

Resolution of singularities holds for excellent schemes, complex spaces, nonarchimedean spaces, p-adic spaces, formal spaces and for stacks.

- This is a consequence of resolution for varieties and schemes, functorial for smooth morphisms (submersions). Moreover
- Włodarczyk showed that if one seriously looks for a resolution functor, one is led to a resolution theorem.

Adendum 2. Order reduction: Example 1

- Consider $Y_1 = \operatorname{Spec} \mathbb{C}[u,x]$ and $D = \{u=0\}$.
- Let $\mathcal{I} = (u^2, x^2)$.
- If one blows up (u,x) the ideal is principalized:

  - on the u-chart $\operatorname{Spec} \mathbb{C}[u,x']$ with $x = x'u$ we have $\mathcal{I}\mathcal{O}_{Y_1'} = (u^2)$,
  - on the x-chart $\operatorname{Spec} \mathbb{C}[u',x]$ with $u' = xu'$ we have $\mathcal{I}\mathcal{O}_{Y'} = (x^2)$,
  - which is exceptional hence monomial.
- This is in fact the only functorial admissible blowing up.

Adendum 2. Order reduction: Example 2

- Consider $Y_2 = \operatorname{Spec} \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$.
- Example 1 is the pullback of this via $v = u^2$.
- Functoriality says: we need to blow up an ideal whose pullback is (u, x).
- This means we need to blow up $(v^{1/2}, x)$.
- What is this? What is its blowup?

Adendum 2. Kummer ideals

Definition

- A Kummer monomial is a monomial in the Kummer-étale topology of Y (like $v^{1/2}$).
- A Kummer monomial ideal is a monomial ideal in the Kummer-étale topology of Y.
- A Kummer center is the sum of a Kummer monomial ideal and the ideal of a log smooth subscheme.
- Locally $(x_1, \ldots, x_k, u_1^{1/d}, \ldots u_\ell^{1/d})$.

Adendum 2. Blowing up Kummer centers

Proposition

Let $\mathcal{J}$ be a Kummer center on a logarithmically smooth Y. There is a universal proper birational $Y' \to Y$ such that Y' is logarithmically smooth and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.

Example 0

$Y = \operatorname{Spec} \mathbb{C}[v]$, with toroidal structure associated to $D = \{v = 0\}$, and $\mathcal{J} = (v^{1/2})$.

- There is no log scheme Y' satisfying the proposition.
- There is a stack $Y' = Y(\sqrt{D})$, the Cadman–Vistoli root stack, satisfying the proposition!

Adendum 2. Example 2 concluded

- Consider $Y_2 = \operatorname{Spec} \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$ and $\mathcal{J} = (v^{1/2}, x)$.
- associated blowing up $Y' \to Y_2$ with charts:
  - $Y'_x := \operatorname{Spec} \mathbb{C}[v, x, v']/(v'x^2 = v)$, where $v' = v/x^2$ (nonsingular scheme).
    - ★ Exceptional $x = 0$, now monomial.
    - ★ $\mathcal{I} = (v, x^2)$ transformed into (x^2), invertible monomial ideal.
    - ★ Kummer ideal $(v^{1/2}, x)$ transformed into monomial ideal (x).
  - The $v^{1/2}$-chart:
    - ★ stack quotient $X'_{v^{1/2}} := \left[\operatorname{Spec} \mathbb{C}[w, y]/\mu_2\right]$,
    - ★ where $y = x/w$ and $\mu_2 = \{\pm 1\}$ acts via $(w, y) \mapsto (-w, -y)$.
    - ★ Exceptional $w = 0$ (monomial).
    - ★ (v, x^2) transformed into invertible monomial ideal $(v) = (w^2)$.
    - ★ $(v^{1/2}, x)$ transformed into invertible monomial ideal (w).

Adendum 2. Proof of proposition

Let $\mathcal{J}$ be a Kummer center on a logarithmically smooth Y. There is a universal proper birational $Y' \to Y$ such that Y' is logarithmically smooth and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.

- There is a stack $\tilde{Y}$ with coarse moduli space Y such that $\tilde{\mathcal{J}} := \mathcal{J}\mathcal{O}_{\tilde{Y}}$ is an ideal.
- Let $\tilde{Y}' \to \tilde{Y}$ be the blowup of $\tilde{\mathcal{J}}$ with exceptional E.
- Let $\tilde{Y}' \to B\mathbb{G}_m$ be the classifying morphism.
- Y' is the relative coarse moduli space of $\tilde{Y}' \to Y \times B\mathbb{G}_m$.
- One shows this is independent of choices. ♠