Moduli techniques in resolution of singularities

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What is resolution of singularities?

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Theorem (Hironaka 1964)

A variety X over a field of characteristic 0 admits a resolution of singularities $X' \to X$, so that the critical locus $E \subset X'$ is a simple normal crossings divisor.[a]

[a] Codim. 1, smooth components meeting transversally - as simple as possible

Always characteristic 0 . . .

Compactifications

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Corollary (Hironaka)

A smooth quasiprojective variety X^0 has a smooth projective compactification X with $D = X \setminus X^0$ a simple normal crossings divisor.

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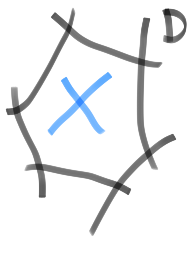

Resolution of families: $\dim B = 1$

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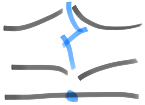

Question

What makes these special?

Log smooth schemes and log smooth morphisms

- A toric variety is a normal variety on which $T = (\mathbb{C}^*)^n$ acts algebraically with a dense free orbit.
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- Étale locally it is defined by equations between monomials.
- A morphism $X \to Y$ is toroidal or log smooth if étale locally it looks like a torus equivariant morphism of toric varieties.
- The inverse image of a monomial [1] is a monomial.

[1] defining equation of part of D_Y

Resolution of families: higher dimensional base

Question

When are the singularities of a morphism $X \to B$ simple?

The best one can hope for, after base change, is a semistable morphism:

Definition (ℵ-Karu 2000)

A log smooth morphism, with B smooth, is *semistable* if locally it is a product of one-parameter semistable families.

$$\begin{aligned} t_1 &= x_1 \cdots x_{l_1} \\ \vdots \quad & \quad \vdots \\ t_m &= x_{l_{m-1}+1} \cdots x_{l_m} \end{aligned}$$

In particular log smooth.

Similar definition by Berkovich, all inspired by de Jong.

Semistable reduction

Theorem (Many credits to be specified)

Let $\pi : X \to B$ be a dominant morphism of varieties in characteristic 0. Let $B^\circ \subset B$ be the locus where π is smooth. There is an alteration[a] *$B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{main}$, which is trivial over B°, such that $X_1 \to B_1$ is semistable.*

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- Major early results by [KKMS 1973], [de Jong 1997].
- Wonderful results in positive and mixed characteristics by de Jong, Gabber, Illusie and Temkin.

Toroidalization and weak semistable reduction

Key results in characteristic 0:

Theorem (Toroidalization, ℵ-Karu 2000, ℵ-K-Denef 2013)

There is a modification $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{main}$ such that $X_1 \to B_1$ is log smooth and flat.

Theorem (Weak semistable reduction, ℵ-Karu 2000)

There is an alteration $B_2 \to B_1$ and a modification $X_2 \to (X_1 \times_{B_1} B_2)$, trivial over B_1°, such that $X_2 \to B_2$ is log smooth, flat, with reduced fibers.

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Theorem (Semistable reduction, Adiprasito-Liu-Temkin 2018)

There is an alteration $B_3 \to B_2$ and a modification $X_3 \to (X_2 \times_{B_2} B_3)$, trivial over B_2°, such that $X_3 \to B_3$ is semistable.

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Passing from toroidalization to weak semistable reduction to semistable reduction was a purely combinatorial question [ℵ-Karu 2000].

Applications of loose semistable reduction

This is already useful for studying families:

Theorem (Karu 2000; K-SB 97, Alexeev 94, BCHM 11)

The moduli space of stable smoothable varieties is projective[a].

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Theorem (Fujino 2017)

Nakayama's numerical logarithmic Kodaira dimension is subadditive in families $X \to B$ with generic fiber F:

$$\kappa_\sigma(X, D_X) \geq \kappa_\sigma(F, D_F) + \kappa_\sigma(B, D_B).$$

Main result

The following result is work-in-progress.

Main result (Functorial toroidalization, $\aleph$-Temkin-Włodarczyk)

Let $X \to B$ be a dominant morphism.

- There are modifications $B_1 \to B$ and $X_1 \to (X \times_B B_1)_{\text{main}}$ such that $X_1 \to B_1$ is log smooth and flat;
- this is compatible with base change $B' \to B$;
- this is functorial, up to base change, with log smooth $X'' \to X$.

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Application:

Theorem (Deng 2018)

The moduli space of minimal complex projective manifolds of general type is Kobayashi hyperbolic.

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- One reduces to principalization of $\mathcal{I}_X$ (Hironaka, Villamayor, Bierstone–Milman).

Theorem (Principalization ...$\aleph$-T-W)

Let $\mathcal{I}$ be an ideal on a log smooth Y. There is a functorial logarithmic morphism $Y' \to Y$, with Y' logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

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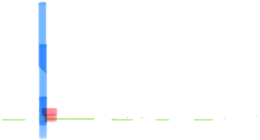

Figure: The ideal (u^2, x^2)
Here u is a monomial but x is not.

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Figure: The ideal (u^2, x^2) and the result of blowing up the origin, $\mathcal{I}_E^2$. Here u is a monomial but x is not.

Logarithmic order

Principalization is done by order reduction, using logarithmic derivatives.

- for a monomial u we use $u\frac{\partial}{\partial u}$.
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$\operatorname{logord}_p(v + u) = \infty$ since $\mathcal{D}^{\leq 1}\mathcal{I} = \mathcal{D}^{\leq 2}\mathcal{I} = \cdots = (u, v)$.

dim $B = 0$: sketch of argument, logord $< \infty$

- In characteristic 0, if $\mathrm{logord}_p(\mathcal{I}) = a < \infty$, then $\mathcal{D}^{\leq a-1}\mathcal{I}$ contains an element x with derivative 1, a maximal contact element.

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There is a functorial logarithmic morphism $Y_1 \to Y$, with Y_1 logarithmically smooth, such that $\mathcal{I}\mathcal{O}_{Y'} = \mathcal{M} \cdot \mathcal{I}_1$ with $\mathcal{M}$ an invertible monomial ideal and

$$\max_p \operatorname{logord}_p(\mathcal{I}_1) < a.$$

Order reduction: Example 1

- Consider $Y_1 = \operatorname{Spec} \mathbb{C}[u, x]$ and $D = \{u = 0\}$.
- Let $\mathcal{I} = (u^2, x^2)$.
- If one blows up (u, x) the ideal is principalized:

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- This is in fact the only functorial admissible blowing up.

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- What is this? What is its blowup?

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- A Kummer monomial ideal is a monomial ideal in the Kummer-étale topology of Y.
- A Kummer center is the sum of a Kummer monomial ideal and the ideal of a log smooth subscheme.
- Locally $(x_1, \ldots, x_k, u_1^{1/d}, \ldots u_\ell^{1/d})$.

Blowing up Kummer centers

Proposition

Let $\mathcal{J}$ be a Kummer center on a logarithmically smooth Y. There is a universal proper birational $Y' \to Y$ such that Y' is logarithmically smooth and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.

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- There is no log scheme Y' satisfying the proposition.

Blowing up Kummer centers

Proposition

Let $\mathcal{J}$ be a Kummer center on a logarithmically smooth Y. There is a universal proper birational $Y' \to Y$ such that Y' is logarithmically smooth and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.

Example 0

$Y = \operatorname{Spec} \mathbb{C}[v]$, with toroidal structure associated to $D = \{v = 0\}$, and $\mathcal{J} = (v^{1/2})$.

- There is no log scheme Y' satisfying the proposition.
- There is a stack $Y' = Y(\sqrt{D})$, the Cadman–Vistoli root stack, satisfying the proposition!

Example 2 concluded

- Consider $Y_2 = \operatorname{Spec} \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$ and $\mathcal{J} = (v^{1/2}, x)$.

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- Let $\mathcal{I} = (v, x^2)$ and $\mathcal{J} = (v^{1/2}, x)$.
- associated blowing up $Y' \to Y_2$ with charts:
  - $Y'_x := \operatorname{Spec} \mathbb{C}[v, x, v']/(v'x^2 = v)$, where $v' = v/x^2$ (nonsingular scheme).
    - Exceptional $x = 0$, now monomial.
    - $\mathcal{I} = (v, x^2)$ transformed into (x^2), invertible monomial ideal.
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  - The $v^{1/2}$-chart:
    - ⋆ stack quotient $X'_{v^{1/2}} := \left[\operatorname{Spec} \mathbb{C}[w, y]/\mu_2\right]$,
    - ⋆ where $y = x/w$ and $\mu_2 = \{\pm 1\}$ acts via $(w, y) \mapsto (-w, -y)$.
    - ⋆ Exceptional $w = 0$ (monomial).
    - ⋆ (v, x^2) transformed into invertible monomial ideal $(v) = (w^2)$.
    - ⋆ $(v^{1/2}, x)$ transformed into invertible monomial ideal (w).

Proof of proposition

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- One shows this is independent of choices. ♠

Key new ingredient: The monomial part of an ideal

Definition

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(1) *In characteristic 0,* $\mathcal{M}(\mathcal{I}) = \mathcal{D}^\infty(\mathcal{I})$. *In particular* $\max_p \operatorname{logord}_p(\mathcal{I}) = \infty$ *if and only if* $\mathcal{M}(\mathcal{I}) \neq 1$.

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(1)⇒(2)

$\mathcal{D}_{Y_0}$ is the pullback of $\mathcal{D}_Y$, so (2) follows from (1) since the ideals have the same generators.

The monomial part of an ideal - proof

Proof of (1), basic affine case.

- Let $\mathcal{O}_Y = \mathbb{C}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ and assume $\mathcal{M} = \mathcal{D}(\mathcal{M})$.

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$$1, u_1 \frac{\partial}{\partial u_1}, \ldots, u_l \frac{\partial}{\partial u_l}$$

commute and have distinct systems of eigenvalues on the eigenspaces $u\,\mathbb{C}[x_1, \ldots, x_n]$, for distinct monomials u.

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The general case requires more commutative algebra.

Arbitrary B

(Work in progress)

Main result (ℵ-T-W)

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{I} \subset \mathcal{O}_Y$ an ideal. There is a log morphism $B' \to B$ and functorial log morphism $Y' \to Y$, with $Y' \to B'$ logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

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Definition

Write $\mathcal{D}^{\leq a}_{Y/B}$ for the sheaf of relative logarithmic differential operators of order $\leq a$. The relative logarithmic order of an ideal $\mathcal{I}$ is the minimum a such that $\mathcal{D}^{\leq a}_{Y/B}\mathcal{I} = (1)$.

The new step

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Monomialization Theorem [ℵ-T-W]

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{M} \subset \mathcal{O}_Y$ an ideal with $\mathcal{D}_{Y/B}\mathcal{M} = \mathcal{M}$. There is a log morphism $B' \to B$ with saturated pullback $Y' \to B'$, such that $\mathcal{M}\mathcal{O}_{Y'}$ a monomial ideal.

After this one can proceed as in the case "dim $B = 0$".

Proof of Monomialization Theorem, special case

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The general case is surprisingly subtle.

Thank you for your attention!