Resolving singularities in families

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Joint work with Michael Temkin and Jarosław Włodarczyk

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Resolution of singularities

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Theorem (Hironaka 1964)

A variety X over a field of characteristic 0 admits a resolution of singularities $X' \to X$, so that the exceptional locus $E \subset X'$ is a simple normal crossings divisor.[a]

[a] Codimension 1, smooth components meeting transversally

Always characteristic 0 ...

Resolution of families: $\dim B = 1$

Question

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Resolution of families: dim $B = 1$

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- If $\dim B = 1$ the simplest one can have by modifying X is $t = \prod x_i^{a_i}$,
- and if one also allows base change, can have $t = \prod x_i$.
  [Kempf–Knudsen–Mumford–Saint-Donat 1973]

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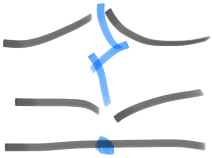

Question

What makes these special?

Log smooth schemes and log smooth morphisms

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- Étale locally it is defined by equations between monomials.
- A morphism $X \to Y$ is toroidal or log smooth if étale locally it looks like a torus equivariant morphism of toric varieties.
- The inverse image of a monomial is a monomial.

Resolution of families: higher dimensional base

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The best one can hope for, after base change, is a semistable morphism, locally a product of stemistable one-parameter families:

Definition (ℵ-Karu 2000)

A log smooth morphism, with B smooth, is *semistable* if locally

$$\begin{aligned} t_1 &= x_1 \cdots x_{l_1} \\ \vdots\ &\quad \vdots \\ t_m &= x_{l_{m-1}+1} \cdots x_{l_m} \end{aligned}$$

In particular log smooth.

Similar definition by Berkovich, all following de Jong.

The semistable reduction problem

Conjecture [ℵ-Karu]

Let $X \to B$ be a dominant morphism of varieties.

- (Loose) There is an alteration $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{\text{main}}$ such that $X_1 \to B_1$ is semistable.

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- Major early results by [KKMS 1973], [de Jong 1997].
- Wonderful results in positive and mixed characteristics by de Jong, Gabber, Illusie and Temkin.

Toroidalization and weak semistable reduction

Back to characteristic 0

Theorem (Toroidalization, ℵ-Karu 2000, ℵ-K-Denef 2013)

There is a modification $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{main}$ such that $X_1 \to B_1$ is log smooth and flat.

Theorem (Weak semistable reduction, ℵ-Karu 2000)

There is an alteration $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{main}$ such that $X_1 \to B_1$ is log smooth, flat, with reduced fibers.

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- proven by [Karu 2000] for families of surfaces and threefolds, and
- whose restriction to rank-1 valuation rings is proven in a preprint by [Karim Adiprasito - Gaku Liu - Igor Pak - Michael Temkin].

Application of weak semistable reduction

(with a whole lot of more input)

Theorem (Viehweg-Zuo 2004)

The moduli space of canonically polarized manifolds is Brody hyperbolic.

Main result

The following result is work-in-progress.

Main result (Functorial toroidalization, ℵ-Temkin-Włodarczyk)

Let $X \to B$ be a dominant log morphism.

- There are log modifications $B_1 \to B$ and $X_1 \to (X \times_B B_1)_{\text{main}}$ such that $X_1 \to B_1$ is log smooth and flat;
- this is compatible with log base change $B' \to B$;
- this is functorial, up to base change, with log smooth $X'' \to X$.

This implies the tight version of the results of semistable reduction type.

Current application of our main result

Theorem (Deng 2018)

The moduli space of minimal complex projective manifolds of general type is Kobayashi hyperbolic.

dim $B = 0$: log resolution via principalization

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Theorem (Principalization ... ℵ-T-W)

Let $\mathcal{I}$ be an ideal on a log smooth Y. There is a functorial logarithmic morphism $Y' \to Y$, with Y' logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

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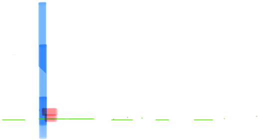

Figure: The ideal (u^2, x^2)
Here u is a monomial but x is not.

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Figure: The ideal (u^2, x^2) and the result of blowing up the origin, $\mathcal{I}_E^2$. Here u is a monomial but x is not.

Logarithmic order

Principalization is done by order reduction, using logarithmic derivatives.

- for a monomial u we use $u\frac{\partial}{\partial u}$.
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$\operatorname{logord}_p(v + u) = \infty$ since $\mathcal{D}^{\leq 1}\mathcal{I} = \mathcal{D}^{\leq 2}\mathcal{I} = \cdots = (u, v)$.

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The monomial part of an ideal - proof

Proof of (1), basic affine case.

- Let $\mathcal{O}_Y = \mathbb{C}[x_1, \ldots, x_n, u_1, \ldots, u_m]$ and assume $\mathcal{M} = \mathcal{D}(\mathcal{M})$.

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$$1, u_1 \frac{\partial}{\partial u_1}, \ldots, u_l \frac{\partial}{\partial u_l}$$

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The general case requires more commutative algebra.

Arbitrary B

(Work in progress)

Main result (ℵ-T-W)

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{I} \subset \mathcal{O}_Y$ an ideal. There is a log morphism $B' \to B$ and functorial log morphism $Y' \to Y$, with $Y' \to B'$ logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

- This is done by relative order reduction, using relative logarithmic derivatives.

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Monomialization Theorem [ℵ-T-W]

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{M} \subset \mathcal{O}_Y$ an ideal with $\mathcal{D}_{Y/B}\mathcal{M} = \mathcal{M}$. There is a log morphism $B' \to B$ with saturated pullback $Y' \to B'$, such that $\mathcal{M}\mathcal{O}_{Y'}$ a monomial ideal.

After this one can proceed as in the case “dim $B = 0$”.

Proof of Monomialization Theorem, special case

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Proof in this special case.

- Every monomial is either $u^\alpha w^k$ or $v^\alpha w^k$.

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The general case is surprisingly subtle.

Order reduction: Example 1

- Consider $Y_1 = \operatorname{Spec} \mathbb{C}[u,x]$ and $D = \{u = 0\}$.
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  - which is exceptional hence monomial.
- This is in fact the only functorial admissible blowing up.

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- What is this? What is its blowup?

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- A Kummer center is the sum of a Kummer monomial ideal and the ideal of a log smooth subscheme.
- Locally $(x_1, \ldots, x_k, u_1^{1/d}, \ldots u_\ell^{1/d})$.

Blowing up Kummer centers

Proposition

Let $\mathcal{J}$ be a Kummer center on a logarithmically smooth Y. There is a universal proper birational $Y' \to Y$ such that Y' is logarithmically smooth and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.

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- There is no log scheme Y' satisfying the proposition.
- There is a stack $Y' = Y(\sqrt{D})$, the Cadman–Vistoli root stack, satisfying the proposition!

Example 2 concluded

- Consider $Y_2 = \operatorname{Spec} \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
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- Consider $Y_2 = \operatorname{Spec} \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$ and $\mathcal{J} = (v^{1/2}, x)$.
- associated blowing up $Y' \to Y_2$ with charts:
  - $Y'_x := \operatorname{Spec} \mathbb{C}[v, x, v']/(v'x^2 = v)$, where $v' = v/x^2$ (nonsingular scheme).
    - ⋆ Exceptional $x = 0$, now monomial.
    - ⋆ $\mathcal{I} = (v, x^2)$ transformed into (x^2), invertible monomial ideal.
    - ⋆ Kummer ideal $(v^{1/2}, x)$ transformed into monomial ideal (x).

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  - The $v^{1/2}$-chart:
    - ★ stack quotient $X'_{v^{1/2}} := \left[\operatorname{Spec} \mathbb{C}[w, y]/\mu_2\right]$,
    - ★ where $y = x/w$ and $\mu_2 = \{\pm 1\}$ acts via $(w, y) \mapsto (-w, -y)$.
    - ★ Exceptional $w = 0$ (monomial).
    - ★ (v, x^2) transformed into invertible monomial ideal $(v) = (w^2)$.
    - ★ $(v^{1/2}, x)$ transformed into invertible monomial ideal (w).

Proof of proposition

Let $\mathcal{J}$ be a Kummer center on a logarithmically smooth Y. There is a universal proper birational $Y' \to Y$ such that Y' is a logarithmically smooth stack and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.

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- One shows this is independent of choices. ♠