Semistable reduction - a progress report

Dan Abramovich
Brown University
Joint work with Michael Tēmkin and Jarosław Włodarczyk

Moduli and Hodge Theory
IMSA, Miami
Outline

- Statement of two results [NTW], [ALT],
- The one-dimensional base case [KKMS]
- Old results and conjectures over larger base [NK]
- Relative desingularization in the age of log stacks

This lays groundwork for Témkin’s lecture tomorrow.
Relatively functorial toroidalization
($\aleph$-Tëmkin-Włodarczyk)

Theorem ($\aleph$TW p2020)

Let $X \to B$ be a dominant morphism of complex varieties. There is a relatively functorial diagram

$$
\begin{array}{c}
X' & \rightarrow & X \\
\downarrow & \downarrow & \downarrow \\
B' & \rightarrow & B
\end{array}
$$

with

- $B' \to B$ and $X' \to B'$ modifications,
- $X' \to B'$ logarithmically smooth.
- In particular,
  - if the generic fiber of $X \to B$ is smooth it is not modified, and
  - a group actions along the fibers of $X \to B$ lifts to $X'$. 
**Theorem (ALT p2018)**

Let $X \to B$ be a generically smooth complex projective family of varieties. There is a diagram

$$
\begin{array}{ccc}
X_1 & \to & X \\
\downarrow & & \downarrow \\
B_1 & \to & B
\end{array}
$$

with

- $B_1 \to B$ an alteration,
- $X_1 \to (X \times_B B_1)_{\text{main}}$ a modification of the main part
- \ldots which is an isomorphism on the generic fiber,
- and such that $X_1 \to B_1$ is **semistable**.
Family resolution

- Both theorems answer the question
  \[ \text{how well can one resolve a family } X \to B \text{ of complex varieties,} \]
  with different notions of what you allow to do to $B$ and $X$, and what you hope to get in the resulting $X' \to B'$.
- The case $\dim B = 0$ is Hironaka’s resolution of singularities.
- The case $\dim B = 1$ is [KKMS]

**Definition**

A morphism $X_1 \to B_1$ of smooth complex varieties with $\dim B_1 = 1$ is **semistable** if in local coordinates it is given by

\[ t = x_1 \cdots x_k. \]
The one-dimensional base case, and what Carlos said

**Theorem (Knudsen-Mumford-Waterman 1973)**

*Given any family $X \to B$ with dim $B = 1$ there is*

$$
\begin{array}{c}
X_1 \to X \\
\downarrow \quad \downarrow \\
B_1 \to B
\end{array}
$$

*with*

- $B_1 \to B$ an alteration and
- $X_1 \to X \times_B B_1$ modification,
- such that $X_1 \to B_1$ is semistable.
The one-dimensional base case, and what Carlos said

Theorem (Knudsen-Mumford-Waterman 1973)

Given any family $X \to B$ with $\dim B = 1$ there is

$$
\begin{array}{c}
X_1 \rightarrow X \\
\downarrow \quad \downarrow \\
B_1 \rightarrow B
\end{array}
$$

with

- $B_1 \to B$ an alteration and
- $X_1 \to X \times_B B_1$ modification,
- such that $X_1 \to B_1$ is semistable.

This gives geometric justification for good Hodge theoretic behavior: given $X \to B$ arbitrary, after base change $B_1 \to B$ and modification $X_1$, the family $X_1 \to B_1$ has unipotent monodromy at every generic point the discriminant $\Delta(X/B) \subset B$. 
What Mumford said

VII

the Borel-Baily "minimal" compactification). We would also like to study semi-stable reduction over a higher dimensional base: viz., given any dominating morphism $f: X \to Y$, replacing $Y$ by any $Y'$ generically finite and proper over $Y$ and $X$ by a blow-up of the component of $X \times_Y Y'$ dominating $Y'$, simplify all the fibres $f': X' \to Y'$ as much as possible while requiring that $X', Y'$ are non-singular and $f$ is flat.
A toroidal embedding $U \subset X$ is an open embedding étale locally isomorphic to the embedding of a torus in a toric variety.

It is the same as a log structure on $X$ log smooth over Spec $k$.

A toroidal morphism between toroidal embeddings $U_i \subset X_i$ is a morphism $X_1 \to X_2$ that is étale locally the pullback of a dominant toric morphism.

It is the same as a log smooth morphism between log smooth schemes.

It is characterized by the fact that the pullback of a monomial is a monomial, and is smooth otherwise.

Once you are log smooth, everything is combinatorial.
Weak toroidalization

The first step is

**Theorem (N-Karu 2000)**

Let $X \to B$ be a dominant morphism of complex varieties. There is a diagram

$\begin{align*}
U_X & \to X' \to X \\
\downarrow & \downarrow & \downarrow \\
U_B & \to B' \to B
\end{align*}$

with $B' \to B$ and $X' \to B'$ modifications, and $X' \to B'$ logarithmically smooth / toroidal.

- The proof used de Jong’s alterations, so could not be made functorial. The generic fiber was modified.
Theorem (Altered semistable reduction, de Jong 1997)

Let $X \rightarrow B$ be a generically smooth complex projective family of varieties. There is a finite group $G$ and a $G$-equivariant diagram

$$
\begin{align*}
U_Y & \hookrightarrow Y \rightarrow X \\
& \downarrow \downarrow \downarrow \\
U_B & \hookrightarrow B_1 \rightarrow B
\end{align*}
$$

with

- $B_1 \rightarrow B$ and $Y \rightarrow X$ alterations,
- $Y/G \rightarrow X$ and $B_1/G \rightarrow B$ birational,

such that $Y \rightarrow B_1$ is semistable.

Consider $\mathcal{X} = [Y/G] \rightarrow X$ and $\mathcal{B} = [B_1/G] \rightarrow B$. 
Updated proof, Step 2 (Bergh-Rydh)

Consider $\mathcal{X} = [Y/G] \to X$ and $\mathcal{B} = [B_1/G] \to B$. $\mathcal{X} \to \mathcal{B}$ is log smooth.

Theorem (Destackification, Bergh-Rydh p2019)

There is a diagram

$$
\begin{array}{ccc}
X' & \leftarrow & \tilde{\mathcal{X}} \to \mathcal{X} \\
\downarrow & & \downarrow \\
B' & \leftarrow & \tilde{\mathcal{B}} \to \mathcal{B}
\end{array}
$$

where $\tilde{\mathcal{X}} \to \mathcal{X}$ and $\tilde{\mathcal{B}} \to \mathcal{B}$ are stack blowup sequences, $\tilde{\mathcal{X}} \to X'$ and $\tilde{\mathcal{B}} \to B'$ coarse moduli spaces, and $X' \to B'$ log smooth.

The resulting diagram

$$
\begin{array}{ccc}
U_X & \hookrightarrow & X' \to X \\
\downarrow & & \downarrow \\
U_B & \hookrightarrow & B' \to B
\end{array}
$$

finishes the proof.
Weakly semistable and semistable morphisms (K–Karu, T. Tsuji)

- A toroidal morphism $X \to B$ is **weakly semistable** if it is flat with reduced fibers.
- This is the same as an integral and saturated morphism of log structures.
- A toroidal morphism is **semistable** if moreover $X$ and $B$ are smooth.
- In local coordinates, we obtain

$$ t_1 = x_1 \cdots x_{l_1} $$

$$ \vdots $$

$$ t_m = x_{l_{m-1}+1} \cdots x_{l_m} $$

- Yes, this is the best one can get (Karu t1999).
Weak Semistable reduction ($\mathbb{N}$–Karu)

**Theorem ($\mathbb{N}$–Karu 2000)**

Let $X \to B$ be a generically smooth complex projective family of varieties. There is a diagram

\[
\begin{array}{ccc}
U_X & \hookrightarrow & X_1 \to X \\
\downarrow & & \downarrow \\
U_B & \hookrightarrow & B_1 \to B
\end{array}
\]

with

- $B_1 \to B$ an alteration,
- $X_1 \to (X \times_B B_1)_{\text{main}}$ a modification of the main part
- and such that $X_1 \to B_1$ is weakly semistable.

By weak toroidalization we may assume $X \to B$ logarithmically smooth.
Recall [KKMS] functor:

\[
\{\text{toroidal embeddings}\} \xrightarrow{X \mapsto \Sigma_X} \{\text{R.P. cone complexes}\}.
\]

It restricts to an equivalence:

\[
\{\text{representable tor. modifications}\} \leftarrow \rightarrow \{\text{subdivisions}\}.
\]

**Theorem (Molcho p2016)**

The functor $\Sigma$ restricts to an equivalence

\[
\{\text{stack toroidal modifications}\} \leftarrow \rightarrow \{\text{lattice altered subdivisions}\}.
\]
Proposition (Molcho p2016)

\( f : X \to B \) is semistable if and only if \( \Sigma(f) : \Sigma_X \to \Sigma_B \) satisfies

- for all \( \sigma \in \Sigma_X \), the image \( \Sigma(f)(\sigma) \) is a cone of \( \Sigma_B \),
- for all \( \sigma \in \Sigma_X \), the image \( \Sigma(f)(N_\sigma) = N_{\Sigma(f)(\sigma)} \).

By the theorem, there is a stack theoretic modification \( \mathcal{B} \to B \) such that the toroidal pullback \( \mathcal{X} \to \mathcal{B} \) is a representable semistable morphism.

Using Kawamata’s trick one replaces \( \mathcal{B} \) by a scheme alteration.
Beyond weak semistable reduction

- In [N–Karu 2000] we conjectured that weakly semistable can be replaced by semistable,
- and reduced the problem to polyhedral combinatorics,
- as recently proved by Adiprasito, Liu and Tämkin.
- This is in parallel to the Knudsen–Mumford–Waterman result.

- We also conjectured that toroidalization can be done more functorially.
- To tell the story we need to go one step back.
Varieties and log structures (K. Kato, Fontaine, Illusie)

- A variety is locally embedded in a smooth variety.
- A log variety is something locally embedded in a toroidal variety.
- The toroidal $U \subset X$ is encoded in the multiplicative submonoid $M_X \subset \mathcal{O}_X$ of functions invertible on $U$.
- In general a log structure $M \to \mathcal{O}_Y$ is a morphism of sheaves of monoids inducing an isomorphism on $\mathcal{O}_Y^\times$.
- A key example is a point on a toric variety.
Resolution and log resolution

- By Hironaka, a variety can be canonically resolved.
- Włodarczyk showed the benefits of **functorial** resolution: if the procedure is functorial for **smooth** morphisms, then gluing and descent is automatic.
- A morphism $X \to B$ has little chance of having a smooth resolution.
- Toroidalization $[\aleph K]$ is precisely **log smooth** resolution.
- To make it functorial we turn to Hironaka–Włodarczyk methods.
Functoriality in log resolution

A logarithmically functorial resolution assigns to a log morphism $X \to B$ a modification $X' \to X$ such that

- $X' \to B$ is log smooth
- If $Y \to X$ is log smooth, with log resolution $Y' \to Y$, then $Y' = Y \times_X X'$ ...
A **logarithmically functorial resolution** assigns to a log morphism $X \to B$ a modification $X' \to X$ such that

- $X' \to B$ is log smooth
- If $Y \to X$ is log smooth, with log resolution $Y' \to Y$, then $Y' = Y \times^\log_X X'$

Either that, or it says “sorry, my friend, please modify $B$ first”.
Functoriality in log resolution

A logarithmically functorial resolution assigns to a log morphism $X \to B$ a modification $X' \to X$ such that

- $X' \to B$ is log smooth
- If $Y \to X$ is log smooth, with log resolution $Y' \to Y$, then $Y' = Y \times_X^{\log} X'$ ...

Either that, or it says “sorry, my friend, please modify $B$ first”. It is a relatively log functorial resolution if moreover

- A good base change $B' \to B$ always exists, and
- $X' \to X$ commutes with base change.
Functoriality in log resolution

A **logarithmically functorial resolution** assigns to a log morphism $X \to B$ a modification $X' \to X$ such that

- $X' \to B$ is log smooth
- If $Y \to X$ is log smooth, with log resolution $Y' \to Y$, then $Y' = Y \times_X^\text{log} X'$...

Either that, or it says “sorry, my friend, please modify $B$ first”.

It is a **relatively log functorial resolution** if moreover

- A good base change $B' \to B$ always exists, and
- $X' \to X$ commutes with base change.

The theorem of \[\mathbb{TW 2020}\] says that a relatively log functorial resolution exists, with the caveat of the next slides.

(We have a draft of a result showing $B' \to B$ can be made functorial when $X \to B$ proper.)
Example: log modification of $B$

- Consider $X = \mathbb{A}^1_{\log} \to B = \mathbb{A}^1$.
- It is not log smooth.
- Modify $B' = \mathbb{A}^1_{\log}$,
- then the log pullback $X \to B'$ is log smooth.
Example: log resolution 1

- Consider $B = \mathbb{A}_\log^1$, $Y = \mathbb{A}_\log^1 \times \mathbb{A}^1$ and $X = V((x - y)(x + y) = V(x^2 - y^2)$.

- To resolve $X$, we blow up the origin $(x, y)$ on $Y$, including the exceptional in the log structure.

- the log proper transform $X' \to B'$ is log smooth.
Example: log resolution 2

- Consider $B = \mathbb{A}^1_{\text{log}}, Y = \mathbb{A}^1_{\text{log}} \times \mathbb{A}^1$ and $X = V(x_1 - y^2)$.
- Its pullback via $x_1 = x^2$ is example 1.
- By log functoriality we must blow up something whose pullback is $(x, y)$.
- In other words, we must blow up $(\sqrt{x_1}, y)$.
- This is a Kummer blow up, whose result is a stack theoretic blowup.
- the log stack proper transform $\mathcal{X}' \rightarrow B'$ is log smooth.
Example 2 computation

- We blow up \((\sqrt{x_1}, y)\):
- Consider \(x_1 = U^2x_1', y = Uy'\),
- with \(\mathbb{G}_m\) action \((x_1', y', U) \mapsto (t^2x_1', ty', t^{-1}U)\).
- The map \(\text{Spec } k[x_1', y', U] \to \text{Spec } k[x_1, y]\) is \(\mathbb{G}_m\)-equivariant,
- leaving \(Z := V(x_1', y')\) invariant.
- Write \(\mathcal{X}' = [(\text{Spec } k[x_1', y', U] \setminus Z)/\mathbb{G}_m].\)
Example 2 computation

- We blow up \((\sqrt{x_1}, y)\):
- Consider \(x_1 = U^2x'_1, \ y = Uy'\),
- with \(\mathbb{G}_m\) action \((x'_1, y', U) \mapsto (t^2x'_1, ty', t^{-1}U)\).
- The map \(\text{Spec } k[x'_1, y', U] \to \text{Spec } k[x_1, y]\) is \(\mathbb{G}_m\)-equivariant,
- leaving \(Z := V(x'_1, y')\) invariant.
- Write \(\mathcal{X}' = [(\text{Spec } k[x'_1, y', U] \setminus Z)/\mathbb{G}_m]\).
- The equation \(x_1 - y^2\) becomes \(U^2(x'_1 - y'^2)\),
- and the proper transform \((x'_1 - y'^2)\) is indeed log smooth over \(\mathbb{A}_\log^1\).
Lesson learned

- So log smooth functoriality requires log stacks.
- With Bergh’s destackification, we get a schematic log resolution as in the theorem,
- which is functorial only for smooth $Y \to X$.
- more to come tomorrow.

Also, Hodge theorists,
- One can have, with good reason, monodromy unipotent everywhere,
- with very nice local equations everywhere,
- and functoriality properties.
The end

Thank you for your attention!