Semistable reduction - a progress report

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Outline

- Statement of two results [ℵTW], [ALT],
- The one-dimensional base case [KKMS]
- Old results and conjectures over larger base [ℵK]
- Relative desingularization in the age of log stacks

This lays groundwork for Tëmkin's lecture tomorrow.

Relatively functorial toroidalization (ℵ-Tëmkin-Włodarczyk)

Theorem (ℵTW p2020)

Let $X \to B$ be a dominant morphism of complex varieties. There is a ***relatively functorial*** *diagram*

$$\begin{array}{ccc} X' & \to & X \\ \downarrow & & \downarrow \\ B' & \to & B \end{array}$$

with

- *$B' \to B$ and $X' \to B'$ modifications,*
- *$X' \to B'$* ***logarithmically smooth****.*
- *In particular,*
  - *if the generic fiber of $X \to B$ is smooth it is not modified, and*
  - *a group actions along the fibers of $X \to B$ lifts to X'.*

Semistable reduction (Adiprasito-Liu-Tëmkin)

Theorem (ALT p2018)

Let $X \to B$ be a generically smooth complex projective family of varieties. There is a diagram

$$\begin{array}{ccc} X_1 & \to & X \\ \downarrow & & \downarrow \\ B_1 & \to & B \end{array}$$

with

- $B_1 \to B$ *an alteration,*
- $X_1 \to (X \times_B B_1)_{main}$ *a modification of the main part*
- *. . . which is an isomorphism on the generic fiber,*
- *and such that* $X_1 \to B_1$ *is semistable.*

Family resolution

- Both theorems answer the question

  how well can one resolve a family $X \to B$ of complex varieties,

  with different notions of what you allow to do to B and X, and what you hope to get in the resulting $X' \to B'$.
- The case $\dim B = 0$ is Hironaka's resolution of singularities.
- The case $\dim B = 1$ is [KKMS]

Definition

A morphism $X_1 \to B_1$ of smooth complex varieties with $\dim B_1 = 1$ is semistable if in local coordinates it is given by

$$t = x_1 \cdots x_k.$$

The one-dimensional base case, and what Carlos said

Theorem (Knudsen-Mumford-Waterman 1973)

Given any family $X \to B$ with $\dim B = 1$ there is

$$\begin{array}{ccc} X_1 & \to & X \\ \downarrow & & \downarrow \\ B_1 & \to & B \end{array}$$

with

- *$B_1 \to B$ an alteration and*
- *$X_1 \to X \times_B B_1$ modification,*
- *such that $X_1 \to B_1$ is semistable.*

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This gives geometric justification for good Hodge theoretic behavior: given $X \to B$ arbitrary, after base change $B_1 \to B$ and modification X_1, the family $X_1 \to B_1$ has unipotent monodromy at every generic point the discriminant $\Delta(X/B) \subset B$.

What Mumford said

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the Borel-Baily "minimal" compactification). We would also like to study semi-stable reduction over a higher dimensional base: viz., given any dominating morphism $f: X \longrightarrow Y$, replacing Y by any Y' generically finite and proper over Y and X by a blow-up of the component of $X \times_Y Y'$ dominating Y', simplify all the fibres $f': X' \longrightarrow Y'$ as much as possible while requiring that X', Y' are non-singular and f is flat.

Toroidal morphisms, log smooth morphisms (KKMS, K. Kato, ℵ–Karu)

- A toroidal embedding $U \subset X$ is an open embedding étale locally isomorphic to the embedding of a torus in a toric variety.
- It is the same as a log structure on X log smooth over $\operatorname{Spec} k$.
- A toroidal morphism between toroidal embeddings $U_i \subset X_i$ is a morphism $X_1 \to X_2$ that is étale locally the pullback of a dominant toric morphism.
- It is the same as a log smooth morphism between log smooth schemes.
- It is characterized by the fact that the pullback of a monomial is a monomial, and is smooth otherwise.
- Once you are log smooth, everything is combinatorial.

Weak toroidalization

The first step is

Theorem (ℵ-Karu 2000)

Let $X \to B$ be a dominant morphism of complex varieties. There is a diagram

$$\begin{array}{ccccc} U_X & \hookrightarrow & X' & \to & X \\ \downarrow & & \downarrow & & \downarrow \\ U_B & \hookrightarrow & B' & \to & B \end{array}$$

with $B' \to B$ and $X' \to B'$ modifications, and $X' \to B'$ logarithmically smooth / toroidal.

- The proof used de Jong's alterations, so could not be made functorial. The generic fiber was modified.

Updated proof, Step 1 (de Jong)

Theorem (Altered semistable reduction, de Jong 1997)

Let $X \to B$ be a generically smooth complex projective family of varieties. There is a finite group G and a G-equivariant diagram

$$\begin{array}{ccccc} U_Y & \hookrightarrow & Y & \to & X \\ \downarrow & & \downarrow & & \downarrow \\ U_B & \hookrightarrow & B_1 & \to & B \end{array}$$

with

- *$B_1 \to B$ and $Y \to X$ alterations,*
- *$Y/G \to X$ and $B_1/G \to B$ birational,*

such that $Y \to B_1$ is semistable.

Consider $\mathcal{X} = [Y/G] \to X$ and $\mathcal{B} = [B_1/G] \to B$.

Updated proof, Step 2 (Bergh-Rydh)

Consider $\mathcal{X} = [Y/G] \to X$ and $\mathcal{B} = [B_1/G] \to B$. $\mathcal{X} \to \mathcal{B}$ is log smooth.

Theorem (Destackification, Bergh-Rydh p2019)

There is a diagram

$$\begin{array}{ccccc} X' & \leftarrow & \tilde{\mathcal{X}} & \rightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ B' & \leftarrow & \tilde{\mathcal{B}} & \rightarrow & \mathcal{B} \end{array}$$

where $\tilde{\mathcal{X}} \to \mathcal{X}$ and $\tilde{\mathcal{B}} \to \mathcal{B}$ are stack blowup sequences, $\tilde{\mathcal{X}} \to X'$ and $\tilde{\mathcal{B}} \to B'$ coarse moduli spaces, and $X' \to B'$ log smooth.

The resulting diagram

$$\begin{array}{ccccc} U_X & \hookrightarrow & X' & \rightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ U_B & \hookrightarrow & B' & \rightarrow & B \end{array}$$

finishes the proof.

Weakly semistable and semistable morphisms (ℵ–Karu, T. Tsuji)

- A toroidal morphism $X \to B$ is weakly semistable if it is flat with reduced fibers.
- This is the same as an integral and saturated morphism of log structures.
- A toroidal morphism is semistable if moreover X and B are smooth.
- In local coordinates, we obtain

$$\begin{aligned} t_1 &= x_1 \cdots x_{l_1} \\ \vdots &\quad \vdots \\ t_m &= x_{l_{m-1}+1} \cdots x_{l_m} \end{aligned}$$

- Yes, this is the best one can get (Karu t1999).

Weak Semistable reduction (ℵ–Karu)

Theorem (ℵ–Karu 2000)

Let $X \to B$ be a generically smooth complex projective family of varieties. There is a diagram

$$\begin{array}{ccccc} U_X & \hookrightarrow & X_1 & \to & X \\ \downarrow & & \downarrow & & \downarrow \\ U_B & \hookrightarrow & B_1 & \to & B \end{array}$$

with

- $B_1 \to B$ *an alteration,*
- $X_1 \to (X \times_B B_1)_{main}$ *a modification of the main part*
- *and such that* $X_1 \to B_1$ *is weakly semistable.*

By weak toroidalization we may assume $X \to B$ logarithmically smooth.

Updated functorial proof (Molcho) part 1

Recall [KKMS] functor:

$$\{\text{toroidal embeddings}\} \xrightarrow{X \mapsto \Sigma_X} \{\text{R.P. cone complexes}\}.$$

It restricts to an equivalence:

$$\{\text{representable tor. modifications}\} \longleftrightarrow \{\text{subdivisions}\}.$$

Theorem (Molcho p2016)

The functor Σ restricts to an equivalence

$$\{\textit{stack toroidal modifications}\} \longleftrightarrow \{\textit{lattice altered subdivisions}\}.$$

Updated functorial proof (Molcho) part 2

Proposition (Molcho p2016)

$f : X \to B$ is semistable if and only if $\Sigma(f) : \Sigma_X \to \Sigma_B$ satisfies

- *for all $\sigma \in \Sigma_X$, the image $\Sigma(f)(\sigma)$ is a cone of Σ_B,*
- *for all $\sigma \in \Sigma_X$, the image $\Sigma(f)(N_\sigma) = N_{\Sigma(f)(\sigma)}$.*

- By the theorem, there is a stack theoretic modification $\mathcal{B} \to B$ such that the toroidal pullback $\mathcal{X} \to \mathcal{B}$ is a representable semistable morphism.
- Using Kawamata's trick one replaces $\mathcal{B}$ by a scheme alteration.

Beyond weak semistable reduction

- In [ℵ–Karu 2000] we conjectured that weakly semistable can be replaced by semistable,
- and reduced the problem to polyhedral combinatorics,
- as recently proved by Adiprasito, Liu and Tëmkin.
- This is in parallel to the Knudsen–Mumford–Waterman result.

- We also conjectured that toroidalization can be done more functorially.
- To tell the story we need to go one step back.

Varieties and log structures (K. Kato, Fontaine, Illusie)

- A variety is locally embedded in a smooth variety.
- A log variety is something locally embedded in a toroidal variety.
- The toroidal $U \subset X$ is encoded in the multiplicative submonoid $M_X \subset \mathcal{O}_X$ of functions invertible on U.
- In general a log structure $M \to \mathcal{O}_Y$ is a morphism of sheaves of monoids inducing an isomorphism on $\mathcal{O}_Y^\times$.
- A key example is a point on a toric variety.

Resolution and log resolution

- By Hironaka, a variety can be canonically resolved.
- Włodarczyk showed the benefits of functorial resolution: if the procedure is functorial for smooth morphisms, then gluing and descent is automatic.
- A morphism $X \to B$ has little chance of having a smooth resolution.
- Toroidalization [ℵK] is precisely log smooth resolution.
- To make it functorial we turn to Hironaka–Włodarczyk methods.

Functoriality in log resolution

A logarithmically functorial resolution assigns to a log morphism $X \to B$ a modification $X' \to X$ such that

- $X' \to B$ is log smooth
- If $Y \to X$ is log smooth, with log resolution $Y' \to Y$, then $Y' = Y \times_X^{\log} X'$ ...

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- A good base change $B' \to B$ always exists, and
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The theorem of [ℵTW 2020] says that a relatively log functorial resolution exists, with the caveat of the next slides.
(We have a draft of a result showing $B' \to B$ can be made functorial when $X \to B$ proper.)

Example: log modification of B

- Consider $X = \mathbb{A}^1_{\text{log}} \to B = \mathbb{A}^1$.
- It is not log smooth.
- Modify $B' = \mathbb{A}^1_{\text{log}}$,
- then the log pullback $X \to B'$ is log smooth.

Example: log resolution 1

- Consider $B = \mathbb{A}^1_{\log}$, $Y = \mathbb{A}^1_{\log} \times \mathbb{A}^1$ and $X = V((x-y)(x+y) = V(x^2 - y^2)$.
- To resolve X, we blow up the origin (x, y) on Y, including the exceptional in the log structure.
- the log proper transform $X' \to B'$ is log smooth.

Example: log resolution 2

- Consider $B = \mathbb{A}^1_{\log}$, $Y = \mathbb{A}^1_{\log} \times \mathbb{A}^1$ and $X = V(x_1 - y^2)$.
- Its pullback via $x_1 = x^2$ is example 1.
- By log functoriality we must blow up something whose pullback is (x, y).
- In other words, we must blow up $(\sqrt{x_1}, y)$.
- This is a Kummer blow up, whose result is a stack theoretic blowup.
- the log stack proper transform $\mathcal{X}' \to B'$ is log smooth.

Example 2 computation

- We blow up $(\sqrt{x_1}, y)$:
- Consider $x_1 \quad = \quad U^2 x_1', \quad y \quad = \quad Uy'$,
- with $\mathbb{G}_m$ action $(x_1', y', U) \mapsto (t^2 x_1', ty', t^{-1}U)$.
- The map $\operatorname{Spec} k[x_1', y', U] \to \operatorname{Spec} k[x_1, y]$ is $\mathbb{G}_m$-equivariant,
- leaving $Z := V(x_1', y')$ invariant.
- Write $\mathcal{X}' = [(\operatorname{Spec} k[x_1', y', U] \smallsetminus Z)/\mathbb{G}_m]$.

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- leaving $Z := V(x_1', y')$ invariant.
- Write $\mathcal{X}' = [(\operatorname{Spec} k[x_1', y', U] \smallsetminus Z)/\mathbb{G}_m]$.
- The equation $x_1 - y^2$ becomes $U^2(x_1' - {y_1'}^2)$,
- and the proper transform $(x_1' - {y_1'}^2)$ is indeed log smooth over $\mathbb{A}^1_{\log}$.

Lesson learned

- So log smooth functoriality requires log stacks.
- With Bergh's destackification, we get a schematic log resolution as in the theorem,
- which is functorial only for smooth $Y \to X$.
- more to come tomorrow.

Also, Hodge theorists,

- One can have, with good reason, monodromy unipotent everywhere,
- with very nice local equations everywhere,
- and functoriality properties.

The end

Thank you for your attention!