Singularities and their resolutions

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On Singularities - Part 1

$x^2 + z^2 = y^3(1-y)^3 \quad y^2z^2 + z^3 - x^2 = 0 \quad (x^2 - y^3)^2 - (z^2 - y^2)^3 = 0$

These are singularities. Look awful, don't they?
Let's get rid of them! (without losing information) - that's resolution of singularities

Algebraic geometry

- My subject: algebraic geometry

  The geometry of sets defined by polynomial equations.

- More specifically: The geometry of subsets $V \subset \mathbb{C}^n$ defined by polynomial equations:

$$V = \{(z_1, \ldots, z_n) | f_1(z_1, \ldots, z_n) = \cdots = f_k(z_1, \ldots, z_n) = 0\},$$

  with $f_i \in \mathbb{C}[z_1, \ldots, z_n]$.

- These sets are called algebraic varieties.[1]

[1] affine

Examples of algebraic varieties

0 $V = \{x = y = 0\} \subset \mathbb{C}^2$: a point.

1 If $a, b \neq 0, 0, c \in \mathbb{C}$, let $V = \{ax + by + c = 0\}$: a line.

2 for $R \in \mathbb{C}$ let $V = \{x^2 + y^2 = R^2\}$.

All my pictures are of $V(\mathbb{R})$ - the real solutions of the equations.
The subject involves geometry, algebra, algebra, number theory, even calculus . . .

Singular and smooth points

The examples above are smooth.
The quintessential example is the graph $w = F(x, y, z)$.

Definition

$\{V = f(x_1, \ldots, x_n) = 0\}$ is singular at p if $\frac{\partial f}{\partial x_i}(p) = 0$ for all i, namely $\nabla f(p) = 0$.
Otherwise smooth[a].

[a] In other words, $\{f = 0\}$ defines a manifold of complex codimension 1.

The implicit function theorem says: $\{f = 0\}$ is smooth if and only if locally it looks like a graph.

do the circle

(In codimension c, the singular locus of $\{f_1 = \cdots = f_k = 0\}$ is the set of points where $d(f_1, \ldots, f_k)$ has rank $< c$.)

Examples of singularities

- $$y^2 = x^3 + x^2$$

board

$$x^2 = y^2 z$$

board

Looks like in general it might be hard to find the singularities.
There is a theorem saying that it is.

Resolution of singularities

Definition

A resolution of singularities $X' \to X$ is a modification[a] with X' nonsingular inducing an isomorphism over the smooth locus of X.

[a]proper birational map

Theorem (Hironaka 1964)

A complex algebraic variety X admits a resolution of singularities $X' \to X$, so that the critical locus $E \subset X'$ is a simple normal crossings divisor.[a]

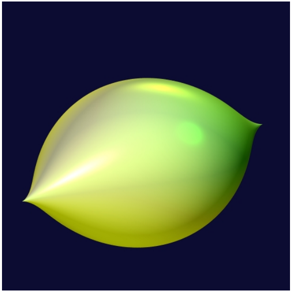
[a]Codimension 1, smooth components meeting transversally

Examples of resolutions

- $$V = \{y^2 = x^2(x+1)\}$$
  - Write $t = y/x$,
  - so $t^2 = x + 1$, and $x = t^2 - 1$
  - so $y = xt = t^3 - t$.
  - get a map $t \mapsto (t^2 - 1, t^3 - t)$ with image V.
- $$V = \{x^2 = y^2 z\}$$
  - Write $t = x/y$,
  - so $z = t^2$,
  - and $x = yt$.
  - get a map $(y, t) \mapsto (yt, y, t^2)$ with image V.

On singularities - Part 2

$x^2 + z^2 = y^3(1-y)^3$ $\quad y^2z^2 + z^3 - x^2 = 0$ $\quad (x^2 - y^3)^2 - (z^2 - y^2)^3 = 0$



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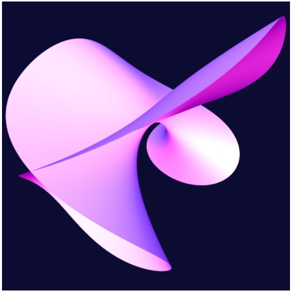

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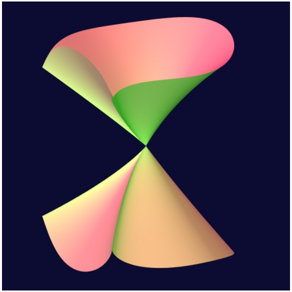


daisy

figrures by Herwig Hauser, `https://imaginary.org/gallery/herwig-hauser-classic`

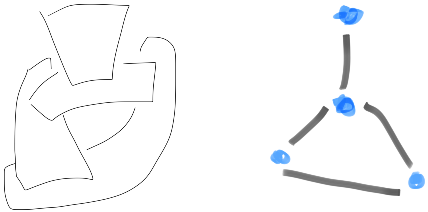
Singularities are beautiful.

Why should we "get rid of them"? try this

`https://imaginary.org/gallery/herwig-hauser-classic`

Example: Stepanov's theorem

If $X' \to X$ a resolution with $E \subset X'$ a simple normal crossings divisor, define $\Delta(E)$ to be the dual complex of E.



Theorem (Stepanov 2006)

The simple homotopy type of $\Delta(E)$ is independent of the resolution $X' \to X$.

Also work by Danilov, Payne, Thuillier, Harper...

Past, present and future

- Alicia Harper was a PhD student at Brown who generalized Stepanov's theorem, answering a question in a paper of Prof. Chan.
- Jonghyun Lee is a Brown undergraduate coding a resolution algorithm appearing in one of my papers.
- Stephen Obinna and Ming-Hao Quek are PhD students at Brown who will prove a generalization of that paper.

The end

Thank you for your attention