# Singularities and their resolutions

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Let's get rid of them! (without losing information) - that's resolution of singularities

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- These sets are called algebraic varieties.[1]

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(In codimension $c$, the singular locus of $\{f_1 = \cdots = f_k = 0\}$ is the set of points where $d(f_1, \ldots, f_k)$ has rank $< c$.)

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There is a theorem saying that it is.

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A resolution of singularities $X' \to X$ is a modification[a] with $X'$ nonsingular inducing an isomorphism over the smooth locus of $X$.

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## Theorem (Hironaka 1964)

*A complex algebraic variety $X$ admits a resolution of singularities $X' \to X$, so that the critical locus $E \subset X'$ is a simple normal crossings divisor.*[a]

[a] Codimension 1, smooth components meeting transversally

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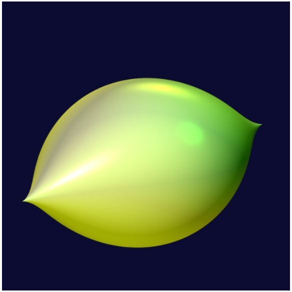

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figrures by Herwig Hauser, https://imaginary.org/gallery/herwig-hauser-classic

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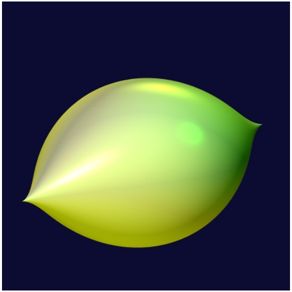

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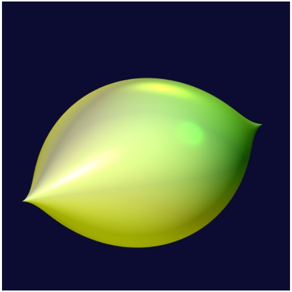

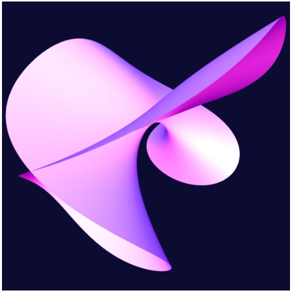

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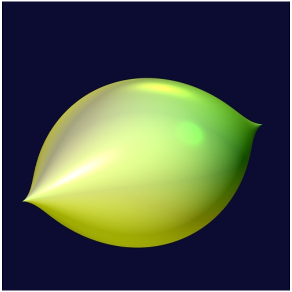

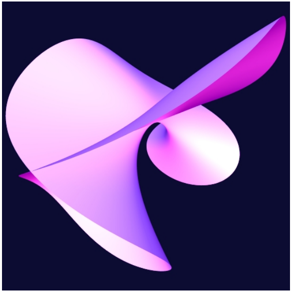

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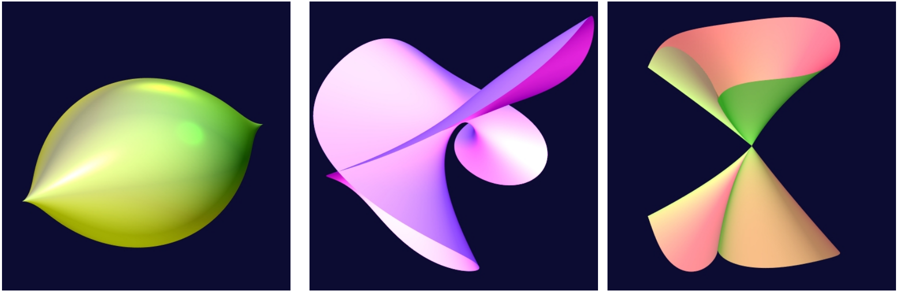

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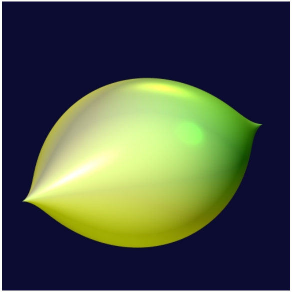

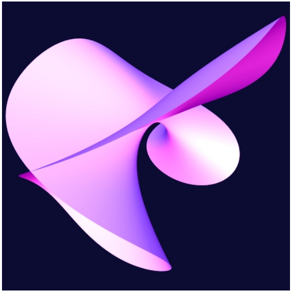

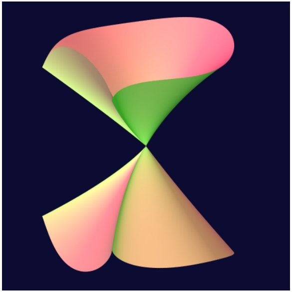

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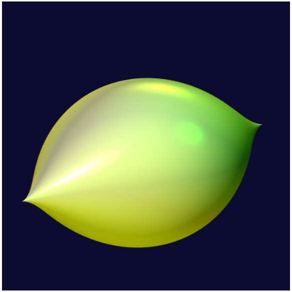

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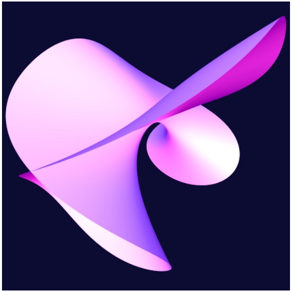

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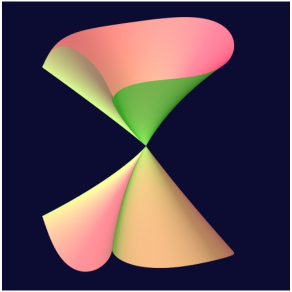

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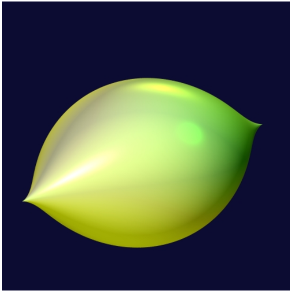

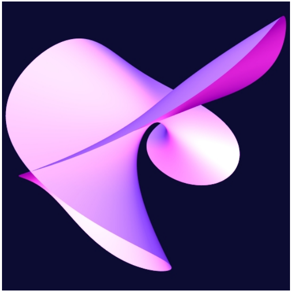

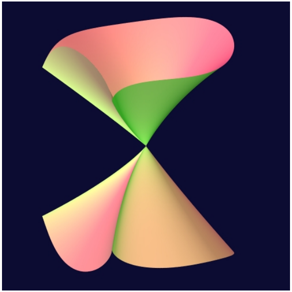

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# Example: Stepanov's theorem

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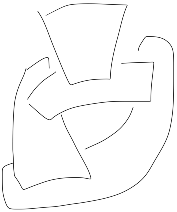

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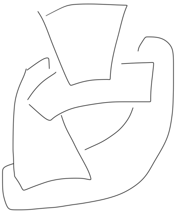

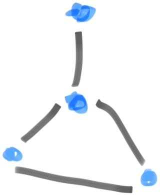

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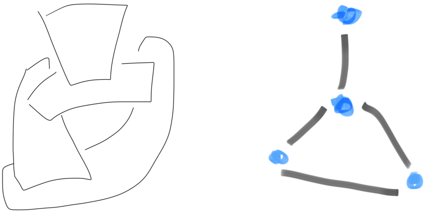

**Theorem (Stepanov 2006)**

*The simple homotopy type of $\Delta(E)$ is independent of the resolution $X' \to X$.*

Also work by Danilov, Payne, Thuillier, Harper...

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- Stephen Obinna and Ming-Hao Quek are PhD students at Brown who will prove a generalization of that paper.

# The end

Thank you for your attention