Singularities and the art of logarithmic stack maintenance

Or:
resolving singularities in Vistoli's workshop

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On machines

Machines are great,

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Resolution of singularities

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A resolution of singularities $X' \to X$ is a proper birational map inducing an isomorphism over the smooth locus of X.

Theorem (Hironaka 1964)

A variety X over a field of characteristic 0 admits a resolution of singularities $X' \to X$, so that the exceptional locus $E \subset X'$ is a simple normal crossings divisor.

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Always characteristic 0 . . .

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- The pair (X, D) is logarithmically smooth, so log geometry should fit naturally.
- Actually in the standard proofs the divisor is a bit of a headache.
- No stacks?!

Resolution does hold for stacks

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An excellent algebraic stack X over a field of characteristic 0 admits a resolution of singularities $X' \to X$.

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An excellent algebraic stack X over a field of characteristic 0 admits a resolution of singularities $X' \to X$.

- This is a consequence of resolution for varieties and schemes, functorial for smooth (actually regular) morphisms.
- Włodarczyk showed that if one seriously looks for a resolution functor, one is led to a resolution.
- Still I want to show that stacks are part of the solution.

Log smooth schemes and log smooth morphisms

Log smooth schemes and log smooth morphisms

- A variety X with divisor D is toroidal or log smooth if étale locally it looks like a toric variety X_σ with its toric divisor $X_\sigma \smallsetminus T$.
- Étale locally it is defined by equations between monomials.
- One records the open $U = X \smallsetminus D$ rather than the divisor.
- (To be honest, the log structure $\mathcal{M} \hookrightarrow \mathcal{O}_X$ associated to this open.)
- A morphism $X \to Y$ is toroidal or log smooth if étale locally it looks like a toric morphism.
- The inverse image of a monomial is a monomial.

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- Let's try to simplify singularities of a family $X \to B$.

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- If $\dim B = 1$ the simplest one can have is $t = \prod x_i^{a_i}$,
- and if one also allows base change, can have $t^k = \prod x_i$, or even $t = \prod x_i$.

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- If $\dim B = 1$ the simplest one can have is $t = \prod x_i^{a_i}$,
- and if one also allows base change, can have $t^k = \prod x_i$, or even $t = \prod x_i$.

- Either way, it is a log smooth morphism of log smooth schemes.
- After base change, it is an integral and saturated morphism: flat with reduced fibers.

Resolution of families: higher dimensional base

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When are the singularities of a morphism simple?

- If $\dim B > 1$ Karu [1999] showed that $t = \prod x_i^{a_i}$ cannot be achieved.

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When are the singularities of a morphism simple?

- If $\dim B > 1$ Karu [1999] showed that $t = \prod x_i^{a_i}$ cannot be achieved.
- The best one can hope for, after base change, is a semistable morphism:

Definition (AK 2000)

A log smooth morphism, with B smooth, is *semistable* if locally

$$\begin{aligned} t_1 &= x_1 \cdots x_{l_1} \\ \vdots \quad & \quad \vdots \\ t_m &= x_{l_{m-1}+1} \cdots x_m \end{aligned}$$

The semistable reduction problem

Conjecture

Let $X \to B$ be a dominant morphism of varieties.

(i) There is an alteration $B_1 \to B$ and a modification $X_1 \to (X \times_B B_1)_{\text{main}}$ such that $X_1 \to B_1$ is semistable.

(ii) If the geometric generic fiber $X_{\bar\eta}$ is smooth, such $X_1 \to B_1$ can be found with $X_{\bar\eta}$ unchanged.

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- (ii) is known for families of curves in mixed characteristics [de Jong 1997]. . .
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Known results:

- (ii) is known for families of curves in mixed characteristics [de Jong 1997]...
- (ii) is known for B a curve in characteristic 0 [KKMS 1973]
- (i) is known for families of threefolds in characteristic 0 [Karu 2000]
- One wants (ii) in order to compactify smooth families.

Toroidalization and weak semistable reduction

Back to characteristic 0

Theorem (Toroidalization, ℵ-Karu 2000, ℵ-K-Denef 2013)

There is an modification $B_1 \to B$ *and a modification* $X_1 \to (X \times_B B_1)_{main}$ *such that* $X_1 \to B_1$ *is log smooth and flat.*

Theorem (Weak semistable reduction, ℵ-Karu 2000)

There is an alteration $B_1 \to B$ *and a modification* $X_1 \to (X \times_B B_1)_{main}$ *such that* $X_1 \to B_1$ *is log smooth, flat, with reduced fibers.*

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- The key is toroidalization.
- Passing from weak semistable reduction to semistable reduction is a purely combinatorial problem [ℵ-Karu 2000],
- proven by Karu for families of threefolds, and
- whose restriction to rank-1 valuation rings is proven in a preprint by [Adiprasito-Liu-Pak-Temkin].

Functoriality

- To reach (ii), Włodarczyk says we need to work functorially.
- We do this functorially for any log base change and for log-smooth morphisms (Temkin).

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Main result (ℵ-Temkin-Włodarczyk)

Let $X \to B$ be a dominant log morphism.

- There are log modifications $B_1 \to B$ and $X_1 \to (X \times_B B_1)_{\text{main}}$ such that $X_1 \to B_1$ is log smooth and flat;
- this is compatible with log base change $B' \to B$;
- this is compatible, up to base change, with log smooth $X'' \to X$.

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- A surprise is awaiting.

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Theorem (...ℵ-T-W)

Let $\mathcal{I}$ be an ideal on a log smooth Y. There is a functorial logarithmic morphism $Y' \to Y$, with Y' logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

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Let $\mathcal{I}$ be an ideal on a log smooth Y. There is a functorial logarithmic morphism $Y' \to Y$, with Y' logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

- This is done by order reduction,
- achieved by blowing up admissible centers.

Example 1

- Consider $Y_1 = \operatorname{Spec} \mathbb{C}[u, x]$ and $D = \{u = 0\}$.
- Let $\mathcal{I} = (u^2, x^2)$.
- If one blows up (u, x) the ideal is principalized:
  - on the u-chart $\operatorname{Spec} \mathbb{C}[u, x']$ with $x = x'u$ we have $\mathcal{I}\mathcal{O}_{Y_1'} = (u^2)$,
  - on the x-chart $\operatorname{Spec} \mathbb{C}[u', x]$ with $u' = xu'$ we have $\mathcal{I}\mathcal{O}_{Y'} = (x^2)$,
  - which is exceptional hence monomial.
- This is in fact the only functorial admissible blowing up.

Example 2

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- What is this? What is its blowup?

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- A Kummer monomial ideal is a monomial ideal in the Kummer-étale topology of Y.
- A Kummer center is the sum of a Kummer monomial ideal and the ideal of a log smooth subscheme.
- Locally $(x_1, \ldots, x_k, u_1^{1/d}, \ldots u_\ell^{1/d})$.

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Apologies: we did not use the infinite root stack.

Blowing up Kummer centers

Proposition

Let $\mathcal{J}$ be a Kummer center on a logarithmically smooth Y. There is a universal proper birational $Y' \to Y$ such that Y' is logarithmically smooth and $\mathcal{J}\mathcal{O}_{Y'}$ is an invertible ideal.

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$Y = \operatorname{Spec}\mathbb{C}[v]$, with toroidal structure associated to $D = \{v = 0\}$, and $\mathcal{J} = (v^{1/2})$.

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- There is no log scheme Y' satisfying the proposition.
- There is a stack $Y' = Y(\sqrt{D})$, the Cadman–Vistoli root stack, satisfying the proposition!

Example 2 concluded

- Consider $Y_2 = \operatorname{Spec} \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$ and $\mathcal{J} = (v^{1/2}, x)$.

Example 2 concluded

- Consider $Y_2 = \operatorname{Spec} \mathbb{C}[v, x]$ and $D = \{v = 0\}$.
- Let $\mathcal{I} = (v, x^2)$ and $\mathcal{J} = (v^{1/2}, x)$.
- associated blowing up $Y' \to Y_2$ with charts:
  - $Y'_x := \operatorname{Spec} \mathbb{C}[x, v, v']/(v'x^2 = v)$, where $v' = v/x^2$ (nonsingular scheme).
    - ⋆ Exceptional $x = 0$, now monomial.
    - ⋆ $\mathcal{I} = (x^2, v)$ transformed into (x^2), invertible monomial ideal.
    - ⋆ Kummer ideal $(x, v^{1/2})$ transformed into monomial ideal (x).

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  - The $v^{1/2}$-chart:
    - stack quotient $X'_{v^{1/2}} := \left[\operatorname{Spec} \mathbb{C}[w,y]/\mu_2\right]$,
    - where $y = x/w$ and $\mu_2 = \{\pm 1\}$ acts via $(w, y) \mapsto (-w, -y)$.
    - Exceptional $w = 0$ (monomial).
    - (x^2, v) transformed into invertible monomial ideal $(v) = (w^2)$.
    - $(x, v^{1/2})$ transformed into invertible monomial ideal (w).

$\dim B = 0$: restatement

Theorem (ℵ-T-W)

Let $\mathcal{I}$ be an ideal on a logarithmically smooth Y. There is a functorial logarithmic morphism $Y' \to Y$, with Y' a logarithmically smooth stack, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

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Write $\mathcal{D}^{\leq a}$ for the sheaf of logarithmic differential operators of order $\leq a$. The logarithmic order of an ideal $\mathcal{I}$ is the minimum a such that $\mathcal{D}^{\leq a}\mathcal{I} = (1)$.

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Proposition (Kollár, ℵ-T-W)

In cahracteristic 0, $\mathcal{M}(\mathcal{I}) = \mathcal{D}^\infty(\mathcal{I})$. *In particular* $\max_p \operatorname{logord}_p(\mathcal{I}) = \infty$ *if and only if* $\mathcal{M}(\mathcal{I}) \neq 1$.

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Proposition (Kollár, ℵ-T-W)

Let $Y_0 \to Y$ be the normalized blowup of $\mathcal{M}(\mathcal{I})$. Then $\mathcal{M} := \mathcal{M}(\mathcal{I})\mathcal{O}_{Y_0} = \mathcal{M}(\mathcal{I}\mathcal{O}_{Y_0})$ is an invertible monomial ideal, and so $\mathcal{I}\mathcal{O}_{Y_0} = \mathcal{I}_0 \cdot \mathcal{M}$ with $\max_p \operatorname{logord}_p(\mathcal{I}_0) < \infty$.

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- In cahracteristic 0, if $\text{logord}_p(\mathcal{I}) = a < \infty$, then $\mathcal{D}^{\leq a-1}\mathcal{I}$ contains an element x with derivative 1.

dim $B = 0$: sketch of argument

- In cahracteristic 0, if $\operatorname{logord}_p(\mathcal{I}) = a < \infty$, then $\mathcal{D}^{\leq a-1}\mathcal{I}$ contains an element x with derivative 1.
- Carefully applying induction on dimension to an ideal on $\{x = 0\}$ gives order reduction:

Proposition (ℵ-T-W)

Let $\mathcal{I}$ be an ideal on a logarithmically smooth Y with

$$\max_p \operatorname{logord}_p(\mathcal{I}) = a.$$

There is a functorial logarithmic morphism $Y_1 \to Y$, with Y_1 logarithmically smooth, such that $\mathcal{I}\mathcal{O}_{Y'} = \mathcal{M}\mathcal{I}_1$ with $\mathcal{M}$ an invertible monomial ideal and

$$\max_p \operatorname{logord}_p(\mathcal{I}_1) < a.$$

Arbitrary B

Main result (ℵ-T-W)

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{I} \subset \mathcal{O}_Y$ an ideal. There is a log morphism $B' \to B$ and functorial log morphism $Y' \to Y$, with $Y' \to B'$ logarithmically smooth, and $\mathcal{I}\mathcal{O}_{Y'}$ an invertible monomial ideal.

- This is done by relative order reduction, using relative logarithmic derivatives.

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Definition

Write $\mathcal{D}^{\leq a}_{Y/B}$ for the sheaf of relative logarithmic differential operators of order $\leq a$. The relative logarithmic order of an ideal $\mathcal{I}$ is the minimum a such that $\mathcal{D}^{\leq a}_{Y/B}\mathcal{I} = (1)$.

The new step

- $\mathrm{relord}_p(\mathcal{I}) = \infty$ if and only if $\mathcal{M} := \mathcal{D}^{\infty}_{Y/B}\mathcal{I}$ is a nonunit monomial ideal along the fibers.
- Equivalently $\mathcal{M} = \mathcal{D}_{Y/B}\mathcal{M}$ is not the unit ideal.

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- Equivalently $\mathcal{M} = \mathcal{D}_{Y/B}\mathcal{M}$ is not the unit ideal.

ℵ-T-W

Let $Y \to B$ a logarithmically smooth morphism of logarithmically smooth schemes, $\mathcal{M} \subset \mathcal{O}_Y$ an ideal with $\mathcal{D}_{Y/B}\mathcal{M} = \mathcal{M}$. There is a log morphism $B' \to B$ with saturated pullback $Y' \to B'$, and $\mathcal{M}\mathcal{O}_{Y'}$ a monomial ideal.