A tale of Algebra and Geometry

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Intersection theory on algebraic stacks and their moduli spaces [Inv. 1989]

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- This is wonderful: it tells you something about varieties, which you can use even if you know nothing about the tools, namely stacks.
- Angelo's thesis leads to many explicit computations, numerous theses, and further work at the foundation of enumerative geometry (180 citations).

Theorem [Invent. Math. 1998]

Assume that κ has characteristic $\neq 2$ and 3. Then
$A^*(\mathcal{M}_2) = \mathbb{Z}[\lambda_1, \lambda_2]/(10\lambda_1, 2\lambda_1^2 - 24\lambda_2)$

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- The next class was given by Angelo, a career-changing event, the first proper introduction to algebraic stacks for many.
- He is basically telling students and professor alike how to seriously think about families and moduli.

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However . . .

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- The student is scared and would have weaseled out.
- But the professor has two bodyguards on both sides, nodding, smiling.
- Angelo has this towering figure, and there is no way the student would escape!

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- But there is an issue: how do you fill up a degenerate elliptic surface across a point?
- The next morning Angelo reveals a beautiful Lemma.

The Purity Lemma [JAMS 2002]

Let $\mathcal{M} \to \mathbf{M}$ be the coarse moduli space of a separated Deligne - Mumford stack, X a separated S_2 surface, P a closed point. Assume that the local fundamental group of $U = X \setminus P$ around P is trivial.

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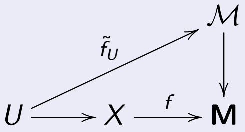
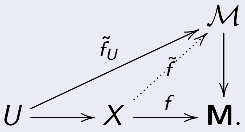
Let $f\colon X \to \mathbf{M}$ be a morphism. Suppose there is a lifting $\tilde{f}_U\colon U \to \mathcal{M}$:

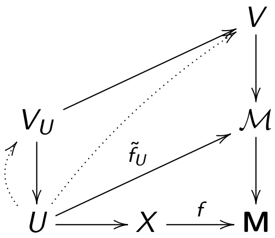


Then the lifting extends to X:



The purity lemma: localization and lifting on U

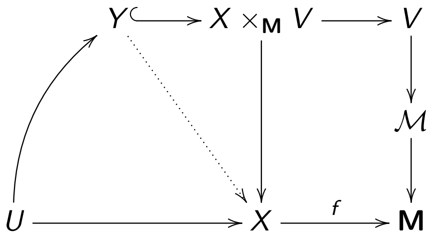
- The problem is étale local, so we may pass to strict henselization.
- We can thus assume U simply connected,
- and $\mathcal{M} = [V/\Gamma]$, with $V \to \mathcal{M}$ finite étale.
- Consider $V_U = U \times_{\mathcal{M}} V$.



- Since $V_U \to U$ is finite étale and U simply connected there is a section $U \to V_U$ composing to a morphism $U \to V$.

The purity lemma: end of proof

- Consider the closure Y of U in $X \times_{\mathbf{M}} V$:



- As $V \to \mathbf{M}$ is finite, $Y \to X$ is finite.
- As $U \to X$ is birational and isomorphism away from codimension 2, $Y \to X$ is also.
- As X is S_2, we have $Y \to X$ an isomorphism.
- $X \to Y \to \cdots \to \mathcal{M}$ is the needed lifting.

Random gems

Theorem [Invent. Math. 1998]

Assume that κ has characteristic $\neq 2$ and 3. Then $\mathcal{M}_2 = [X/GL_2]$, where X is the space of smooth degree 6 binary forms (and the action is twisted!).

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Let X be a Deligne–Mumford quotient stack over a field having a qusiprojective coarse moduli space. Then X has a finite flat lci cover $Z \to X$ by a quasiprojective scheme which is as smooth as X.

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Theorem 1.2 [Brosnan-Reichstein-Vistoli 2009]

The essential dimension of $\mathcal{M}_2$ is 5.

This is an interim report.

More to come!