BIRATIONAL GEOMETRY USING WEIGHTED BLOWING UP

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Abstract. This is an exposition of ideas appearing in [ATW19], discussing in addition the extent to which one can address other aspects of birational geometry using weighted blowings up.

1. Introduction

1.1. **The place of resolution.** Resolution of singularities, when available, is one of the most powerful tool at the hands of an algebraic geometer. One would wish it to have a completely intuitive, natural proof. Many such proofs exist for curves, see [Kol07, Chapter 1]. At least one completely conceptual proof exists for surfaces in characteristic > 2 and for threefolds of characteristic > 5, see [Cut09]. It is fair to say that with most other cases, including Hironaka's monumental proof of resolution of singularities in characteristic zero [Hir64], one feels that varieties resist our resolution efforts, and one has to resort to extreme measure to force resolution upon them. On the other hand, one feels that it should not be so: myriad applications show that it is in the best interest of any variety to be resolved.

1.2. **This note.** The purpose of this note is to recall how this "resistance" comes about, and to outline a new approach, where we are able to prove resolution of singularities in characteristic 0 with the complete cooperation of our varieties, by infusing the theory with a bit of modern moduli theory, specifically the theory of algebraic stacks. The main result, explained throughout this note, is the following:

Theorem 1.2.1 (Weighted Hironaka). *There is a procedure F associating to a singular subvariety $X \subset Y$ embedded with pure codimension c in a smooth variety Y over a field of characteristic 0, a center $\bar{J}$ with blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that* $\operatorname{maxinv}(X') < \operatorname{maxinv}(X)$. *In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n smooth.*

For this purpose we systematically use stack-theoretic weighted blowings up. As is well known, the use of classical blowings up of smooth centers has been of great value in birational geometry. We end this note by discussing to what extent stack-theoretic weighted blowings up can be as useful.

Theorem 1.2.1 is our main theorem in [ATW19]. An almost identical result is given concurrently in [McQ19]. The methods employed are quite different.

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2. Curves and surfaces

Our aim is *embedded resolution*, namely to resolve a signular subvariety $X \subset Y$ of a smooth variety Y by applying birational transformations on Y.

2.1. **How to resolve a curve?** To resolve a singular curve C

(1) find a singular point $x \in C$, and then
(2) blow it up.

This procedure always works:

Fact (See [Har77, Theorem V.3.8]). *$p_a(C)$ gets smaller in each such step,*

hence the procedure ends with a smooth curve.

2.2. **How to resolve a surface?** Surfaces are more complex, as their singularities can reside on either curves or points. To resolve a singular surface S one wants to

(1) find the worst singular locus $C \subset S$, and then
(2) show that C is smooth, and blow it up.

However:

Fact. *This in general* does not *get better.*

2.3. **Example: Whitney's umbrella.** Consider $S = V(x^2 - y^2 z)$ The worst singularity is the origin. In the z chart we get $x = x_3 z,\ y = y_3 z$, giving $x_3^2 z^2 - y_3^2 z^3 = 0$, or

$$z^2(x_3^2 - y_3^2 z) = 0.$$

The first term is exceptional, which we may ignore. However the second is the same as S. It appears that we gained nothing.

2.4. **How to resolve a surface - classical approach.** Of course surface resolution can be achieved. The standard algorithms in characteristic 0 - which applies in arbitrary dimension - calls for recording the exceptional divisors. Thus after the first blowing up the equation $x_3^2 - y_3^2 z$ has a distinguished coordinate z, which should be thought of as an improvement.

But even surfaces do not like this approach and resist it kicking and screaming: the exceptional divisors get in the way of the standard natural algorithm, which uses hypersurfaces of maximal contact recalled later on. In short, such hypersurfaces are not necessarily transverse to the exceptional divisors, giving no end of trouble.

By what seems like pure luck, one can introduce an auxiliary subroutine of resolution to throw exceptional divisors out of the picture. In terms of singularity invariants, it is not enough to record the exceptional locus, but also some of its history - or the "state of the algorithm" - is needed. Of course this is all counterintuitive - we introduce these divisors to make progress, and yet we introduce a procedure to get them out of the way - but somehow this nevertheless works.

3. Explaining the main result

Coming back to Theorem 1.2.1, our first goal is to explain this result and how it could possibly avoid the complications discussed in the previous section.

3.1. **Functoriality.** First things first: here *procedure* means *a functor for smooth surjective morphisms:* if $f : Y_1 \twoheadrightarrow Y$ is smooth then $J_1 = f^{-1}J$ and $Y_1' = Y_1 \times_Y Y'$, and X' can be taken to be the proper transform (in the course of the proof we actually use the so called *weak transform* instead).

The final result $(X_n \subset Y_n)$ is actually functorial for smooth but not necessarily surjective morphisms - only the number n is not.

Functoriality has great value beyond elegance - it guarantees that resolution is equivariant under automorphisms, it is compatible with localization, and in particular smooth points are not disturbed.

Functoriality is present in Hironaka's later work (under the term "canonical resolution"), and is clarified in the works of Villamayor [Vil89] and Bierstone–Milman [BM97, BM08]. Włodarczyk [Wło05] was the first to show that functoriality is a powerful tool for the proof itself, a method we employ here. Indeed, the result of our blowing up is an algebraic stack, but functoriality allows us to replace it by a presentation by schemes, so we can have the input of the theorem be a scheme.

We proceed with an overview of the concepts we use, exhibiting them in examples, and then defining them more thoroughly later on.

3.2. **Preview on invariants.** We use singularity invariants to guide the procedure. For $p \in X$ we define

$$\mathrm{inv}_p(X) \in \Gamma \subset \quad \mathbb{Q}_{\geq 0}^{\leq n},$$

and show

Theorem 3.2.1 ([ATW19, Theorem 5.2.1])**.** *(1) Γ is a well-ordered subset with respect to the lexicographical ordering,*

(2) $\mathrm{inv}_p(X)$ is lexicographically upper-semi-continuous, and

(3) $p \in X$ is smooth if and only if $\mathrm{inv}_p(X) = \min \Gamma$.

We define $\mathrm{maxinv}(X) = \max_p \mathrm{inv}_p(X)$.

Example 3.2.2. *$inv_p(V(x^2 - y^2z)) = (2, 3, 3)$*

The idea is: the variable x appears in the equation $x^2 - y^2z$ in the monomial of lowest possible degree 2, and the variables y and z appear in a monomial of the next lowest degree 3.

Remark 3.2.3. These invariants have been in our arsenal for ages. All the great works of the last three decades on resolution in characteristic 0 use this invariant, with additional information interspersed within it.

3.3. **Preview of centers.** We use centers denoted J for comparing singularities and the notion of *admissibility*, and associated *reduced centers* denoted $\bar{J}$ for blowing up.

As in the example above, invariants are determined using associated local parameters. If $\mathrm{inv}_p(X) = \mathrm{maxinv}(X) = (a_1, \ldots, a_k)$ then, locally at p, we have

$$J = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

Now we normalize the center by a rescaling procedure: write

$$(a_1, \ldots, a_k) = \ell(1/w_1, \ldots, 1/w_k)$$

with $w_i, \ell \in \mathbb{N}$ and $\gcd(w_1, \ldots, w_k) = 1$. We set the *reduced center* to be

$$\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k}).$$

Defining the center J requires a new formalism: first, we will see that the parameters x_i require a choice, while we claim below the center is uniquely defined. Second, the center is a beast involving fractional powers of parameters, something that goes beyond the familiar world of ideals. We will explain these points later in this note.

Example 3.3.1. For $X = V(x^2 - y^2 z)$ we have $J = (x^2, y^3, z^3)$ and $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$.

This example is a bit too easy, since the variables x, y and z were staring us in the face, as the ideal is binomial. We will revisit this issue below where a different example will be the guide.

Remark 3.3.2. The center J has been staring in our face for a while. If one interprets J in terms of Newton polyhedra, it appears in section 1 of Youssin's thesis [You90]. Youssin's construction is a simplified variant of Hironaka's *characteristic polyhedron of a singularity*, see [Hir67]

3.4. **Example: blowing up Whitney's umbrella $x^2 = y^2 z$.** The blowing up $Y' \to Y$ makes $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$ principal. Explicitly: the z chart has $x = w^3 x_3, y = w^2 y_3, z = w^2$ with chart

$$Y' \;=\; [\,\mathrm{Spec}\,\mathbb{C}[x_3, y_3, w]\;/\;(\pm 1)\,],$$

on which (± 1) acts by $(x_3, y_3, w) \mapsto (-x_3, y_3, -w)$.

The transformed equation is

$$w^6(x_3^2 - y_3^2),$$

and the invariant of the proper transform $(x_3^2 - y_3^2)$ is $(2, 2) < (2, 3, 3)$.

Remark 3.4.1. In fact, people studying explicit birational geometry, as well as people studying explicit moduli spaces of surfaces, have known all along that X begs to be blown up like this.

3.5. **Definition of the weighted blowing up $Y' \to Y$.** We are now ready to define our blowing up in general.

Let $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. Define the graded algebra

$$\mathcal{A}_{\bar{J}} \quad \subset \quad \mathcal{O}_Y[T]$$

as the *integral closure* of the image of

$$\begin{aligned} \mathcal{O}_Y[Y_1, \ldots, Y_n] &\longrightarrow \mathcal{O}_Y[T] \\ Y_i \quad &\longmapsto x_i T^{w_i}. \end{aligned}$$

Let

$$S_0 \subset \mathrm{Spec}_Y\, \mathcal{A}_{\bar{J}}, \quad S_0 = V((\mathcal{A}_{\bar{J}})_{>0})$$

be the vertex of the spectrum. Then

$$Bl_{\bar{J}}(Y) \;:=\; \mathcal{P}roj_Y \mathcal{A}_{\bar{J}} \;:=\; \left[(\mathrm{Spec}\,\mathcal{A}_{\bar{J}} \smallsetminus S_0)\;/\;\mathbb{G}_m\right].$$

This is the analogue of the definition of usual blowing up one can find in [Har77, Example 7.12.1], where Y_i is sent to $x_i T$, namely placed in degree 1, whereas here it is placed in degree w_i. Unlike Hartshorne's description, we take the stack theoretic quotient rather than the scheme theoretic quotient. This is critical if one is to have a smooth ambient space after blowing up.

3.6. **Description of $Y' \to Y$.** Just like regular blowing up, a weighted blowing up has a local description in terms of charts. The x_1-chart is

$$[\operatorname{Spec} k[u, x_2', \ldots, x_n'] \;/\; \boldsymbol{\mu}_{w_1}],$$

with $x_1 = u^{w_1}$ and $x_i = u^{w_i} x_i'$ for $2 \leq i \leq k$, and induced action:

$$(u, x_2', \ldots, x_n') \;\mapsto\; (\zeta u\,,\, \zeta^{-w_2} x_2'\,,\, \ldots\,,\, \zeta^{-w_k} x_k',\; x_{k+1}', \ldots, x_n').$$

For our discussion at the end of this paper it is also useful to have a local toric description of the blowing up. We follow [BCS05]. For simplicity let us assume Y is affine space, corresponding to the cone $\sigma = \mathbb{R}^n_{\geq 0}$ with lattice $\mathbb{N}^n$.

Then Y' is the toric stack corresponding to the star subdivision $\Sigma := v_{\bar{J}} \star \sigma$ along

$$v_{\bar{J}} \;=\; (w_1, \ldots, w_k, 0, \ldots, 0),$$

with the cone

$$\sigma_i = \langle v_{\bar{J}}, e_1, \ldots, \hat{e}_i, \ldots, e_n \rangle$$

endowed with the sublattice $N_i \subset N$ generated by the elements

$$v_{\bar{J}}, e_1, \ldots, \hat{e}_i, \ldots, e_n,$$

for all $i = 1, \ldots, k$.

The coarse moduli space of Y' is simply the toric *variety* corresponding to the star subdivision, without the auxiliary sublattices.

3.7. Determining the center J associated to an ideal $\mathcal{I}$ in examples.

3.7.1. *An example with fractional powers.* Consider $X = V(x^5 + x^3 y^3 + y^8)$ at $p = (0, 0)$; write $\mathcal{I} := \mathcal{I}_X$. Define $a_1 = \operatorname{ord}_p \mathcal{I} = 5$, and choose x_1 to be any variable appearing in a degree-a_1 term, for instance x.

This determines the beginning of our center $J_{\mathcal{I}} = (x^5,\ y^{\star}\ )$.

We note that if we were to change variables, we could use $x + y^2$ instead, so there is definitely a choice involved.

To balance x^5 with $x^3 y^3$ we need x^2 and y^3 to have the same weight, implying that x^5 and $y^{15/2}$ have the same weight.

If we were to balance with the term y^8 we would have taken y^8 instead. Since $15/2 < 8$ the choice $y^{15/2}$ dominates, and we use

$$J_{\mathcal{I}} = (x^5, y^{15/2}) \quad \text{ and } \quad \bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2}).$$

3.7.2. *A related example.* If instead we took $X = V(x^5 + x^3 y^3 + y^7)$, then since $7 < 15/2$ we would use

$$J_{\mathcal{I}} = (x^5, y^7) \quad \text{ and } \quad \bar{J}_{\mathcal{I}} = (x^{1/7}, y^{1/5}).$$

3.8. **Describing the blowing up in the new examples.**

(1) Considering $X = V(x^5 + x^3y^3 + y^8)$ at $p = (0,0)$,

- the x-chart has $x = u^3, y = u^2y_1$ with $\boldsymbol{\mu}_3$-action, and equation

$$u^{15}(1 + y_1^3 + uy_1^8)$$

with smooth proper transform.

- The y-chart has $y = v^2, x = v^3x_1$ with $\boldsymbol{\mu}_2$-action, and equation

$$v^{15}(x_1^5 + x_1^3 + u)$$

with smooth proper transform.

(2) Considering $X = V(x^5 + x^3y^3 + y^7)$ at $p = (0,0)$,

- the x-chart has $x = u^7, y = u^5y_1$ with $\boldsymbol{\mu}_7$-action, and equation

$$u^{35}(1 + uy_1^3 + y_1^7)$$

with smooth proper transform.

- The y-chart has $y = v^5, x = v^7x_1$ with $\boldsymbol{\mu}_5$-action, and equation

$$v^{35}(x_1^5 + ux_1^3 + 1)$$

with smooth proper transform.

3.9. **Coefficient ideals.** We need a mechanism for induction on dimension. The first example shows clearly that one can't just restrict the ideal $\mathcal{I}$ to $\{x_1 = 0\}$, since this loses information of monomials mixing x_1 and other variables. These mixed monomials are revealed by taking derivatives of the ideal $\mathcal{I}$.

We thus must restrict to $x_1 = 0$ the data of all

$$\mathcal{I},\ \mathcal{D}\mathcal{I},\ \ldots,\ \mathcal{D}^{a_1-1}\mathcal{I}$$

with corresponding weights

$$a_1,\ a_1 - 1,\ \ldots,\ 1.$$

We combine these in

$$C(\mathcal{I}, a_1)\ :=\ \sum f\left(\mathcal{I}, \mathcal{D}\mathcal{I}, \ldots, \mathcal{D}^{a_1-1}\mathcal{I}\right),$$

where f runs over monomials $f = t_0^{b_0} \cdots t_{a_1-1}^{b_{a_1-1}}$ with weights

$$\sum b_i(a_1 - i) \geq a_1!.$$

We now define the restricted coefficient ideal $\mathcal{I}[2] = C(\mathcal{I}, a_1)|_{x_1=0}$.

We note however that $\mathcal{I}[2]$ naturally has weight $a_1!$, whereas $\mathcal{I}$ has weight a_1. We need to compensate for this by "rescaling" $\mathcal{I}[2]$ down to degree a_1. This is the source of fractional invariants and fractional powers in our centers.

The coefficient ideal we use here was introduced in [Kol07, §3.54]. It is a variant of constructions appearing in [Vil89, BM97, Wło05].

3.10. **Defining $J_\mathcal{I}$ in general.**

Definition 3.10.1. Let $a_1 = \mathrm{ord}_p\mathcal{I}$, with x_1 a regular element in $\mathcal{D}^{a_1-1}\mathcal{I}$ - a maximal contact element. Suppose $\mathcal{I}[2]$ has invariant $\mathrm{inv}_p(\mathcal{I}[2])$ defined with parameters $\bar{x}_2, \ldots, \bar{x}_k$ on $\{x_1 = 0\}$, with lifts $x_2, \ldots, x_k$ in $\mathcal{O}_Y$. Set

$$\mathrm{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k) := \left(a_1, \frac{\mathrm{inv}_p(\mathcal{I}[2])}{(a_1-1)!}\right)$$

and

$$J_\mathcal{I} = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

Example 3.10.2. (1) for $X = V(x^5 + x^3y^3 + y^8)$ we have $\mathcal{I}[2] = (y)^{180}$, so

$$J_\mathcal{I} = (x^5, y^{180/24}) = (x^5, y^{15/2}).$$

(2) for $X = V(x^5 + x^3y^3 + y^7)$ we have $\mathcal{I}[2] = (y)^{7\cdot 24}$, so $J_\mathcal{I} = (x^5, y^7)$.

3.11. **What is J?** We have postponed the question - what kind of beast is J? It needs to allow for rational powers, and differeent choices of parameters giving rise to $J_1 = (x^{5/2}, y^{5/2})$ and $J_2 = ((x+y)^{5/2}, (x-5y)^{5/2}))$ must have $J_1 = J_2$. We follow ideas permeating birational geometry to resolve this issue.

Definition 3.11.1. Consider the Zariski-Riemann space $\mathbf{ZR}(X)$ with its sheaf of ordered groups Γ, and associated sheaf of rational ordered group $\Gamma \otimes \mathbb{Q}$.

- A valuative $\mathbb{Q}$-ideal is
$$\gamma \in H^0\left(\mathbf{ZR}(X), (\Gamma \otimes \mathbb{Q})_{\geq 0})\right).$$
- Each valuative $\mathbb{Q}$-ideal induces a "shadow" ideal on Y by $\mathcal{I}_\gamma := \{f \in \mathcal{O}_X : v(f) \geq \gamma_v \forall v\}$.
- Conversely, any coherent ideal $\mathcal{I}$ induces a valuative $\mathbb{Q}$-ideal denoted $v(\mathcal{I}) := (\min v(f) : f \in \mathcal{I})_v$.

A center is in particular a valuative $\mathbb{Q}$-ideal.

4. Elements of the proof

Now that we have defined our terms, we briefly describe key elements of the proof.

4.1. **Homogeneity.** Let $\mathcal{I} \subset \mathcal{O}_Y$ and assume $x_1 \in \mathcal{D}^{\leq a-1}\mathcal{I}$ is a maximal contact element at $p \in Y$. The ideals $C(\mathcal{I}, a_1)$ is MC-invariant in the sense of [Kol07, §3.53], hence it is *homogeneous* in the sense of [Wło05]:

Theorem 4.1.1 ([Wło05, Lemma 3.5.5], [Kol07, Theorem 3.92])**.** *Let x_1, x_1' be maximal contact elements at p, and $x_2, \ldots, x_n \in \mathcal{O}_{Y,p}$ such that $(x_1, x_2, \ldots, x_n)$ and $(x_1', x_2, \ldots, x_n)$ are both regular sequences. There is a scheme $\tilde{Y}$ with point $\tilde{p} \in \tilde{Y}$ and two morphisms $\phi, \phi' : \tilde{Y} \to Y$ with $\phi(\tilde{p}) = \phi'(\tilde{p}) = p$, both étale at p, satisfying*

(1) $\phi^* x_1 = \phi'^* x_1'$,
(2) $\phi^* x_i = \phi'^* x_i$ for $i = 2, \ldots, n$, and
(3) $\phi^* C(\mathcal{I}, a_1) = \phi'^* C(\mathcal{I}, a_1)$.

This in particular implies that replacing x_1 by x_1' while keeping the other parameters intact preserves the whole procedure.

4.2. **Formal decomposition.** A useful close cousin of homogeneity is a convenient formal decomposition of coefficient ideals, obtained by diagonalizing logarithmic differential operators:

Lemma 4.2.1 ([ATW19, Lemma 4.4.1]). *If $\mathrm{ord}_p(\mathcal{I}) = a_1$ and x_1 a corresponding maximal contact, then in $\mathbb{C}[\![x_1, \ldots, x_n]\!]$ we have*

$$C(\mathcal{I}, a) = (x_1^{a_1!}) + (x_1^{a_1!-1}\tilde{\mathcal{C}}_1) + \cdots + (x_1\tilde{\mathcal{C}}_{a_1!-1}) + \tilde{\mathcal{C}}_{a_1!},$$

where

$$\mathcal{C}_{a_1!} \subset (x_2, \ldots, x_n)^{a!} \subset k[\![x_2, \ldots, x_n]\!],$$

where $\mathcal{C}_{j-1} := \mathcal{D}^{\leq 1}(\mathcal{C}_j)$ satisfy $\mathcal{C}_k\mathcal{C}_l \subset \mathcal{C}_{k+l}$, and $\tilde{\mathcal{C}}_j = \mathcal{C}_j k[\![x_1, \ldots, x_n]\!]$.

4.3. **Admissibility and coefficient ideals.** Admissibility of centers is a notion used throughout resolution of singularities. The key point is that if one blows up an admissible center, some measurement of the singularity does not get worse.

Definition 4.3.1. J is $\mathcal{I}$-admissible if $v(J) \leq v(\mathcal{I})$.

Lemma 4.3.2 ([ATW19, Section 5.3.1]). *This is equivalent to $\mathcal{I}\mathcal{O}_{Y'} = E^\ell \mathcal{I}'$, with $E \subset \mathcal{O}_{Y'}$ the ideal of the exceptional divisor, $J = \bar{J}^\ell$ and $\mathcal{I}'$ an ideal.*

Indeed, on Y' the center J becomes E^ℓ, in particular principal.

Proposition 4.3.3 ([ATW19, Lemma 5.3.7]). *A center J is $\mathcal{I}$-admissible if and only if $J^{(a_1-1)!}$ is $C(\mathcal{I}, a_1)$- admissible.*

This is a consequence of formal decomposition.

4.4. **The key theorems.** It is now clear what remains to prove:

Theorem 4.4.1 ([ATW19, Theorem 5.2.1 and 5.6.1]). *The center $J_{\mathcal{I}}$ is well-defined and functorial.*

Theorem 4.4.2 ([ATW19, Theorem 5.4.1]). *The center $J_{\mathcal{I}}$ is $\mathcal{I}$-admissible.*

Theorem 4.4.3 ([ATW19, Theorem 5.5.1]).
(1) $C(\mathcal{I}, a_1)\mathcal{O}_{Y'} = E^{\ell'} C'$ with $\mathrm{inv}_{p'} C' < \mathrm{inv}_p(C(\mathcal{I}, a_1))$.
(2) $\mathcal{I}\mathcal{O}_{Y'} = E^\ell \mathcal{I}'$ with $\mathrm{inv}_{p'}\mathcal{I}' < \mathrm{inv}_p(\mathcal{I})$.

Remarkably, each of these theorems follows with little effort from homegeneity, formal decomposition, and induction.

5. Birational geometry and blowing up

5.1. **Birational geometry using smooth blowing up.** Smooth blowings up are remarkably useful in biarational geometry in characteristic 0. One can point out to two salient features making them useful:

(1) One can describe the change of geometry resulting from a smooth blowing up $Y' = Bl_Z(Y)$ of a smooth variety Y, describing the geometry of Y' in terms of that of Y, Z, and the position of Z in Y.
(2) If any two smooth varieties Y_1, Y_2 are related by a proper birational map, they are in fact connected through smooth blowings up.

Item (2) is Weak Factorization, see [Wło03, AKMW02]. It directly implies structural result, such as Bittner's theorem on the Grothendieck ring of varieties [Bit04, Theorem 3.1]. Examples of item (1) include the formula [Ful98, Proposition 6.7 and Example 8.3.9] for the intersection theory of Y', and Bondal and Orlov's semiorthogonal decomposition of $D(Y')$ [BO02, Theorem 4.2]. Together (1) and (2) imply results on birational invariance of certain biregular invariants, for instance Borisov and Libgober's results on elliptic genera [BL05].

5.2. **Birational geometry using weighted blowing up.** The situation of weighted blowings up is similar but incomplete:

(1) There is no doubt one can describe the change of geometry resulting from a weighted blowing up $Y' = Bl_J(Y)$.
(2) It follows from Weak Factorization and Bergh's destackification [Ber17] that if two smooth stackss Y_1, Y_2 are related by a proper birational map, they are in fact connected through weighted blowings up.

Regrding (2), indeed the paper [Wło05] preceding weak factorization provides a factorization in weighted blowings up and down for smooth varieties. Hu [Hu04] showed how to directly extend such factorization results to varieties with orbifold singularities.

As an example for (1), Kawamata [Kaw06, Section 5] generalized the theorem of Bondal and Orlov in great generality. It appears that the only reason other aspects are not readily available is that people did not have weighted blowings up in mind!

One point of caution though: weighted blowings up exhibit codimension-1 phenomena, as they include root constructions $Y' = Y(\sqrt[r]{D})$ along smooth divisors D. For instance, the plurigenera of Y' are in general bigger than those of Y.

5.3. **Strong factorization of toric mas.** Oda [OM75] conjectured that if Y, Y' are smooth toric varieties retaled by a proper toric birational map, then there is a sequence of smooth toric blowings up $Y'' \dashrightarrow Y$ such that $Y'' \dashrightarrow Y'$ is also a morphism factoring as a sequence of smooth blowings up. This is still a conjecture even in dimension 3. The paper [DSK11] describes an algorithm which, if it terminates, provides such a factorization in general. The algorithm was shown to terminate on millions of cases in dimension 3.

The same algorithm should apply for birational toric stacks and weighted blowings up, a more general, and therefore harder, case. However Ewald [Ewa86] showed that any two *three-dimentional* fans Σ_1, Σ_2 with the same support are related via a sequence of star subdivisions, if these are allowed to be centered at points which are not unimodular barycenters. This imediately implies the following:

Corollary 5.3.1 (The weighted strong Oda's conjecture for threefolds)**.** *Let Y, Y' be smooth* three dimensional *toric stacks related by a proper toric birational map. Then there is a sequence of weighted blowings up $Y'' \dashrightarrow Y$ such that $Y'' \dashrightarrow Y'$ is also a morphism factoring as a sequence of weighted blowings up.*

Ewald's factorization algorithm is a greedy algorithm using the three-dimensional situation. It is not known if it can be generalized to higher dimensions:

Conjecture 5.3.2 (The weighted strong Oda's conjecture)**.** *Let Y, Y' be smooth toric stacks of dimension > 3 related by a proper toric birational map. Then there is a sequence of weighted blowings up $Y'' \dashrightarrow Y$ such that $Y'' \dashrightarrow Y'$ is also a morphism factoring as a sequence of weighted blowings up.*

This is evidently *weaker* than the usual strong Oda's conjecture. Given that Oda's conjecture is unsolved in decades, one might give this possibly easier question a try.

Note that even here weighted blowings up result in codimension-1 phenomena through the back door: consider the cone $\sigma = \mathbb{R}_{\geq 0}$ with the lattice $\mathbb{N}^2$. Let Σ_1 be the standard subdivision along $(1,1)$, and let Σ_2 be the stacky fan obtained by subdividing along $(1,2)$, with lattices generated by edge generators. The stellar subdivision Σ_1' of Σ_1 along $(1,2)$ is not the same as the stellar subdivision Σ_2' of Σ_2 along the ray of $(1,1)$, since this ray is generated by $(2,2)$ in Σ_2! Rather $\Sigma_2' \to \Sigma_1'$ is the lattice alteration of the corresponding root construction.

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