# Resolution by weighted blowing up

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Rational points on irrational varieties

IHP, June 28, 2019

# How to resolve a curve

To resolve a singular curve $C$

(1) find a singular point $x \in C$,

(2) blow it up.

**Fact**

*$p_a$ gets smaller.*

# How to resolve a surface

To resolve a singular surface $S$ one wants to

(1) find the worst singular locus $C \in S$,

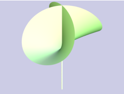
(2) $C$ is smooth - blow it up.

**Fact**

*This in general does not get better.*

# Example: Whitney's umbrella

Consider $S = V(x^2 - y^2 z)$ (image by Eleonore Faber).



(1) The worst singularity is the origin.

(2) In the $z$ chart we get
$x = x_3 z,\ y = y_3 z,$ giving
$x_3^2 z^2 - y_3^2 z^3 = 0,$ or $z^2(x_3^2 - y_3^2 z) = 0.$
The first term is exceptional, the second is the same as $S$.

Classical solution:

(a) Remember exceptional divisors (this is OK)

(b) Remember their history (this is a pain)

# Main result

Nevertheless:

**Theorem ($\aleph$-T-W, MM, "weighted Hironaka", characteristic 0)**

*There is a procedure $F$ associating to a singular subvariety $X \subset Y$ embedded with pure codimension $c$ in a smooth variety $Y$, a center $\bar{J}$ with blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\operatorname{maxinv}(X') < \operatorname{maxinv}(X)$. In particular, for some $n$ the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of $F$ has $X_n$ smooth.*

Here

*procedure*

means

*a functor for smooth surjective morphisms:*

if $f : Y_1 \twoheadrightarrow Y$ smooth then $J_1 = f^{-1}J$ and $Y_1' = Y_1 \times_Y Y'$.

# Preview on invariants

For $p \in X$ we define

$$\mathrm{inv}_p(X) \in \Gamma \subset \quad \mathbb{Q}_{\geq 0}^{\leq n},$$

with $\Gamma$ well-ordered, and show

## Proposition

- *it is lexicographically upper-semi-continuous, and*
- $p \in X$ *is smooth* $\Leftrightarrow \mathrm{inv}_p(X) = \min \Gamma$.

We define $\mathrm{maxinv}(X) = \max_p \mathrm{inv}_p(X)$.

## Example

$inv_p(V(x^2 - y^2 z)) = (2, 3, 3)$

## Remark

These invariants have been in our arsenal for ages.

# Preview of centers

If $\text{inv}_p(X) = \text{maxinv}(X) = (a_1, \ldots, a_k)$ then, locally at $p$

$$J = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

Write $(a_1, \ldots, a_k) = \ell(1/w_1, \ldots, 1/w_k)$ with $w_i, \ell \in \mathbb{N}$ and $\gcd(w_1, \ldots, w_k) = 1$. We set

$$\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k}).$$

**Example**

For $X = V(x^2 - y^2 z)$ we have $J = (x^2, y^3, z^3)$; $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$.

**Remark**

$J$ has been staring in our face for a while.

# Example: blowing up Whitney's umbrella $x^2 = y^2 z$

The blowing up $Y' \to Y$ makes $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$ principal. Explicitly:

- The $z$ chart has $x = w^3 x_3, y = w^2 y_3, z = w^2$ with chart

  $$Y' \ = \ [\,\mathrm{Spec}\,\mathbb{C}[x_3, y_3, w] \ / \ (\pm 1)\,],$$

  with action of $(\pm 1)$ given by $(x_3, y_3, w) \mapsto (-x_3, y_3, -w)$.
  The transformed equation is

  $$w^6 (x_3^2 - y_3^2),$$

  and the invariant of the proper transform $(x_3^2 - y_3^2)$ is $(2, 2) < (2, 3, 3)$.

In fact, $X$ has begged to be blown up like this all along.

# Definition of $Y' \to Y$

Let $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. Define the graded algebra

$$\mathcal{A}_{\bar{J}} \quad \subset \quad \mathcal{O}_Y[T]$$

as the integral closure of the image of

$$\begin{array}{ccc} \mathcal{O}_Y[Y_1, \ldots, Y_n] & \longrightarrow & \mathcal{O}_Y[T] \\ Y_i & \longmapsto & x_i T^{w_i}. \end{array}$$

Let

$$S_0 \subset \operatorname{Spec}_Y \mathcal{A}_{\bar{J}}, \quad S_0 = V((\mathcal{A}_{\bar{J}})_{>0}).$$

Then

$$Bl_{\bar{J}}(Y) \ := \ \mathcal{P}roj_Y \mathcal{A}_{\bar{J}} \ := \quad \left[ (\operatorname{Spec} \mathcal{A}_{\bar{J}} \smallsetminus S_0) \ / \ \mathbb{G}_m \right].$$

# Description of $Y' \to Y$

- **Charts:** The $x_1$-chart is

$$[\operatorname{Spec} k[u, x_2', \dots, x_n'] \;/\; \boldsymbol{\mu}_{w_1}],$$

with $x_1 = u^{w_1}$ and $x_i = u^{w_i} x_i'$ for $2 \le i \le k$, and induced action:

$$(u, x_2', \dots, x_n') \;\mapsto\; (\zeta u\,,\, \zeta^{-w_2} x_2'\,,\, \dots\,,\, \zeta^{-w_k} x_k',\; x_{k+1}', \dots, x_n').$$

- **Toric stack:** $Y'$ corresponds to the star subdivision $\Sigma := v_{\bar{J}} \star \sigma$ along

$$v_{\bar{J}} \;=\; (w_1, \dots, w_k, 0, \dots, 0),$$

with a natural toric stack structure.

# Examples: Defining $J$

(1) Consider $X = V(x^5 + x^3y^3 + y^8)$ at $p = (0,0)$; write $\mathcal{I} := \mathcal{I}_X$.

- Define $a_1 = \operatorname{ord}_p \mathcal{I} = 5$,
- and $x_1$ = any variable appearing in a degree-$a_1$ term $= x$.
- So $J_\mathcal{I} = (x^5,\ y^\star\ )$.
- To balance $x^5$ with $x^3y^3$ we need $x^2$ and $y^3$ to have the same weight, so $x^5$ and $y^{15/2}$ have the same weight.
- Since $15/2 < 8$ we use

$$J_\mathcal{I} = (x^5, y^{15/2}) \quad \text{and} \quad \bar{J}_\mathcal{I} = (x^{1/3}, y^{1/2}).$$

(2) If instead we took $X = V(x^5 + x^3y^3 + y^7)$, then since $7 < 15/2$ we would use

$$J_\mathcal{I} = (x^5, y^7) \quad \text{and} \quad \bar{J}_\mathcal{I} = (x^{1/7}, y^{1/5}).$$

# Examples: describing the blowing up

(1) Considering $X = V(x^5 + x^3y^3 + y^8)$ at $p = (0,0)$,

- the $x$-chart has $x = u^3, y = u^2y_1$ with $\boldsymbol{\mu}_3$-action, and equation
$$u^{15}(1 + y_1^3 + uy_1^8)$$
with smooth proper transform.
- The $y$-chart has $y = v^2, x = v^3x_1$ with $\boldsymbol{\mu}_2$-action, and equation
$$v^{15}(x_1^5 + x_1^3 + u)$$
with smooth proper transform.

(2) Considering $X = V(x^5 + x^3y^3 + y^7)$ at $p = (0,0)$,

- the $x$-chart has $x = u^7, y = u^5y_1$ with $\boldsymbol{\mu}_7$-action, and equation
$$u^{35}(1 + uy_1^3 + y_1^7)$$
with smooth proper transform.
- The $y$-chart has $y = v^5, x = v^7x_1$ with $\boldsymbol{\mu}_5$-action, and equation
$$v^{35}(x_1^5 + ux_1^3 + 1)$$
with smooth proper transform.

# Coefficient ideals

We must restrict to $x_1 = 0$ the data of all

$$\mathcal{I},\ \mathcal{D}\mathcal{I},\ \ldots,\ \mathcal{D}^{a_1-1}\mathcal{I}$$

with corresponding weights

$$a_1,\ a_1 - 1,\ \ldots,\ 1.$$

We combine these in

$$C(\mathcal{I}, a_1)\ :=\ \sum f\left(\mathcal{I}, \mathcal{D}\mathcal{I}, \ldots, \mathcal{D}^{a_1-1}\mathcal{I}\right),$$

where $f$ runs over monomials $f = t_0^{b_0} \cdots t_{a_1-1}^{b_{a_1-1}}$ with weights

$$\sum b_1(a_1 - i) \geq a_1!.$$

Define $\mathcal{I}[2] = C(\mathcal{I}, a_1)|_{x_1=0}$.

# Defining $J_{\mathcal{I}}$

## Definition

Let $a_1 = \mathrm{ord}_p\mathcal{I}$, with $x_1$ a regular element in $\mathcal{D}^{a_1-1}\mathcal{I}$ - a maximal contact. Suppose $\mathcal{I}[2]$ has invariant $\mathrm{inv}_p(\mathcal{I}[2])$ defined with parameters $\bar{x}_2, \ldots, \bar{x}_k$, with lifts $x_2, \ldots, x_k$. Set

$$\mathrm{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k) := \left(a_1, \frac{\mathrm{inv}_p(\mathcal{I}[2])}{(a_1-1)!}\right)$$

and

$$J_{\mathcal{I}} = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

## Example

(1) for $X = V(x^5 + x^3y^3 + y^8)$ we have $\mathcal{I}[2] = (y)^{180}$, so $J_{\mathcal{I}} = (x^5, y^{180/24}) = (x^5, y^{15/2})$.

(2) for $X = V(x^5 + x^3y^3 + y^7)$ we have $\mathcal{I}[2] = (y)^{7\cdot 24}$, so $J_{\mathcal{I}} = (x^5, y^7)$.

# What is $J$?

## Definition

Consider the Zariski-Riemann space $\mathbf{ZR}(X)$ with its sheaf of ordered groups $\Gamma$, and associated sheaf of rational ordered group $\Gamma \otimes \mathbb{Q}$.

- A valuative $\mathbb{Q}$-ideal is

$$\gamma \in H^0\left(\mathbf{ZR}(X), (\Gamma \otimes \mathbb{Q})_{\geq 0}\right).$$

- $\mathcal{I}_\gamma := \{f \in \mathcal{O}_X : v(f) \geq \gamma_v \forall v\}$.
- $v(\mathcal{I}) := (\min v(f) : f \in \mathcal{I})_v$.

A center is in particular a valuative $\mathbb{Q}$-ideal.

# Admissibility and coefficient ideals

**Definition**

$J$ is $\mathcal{I}$-admissible if $v(J) \leq v(\mathcal{I})$.

**Lemma**

*This is equivalent to* $\mathcal{I}\mathcal{O}_{Y'} = E^\ell \mathcal{I}'$, *with* $J = \bar{J}^\ell$ *and* $\mathcal{I}'$ *an ideal.*

Indeed, on $Y'$ the center $J$ becomes $E^\ell$, in particular principal.

**Proposition**

*$J$ is $\mathcal{I}$-admissible if and only if $J^{(a_1-1)!}$ is $C(\mathcal{I}, a_1)$- admissible.*

This is a consequence of the following technical, but known, lemma.

# Structure of coefficient ideals

**Lemma**

*If* $\operatorname{ord}_p(\mathcal{I}) = a_1$ *and* $x_1$ *a corresponding maximal contact, then in* $\mathbb{C}[\![x_1, \ldots, x_n]\!]$ *we have*

$$C(\mathcal{I}, a) = (x_1^{a_1!}) + (x_1^{a_1!-1}\tilde{\mathcal{C}}_1) + \cdots + (x_1\tilde{\mathcal{C}}_{a_1!-1}) + \tilde{\mathcal{C}}_{a_1!},$$

*where*

$$\mathcal{C}_{a_1!} \subset (x_2, \ldots, x_n)^{a!} \subset k[\![x_2, \ldots, x_n]\!],$$

*where* $\mathcal{C}_{j-1} := \mathcal{D}^{\leq 1}(\mathcal{C}_j)$ *satisfy* $\mathcal{C}_k\mathcal{C}_l \subset \mathcal{C}_{k+l}$*, and* $\tilde{\mathcal{C}}_j = \mathcal{C}_j k[\![x_1, \ldots, x_n]\!]$*.*

The lemma and proposition are proven by looking at monomials.

# The key theorems

**Theorem**

*The invariant is well-defined, USC, functorial.*

**Theorem**

*The center is well-defined.*

**Theorem**

$J_{\mathcal{I}}$ *is* $\mathcal{I}$*-admissible.*

**Theorem**

$C(\mathcal{I}, a_1)\mathcal{O}_{Y'} \;=\; E^{\ell'} C'$ *with* $\mathrm{inv}_{p'} C' \;<\; \mathrm{inv}_p(C(\mathcal{I}, a_1))$.

**Theorem**

$\mathcal{I}\mathcal{O}_{Y'} = E^{\ell}\mathcal{I}'$ *with* $\mathrm{inv}_{p'}\mathcal{I}' \;<\; \mathrm{inv}_p(\mathcal{I})$.

# The end

Thank you for your attention