Resolution and logarithmic resolution by weighted blowing up

Dan Abramovich, Brown University

Work with Michael Tëmkin and Jarosław Włodarczyk
and work by Ming Hao Quek

Also parallel work by M. McQuillan

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Example: Whitney's umbrella

"You can't just blow up the worst singular locus"

Consider $X = V(x^2 - y^2 z)$

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- The worst singularity is the origin.
- In the z chart we get
  $x = x'z, \ y = y'z,$ giving
  $x'^2 z^2 - y'^2 z^3 = 0, \quad \text{or} \quad z^2(x'^2 - y'^2 z) = 0.$

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  The first term is exceptional, the second is the same as X.

Two theorems

Nevertheless:

Theorem (ℵ-T-W, McQuillan, 2019, characteristic 0)

There is a functor F associating to a singular subvariety[a] $X \subset Y$ of a smooth variety Y, a center $\bar{J}$ with stack theoretic weighted blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\text{maxinv}(X') < \text{maxinv}(X)$. In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n smooth.

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There is a functor F associating to a singular subvariety[a] $X \subset Y$ of a smooth variety Y, a center $\bar{J}$ with stack theoretic weighted blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\operatorname{maxinv}(X') < \operatorname{maxinv}(X)$. In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n smooth.

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Theorem (Quek, 2020, characteristic 0)

There is a functor F associating to a logarithmically singular subvariety[a] $X \subset Y$ of a logarithmically smooth variety Y, a logarithmic center $\bar{J}$ with stack theoretic logarithmic blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\operatorname{maxloginv}(X') < \operatorname{maxloginv}(X)$. In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n logarithmically smooth.

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Context: families

Hironaka's theorem resolves varieties. What can you do with families of varieties $X \to B$?

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Logarithmically smooth $=$ toroidal:

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Theorem (ℵ-Karu, 2000)

There is a modification $X' \to B'$ which is logarithmically smooth.

Logarithmically smooth = toroidal:

- A toroidal morphism $X \to B$ of toroidal embeddings is étale locally isomorphic to a torus equivariant dominant morphism.

Examples of toroidal morphisms

e.g.

- $$\operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^2) \quad \to \quad \operatorname{Spec} \mathbb{C},$$
- $$\operatorname{Spec} \mathbb{C}[x] \quad \to \quad \operatorname{Spec} \mathbb{C}[x^2],$$
- toric blowups

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Given $X \to B$ there is a relatively functorial logarithmically smooth modification $X' \to B'$.

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Theorem (ℵ-T-W 2020)

Given $X \to B$ there is a relatively functorial logarithmically smooth modification $X' \to B'$.

- This respects $\mathrm{Aut}_B X$.
- Does not modify log smooth fibers.

Context: principalization

- Following Hironaka, the above theorem is based on embedded methods:

Theorem ($\aleph$-T-W 2020)

Given $Y \to B$ logarithmically smooth and $\mathcal{I} \subset \mathcal{O}_Y$, there is a relatively functorial logarithmically smooth modification $Y' \to B'$ such that $\mathcal{I}\mathcal{O}_{Y'}$ is monomial.

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- This is done by a sequence of logarithmic modifications, where in each step E becomes part of the divisor $D_{Y'}$.

Example 1

- $Y = \operatorname{Spec} k[x, u]; \quad D_Y = V(u); \quad B = \operatorname{Spec} k;$

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- Blow up $J = (x,u)$
- $\mathcal{I}\mathcal{O}_{Y'} = \mathcal{O}(-2E)$

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- $f : Y \to Y_0 \quad f^*v = u^2$ so $\mathcal{I} = f^*\mathcal{I}_0$

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- By functoriality blow up J_0 so that $f^*J_0 = J = (x,u)$.
- Blow up $J_0 = (x, \sqrt{v})$
- Whatever J_0 is, the blowup is a stack.

Example 1/2: charts

- x chart: $v = v'x^2$:

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$$(x^2, v) = ({x'}^2 w^2, w^2) = (w^2)$$

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- The schematic quotient of the above is not toroidal.

Resolution again

Theorem (ℵ-T-W, McQuillan, characteristic 0)

There is a functor F associating to a singular subvariety $X \subset Y$ of a smooth variety Y, a center $\bar{J}$ with stack theoretic weighted blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\mathrm{maxinv}(X') < \mathrm{maxinv}(X)$. In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n smooth.

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Example

- For $X = V(x^2 - y^2 z)$ we have $inv_p(X) = (2, 3, 3)$
- We read it from the degrees of terms.
- The center is:
  $J = (x^2, y^3, z^3)$; $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$.

Example: blowing up Whitney's umbrella $x^2 = y^2 z$

The blowing up $Y' \to Y$ makes $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$ principal. Explicitly:

- The z chart has $x = w^3 x', y = w^2 y', z = w^2$ with chart

$$Y' \ = \ [\operatorname{Spec} \mathbb{C}[x', y', w] \ / \ (\pm 1)\,],$$

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  The transformed equation is

  $$w^6 ({x'}^2 - {y'}^2),$$

  and the invariant of the proper transform ${x'}^2 - {y'}^2$ is $(2, 2) < (2, 3, 3)$.

Order (following Kollár's book)

We fix Y smooth and $\mathcal{I} \subset \mathcal{O}_Y$.

Definition

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- The invariant starts with $a_1 = \operatorname{ord}_p(\mathcal{I})$.

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Proposition

The order is upper semicontinuous.

Proof.

$V(\mathcal{D}^{a-1}\mathcal{I}) = \{p : \mathrm{ord}_p(\mathcal{I}) \geq a\}$. ♠

Maximal contact (following Kollár's book)

Definition (Giraud, Hironaka)

A regular parameter $x_1 \in \mathcal{D}^{a_1-1}\mathcal{I}_p$ is called a maximal contact element.

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Lemma (Giraud, Hironaka)

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Example

For $\mathcal{I} = (x^2 - y^2 z)$ we have $\text{ord}_p\mathcal{I} = 2$ with $x_1 = x$
(or αx+h.o.t. in $\mathcal{D}(\mathcal{I})$).

Coefficient ideals (treated following Kollár)

We must restrict to $x_1 = 0$ the data of all

$$\mathcal{I},\ \mathcal{D}\mathcal{I},\ \ldots,\ \mathcal{D}^{a_1-1}\mathcal{I}$$

with corresponding weights $a_1,\ a_1 - 1,\ \ldots,\ 1.$

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We combine these in

$$C(\mathcal{I}, a_1)\ :=\ \sum f\left(\mathcal{I}, \mathcal{D}\mathcal{I}, \ldots, \mathcal{D}^{a_1-1}\mathcal{I}\right),$$

where f runs over monomials $f = t_0^{b_0}\cdots t_{a_1-1}^{b_{a_1-1}}$ with weights

$$\sum b_i(a_1 - i) \geq a_1!.$$

Define $\mathcal{I}[2] = C(\mathcal{I}, a_1)|_{x_1=0}$.

Defining $J_{\mathcal{I}}$

Again $a_1 = \mathrm{ord}_p \mathcal{I}$ and x_1 maximal contact.
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Definition

Suppose $\mathcal{I}[2]$ has invariant $\mathrm{inv}_p(\mathcal{I}[2])$ defined with parameters $\bar{x}_2, \ldots, \bar{x}_k$, with lifts $x_2, \ldots, x_k$.

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$$\text{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k) := \left(a_1, \frac{\text{inv}_p(\mathcal{I}[2])}{(a_1 - 1)!}\right)$$

and

$$J_{\mathcal{I}} = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

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Write $(a_1, \ldots, a_k) = \ell(1/w_1, \ldots, 1/w_k)$ with $w_i, \ell \in \mathbb{N}$ and $\gcd(w_1, \ldots, w_k) = 1$. We set

$$\bar{J}_{\mathcal{I}} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k}).$$

Examples of $J_{\mathcal{I}}$

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(0) for $X = V(x^2 + y^2 z)$ we have $\mathcal{I}[2] = (y^2 z)$, leading to $J_{\mathcal{I}} = (x^2, y^3, z^3)$, $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2}, z^{1/2})$

(1) for $X = V(x^5 + x^3 y^3 + y^8)$

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(1) for $X = V(x^5 + x^3 y^3 + y^8)$ we have $\mathcal{I}[2] = (y)^{180}$, so $J_{\mathcal{I}} = (x^5, y^{180/24}) = (x^5, y^{15/2})$, $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2})$.

Implementation: Jonghyun Lee, Anne Frühbis-Krüger.

Properties of the invariant

Proposition

- inv_p *is well-defined.*
- inv_p *is upper-semi-continuous.*
- inv_p *is functorial.*
- inv_p *takes values in a well-ordered set.*[a]

[a] see slides "invariant"

We define $\mathrm{maxinv}(X) = \max_p \mathrm{inv}_p(X)$.

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We define $\mathrm{maxinv}(X) = \max_p \mathrm{inv}_p(X)$.
The invariant is well defined because of the MC-invariance property of $C(\mathcal{I}, a_1)$. The rest is induction!

Definition and description of $Y' \to Y$

$Y' = \mathcal{P}roj_Y(\oplus \mathcal{I}_{\bar{J}^n})$, the stack-theoretic $\mathcal{P}roj$,[1]

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Definition and description of $Y' \to Y$

$Y' = \mathcal{P}roj_Y(\oplus \mathcal{I}_{\bar{J}^n})$, the stack-theoretic $\mathcal{P}roj$,[1]
explicitly:

- The x_1-chart is
$$[\operatorname{Spec} k[u, x_2', \ldots, x_n'] \;/\; \boldsymbol{\mu}_{w_1}],$$
with $x_1 = u^{w_1}$ and $x_i = u^{w_i} x_i'$ for $2 \le i \le k$, and induced action:
$$(u, x_2', \ldots, x_n') \;\mapsto\; (\zeta u\,,\, \zeta^{-w_2} x_2'\,,\, \ldots\,,\, \zeta^{-w_k} x_k',\; x_{k+1}', \ldots, x_n').$$

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What is J?

Definition

Consider the Zariski-Riemann space $\mathbf{ZR}(Y)$ with its sheaf of ordered groups Γ, and associated sheaf of rational ordered group $\Gamma \otimes \mathbb{Q}$.

- A valuative $\mathbb{Q}$-ideal is

$$\gamma \in H^0\left(\mathbf{ZR}(Y), (\Gamma \otimes \mathbb{Q})_{\geq 0}\right).$$

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- $\mathcal{I} \quad \mapsto \quad v(\mathcal{I}) := (\min v(f) : f \in \mathcal{I})_v.$

A center $(x_1^{a_1}, \ldots, x_k^{a_k})$ is in particular a valuative $\mathbb{Q}$-ideal.

$$\left(\min_i \{a_i v(x_i)\}\right)_v .$$

It is also an idealistic exponent or graded sequence of ideals.

Admissibility and coefficient ideals

Definition

J is $\mathcal{I}$-admissible if $J \leq v(\mathcal{I})$.[a]

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Proposition

A center J is $\mathcal{I}$-admissible if and only if $J^{(a_1-1)!}$ is $C(\mathcal{I}, a_1)$-admissible.

The key theorems

Theorem

- $\mathrm{inv}_p(\mathcal{I})$ *is the maximal invariant of an* $\mathcal{I}$*-admissible center.*
- $J_{\mathcal{I}}$ *is well-defined: it is the unique admissible center of maximal invariant.*[a]

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Theorem

- $C(\mathcal{I}, a_1)\mathcal{O}_{Y'} \;=\; E^{\ell'} C'$ *with* $\mathrm{inv}_{p'} C' \;<\; \mathrm{inv}_p(C(\mathcal{I}, a_1))$.
- $\mathcal{I}\mathcal{O}_{Y'} = E^{\ell}\mathcal{I}'$ *with* $\mathrm{inv}_{p'}\mathcal{I}' \;<\; \mathrm{inv}_p(\mathcal{I})$.[a]

[a] Slides "principaliztion"

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- Instead work with logarithmic derivative in z.
- $\operatorname{maxloginv}(\mathcal{I}') = (3,3,\infty)$ with center $(y^3, x^3, z^{3/2})$ and reduced logarithmic center $(y, x, z^{1/2})$.
- This reduces logarithmic invariants respecting logarithmic, hence exceptional, divisors.

The end

Thank you for your attention