Resolution and logarithmic resolution by weighted blowing up

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Work with Michael Tëmkin and Jarosław Włodarczyk
and work by Ming Hao Quek

Also parallel work by M. McQuillan

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How to resolve

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Fact

This works for curves but not in general.

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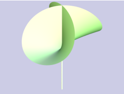

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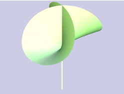

- The worst singularity is the origin.
- In the z chart we get
  $x = x'z,\ y = y'z,$ giving
  $x'^2 z^2 - y'^2 z^3 = 0, \quad \text{or} \quad z^2(x'^2 - y'^2 z) = 0.$

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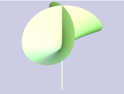

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  $x = x'z,\ y = y'z,$ giving
  $x'^2 z^2 - y'^2 z^3 = 0,$ or $z^2(x'^2 - y'^2 z) = 0.$
  The first term is exceptional, the second is the same as X.

Two theorems

Nevertheless:[1]

Theorem ($\aleph$-T-W, McQuillan, 2019, characteristic 0)

There is a functor F associating to a singular subvariety[a] $X \subset Y$ of a smooth variety Y, a center $\bar{J}$ with stack theoretic weighted blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\operatorname{maxinv}(X') < \operatorname{maxinv}(X)$. In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n smooth.

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Theorem (Quek, 2020, characteristic 0)

There is a functor F associating to a logarithmically singular subvariety[a] $X \subset Y$ of a logarithmically smooth variety Y, a logarithmic center $\bar{J}$ with stack theoretic logarithmic blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\operatorname{maxloginv}(X') < \operatorname{maxloginv}(X)$. In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n logarithmically smooth.

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- The z chart has $x = w^3 x', y = w^2 y', z = w^2$ with chart

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  with action of (± 1) given by $(x', y', w) \mapsto (-x', y', -w)$.
  The transformed equation is

$$w^6 (x'^2 - y'^2),$$

  and the invariant of the proper transform $x'^2 - y'^2$ is $(2,2) < (2,3,3)$.

Order (following Kollár's book)

We fix Y smooth and $\mathcal{I} \subset \mathcal{O}_Y$.

Definition

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Proposition

The order is upper semicontinuous.

Proof.

$V(\mathcal{D}^{a-1}\mathcal{I}) = \{p : \mathrm{ord}_p(\mathcal{I}) \geq a\}$. ♠

Maximal contact (following Kollár's book)

Definition (Giraud, Hironaka)

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Example

For $\mathcal{I} = (x^2 - y^2 z)$ we have $\mathrm{ord}_p\mathcal{I} = 2$ with $x_1 = x$
(or αx+h.o.t. in $\mathcal{D}(\mathcal{I})$).

Coefficient ideals (treated following Kollár)

We must restrict to $x_1 = 0$ the data of all

$$\mathcal{I},\ \mathcal{D}\mathcal{I},\ \ldots,\ \mathcal{D}^{a_1-1}\mathcal{I}$$

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We combine these in

$$C(\mathcal{I}, a_1)\ :=\ \sum f\left(\mathcal{I}, \mathcal{D}\mathcal{I}, \ldots, \mathcal{D}^{a_1-1}\mathcal{I}\right),$$

where f runs over monomials $f = t_0^{b_0} \cdots t_{a_1-1}^{b_{a_1-1}}$ with weights

$$\sum b_i(a_1 - i) \geq a_1!.$$

Define $\mathcal{I}[2] = C(\mathcal{I}, a_1)|_{x_1=0}$.

Defining $\mathrm{inv}_p(\mathcal{I})$ and $J_{\mathcal{I}}$

Again $a_1 = \mathrm{ord}_p\mathcal{I}$ and x_1 maximal contact.
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$$\mathrm{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k) := \left(a_1, \frac{\mathrm{inv}_p(\mathcal{I}[2])}{(a_1 - 1)!}\right)$$

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$$J_{\mathcal{I}} = (x_1^{a_1}, x_2^{a_2}, \ldots, x_k^{a_k}).$$

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Write $(a_1, \ldots, a_k) = \ell(1/w_1, \ldots, 1/w_k)$ with $w_i, \ell \in \mathbb{N}$ and $\gcd(w_1, \ldots, w_k) = 1$. We set

$$\bar{J}_{\mathcal{I}} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k}).$$

What is J?

Definition

Consider the Zariski-Riemann space $\mathbf{ZR}(Y)$ with its sheaf of ordered groups Γ, and associated sheaf of rational ordered group $\Gamma \otimes \mathbb{Q}$.

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- $\mathcal{I}_\gamma := \{f \in \mathcal{O}_Y : v(f) \geq \gamma_v \forall v\}$.
- $\mathcal{I} \quad \mapsto \quad v(\mathcal{I}) := (\min v(f) : f \in \mathcal{I})_v$.

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A center $(x_1^{a_1}, \ldots, x_k^{a_k})$ is in particular a valuative $\mathbb{Q}$-ideal.

$$\left(\min_i \{a_i v(x_i)\}\right)_v.$$

It is also an idealistic exponent or graded sequence of ideals.

Examples of $J_{\mathcal{I}}$

$$\mathrm{inv}_p(\mathcal{I}) = (a_1, \ldots, a_k) := \left(a_1, \frac{\mathrm{inv}_p(\mathcal{I}[2])}{(a_1-1)!}\right), \quad \text{with} \quad J_{\mathcal{I}} = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

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Example

(1) for $X = V(x^2 + y^2 z)$ we have $\mathcal{I}[2] = (y^2 z)$, leading to $J_{\mathcal{I}} = (x^2, y^3, z^3)$, $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2}, z^{1/2})$

(2) for $X = V(x^5 + x^3 y^3 + y^8)$

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(2) for $X = V(x^5 + x^3 y^3 + y^8)$ we have $\mathcal{I}[2] = (y)^{180}$, so $J_{\mathcal{I}} = (x^5, y^{180/24}) = (x^5, y^{15/2})$, $\bar{J}_{\mathcal{I}} = (x^{1/3}, y^{1/2})$.

Implementation: Jonghyun Lee, Anne Frühbis-Krüger.

Definition of $Y' \to Y$

Let $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. Define the graded algebra

$$\mathcal{A}_{\bar{J}} \quad \subset \quad \mathcal{O}_Y[T]$$

as the integral closure of the image of

$$\begin{array}{ccc} \mathcal{O}_Y[Y_1, \ldots, Y_n] & \longrightarrow & \mathcal{O}_Y[T] \\ Y_i & \longmapsto & x_i T^{w_i}. \end{array}$$

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Let

$$S_0 \subset \operatorname{Spec}_Y \mathcal{A}_{\bar{J}}, \quad S_0 = V((\mathcal{A}_{\bar{J}})_{>0}).$$

Then

$$Bl_{\bar{J}}(Y) \; := \; \mathcal{P}roj_Y \mathcal{A}_{\bar{J}} \; := \quad \left[(\operatorname{Spec} \mathcal{A}_{\bar{J}} \smallsetminus S_0) \; / \; \mathbb{G}_m \right].$$

Local description of $Y' \to Y$

$Y' = \mathcal{P}roj_Y\,(\oplus \mathcal{I}_{\bar{J}^n})$, the stack-theoretic $\mathcal{P}roj$,[2]

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$Y' = \mathcal{P}roj_Y(\oplus \mathcal{I}_{\bar{J}^n})$, the stack-theoretic $\mathcal{P}roj$,[2]
explicitly:

- The x_1-chart is
$$[\operatorname{Spec} k[u, x_2', \dots, x_n'] \,/\, \boldsymbol{\mu}_{w_1}],$$
with $x_1 = u^{w_1}$ and $x_i = u^{w_i} x_i'$ for $2 \le i \le k$, and induced action:
$$(u, x_2', \dots, x_n') \;\mapsto\; (\zeta u\,,\, \zeta^{-w_2} x_2'\,,\, \dots\,,\, \zeta^{-w_k} x_k',\; x_{k+1}', \dots, x_n').$$

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Properties of the invariant

Proposition

- inv_p *is well-defined.*
- inv_p *is upper-semi-continuous.*
- inv_p *is functorial.*
- inv_p *takes values in a well-ordered set.*[a]

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We define $\mathrm{maxinv}(X) = \max_p \mathrm{inv}_p(X)$.
The invariant is well defined because of the MC-invariance property of $C(\mathcal{I}, a_1)$. The rest is induction!

Admissibility and coefficient ideals

Definition

J is $\mathcal{I}$-admissible if $J \leq v(\mathcal{I})$.[a]

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Proposition

A center J is $\mathcal{I}$-admissible if and only if $J^{(a_1-1)!}$ is $C(\mathcal{I}, a_1)$-admissible.

The key theorems

Theorem

- $\mathrm{inv}_p(\mathcal{I})$ *is the maximal invariant of an* $\mathcal{I}$*-admissible center.*
- $J_{\mathcal{I}}$ *is well-defined: it is the unique admissible center of maximal invariant.*[a]

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Theorem

- $C(\mathcal{I}, a_1)\mathcal{O}_{Y'} \;=\; E^{\ell'} C'$ *with* $\mathrm{inv}_{p'} C' \;<\; \mathrm{inv}_p(C(\mathcal{I}, a_1))$.
- $\mathcal{I}\mathcal{O}_{Y'} = E^{\ell}\mathcal{I}'$ *with* $\mathrm{inv}_{p'}\mathcal{I}' \;<\; \mathrm{inv}_p(\mathcal{I})$.[a]

[a] Slides "principaliztion"

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- Instead work with logarithmic derivative in z.
- $\text{maxloginv}(\mathcal{I}') = (3, 3, \infty)$ with center $(y^3, x^3, z^{3/2})$ and reduced logarithmic center $(y, x, z^{1/2})$.

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- $\mathrm{maxloginv}(\mathcal{I}') = (3, 3, \infty)$ with center $(y^3, x^3, z^{3/2})$ and reduced logarithmic center $(y, x, z^{1/2})$.
- This reduces logarithmic invariants respecting logarithmic, hence exceptional, divisors.

The end

Thank you for your attention