Definition of $Y' \to Y$

Let $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. Define the graded algebra

$$\mathcal{A}_{\bar{J}} \quad = \quad \bigoplus (\mathcal{I}_{\bar{J}^m}) T^m \subset \quad \mathcal{O}_Y[T],$$

with monomial ideals

$$(\mathcal{A}_{\bar{J}})_m \quad = \quad \left(x_1^{b_1} \cdots x_n^{b_n} \ : \ \sum b_i w_i \geq m \right). \quad \text{Note}(\mathcal{A}_{\bar{J}})_0 = \mathcal{O}_Y.$$

It is the integral closure of the image of

$$\begin{array}{ccc} \mathcal{O}_Y[Y_1, \ldots, Y_n] & \longrightarrow & \mathcal{O}_Y[T] \\ Y_i & \longmapsto & x_i T^{w_i}. \end{array}$$

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Let $S_0 \subset \operatorname{Spec}_Y \mathcal{A}_{\bar J}, \quad S_0 = V((\mathcal{A}_{\bar J})_{>0})$. Then

$$Bl_{\bar J}(Y) \ := \ \mathcal{P}roj_Y \mathcal{A}_{\bar J} \ := \quad \left[(\operatorname{Spec} \mathcal{A}_{\bar J} \smallsetminus S_0) \ / \ \mathbb{G}_m \right].$$

Description of $Y' \to Y$: charts

- The x_1-chart is
$$[\operatorname{Spec} k[u, x_2', \ldots, x_n'] \;/\; \boldsymbol{\mu}_{w_1}],$$
with $x_1 = u^{w_1}$ and $x_i = u^{w_i} x_i'$ for $2 \leq i \leq k$, and induced action:
$$(u, x_2', \ldots, x_n') \;\mapsto\; (\zeta u\,,\, \zeta^{-w_2} x_2'\,,\, \ldots\,,\, \zeta^{-w_k} x_k',\; x_{k+1}', \ldots, x_n').$$

Description of $Y' \to Y$: toric DM stack

- Y' corresponds to the star subdivision $\Sigma := v_{\bar{J}} \star \sigma$ along

$$v_{\bar{J}} \;=\; (w_1, \ldots, w_k, 0, \ldots, 0),$$

with a natural toric stack structure, with the cone

$$\sigma_i = \langle v_{\bar{J}}, e_1, \ldots, \hat{e}_i, \ldots, e_n \rangle$$

endowed with the sublattice $N_i \subset N$ generated by the elements

$$v_{\bar{J}}, e_1, \ldots, \hat{e}_i, \ldots, e_n,$$

for all $i = 1, \ldots, k$.

Description of $Y' \to Y$: quotient of the Cox construction

- Consider $\operatorname{Spec} k[x_1, \ldots, x_n, T]$ with $\mathbb{G}_m$ action with weights $(w_1, \ldots, w_n, -1)$.
  Let U be the open set where at least one of the x_i is a unit.
  Then $Y' = [U/\mathbb{G}_m]$.
  It is an example of a *fantastack* [Geraschenko-Satriano], the stack quotient of a Cox construction.