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Hironaka's theorem resolves varieties. What can you do with families of varieties $X \to B$?

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- A toroidal embedding $U_X \subset X$ is an open embedding étale locally isomorphic to toric $T \subset V$.
- A toroidal morphism $X \to B$ of toroidal embeddings is étale locally isomorphic to a torus equivariant dominant morphism.

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A toric morphism $X \to B$ of toric varieties is a torus equivariant morphism.e.g.

- $$\operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^2) \quad \to \quad \operatorname{Spec} \mathbb{C},$$
- $$\operatorname{Spec} \mathbb{C}[x] \quad \to \quad \operatorname{Spec} \mathbb{C}[x^2],$$
- toric blowups

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Theorem ($\aleph$-T-W 2020)

Given $X \to B$ there is a relatively functorial logarithmically smooth modification $X' \to B'$.

- This respects $\mathrm{Aut}_B X$.
- Does not modify log smooth fibers.

Context: principalization

- Following Hironaka, the above theorem is based on embedded methods:

Theorem (ℵ-T-W 2020)

Given $Y \to B$ logarithmically smooth and $\mathcal{I} \subset \mathcal{O}_Y$, there is a relatively functorial logarithmically smooth modification $Y' \to B'$ such that $\mathcal{I}\mathcal{O}_{Y'}$ is monomial.

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- This is done by a sequence of logarithmic modifications, where in each step E becomes part of the divisor $D_{Y'}$.

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- Blow up $J = (x, u)$
- $\mathcal{I}\mathcal{O}_{Y'} = \mathcal{O}(-2E)$

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- $f : Y \to Y_0 \quad f^* v = u^2 \quad \text{so} \quad \mathcal{I} = f^* \mathcal{I}_0$

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- By functoriality blow up J_0 so that $f^*J_0 = J = (x, u)$.
- Blow up $J_0 = (x, \sqrt{v})$
- Whatever J_0 is, the blowup is a stack.

Example 1/2: charts

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- The schematic quotient of the above is not toroidal.