Formal decomposition

Following Encinas–Villamayor, consider the algebra $\mathcal{G} = \oplus \mathcal{G}_i$ generated by $\mathcal{D}^j(\mathcal{I})$ in degree $a_1 - j$, for $0 \leq j \leq a_1 - 1$. We have $\mathcal{G}_{a_1!} = C(\mathcal{I}, a_1)$.
Writing formally $Y = \operatorname{Spec} k[\![x_1, \ldots, x_n]\!]$, with $H = V(x_1)$ maximal contact, we consider $\pi : Y \to H$.
Let $\tilde{\mathcal{G}}_i = \pi^*(\mathcal{G}_i|_H)$.

Proposition (Formal decomposition)

$$C(\mathcal{I}, a_1) = (x_1^{a_1!}) + (x_1^{a_1!-1})\tilde{\mathcal{G}}_1 + \cdots + (x_1)\tilde{\mathcal{G}}_{a_1!-1} + \tilde{\mathcal{G}}_{a_1!}.$$

This is proven by decomposing into eigenspaces for $x_1 \frac{\partial}{\partial x_1}$.

Proposition ($\mathcal{D}$-balanced property (Kollár))

$\tilde{\mathcal{G}}_{a_1!-j}^{a_1!} \subset \tilde{\mathcal{G}}_{a_1!}^{a_1!-j}$.

The center is admissible

Theorem

$J_{\mathcal{I}}$ is $\mathcal{I}$-admissible.

- This is equivalent to $J^{(a_1-1)!}$ is $C(\mathcal{I}, a_1)$-admissible.
- One checks that $J^{(a_1-1)!}$ is admissible for each term in the formal decomposition.
- Hence it is admissible.

The unique admissibility theorem

Theorem

$J_{\mathcal{I}} = (x_1^{a_1}, \ldots, x_k^{a_k})$ *is the unique admissible center of maximal invariant.*

- First if $J' = ({x'_1}^{b_1}, \ldots, {x_m}'^{a_m})$ is admissible one sees that $b_1 \leq a_1$, otherwise $v_J(f) < 1$ for $f \in \mathcal{I}$ of order a_1.
- Assume now $(b_1, \ldots, b_m) > (a_1 \ldots, a_k)$. So $a_1 = b_1$.
- With a bit more work one may assume $J' = ({x_1}^{a_1}, {x'_2}^{b_2}, \ldots, {x_m}'^{b_m})$, with $x'_i \in k[\![x_2, \ldots, x_n]\!]$.
- Consider the formal completion. Induction gives $(a_1 - 1)!(a_2, \ldots, a_k)$ is the maximal invariant of $\tilde{\mathcal{G}}_{a_1!}$, with unique center $(x_2^{a_2}, \ldots, x_k^{a_k})^{(a_1-1)!}$.
- On the other hand $({x'_1}^{a_1}, {x'_2}^{b_2}, \ldots, {x_m}'^{b_m})^{(a_1-1)!} \leq v(\tilde{\mathcal{G}}_{a_1!})$, giving equality throughout.