Resolution by weighted blowing up

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Joint work with Michael Tëmkin and Jarosław Włodarczyk

Also parallel work by M. McQuillan with G. Marzo

Rational points on irrational varieties

Columbia, September 13, 2019

How to resolve a curve

To resolve a singular curve C

(1) find a singular point $x \in C$,

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Fact

p_a gets smaller.

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To resolve a singular surface S one wants to

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Fact

This in general does not get better.

Example: Whitney's umbrella

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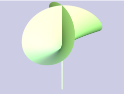

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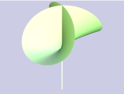

(1) The worst singularity is the origin.

(2) In the z chart we get
$x = x_3 z, \ y = y_3 z,$ giving
$x_3^2 z^2 - y_3^2 z^3 = 0,$ or $z^2(x_3^2 - y_3^2 z) = 0.$

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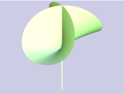

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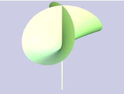

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Classical solution:

(a) Remember exceptional divisors (this is OK)

(b) Remember their history (this is a pain)

Main result

Nevertheless:

Theorem (ℵ-T-W, MM, "weighted Hironaka", characteristic 0)

There is a procedure F associating to a singular subvariety $X \subset Y$ embedded with pure codimension c in a smooth variety Y, a center $\bar{J}$ with blowing up $Y' \to Y$ and proper transform $(X' \subset Y') = F(X \subset Y)$ such that $\operatorname{maxinv}(X') < \operatorname{maxinv}(X)$. In particular, for some n the iterate $(X_n \subset Y_n) := F^{\circ n}(X \subset Y)$ of F has X_n smooth.

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Here

procedure

means

a functor for smooth surjective morphisms:

if $f : Y_1 \twoheadrightarrow Y$ smooth then $J_1 = f^{-1}J$ and $Y_1' = Y_1 \times_Y Y'$.

Preview on invariants

For $p \in X$ we define
$$\operatorname{inv}_p(X) \in \Gamma \subset \quad \mathbb{Q}_{\geq 0}^{\leq n},$$
with Γ well-ordered, and show

Proposition

- *it is lexicographically upper-semi-continuous, and*
- $p \in X$ *is smooth* $\Leftrightarrow \operatorname{inv}_p(X) = \min \Gamma$.

We define $\operatorname{maxinv}(X) = \max_p \operatorname{inv}_p(X)$.

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Remark

These invariants have been in our arsenal for ages.

Preview of centers

If $\text{inv}_p(X) = \text{maxinv}(X) = (a_1, \ldots, a_k)$ then, locally at p

$$J = (x_1^{a_1}, \ldots, x_k^{a_k}).$$

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Write $(a_1, \ldots, a_k) = \ell(1/w_1, \ldots, 1/w_k)$ with $w_i, \ell \in \mathbb{N}$ and $\gcd(w_1, \ldots, w_k) = 1$. We set

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Remark

$\bar{J}$ has been staring in our face for a while.

Example: blowing up Whitney's umbrella $x^2 = y^2 z$

The blowing up $Y' \to Y$ makes $\bar{J} = (x^{1/3}, y^{1/2}, z^{1/2})$ principal. Explicitly:

- The z chart has $x = w^3 x_3, y = w^2 y_3, z = w^2$ with chart

$$Y' = [\operatorname{Spec} \mathbb{C}[x_3, y_3, w] \,/\, (\pm 1)],$$

with action of (± 1) given by $(x_3, y_3, w) \mapsto (-x_3, y_3, -w)$.

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In fact, X has begged to be blown up like this all along.

Definition of $Y' \to Y$

Let $\bar{J} = (x_1^{1/w_1}, \ldots, x_k^{1/w_k})$. Define the graded algebra

$$\mathcal{A}_{\bar{J}} \quad \subset \quad \mathcal{O}_Y[T]$$

as the integral closure of the image of

$$\begin{array}{ccc} \mathcal{O}_Y[Y_1, \ldots, Y_n] & \longrightarrow & \mathcal{O}_Y[T] \\ Y_i & \longmapsto & x_i T^{w_i}. \end{array}$$

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Let

$$S_0 \subset \operatorname{Spec}_Y \mathcal{A}_{\bar{J}}, \quad S_0 = V((\mathcal{A}_{\bar{J}})_{>0}).$$

Then

$$Bl_{\bar{J}}(Y) \ := \ \mathcal{P}roj_Y \mathcal{A}_{\bar{J}} \ := \quad \left[(\operatorname{Spec} \mathcal{A}_{\bar{J}} \smallsetminus S_0) \ / \ \mathbb{G}_m \right].$$

Description of $Y' \to Y$

- **Charts:** The x_1-chart is

$$[\operatorname{Spec} k[u, x_2', \ldots, x_n'] \;/\; \boldsymbol{\mu}_{w_1}],$$

with $x_1 = u^{w_1}$ and $x_i = u^{w_i} x_i'$ for $2 \leq i \leq k$, and induced action:

$$(u, x_2', \ldots, x_n') \;\mapsto\; (\zeta u\,,\, \zeta^{-w_2} x_2'\,,\, \ldots\,,\, \zeta^{-w_k} x_k',\; x_{k+1}', \ldots, x_n').$$

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- **Toric stack:** Y' corresponds to the star subdivision $\Sigma := v_{\bar{J}} \star \sigma$ along

$$v_{\bar{J}} \;=\; (w_1, \ldots, w_k, 0, \ldots, 0),$$

with a natural toric stack structure.

Examples: Defining J

(1) Consider $X = V(x^5 + x^3y^3 + y^8)$ at $p = (0,0)$; write $\mathcal{I} := \mathcal{I}_X$.

- Define $a_1 = \mathrm{ord}_p\mathcal{I} = 5$,
- and $x_1 =$ any variable appearing in a degree-a_1 term $= x$.
- So $J_{\mathcal{I}} = (x^5,\ y^{\star}\ )$.

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(2) If instead we took $X = V(x^5 + x^3y^3 + y^7)$, then since $7 < 15/2$ we would use

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Examples: describing the blowing up

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(2) Considering $X = V(x^5 + x^3y^3 + y^7)$ at $p = (0, 0)$,

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$$u^{35}(1 + uy_1^3 + y_1^7)$$
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Coefficient ideals

We must restrict to $x_1 = 0$ the data of all

$$\mathcal{I},\ \mathcal{D}\mathcal{I},\ \ldots,\ \mathcal{D}^{a_1-1}\mathcal{I}$$

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We combine these in

$$C(\mathcal{I}, a_1)\ :=\ \sum f\left(\mathcal{I}, \mathcal{D}\mathcal{I}, \ldots, \mathcal{D}^{a_1-1}\mathcal{I}\right),$$

where f runs over monomials $f = t_0^{b_0} \cdots t_{a_1-1}^{b_{a_1-1}}$ with weights

$$\sum b_1(a_1 - i) \geq a_1!.$$

Define $\mathcal{I}[2] = C(\mathcal{I}, a_1)|_{x_1=0}$.

Defining $J_{\mathcal{I}}$

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Example

(1) for $X = V(x^5 + x^3y^3 + y^8)$ we have $\mathcal{I}[2] = (y)^{180}$, so $J_{\mathcal{I}} = (x^5, y^{180/24}) = (x^5, y^{15/2})$.

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(2) for $X = V(x^5 + x^3y^3 + y^7)$ we have $\mathcal{I}[2] = (y)^{7\cdot 24}$, so $J_{\mathcal{I}} = (x^5, y^7)$.

What is J?

Definition

Consider the Zariski-Riemann space $\mathbf{ZR}(Y)$ with its sheaf of ordered groups Γ, and associated sheaf of rational ordered group $\Gamma \otimes \mathbb{Q}$.

- A valuative $\mathbb{Q}$-ideal is

$$\gamma \in H^0\left(\mathbf{ZR}(Y), (\Gamma \otimes \mathbb{Q})_{\geq 0}\right).$$

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A center is in particular a valuative $\mathbb{Q}$-ideal.

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This is equivalent to $\mathcal{I}\mathcal{O}_{Y'} = E^\ell \mathcal{I}'$, with $J = \bar{J}^\ell$ and $\mathcal{I}'$ an ideal.

Indeed, on Y' the center J becomes E^ℓ, in particular principal.

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Proposition

J *is* $\mathcal{I}$*-admissible if and only if* $J^{(a_1-1)!}$ *is* $C(\mathcal{I}, a_1)$*- admissible.*

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$J_{\mathcal{I}}$ *is* $\mathcal{I}$*-admissible.*

Theorem

$C(\mathcal{I}, a_1)\mathcal{O}_{Y'} \;=\; E^{\ell'} C'$ *with* $\mathrm{inv}_{p'} C' \;<\; \mathrm{inv}_p(C(\mathcal{I}, a_1))$.

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Theorem

The center is well-defined.

Theorem

$J_{\mathcal{I}}$ *is* $\mathcal{I}$*-admissible.*

Theorem

$C(\mathcal{I}, a_1)\mathcal{O}_{Y'} \ = \ E^{\ell'} C'$ *with* $\mathrm{inv}_{p'} C' \ < \ \mathrm{inv}_p(C(\mathcal{I}, a_1))$.

Theorem

$\mathcal{I}\mathcal{O}_{Y'} = E^{\ell}\mathcal{I}'$ *with* $\mathrm{inv}_{p'}\mathcal{I}' \ < \ \mathrm{inv}_p(\mathcal{I})$.

The end

Thank you for your attention