# FOCUSING QUANTUM MANY-BODY DYNAMICS: THE RIGOROUS DERIVATION OF THE 1D FOCUSING CUBIC NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We consider the dynamics of $N$ bosons in one dimension. We assume that the pair interaction is attractive and given by $N^{\beta-1}V(N^{\beta}\cdot)$ where $\int V \leqslant 0$. We develop new techniques in treating the $N-$body Hamiltonian so that we overcome the difficulties generated by the attractive interaction and establish new energy estimates. We also prove the optimal 1D collapsing estimate which reduces the regularity requirement in the uniqueness argument by half a derivative. We derive rigorously the one dimensional focusing cubic NLS with a quadratic trap as the $N \to \infty$ limit of the $N$-body dynamic and hence justify the mean-field limit and prove the propagation of chaos for the focusing quantum many-body system.

## Contents



## 1. Introduction

In 1925, Einstein predicted that, at low temperatures, non-interacting bosons in a gas could all reside in the same quantum state. This peculiar gaseous state in trapped interacting

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atomic clouds, a Bose-Einstein condensate (BEC), was produced in the laboratory for the first time in 1995 using the laser-cooling methods [4, 21]. E. A. Cornell, W. Ketterle, and C. E. Wieman were awarded the 2001 Nobel Prize in physics for observing BEC. Many similar successful experiments [20, 35, 47] were performed later. These condensates exhibit quantum phenomena on a large scale, and investigating them has become one of the most active areas of contemporary research.

Let $t \in \mathbb{R}$ be the time variable and $\mathbf{x}_N = (x_1, x_2, ..., x_N) \in \mathbb{R}^{nN}$ be the position vector of $N$ particles in $\mathbb{R}^n$. Then BEC naively means that the $N$-body wave function $\psi_N(t, \mathbf{x}_N)$ satisfies

$$\psi_N(t, \mathbf{x}_N) \sim \prod_{j=1}^{N} \phi(t, x_j)$$

up to a phase factor solely depending on $t$, for some one particle state $\phi$. In other words, every particle is in the same quantum state. Equivalently, there is the Penrose-Onsager formulation of BEC: if we define $\gamma_N^{(k)}$ to be the $k$-particle marginal densities associated with $\psi_N$ by

$$\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k) = \int \psi_N(t, \mathbf{x}_k, \mathbf{x}_{N-k})\overline{\psi_N}(t, \mathbf{x}'_k, \mathbf{x}_{N-k}) d\mathbf{x}_{N-k}, \quad \mathbf{x}_k, \mathbf{x}'_k \in \mathbb{R}^{nk} \tag{1.1}$$

then, equivalently, BEC means

$$\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k) \sim \prod_{j=1}^{k} \phi(t, x_j)\bar{\phi}(t, x'_j). \tag{1.2}$$

It is widely believed that the one particle state $\phi$ in (1.2), also called the condensate wave function since it describes the whole condensate, satisfies the cubic nonlinear Schrödinger equation (NLS)

$$i\partial_t \phi = L\phi + \mu |\phi|^2 \phi,$$

where $L$ is the Laplacian $-\triangle$ or the Hermite operator $-\triangle + \omega^2 |x|^2$. Such a belief is one of the motivations for studying the cubic NLS. Here, the nonlinear term $\mu |\phi|^2 \phi$ represents a mean-field approximation of the pair interactions between the particles: a repelling interaction gives a positive $\mu$ while an attractive interaction yields a $\mu < 0$. Gross and Pitaevskii proposed such a description of the many-body effect. Naturally, the validity of the cubic NLS needs to be established rigorously from the many body system which it is supposed to characterize because it is a phenomenological mean-field type equation.

In a series of works [40, 1, 22, 24, 25, 26, 27, 10, 16, 11, 17, 6, 18, 31], it has been proven rigorously that, for a repelling interaction potential with suitable assumptions, relation (1.2) holds, moreover, the one-particle state $\phi$ satisfies the defocusing cubic NLS ($\mu > 0$).

It is then natural to wonder, whether BEC happens (whether relation (1.2) holds) when the interaction potential is attractive, and whether the condensate wave function $\phi$ satisfies a focusing cubic NLS ($\mu < 0$) if relation (1.2) does hold. In contemporary experiments, both positive [36, 48] and negative [20] results exist. To present the mathematical interpretations of the experiments, we investigate the procedure of laboratory experiments of BEC subject to attractive interactions according to [20, 36, 48].

Step A. Confine a large number of bosons, whose interactions are originally *repelling*, inside a trap. Reduce the temperature of the system so that the many-body system reaches its ground state. It is expected that this ground state is a BEC state / factorized state. This step corresponds to the following mathematical problem.

**Problem 1.** *Show that if $\psi_{N,0}$ is the ground state of the N-body Hamiltonian $H_{N,0}$ defined by*

$$H_{N,0}=\sum_{j=1}^{N}\left(-\frac{1}{2}\triangle_{x_j}+\frac{\omega_0^2}{2}\left|x_j\right|^2\right)+\frac{1}{N}\sum_{1\leqslant i<j\leqslant N}N^{n\beta}V_0\left(N^{\beta}\left(x_i-x_j\right)\right) \tag{1.3}$$

*where $V_0\geqslant 0$, then the marginal densities $\left\{\gamma_{N,0}^{(k)}\right\}$ associated with $\psi_{N,0}$, defined in (1.1), satisfy relation* (1.2).

Here, the factor $1/N$ is to make sure that the interactions are proportional to the number of particles, the pair interaction $N^{n\beta}V_0(N^{\beta}\cdot)$ is an approximation to the Dirac $\delta$ function so that it matches the Gross-Pitaevskii description of BEC that the many-body effect should be modeled by a strong on-site self-interaction, and the quadratic potential $\omega_0^2\left|x\right|^2$ represents the trapping since [20, 36, 48] and many other experiments of BEC use the harmonic trap and measure the strength of the trap with $\omega_0$. This step is exactly the same as the preparation of experiments with repelling interactions and satisfactory answers to Problem 1 have been given in [40].

Step B. Strengthen the trap (increase $\omega_0$) to make the interaction attractive and observe the evolution of the many-body system. This technique which continuously controls the sign and the size of the interaction in a certain range is called the Feshbach resonance.[1] The system is then time dependent. In order to observe BEC, the factorized structure obtained in Step A must be preserved in time. Assuming this to be the case, we then reset the time so that $t=0$ represents the point at which this Feshbach resonance phase is complete. The subsequent evolution should then be governed by a focusing time-dependent $N$-body Schrödinger equation with an attractive pair interaction $V$ subject to an asymptotically factorized initial datum. Moreover, the confining strength is different from Step A, and we denote it by $\omega$. A mathematically precise statement is the following:

**Problem 2.** *Let $\psi_N\left(t,\mathbf{x}_N\right)$ be the solution to the $N-$body Schrödinger equation*

$$i\partial_t\psi_N=\sum_{j=1}^{N}\left(-\frac{1}{2}\triangle_{x_j}+\frac{\omega^2}{2}\left|x_j\right|^2\right)\psi_N+\frac{1}{N}\sum_{1\leqslant i<j\leqslant N}N^{n\beta}V\left(N^{\beta}\left(x_i-x_j\right)\right)\psi_N \tag{1.4}$$

*where $V\leqslant 0$, with $\psi_{N,0}$ from Step A as initial datum. Prove that the marginal densities $\left\{\gamma_N^{(k)}(t)\right\}$ associated with $\psi_N\left(t,\mathbf{x}_N\right)$ satisfies relation* (1.2).[2]

[1] See [20, Fig.1], [36, Fig.2], or [48, Fig.1] for graphs of the relation between $\omega$ and $V$.

[2] Since $\omega\neq\omega_0$, $V\neq V_0$, one could not expect that $\psi_{N,0}$, the ground state of (1.3), is close to the ground state of (1.4).

In the experiment by Cornell and Wieman et.al [20], once the interaction is tuned attractive, the condensate suddenly shrinks to below the resolution limit, then after $\sim 5ms$, the many-body system blows up. That is, there is no BEC once the interaction becomes attractive. Moreover, there is no condensate wave function due to the absence of the condensate. Whence, the current NLS theory, which is about the condensate wave function when there is a condensate, cannot explain this $5ms$ of time or the blow up. This is currently an open problem in the study of quantum many systems.

In [36, 48], the particles are confined in a strongly anisotropic cigar-shape trap. That is, the confinement is very strong in two spatial directions to simulate a 1D system. In this case, the experiment is a success in the sense that one obtains a persistent BEC after the interaction is switched to attractive. Moreover, a soliton is observed in [36] and a soliton train is observed in [48]. The solitons in [36, 48] have different motion patterns.

In this paper, we consider the 1D model in [36, 48]: we take $n = 1$ in (1.4). We derive rigorously the 1D cubic focusing NLS from a 1D quantum many-body system. We establish the following theorem.

**Theorem 1.1** (Main Theorem)**.** *Assume that the pair interaction $V$ is an even Schwartz class function, which has a nonpositive integration, that is, $\int_{\mathbb{R}} V(x)dx \leqslant 0$, but may not be negative everywhere. Let $\psi_N(t, \mathbf{x}_N)$ be the $N-$body Hamiltonian evolution $e^{itH_N}\psi_N(0)$, where*

$$H_N = \sum_{j=1}^{N}\left(-\frac{1}{2}\partial_{x_j}^2 + \frac{\omega^2}{2}x_j^2\right) + \frac{1}{N}\sum_{1\leqslant i<j\leqslant N} N^{\beta}V\left(N^{\beta}(x_i - x_j)\right) \tag{1.5}$$

*for some $\omega \in \mathbb{R}$ which could be zero and for some $\beta \in (0,1)$, and let $\left\{\gamma_N^{(k)}\right\}$ be the family of marginal densities associated with $\psi_N$. Suppose that the initial datum $\psi_N(0)$ verifies the following conditions:*

*(a) the initial datum is normalized, that is*

$$\|\psi_N(0)\|_{L^2} = 1,$$

*(b) the initial datum is asymptotically factorized, in the sense that,*

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_N^{(1)}(0, x_1; x_1') - \phi_0(x_1)\overline{\phi_0}(x_1')\right| = 0, \tag{1.6}$$

*for some one particle wave function $\phi_0$ s.t. $\left\|\left(1 - \partial_x^2 + \omega^2 x^2\right)^{\frac{1}{2}}\phi_0\right\|_{L^2(\mathbb{R})} < \infty$.*

*(c) the initial datum has finite kinetic energy and variance each particle*[3]

$$\sup_{j,N}\left\langle \psi_N(0), \left(-\partial_{x_j}^2 + \omega^2 x_j^2\right)\psi_N(0)\right\rangle < \infty. \tag{1.7}$$

*Then $\forall t \geqslant 0$, $\forall k \geqslant 1$, we have the convergence in the trace norm or the propagation of chaos that*

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') - \prod_{j=1}^{k}\phi(t, x_j)\overline{\phi}(t, x_j')\right| = 0,$$

[3]Finite variance can be dropped when $\omega$ is zero.

*where $\phi(t,x)$ is the solution to the 1D focusing cubic NLS*

$$\begin{aligned} i\partial_t\phi &= \left(-\frac{1}{2}\partial_{x_j}^2+\frac{\omega^2}{2}x_j^2\right)\phi - b_0\left|\phi\right|^2\phi \text{ in } \mathbb{R}^{1+1} \\ \phi(0,x) &= \phi_0(x) \end{aligned} \tag{1.8}$$

*and the coupling constant $b_0=\left|\int_{\mathbb{R}} V(x)dx\right|$.*

Theorem 1.1 is equivalent to the following theorem.

**Theorem 1.2** (Main Theorem)**.** *Assume that the pair interaction $V$ is an even Schwartz class function, which has a nonpositive integration, that is, $\int_{\mathbb{R}} V(x)dx\leqslant 0$, but may not be negative everywhere. Let $\psi_N(t,\mathbf{x}_N)$ be the $N-body$ Hamiltonian evolution $e^{itH_N}\psi_N(0)$ with $H_N$ given by (1.5) for some $\omega\in\mathbb{R}$ which could be zero and for some $\beta\in(0,1)$, and let $\left\{\gamma_N^{(k)}\right\}$ be the family of marginal densities associated with $\psi_N$. Suppose that the initial datum $\psi_N(0)$ is normalized and asymptotically factorized in the sense of (a) and (b) in Theorem 1.1 and verifies the following energy condition:*

*(c') there is a $C>0$ independent of $N$ or $k$ such that*

$$\left\langle \psi_N(0), H_N^k\psi_N(0)\right\rangle < C^kN^k,\ \forall k\geqslant 1.^4 \tag{1.9}$$

*Then $\forall t\geqslant 0$, $\forall k\geqslant 1$, we have the convergence in the trace norm or the propagation of chaos that*

$$\lim_{N\to\infty}\operatorname{Tr}\left|\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')-\prod_{j=1}^k\phi(t,x_j)\overline{\phi}(t,x_j')\right|=0,$$

*where $\phi(t,x)$ is the solution to the 1D focusing cubic NLS (1.8).*

The equivalence of Theorems 1.1 and 1.2 for asymptotically factorized initial data has been used in all defocusing works. In the main part of this paper, we prove Theorem 1.2 in full detail. For completeness, we discuss briefly how to deduce Theorem 1.1 from Theorem 1.2 in Appendix B.

To our knowledge, Theorems 1.1 and 1.2 offer the first rigorous derivation of the focusing cubic NLS (1.8) from the $N$-body dynamic (1.4).[5] The main tool used in establishing Theorem 1.2 is the analysis of the focusing Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy (BBGKY) hierarchy of $\left\{\gamma_N^{(k)}\right\}_{k=1}^N$ as $N\to\infty$. With our definition, the sequence of the marginal

[4]Here, the energies $\left\langle\psi_N(0),H_N^k\psi_N(0)\right\rangle$ are allowed to be negative. Estimate (4.1), in which we use (1.9), does not depend on the signs of $\left\langle\psi_N(0),H_N^k\psi_N(0)\right\rangle$. This is not surprising because we are working in one dimension.

[5]If one replaces the Hermite operator with $-\triangle$ or $(1-\triangle)^{\frac{1}{2}}$ in (1.4), assumes the Coulomb interaction, and let $\beta=0$, then there are works by Erdös and Yau [23], Michelangeli and Schlein [42], and Ammari and Nier [2, 3] which derive the corresponding focusing Hartree equations. The techniques in treating the three dimensional focusing Coulomb potential do not apply here. We need new ideas to handle the attractive delta-potential, because it cannot be squared (the square of the Coulomb potential, on the other hand, is bounded by the kinetic energy). For works on defocusing Hartree dynamic ($\beta=0$), see [28, 39, 44, 32, 33, 15, 8].

densities $\left\{\gamma_N^{(k)}\right\}_{k=1}^{N}$ associated with $\psi_N$ solves the 1D BBGKY hierarchy with a quadratic trap

$$
\begin{aligned}
(1.10)\quad i\partial_t\gamma_N^{(k)} &= \left[-\frac{1}{2}\triangle_{\mathbf{x}_k}+\omega^2\frac{\left|\mathbf{x}_k\right|^2}{2},\gamma_N^{(k)}\right]+\frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\left[N^{\beta}V\left(N^{\beta}\left(x_i-x_j\right)\right),\gamma_N^{(k)}\right] \\
&\quad +\frac{N-k}{N}\sum_{j=1}^{k}\operatorname{Tr}_{k+1}\left[N^{\beta}V\left(N^{\beta}\left(x_j-x_{k+1}\right)\right),\gamma_N^{(k+1)}\right].
\end{aligned}
$$

In the classical setting, deriving mean-field type equations by studying the limit of the BBGKY hierarchy was proposed by Kac and demonstrated by Landford's work on the Boltzmann equation. In the quantum setting, the usage of the BBGKY hierarchy was suggested by Spohn [46] and has been proven to be successful by Elgart, Erdös, Schlein, and Yau in their fundamental papers [22, 24, 25, 26, 27][6] which rigorously derives the 3D cubic defocusing NLS from a 3D quantum many-body dynamic without trapping. The Elgart-Erdös-Schlein-Yau program[7] consists of two principal parts: in one part, they consider the sequence of the marginal densities $\left\{\gamma_N^{(k)}\right\}$ and prove that its appropriate limit as $N\to\infty$ solves the 3D defocusing Gross-Pitaevskii (GP) hierarchy

$$
(1.11)\qquad i\partial_t\gamma^{(k)}=\left[-\frac{1}{2}\triangle_{\mathbf{x}_k},\gamma^{(k)}\right]+b_0\sum_{j=1}^{k}\operatorname{Tr}_{k+1}[\delta(x_j-x_{k+1}),\gamma^{(k+1)}],\ b_0\geqslant 0.
$$

In another part, they show that hierarchy (1.11) has a unique solution which is therefore a completely factorized state. However, the uniqueness theory for hierarchy (1.11) is surprisingly delicate due to the fact that it is a system of infinitely many coupled equations over an unbounded number of variables. In [38], by assuming a space-time bound on the limit of $\left\{\gamma_N^{(k)}\right\}$, Klainerman and Machedon gave another uniqueness theorem regarding (1.11) through a collapsing estimate originating from the multilinear Strichartz estimates and a board game argument inspired by the Feynman graph argument in [25].

Later, the method in Klainerman and Machedon [38] was taken up by Kirkpatrick, Schlein, and Staffilani [37], who derived the 2D cubic defocusing NLS from the 2D quantum many-body dynamic; by Chen and Pavlović [9, 10], who considered the 1D and 2D 3-body repelling interaction problem and the general existence theory of hierarchy (1.11); by X.C. [16, 17], who investigated the defocusing problem with trapping in 2D and 3D; and by X.C. and J.H. [18], who proved the effectiveness of the 3D to 2D reduction problem. In [12, 13], Chen, Pavlović and Tzirakis worked out the virial and Morawetz identities for hierarchy (1.11) and showed the blow up for hierarchy (1.11) in 2D and 3D in the case of negative energy initial data and negative $b_0$. In [29], Gressman, Sohinger, and Staffilani have obtained a uniqueness theorem of solution to hierarchy (1.11) in 3D subject to periodic boundary condition.

Recently, in [11], for the 3D defocusing problem without traps, Chen and Pavlović showed that, for $\beta\in(0,1/4)$, the limit of the BBGKY sequence satisfies the space-time bound

[6]Around the same time, there was the 1D defocusing work [1].

[7]See [6, 31, 43] for different approaches.

assumed by Klainerman and Machedon [38] as $N \to \infty$. In [17], X.C. extended and simplified their method to study the 3D trapping problem for $\beta \in (0, 2/7]$. X.C. and J.H. [19] then extended the $\beta \in (0, 2/7]$ result by X.C. to $\beta \in (0, 2/3)$ using $X_b$ spaces and Littlewood-Paley theory.

We use the Klainerman-Machedon framework for the uniqueness argument in this paper. While the known uniqueness theorems [1, 9, 10] regarding the 1D GP hierarchy need $H^{\frac{1}{2}+\varepsilon}$ smoothness, i.e. more than the continuity in 1D, our Theorem 3.1 requires merely $H^{\varepsilon}$ regularity to establish uniqueness. To achieve this reduction, we prove the optimal 1D collapsing estimate which has been open for a while.

**Theorem 1.3.** [8] *Let* $U^{(k)}(\tau) = \prod_{j=1}^{k} e^{i\tau\partial_{y_j}^2} e^{-i\tau\partial_{y_j'}^2}$ *and* $R_{\varepsilon}^{(k)} = \prod_{j=1}^{k} \langle \partial_{y_j} \rangle^{\varepsilon} \left\langle \partial_{y_j'} \right\rangle^{\varepsilon}$. *Define the collision operator* $B_{j,k+1}$ *by*

$$B_{j,k+1} u^{(k+1)} = \mathrm{Tr}_{k+1} \left[ \delta \left( y_j - y_{k+1} \right), u^{(k+1)} \right].$$

*Given any finite time* $T$ *and any* $\varepsilon > 0$, *there is a constant* $C_T > 0$ *independent of* $j, k$ *and* $\phi^{(k+1)}$, *such that*

$$\left\| R_{\varepsilon}^{(k)} B_{j,k+1} U^{(k+1)}(\tau) \phi^{(k+1)} \right\|_{L_T^2 L_{\mathbf{y},\mathbf{y}'}^2} \leqslant C_T \left\| R_{\varepsilon}^{(k+1)} \phi^{(k+1)} \right\|_{L_{\mathbf{y},\mathbf{y}'}^2}.$$

*This estimate is optimal in the sense that it fails whenever* $T = \infty$ *or* $\varepsilon = 0$.

It is surprising that the 1D scale-invariant global-in-time collapsing estimate ($T = \infty$ and $\varepsilon = 0$) fails while the scale-invariant global-in-time estimates are true in 2D [16, 5] and 3D [38]. The failure of Theorem 1.3 when $T = \infty$ for any $\epsilon \geq 0$ indicates that we do not have enough decay in time in 1D. Since collapsing estimates like Theorem 1.3 determine many features of the corresponding GP hierarchies, we wonder if this is related to the fact that there is no $L^2$ small data scattering theory for the ordinary 1D focusing GP hierarchy

$$i\partial_t \gamma^{(k)} = \left[ -\frac{1}{2} \triangle_{\mathbf{x}_k}, \gamma^{(k)} \right] - b_0 \sum_{j=1}^{k} \left[ \delta(x_j - x_{k+1}), \gamma^{(k+1)} \right].$$

Specifically, we can write down a tensor product of 1D NLS solitons arbitrarily small in any unweighted $H^s$ norm, $s \geq 0$, which shows the lack of small data scattering for GP. If we conjecture that in a general setting, scale-invariant global-in-time collapsing estimates from [38, 16, 5] could be part of a proof of small data scattering, then the above mentioned lack of scattering in 1D implies the nonexistence of global-in-time collapsing estimates in 1D. This heuristically implies the optimality of Theorem 1.3. On the other hand, the known global-in-time collapsing estimates in 2D and 3D could eventually be used to prove small data scattering for 2D and 3D GP. All of these remarks pertain to unweighted Sobolev spaces at or above the critical (scale invariant) level; in the setting of weighted Sobolev spaces, small-data scattering for 1D cubic NLS is known [34].

[8] For more estimates of this type, see [38, 37, 30, 14, 16, 5, 29]

Theorem 1.3 also reduces the regularity requirement by $1/2$ for the current local existence theory [9] of the GP hierarchy (1.11) subject to general initial data in 1D. In fact, plugging Theorem 1.3 into [9] yields the following corollary.

**Corollary 1.1.** *For every initial data in $H^{\varepsilon}$ which is not necessarily factorized, there is a time $T>0$ such that there exists a unique solution in $H^{\varepsilon}$ for $t\in[0,T]$ to the GP hierarchy (1.11) in 1D regardless of the sign of $b_0$.*

1.1. **Organization of the Paper.** We first review the lens transform and its relevant properties in §2. It aids in the proof of the main theorem in the sense that it links the analysis of $-\partial_x^2+\omega^2x^2$ to the analysis of $-\partial_y^2$ which is easier to deal with using the Fourier transform. With the lens transform, we then outline the proof of our main theorem, Theorem 1.2, in §3. The components of the proof are in §4, 5, and 6.

In §4, we prove the needed energy estimate for the focusing $N$-body Schrödinger evolution. The key obstacle here, compared to earlier versions of such estimates in the defocusing works [1, 22, 24, 25, 26, 27, 10, 16, 11, 17, 18], is to accommodate the negativity of the potential. We first observe a new decomposition of the Hamiltonian $H_N$ given by

$$N^{-1}H_N+\|V\|_{L^1}^2+1=\frac{1}{2N(N-1)}\sum_{1\leq i,j\leq N}H_{+ij}$$

where

$$H_{+ij}=S_i^2+S_j^2+\frac{N-1}{N}N^{\beta}V(N^{\beta}(x_i-x_j))+2\|V\|_{L^1}^2$$

and

$$S_j=(1-\tfrac{1}{2}\partial_{x_j}^2+\tfrac{1}{2}\omega^2x_j^2)^{1/2}$$

In the expansion of $(N^{-1}H_N+\|V\|_{L^1}^2+1)^k$, the terms that occur most frequently are of the form

$$H_{+i_1j_1}\cdots H_{+i_kj_k}$$

with all $i_1,j_1,\ldots,i_k,j_k$ distinct. Since these operators pairwise commute, we can exploit the positivity of each $H_{+ij}$. In particular, we have

$$H_{+ij}\geq\tfrac{1}{2}(S_i^2+S_j^2)$$

We justify the above heuristic by induction.

In §5, we use the energy estimates derived in §4 and duality to prove weak* compactness and convergence of the corresponding BBGKY hierarchy. This follows the similar procedure in the defocusing works.

Finally, in §6, we prove Theorem 1.3, the optimal 1D collapsing estimate. As discussed previously, we need to include a time-localization. On the Fourier side, the time localization mollifies the resulting surface measure and makes it integrable. Without the time localization, the surface measure remains unmollified and is not integrable, and the estimate fails. The optimality statement essentially follows.

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## 2. Lens Transform

In this section, we review the lens transform and its relevant properties. Everything here comes from [17]. (See also [7, 16].) We include it solely for completeness. The lens transform aids in the proof of the main theorem in the sense that it links the analysis of $-\partial_x^2+\omega^2x^2$ to the analysis of $-\partial_y^2$ which is a better understood operator. We remark that the lens transform is exactly the identity when $\omega=0$ i.e. this section is trivial when $\omega=0$.

We denote $(t,x)$ the space-time on the Hermite side and $(\tau,y)$ the space-time on the Laplacian side. We define the lens transform in Definitions 1 and 2. We then explain how the lens transform acts on the BBGKY hierarchy and the GP hierarchy via Lemmas 2.1 and 2.2. Finally, we relate the trace norms and the energies of the two sides of the lens transform through Lemmas 2.3 and 2.4.

**Definition 1** ([17])**.** *Let $\mathbf{x}_N,\mathbf{y}_N\in\mathbb{R}^N$. We define the lens transform for $L^2$ functions $M_N:L^2(d\mathbf{y}_N)\to L^2(d\mathbf{x}_N)$ and its inverse by*

$$\begin{aligned}\left(M_Nu_N\right)(t,\mathbf{x}_N) &= \frac{e^{-i\omega\tan\omega t\frac{|\mathbf{x}_N|^2}{2}}}{(\cos\omega t)^{\frac{N}{2}}}u_N(\frac{\tan\omega t}{\omega},\frac{\mathbf{x}_N}{\cos\omega t})\\ \left(M_N^{-1}\psi_N\right)(\tau,\mathbf{y}_N) &= \frac{e^{i\frac{\omega^2\tau}{1+\omega^2\tau^2}\frac{|\mathbf{y}_N|^2}{2}}}{(1+\omega^2\tau^2)^{\frac{N}{4}}}\psi_N(\frac{\arctan(\omega\tau)}{\omega},\frac{\mathbf{y}_N}{\sqrt{1+\omega^2\tau^2}}).\end{aligned}$$

*$M_N$ is unitary by definition and the variables are related by*

$$\tau=\frac{\tan\omega t}{\omega},\ \mathbf{y}_N=\frac{\mathbf{x}_N}{\cos\omega t}.$$

**Definition 2** ([17])**.** *Let $\mathbf{x}_k,\mathbf{x}_k',\mathbf{y}_k,\mathbf{y}_k'\in\mathbb{R}^k$. We define the lens transform for Hilbert-Schmidt kernels $T_k:L^2(d\mathbf{y}_kd\mathbf{y}_k')\to L^2(d\mathbf{x}_kd\mathbf{x}_k')$ and its inverse by*

$$\begin{aligned}&\left(T_ku^{(k)}\right)(t,\mathbf{x}_k;\mathbf{x}_k')\\ &= \frac{e^{-i\omega\tan\omega t\frac{\left(|\mathbf{x}_k|^2-|\mathbf{x}_k'|^2\right)}{2}}}{(\cos\omega t)^k}u^{(k)}(\frac{\tan\omega t}{\omega},\frac{\mathbf{x}_k}{\cos\omega t};\frac{\mathbf{x}_k'}{\cos\omega t})\end{aligned}$$

$$\begin{aligned}&\left(T_k^{-1}\gamma^{(k)}\right)(\tau,\mathbf{y}_k;\mathbf{y}_k')\\ &= \frac{e^{i\frac{\omega^2\tau}{1+\omega^2\tau^2}\frac{\left(|\mathbf{y}_k|^2-|\mathbf{y}_k'|^2\right)}{2}}}{(1+\omega^2\tau^2)^{\frac{k}{2}}}\gamma^{(k)}(\frac{\arctan(\omega\tau)}{\omega},\frac{\mathbf{y}_k}{\sqrt{1+\omega^2\tau^2}};\frac{\mathbf{y}_k'}{\sqrt{1+\omega^2\tau^2}}).\end{aligned}$$

*$T_k$ is unitary by definition as well and the variables are again related by*

$$\tau=\frac{\tan\omega t}{\omega},\ \mathbf{y}_k=\frac{\mathbf{x}_k}{\cos\omega t}\ \text{and}\ \mathbf{y}_k'=\frac{\mathbf{x}_k'}{\cos\omega t}.$$

*In particular, if* $u_N(\tau,\mathbf{y}_N) = M_N^{-1}(\psi_N)$, *then* $\left\{u_N^{(k)} = T_k^{-1}\gamma_N^{(k)}\right\}$ *is exactly the family of marginal densities associated with* $u_N$.

**Lemma 2.1** ([17]). *Write* $V_N(x) = N^\beta V\left(N^\beta x\right)$. $\left\{\gamma_N^{(k)}\right\}$ *solves the 1D BBGKY hierarchy with a quadratic trap (1.10) in* $[-T_0,T_0]$ *if and only if* $\left\{u_N^{(k)} = T_k^{-1}\gamma_N^{(k)}\right\}$ *solves the hierarchy*

$$\begin{aligned} i\partial_\tau u_N^{(k)} &= \left[-\frac{1}{2}\triangle_{\mathbf{y}_k},\gamma_N^{(k)}\right] + \frac{1}{(1+\omega^2\tau^2)}\frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\left[V_N(\frac{y_i-y_j}{(1+\omega^2\tau^2)^{\frac{1}{2}}}),u_N^{(k)}\right] \\ &\quad + \frac{N-k}{N}\frac{1}{(1+\omega^2\tau^2)}\sum_{j=1}^{k}\mathrm{Tr}_{k+1}\left[V_N\left(\frac{y_j-y_{k+1}}{(1+\omega^2\tau^2)^{\frac{1}{2}}}\right),u_N^{(k+1)}\right] \end{aligned} \tag{2.1}$$

*in* $\left[-\frac{\tan\omega T_0}{\omega},\frac{\tan\omega T_0}{\omega}\right]$.

**Lemma 2.2** ([17]). $\left\{\gamma^{(k)}\right\}$ *solves the 1D focusing GP hierarchy with a quadratic trap*

$$i\partial_t\gamma^{(k)} = \left[-\frac{1}{2}\triangle_{\mathbf{x}_k}+\omega^2\frac{|\mathbf{x}_k|^2}{2},\gamma^{(k)}\right] - b_0\sum_{j=1}^{k}\left[\delta(x_j-x_{k+1}),\gamma^{(k+1)}\right], \tag{2.2}$$

*in* $[-T_0,T_0]$ *if and only if* $\left\{u^{(k)} = T_k^{-1}\gamma^{(k)}\right\}$ *solves the focusing hierarchy*

$$i\partial_\tau u^{(k)} = \left[-\frac{1}{2}\triangle_{\mathbf{y}_k},u^{(k)}\right] - \frac{1}{(1+\omega^2\tau^2)^{\frac{1}{2}}}b_0\sum_{j=1}^{k}\left[\delta(y_j-y_{k+1}),u^{(k+1)}\right], \tag{2.3}$$

*in* $\left[-\frac{\tan\omega T_0}{\omega},\frac{\tan\omega T_0}{\omega}\right]$.

**Lemma 2.3** ([17]). *If* $K(\mathbf{y}_k,\mathbf{y}_k')$ *is the kernel of a self-adjoint trace class operator on* $L^2\left(\mathbb{R}^k\right)$, *then the eigenvectors of the kernel* $(T_kK)(\mathbf{x}_k,\mathbf{x}_k')$ *are exactly the lens transform of the eigenvectors of the kernel* $K(\mathbf{y}_k,\mathbf{y}_k')$ *with the same eigenvalues. In particular, we have*

$$\mathrm{Tr}\,|T_kK| = \mathrm{Tr}\,|K|.$$

**Lemma 2.4** ([17]). *There is a* $C>0$ *such that*

$$\left\langle u_N(\tau),\prod_{j=1}^{k}\left(1-\partial_{y_j}^2\right)u_N(\tau)\right\rangle \leqslant C^k\left\langle\psi_N(t),\prod_{j=1}^{k}\left(1-\frac{1}{2}\partial_{x_j}^2+\frac{1}{2}\omega^2x_j^2\right)\psi_N(t)\right\rangle$$

*for all* $\psi_N(t,\mathbf{x}_N)$, *where* $u_N(\tau,\mathbf{y}_N) = M_N^{-1}(\psi_N)$. *In particular, if* $u_N^{(k)} = T_k^{-1}\gamma_N^{(k)}$, *we have*

$$\begin{aligned} &\mathrm{Tr}\left(\prod_{j=1}^{k}\left(1-\partial_{y_j}^2\right)^{\frac{1}{2}}\right)u_N^{(k)}(\tau)\left(\prod_{j=1}^{k}\left(1-\partial_{y_j}^2\right)^{\frac{1}{2}}\right) \\ \leqslant\ & C^k\,\mathrm{Tr}\left(\prod_{j=1}^{k}\left(1-\frac{1}{2}\partial_{x_j}^2+\frac{1}{2}\omega^2x_j^2\right)^{\frac{1}{2}}\right)\gamma_N^{(k)}(t)\left(\prod_{j=1}^{k}\left(1-\frac{1}{2}\partial_{x_j}^2+\frac{1}{2}\omega^2x_j^2\right)^{\frac{1}{2}}\right). \end{aligned}$$

**Notation 1.** *From here on out, to make formulas shorter, we write*

$$L_j = \left(1-\partial_{y_j}^2\right)^{\frac{1}{2}},\ S_j = \left(1-\frac{1}{2}\partial_{x_j}^2+\frac{1}{2}\omega^2 x_j^2\right)^{\frac{1}{2}},$$

$$L^{(k)} = \prod_{j=1}^{k} L_j,\ S^{(k)} = \prod_{j=1}^{k} S_j,$$

$$g(\tau) = \left(1+\omega^2\tau^2\right)^{-\frac{1}{2}}, V_{N,\tau}(y) = N^{\beta} g(\tau) V(N^{\beta} g(\tau) y).$$

*The only properties we need are* $0 < g(\tau) \leqslant 1$, *and* $\int V_{N,\tau}(y)dy = b_0$.

## 3. Proof of the Main Theorem (Theorem 1.2)

We start by introducing an appropriate topology on the density matrices as was previously done in [22, 23, 24, 25, 26, 27, 37, 10, 16, 17, 18, 19]. Denote the spaces of compact operators and trace class operators on $L^2\left(\mathbb{R}^k\right)$ as $\mathcal{K}_k$ and $\mathcal{L}_k^1$, respectively. Then $(\mathcal{K}_k)' = \mathcal{L}_k^1$. By the fact that $\mathcal{K}_k$ is separable, we select a dense countable subset $\{J_i^{(k)}\}_{i\geqslant 1} \subset \mathcal{K}_k$ in the unit ball of $\mathcal{K}_k$ (so $\|J_i^{(k)}\|_{\mathrm{op}} \leqslant 1$ where $\|\cdot\|_{\mathrm{op}}$ is the operator norm). For $\gamma^{(k)}, \tilde{\gamma}^{(k)} \in \mathcal{L}_k^1$, we then define a metric $d_k$ on $\mathcal{L}_k^1$ by

$$d_k(\gamma^{(k)}, \tilde{\gamma}^{(k)}) = \sum_{i=1}^{\infty} 2^{-i} \left|\mathrm{Tr}\, J_i^{(k)}\left(\gamma^{(k)} - \tilde{\gamma}^{(k)}\right)\right|.$$

A uniformly bounded sequence $\gamma_N^{(k)} \in \mathcal{L}_k^1$ converges to $\gamma^{(k)} \in \mathcal{L}_k^1$ with respect to the weak* topology if and only if

$$\lim_{N\to\infty} d_k(\gamma_N^{(k)}, \gamma^{(k)}) = 0.$$

For fixed $T > 0$, let $C\left([0,T], \mathcal{L}_k^1\right)$ be the space of functions of $t \in [0,T]$ with values in $\mathcal{L}_k^1$ which are continuous with respect to the metric $d_k$. On $C\left([0,T], \mathcal{L}_k^1\right)$, we define the metric

$$\hat{d}_k(\gamma^{(k)}(\cdot), \tilde{\gamma}^{(k)}(\cdot)) = \sup_{t\in[0,T]} d_k(\gamma^{(k)}(t), \tilde{\gamma}^{(k)}(t)),$$

and denote by $\tau_{prod}$ the topology on the space $\oplus_{k\geqslant 1} C\left([0,T], \mathcal{L}_k^1\right)$ given by the product of topologies generated by the metrics $\hat{d}_k$ on $C\left([0,T], \mathcal{L}_k^1\right)$.

With the above topology on the space of marginal densities, we can begin the proof of Theorem 1.2. We divide the proof into four steps.

Step I (Energy estimate) Before we apply the lens transform to our problem, we first establish, through an elaborate calculation in Theorem 4.1, that one can absorb the negativity of the interaction in (1.5). Henceforth we transform the energy condition (1.9) into a $H^1$ type bound. Due to the fact that the quantity $\left\langle \psi_N(0), H_N^k \psi_N(0)\right\rangle$ in (1.9) is conserved by the evolution, we deduce the *a priori* bound on the marginal densities

$$\sup_t \mathrm{Tr}\, S^{(k)} \gamma_N^{(k)}(t) S^{(k)} \leqslant C^k.$$

In Corollary 4.1, we then combine the above bound and Lemma 2.4 to obtain the $H^1$ bound

$$\sup_{\tau\in\left[-\frac{\tan\omega T_0}{\omega},\frac{\tan\omega T_0}{\omega}\right]}\operatorname{Tr}L^{(k)}u_N^{(k)}(\tau)L^{(k)}\leqslant C^k,\text{ if }T_0<\frac{\pi}{2\omega},\tag{3.1}$$

where $u_N^{(k)}=T_k^{-1}\gamma_N^{(k)}$.

Step II (Compactness and Convergence) Fix $T_0<\frac{\pi}{2\omega}$ and employ (3.1), we prove, in Theorem 5.1, that the sequence $\Gamma_N(\tau)=\left\{u_N^{(k)}\right\}_{k=1}^N$ which satisfies the 1D BBGKY hierarchy (2.1) is compact with respect to the product topology $\tau_{prod}$. Moreover, we prove, in Theorem 5.2, that if $\Gamma(\tau)=\left\{u^{(k)}\right\}_{k=1}^\infty$ is a limit point of $\Gamma_N(\tau)$ with respect to the product topology $\tau_{prod}$, then $\Gamma(\tau)$ is a solution to the focusing GP hierarchy (2.3) subject to initial data $u^{(k)}(0)=|\phi_0\rangle\langle\phi_0|^{\otimes k}$ and the coupling constant is given by $b_0=\left|\int V(x)\,dx\right|$. This is a well-known argument used in [22, 23, 24, 25, 26, 27, 37, 10, 16, 17, 18], we include the proof in §5 for completeness since it is the first time such an argument is used in the focusing setting.

Step III (Uniqueness) When $u^{(k)}(0)=|\phi_0\rangle\langle\phi_0|^{\otimes k}$, we know that there is a special solution to the focusing GP hierarchy (2.3), namely

$$u^{(k)}(\tau,\mathbf{y}_k,\mathbf{y}_k')=\prod_{j=1}^k\tilde{\phi}(\tau,y_j)\overline{\tilde{\phi}(\tau,y_j')}\tag{3.2}$$

where $\tilde{\phi}$ solves

$$\begin{aligned}i\partial_\tau\tilde{\phi}&=-\partial_y^2\tilde{\phi}-g(\tau)b_0\left|\tilde{\phi}\right|^2\tilde{\phi}\\ \tilde{\phi}(0,y)&=\phi_0.\end{aligned}\tag{3.3}$$

A suitable uniqueness theorem regarding (2.3) will then identify all limit points of $\Gamma_N(\tau)$ obtained in Step II with (3.2) for us. The Klainerman-Machedon scheme, introduced in [38] and used in [37, 10, 16, 17, 18, 19], transforms Theorem 1.3 into the following uniqueness theorem.

**Theorem 3.1.** *Let $R_\varepsilon^{(k)}$ and $B_{j,k+1}$ be defined in Theorem 1.3. Suppose that $\left\{u^{(k)}\right\}_{k=1}^\infty$ solves the 1D focusing GP hierarchy (2.3) subject to zero initial data and the space-time bound*

$$\int_0^T\left\|R_\varepsilon^{(k)}B_{j,k+1}u^{(k+1)}(\tau,\cdot;\cdot)\right\|_{L^2_{\mathbf{y},\mathbf{y}'}}d\tau\leqslant C^k\tag{3.4}$$

*for some $\varepsilon,C>0$ and all $1\leqslant j\leqslant k$. Then $\forall k,\tau\in[0,T]$,*

$$\left\|R_\varepsilon^{(k)}u^{(k)}(\tau,\cdot;\cdot)\right\|_{L^2_{\mathbf{y},\mathbf{y}'}}=0.$$

*Proof.* Once we prove Theorem 1.3, Theorem 3.1 follows from the proof of [17, Theorem 6] line by line. □

To apply Theorem 3.1, we need to check (3.4). As the spatial dimension is one, the following trace lemma and (3.1) takes care of (3.4) for us.[9]

[9]Verifying (3.4) in 3D is highly nontrivial and is merely partially solved so far. See [11, 17, 19]

**Lemma 3.1** ([10, Theorem 4.3]). *For $\alpha > \frac{1}{2}$, we have*

$$\left\| R_\alpha^{(k)} B_{j,k+1} u^{(k+1)} \right\|_{L^2_{\mathbf{y},\mathbf{y}'}} \leqslant C \left\| R_\alpha^{(k+1)} u^{(k+1)} \right\|_{L^2_{\mathbf{y},\mathbf{y}'}}.$$

Step IV (Conclusion) By Step III, the compact sequence $\{\Gamma_N(\tau)\}$ has only one limit point, thus it converges, that is, as trace class operators kernels,

$$u_N^{(k)}(\tau) \to \prod_{j=1}^k \tilde{\phi}(\tau, y_j)\overline{\tilde{\phi}(\tau, y_j')} \text{ weak* as } N \to \infty,\ \forall \tau \in \left[0, \frac{\tan \omega T_0}{\omega}\right].$$

Notice that the above weak* limit is an orthogonal projection, the argument in the bottom of [27, p. 296] which uses the Grümm's convergence theorem [45, Theorem 2.19][10] then implies the strong convergence in trace norm

$$\lim_{N\to\infty} \operatorname{Tr} \left| u_N^{(k)}(\tau, \mathbf{y}_k, \mathbf{y}_k') - \prod_{j=1}^k \tilde{\phi}(\tau, y_j)\overline{\tilde{\phi}(\tau, y_j')} \right| = 0,\ \forall \tau \in \left[0, \frac{\tan \omega T_0}{\omega}\right].$$

Recall $\gamma_N^{(k)} = T_k u_N^{(k)}$ and $\phi = M_1 \tilde{\phi}$, we utilize Lemma 2.3 and infer that

$$\lim_{N\to\infty} \operatorname{Tr} \left| \gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k') - \prod_{j=1}^k \phi(t, x_j)\overline{\phi(t, x_j')} \right| = 0,\ \forall t \in [0, T_0],$$

where $\phi$ solves (1.8). So far, we have proved Theorem 1.2 for every $T_0 < \frac{\pi}{2\omega}$, a bootstrapping argument then establishes Theorem 1.2 for all time. Thence we conclude the proof of Theorem 1.2.

## 4. Energy Estimates for the Focusing $N$-body Hamiltonian

**Theorem 4.1.** *Let $H_N$ be defined as in (1.5). For every $k$, there exists $N_0(k)$ such that, we have*

$$\langle \psi, \left(H_N + N \left\|V\right\|_{L^1}^2 + N\right)^k \psi \rangle \geq 2^{-k} N^k \|S^{(k)}\psi\|_{L^2}^2,$$

*for all $N \geq N_0(k)$ and $\psi \in L_s^2(\mathbb{R}^N)$ with $\left\|\psi\right\|_{L^2} = 1$.*

We prove Theorem 4.1 in §4.1. At the moment, we present the following corollary of Theorem 4.1.

**Corollary 4.1.** *Let $\psi_N(t, \mathbf{x}_N) = e^{itH_N}\psi_N(0)$ for some $\beta \in (0,1)$ subject to initial $\psi_N(0)$ which satisfies energy condition (1.9). If $u_N(\tau, \mathbf{y}_N) = M_N^{-1}\psi_N$, where $M_N^{-1}$ is the inverse lens transform for functions in Definition 1, then there is a $C \geqslant 0$, for all $k \geqslant 0$, there exists $N_0(k)$ such that*

$$\left\langle u_N(\tau), \prod_{j=1}^k L_j^2 u_N(\tau) \right\rangle \leqslant C^k,$$

[10] One can also use the argument in [17, Appendix A].

*for all $N \geqslant N_0$ and all $\tau \in \left[-\frac{\tan\omega T_0}{\omega}, \frac{\tan\omega T_0}{\omega}\right]$ provided that $T_0 < \frac{\pi}{2\omega}$. Thus, for $u_N^{(k)} = T_k^{-1}\gamma_N^{(k)}$, the inverse lens transform of $\gamma_N^{(k)}$,*

$$\sup_{\tau\in\left[-\frac{\tan\omega T_0}{\omega}, \frac{\tan\omega T_0}{\omega}\right]} \operatorname{Tr} L^{(k)} u_N^{(k)}(\tau)\, L^{(k)} \leqslant C^k$$

*where the inverse lens transform for kernels is given by Definition 2.*

*Proof.* By Lemma 2.4, we have

$$\left\langle u_N(\tau), \prod_{j=1}^{k} L_j^2 u_N(\tau) \right\rangle \leqslant C^k \left\langle \psi_N(t), \prod_{j=1}^{k} S_j^2 \psi_N(t) \right\rangle.$$

With Theorem 4.1 and the conservation law, we get

$$\begin{aligned} \left\langle u_N(\tau), \prod_{j=1}^{k} L_j^2 u_N(\tau) \right\rangle &\leqslant \frac{C^k}{N^k} \langle \psi_N(t), \left(H_N + N\|V\|_{L^1}^2 + N\right)^k \psi_N(t)\rangle \\ &= \frac{C^k}{N^k} \langle \psi_N(0), \left(H_N + N\|V\|_{L^1}^2 + N\right)^k \psi_N(0)\rangle \end{aligned}$$

The binomial theorem and (1.9) give

$$\begin{aligned} &\langle \psi_N(0), \left(H_N + N\|V\|_{L^1}^2 + N\right)^k \psi_N(0)\rangle \\ =&\ \sum_{j=0}^{k} \binom{k}{j} \left(N\|V\|_{L^1}^2 + N\right)^j \langle \psi_N(0), H_N^{k-j} \psi_N(0)\rangle \\ \leqslant&\ \sum_{j=0}^{k} \binom{k}{j} \left(N\|V\|_{L^1}^2 + N\right)^j C^{k-j} N^{k-j} \\ =&\ \left(CN + N\|V\|_{L^1}^2 + N\right)^k \\ \leqslant&\ C^k N^k. \end{aligned} \tag{4.1}$$

Thus

$$\left\langle u_N(\tau), \prod_{j=1}^{k} L_j^2 u_N(\tau) \right\rangle \leqslant \frac{C^k}{N^k} C^k N^k \leqslant C^k,$$

as claimed. □

4.1. **Proof of Theorem 4.1.** For convenience, we let $\alpha = \|V\|_{L^1}^2$ and rewrite the desired estimate as

$$\langle \psi, \left(N^{-1}H_N + 1 + \alpha\right)^k \psi\rangle \geqslant 2^{-k} \|S^{(k)}\psi\|_{L^2}^2. \tag{4.2}$$

Note that estimate (4.2) is trivial for $k = 0$. To establish estimate (4.2) for general $k$, we first prove the $k = 1$ case which is already nontrivial in §4.1.1, we then prove estimate (4.2) for $k + 2$ assuming that it holds for $k$ in §4.1.2, thence a two-step induction based on the $k = 0$ and $k = 1$ cases proves (4.2) for all $k$.

The only technical tool we need is the 1D estimate: for $f(x)$

$$\|f\|_{L_x^\infty} \leq \|f'\|_{L_x^1} \tag{4.3}$$

which is a direct consequence of the fundamental theorem of calculus. We also utilize the ordinary Sobolev estimates:

$$\|f\|_{L^\infty_x} \leqslant C\|S_x f\|_{L^2_x} \text{ for } f(x), \tag{4.4}$$

$$\|f\|_{L^\infty_{xy}} \leqslant C\|S_x S_y f\|_{L^2_{xy}} \text{ for } f(x,y). \tag{4.5}$$

when the sizes of the controlling constants do not matter.[11] We will use the shorthand $L^1_c L^\infty_{x_2}$ for $L^1_{x_1 x_3 x_4 \cdots x_N} L^\infty_{x_2}$. Here, $c$ stands for "complementary coordinates".

4.1.1. *The $k=1$ Case.* Recall $V_N(x) = N^\beta V(N^\beta x)$ and

$$S_j = \left(1 - \frac{1}{2}\partial^2_{x_j} + \frac{1}{2}\omega^2 x_j^2\right)^{\frac{1}{2}}.$$

We write

$$H_N + N = \sum_{j=1}^N S_j^2 + \frac{1}{2N} \sum_{\substack{i,j=1,\ldots,N \\ i\neq j}} V_N(x_i - x_j).$$

We next introduce a convenient decomposition of $H_N + N$. Let

$$H_{ij} = S_i^2 + S_j^2 + \frac{N-1}{N} V_N(x_i - x_j)$$

Note that $H_{ij} = H_{ji}$ because $V$ is even, and

$$H_N + N = \frac{1}{2(N-1)} \sum_{\substack{i,j=1,\ldots,N \\ i\neq j}} H_{ij}.$$

It follows that

$$N^{-1} H_N + 1 + \alpha = \frac{1}{2N(N-1)} \sum_{\substack{i,j=1,\ldots,N \\ i\neq j}} (H_{ij} + 2\alpha) \tag{4.6}$$

**Lemma 4.1.** *Recall $\alpha = \|V\|^2_{L^1}$, we have*

$$(H_{12} + 2\alpha) \geqslant \tfrac{1}{2}(S_1^2 + S_2^2)$$

*Proof.* Apply the well-known change of variable $y_1 = x_1 - x_2$, $y_2 = x_1 + x_2$ which is also compatible with the Hermite operator, then

$$\begin{aligned} H_{12} &= 2 - \partial^2_{y_1} - \partial^2_{y_2} + \omega^2 |y_1|^2 + \omega^2 |y_2|^2 + (1 - N^{-1}) V_N(y_1) \\ &= K_{y_1} + 2 - \partial^2_{y_2} + \omega^2 |y_1|^2 + \omega^2 |y_2|^2 \end{aligned}$$

where

$$K_y = -\partial_y^2 + (1 - N^{-1}) V_N(y).$$

[11] The only place in which we apply (4.3) is the proof of Lemma 4.1. We use (4.3) to determine $\alpha = \|V\|^2_{L^1}$. With the elementary inequality: $|ab| \leqslant \varepsilon a^2 + \varepsilon^{-1} b^2$, one can use (4.4) instead and get another $\alpha$, namely, $\alpha = C\|V\|^2_{L^1}$ for some $C$ depending on the controlling constant in (4.4). We are not using (4.4) because we would like to give an exact $\alpha$ and keep track of one less constant.

We claim that

$$(K+2\alpha)\geq -\frac{1}{2}\partial_y^2 \tag{4.7}$$

Indeed,

$$\begin{aligned}
\langle K\phi,\phi\rangle &\geqslant \|\phi'\|_{L^2}^2-\|V_N\|_{L^1}\||\phi|^2\|_{L_y^\infty}\\
&\geqslant \|\phi'\|_{L^2}^2-\|V\|_{L^1}\|\partial_y(|\phi|^2)\|_{L_y^1}\\
&\geqslant \|\phi'\|_{L^2}^2-2\|V\|_{L^1}\|\phi'\|_{L^2}\|\phi\|_{L^2}\\
&\geqslant \|\phi'\|_{L^2}^2-(\frac{1}{2}\|\phi'\|_{L^2}^2+2\|V\|_{L^1}^2\|\phi\|_{L^2}^2)\\
&=\frac{1}{2}\|\phi'\|_{L^2}^2-2\|V\|_{L^1}^2
\end{aligned}$$

from which (4.7) follows.

We clearly have

$$2-\partial_{y_2}^2+\omega^2|y_1|^2+\omega^2|y_2|^2\geqslant 1-\frac{1}{2}\partial_{y_2}^2+\frac{1}{2}\omega^2|y_1|^2+\frac{1}{2}\omega^2|y_2|^2.$$

By this and (4.7), we have

$$\begin{aligned}
H_{12}+2\alpha &= (K_{y_1}+2\alpha+2-\partial_{y_2}^2+\omega^2|y_1|^2+\omega^2|y_2|^2)\\
&\geqslant -\frac{1}{2}\partial_{y_1}^2+1-\frac{1}{2}\partial_{y_2}^2+\frac{1}{2}\omega^2|y_1|^2+\frac{1}{2}\omega^2|y_2|^2\\
&= 1-\frac{1}{4}\partial_{x_1}^2-\frac{1}{4}\partial_{x_2}^2+\frac{1}{4}\omega^2|x_1|^2+\frac{1}{4}\omega^2|x_2|^2\\
&= \tfrac{1}{2}(S_1^2+S_2^2).
\end{aligned}$$

□

In light of (4.6), symmetry, and Lemma 4.1, we readily see that

$$\begin{aligned}
2\langle\psi,(N^{-1}H_N+1+\alpha)\psi\rangle &= \langle\psi,(H_{12}+2\alpha)\psi\rangle\\
&\geqslant \tfrac{1}{2}\langle\psi,(S_1^2+S_2^2)\psi\rangle\\
&= \|S_1\psi\|_{L^2}^2.
\end{aligned}\tag{4.8}$$

Thus we have proved (4.2) for $k=1$.

4.1.2. *The $k+2$ Case.* For convenience, let us introduce some notation. For any function $f$, let

$$f_{Nij}=N^\beta f(N^\beta(x_i-x_j))$$

Also, let

$$H_{+ij}=H_{ij}+2\alpha=S_i^2+S_j^2+(1-N^{-1})V_{Nij}+2\alpha \tag{4.9}$$

Then (4.6) can be written more compactly as

$$N^{-1}H_N+1+\alpha=\frac{1}{2N(N-1)}\sum_{\substack{i,j=1,\dots,N\\ i\neq j}}H_{+ij}=\frac{1}{N(N-1)}\sum_{1\leq i<j\leq N}H_{+ij} \tag{4.10}$$

Before delving into the proof of the $k+2$ case, we give an idea for why (4.2) is true for all $k$. Note that we have

$$2^k(N^{-1}H_N+1+\alpha)^k = \frac{1}{N^k(N-1)^k}\sum_{\substack{1\le i_1,j_1,\dots,i_k,j_k\le N\\ i_1\neq i_2,\dots,i_k\neq j_k}} H_{+i_1j_1}\cdots H_{+i_kj_k}$$

The dominant term in this expression occurs when all indices $i_1, j_1, \dots, i_k, j_k$ are distinct, since it occurs with frequency $\sim N^{2k}$. The other terms occur with lower frequency – for example, the terms in which exactly two of the indices are equal and all others are distinct occur with frequency $\sim N^{2k-1}$. By symmetry, the terms in which all indices are distinct can be rearranged so that formally, we have

$$2^k\langle\psi,(N^{-1}H_N+1+\alpha)^k\psi\rangle \approx \langle H_{+12}\cdots H_{+(2k-1)(2k)}\psi,\psi\rangle \tag{4.11}$$

Moreover, by symmetry

$$2^{-k}\langle\psi,\prod_{i=1}^k(S_{2i-1}^2+S_{2i}^2)\psi\rangle = \|S^{(k)}\psi\|_{L^2}^2 \tag{4.12}$$

Since $H_{+ij}\ge 2^{-1}\left(S_i^2+S_j^2\right)$ for each $i,j$ by Lemma 4.1, Lemma A.2 implies

$$H_{+12}\cdots H_{+(2k-1)(2k)} \ge 2^{-k}\prod_{i=1}^k(S_{2i-1}^2+S_{2i}^2)$$

This, together with (4.11) and (4.12) suggest (not rigorously) that a statement like (4.2) should hold.

We now establish (4.2) for $k+2$ rigorously, assuming it holds for $k$. To be precise, we will prove that, if (4.2) holds for $k$, then

$$\begin{aligned}&2^{k+2}\langle\psi,(N^{-1}H_N+\alpha+1)^{k+2}\psi\rangle\\ \geqslant\ &\left(1-C_{k+2}N^{\beta-1}\right)\left(\left\|S^{(k+2)}\psi\right\|_{L^2}^2+N^{-1}\left\|S_1S^{(k+1)}\psi\right\|_{L^2}^2\right)\end{aligned} \tag{4.13}$$

We remind the reader that we already have the $k=0$ case which is trivial and the $k=1$ case proved in §4.1.1, thus (4.2) is proved for all $k$ once we prove that (4.13) holds as long as (4.2) is true for $k$.

Using the induction hypothesis, we arrive at

$$\begin{aligned}&2^{k+2}\langle\psi,(N^{-1}H_N+\alpha+1)^{k+2}\psi\rangle\\ =\ &4(2^k\langle(N^{-1}H_N+\alpha+1)\psi,(N^{-1}H_N+\alpha+1)^k(N^{-1}H_N+\alpha+1)\psi\rangle)\\ \geqslant\ &4\langle S^{(k)}(N^{-1}H_N+\alpha+1)\psi,S^{(k)}(N^{-1}H_N+\alpha+1)\psi\rangle.\end{aligned} \tag{4.14}$$

We start with the following decomposition of the rightmost sum in (4.10):

$$(N^{-1}H_N+\alpha+1) = \frac{1}{N(N-1)}\sum_{\substack{1\le i<j\le N\\ i\le k}} H_{+ij} + \frac{1}{N(N-1)}\sum_{\substack{1\le i<j\le N\\ i>k}} H_{+ij}. \tag{4.15}$$

Note that in the first term $i\le k$ and $j$ can be either $\le k$ or $>k$. We have ordered the indices $i_1<j_1$ and $i_2<j_2$ for convenience. In the unordered setting, the above decomposition would be characterized as follows: the first sum consists of terms in which *at least* one index is $\le k$,

and the second term consists of terms in which both indices are $> k$. The decomposition (4.15) is similar to the one used in the [22, Proposition 1], although the authors of [22] do not use the $H_{ij}$ decomposition of the Hamiltonian. There are $\sim N$ terms in the first sum and $\sim N^2$ terms in the second sum. Note that in the $k = 0$ case, the decomposition (4.15) contains only the second term since the first term is an empty sum.

Plug the decomposition (4.15) into the end of (4.14) to obtain

$$2^{k+2}\langle\psi, (N^{-1}H_N + \alpha + 1)^{k+2}\psi\rangle \geq A_1 + A_2 + A_3$$

where

$$A_1 = \frac{4}{N^2(N-1)^2} \sum_{\substack{1\leq i_1<j_1\leq N \\ 1\leq i_2<j_2\leq N \\ \text{such that } i_1>k,\, i_2>k}} \langle S^{(k)}H_{+i_1j_1}\psi,\ S^{(k)}H_{+i_2j_2}\psi\rangle,$$

$$A_2 = \frac{4}{N^2(N-1)^2} \sum_{\substack{1\leq i_1<j_1\leq N \\ 1\leq i_2<j_2\leq N \\ \text{such that } i_1\leq k,\, i_2>k}} 2\,\mathrm{Re}\langle S^{(k)}H_{+i_1j_1}\psi,\ S^{(k)}H_{+i_2j_2}\psi\rangle,$$

$$\begin{aligned} A_3 &= \frac{4}{N^2(N-1)^2} \sum_{\substack{1\leq i_1<j_1\leq N \\ 1\leq i_2<j_2\leq N \\ \text{such that } i_1\leq k,\, i_2\leq k}} \langle S^{(k)}H_{+i_1j_1}\psi,\ S^{(k)}H_{+i_2j_2}\psi\rangle \\ &= \frac{4}{N^2(N-1)^2}\langle S^{(k)} \sum_{\substack{1\leq i<j\leq N \\ i\leqslant k}} H_{+ij}\psi,\ \sum_{\substack{1\leq i<j\leq N \\ i\leqslant k}} S^{(k)}H_{+ij}\psi\rangle \geqslant 0. \end{aligned}$$

Since $A_3 \geq 0$, we drop this term to obtain

$$2^{k+2}\langle\psi, (N^{-1}H_N + \alpha + 1)^{k+2}\psi\rangle \geq A_1 + A_2. \tag{4.16}$$

Note that $A_1$ contains $\sim N^4$ terms and the cross term $A_2$ contains $\sim N^3$ terms.[12] In other words, $A_1$ is the dominant term and $A_2$ is the error term. In below, we deal with $A_1$ and $A_2$ one by one.

In $A_1$, we can commute both terms $H_{+i_1j_1}$ and $H_{+i_2j_2}$ with $S^{(k)}$. Then

$$A_1 = \frac{4}{N^2(N-1)^2} \sum_{\substack{1\leq i_1<j_1\leq N \\ 1\leq i_2<j_2\leq N \\ \text{such that } i_1>k,\, i_2>k}} \langle S^{(k)}\psi,\ H_{+i_1j_1}H_{+i_2j_2}S^{(k)}\psi\rangle \tag{4.17}$$

We decompose

$$A_1 = A_{11} + A_{12} + A_{13} \tag{4.18}$$

where

- $A_{11}$ consists of those terms for which all indices $i_1$, $j_1$, $i_2$, $j_2$ are different. There are $\frac{1}{4}a_{N,k}N^4$ such terms, where

$$a_{N,k} \stackrel{\text{def}}{=} N^{-4}(N-k)(N-k-1)(N-k-2)(N-k-3)$$

[12] In the case $k = 0$, $A_2 = 0$ and $A_3 = 0$ since they are both empty sums.

- $A_{12}$ consists of those terms for which exactly one pair of indices $i_1$, $j_1$, $i_2$, $j_2$ are the same. There are $b_N N^3$ such terms, where

$$b_{N,k} \stackrel{\text{def}}{=} N^{-3}(N-k)(N-k-1)(N-k-2)$$

- $A_{13}$ consists of those terms for which exactly two pairs of indices $i_1$, $j_1$, $i_2$, $j_2$ are the same. There are $\frac{1}{2}c_N N^2$ such terms, where

$$c_{N,k} \stackrel{\text{def}}{=} N^{-2}(N-k)(N-k-1)$$

Note that

$$1 - C_k N^{-1} \leq a_{N,k} \leq 1 + C_k N^{-1}$$

for some[13] $C_k$, and similarly for $b_{N,k}$, $c_{N,k}$. Likewise, the coefficient in (4.17) satisfies

$$4N^{-4}(1 - CN^{-1}) \leq \frac{4}{N^2(N-1)^2} \leq 4N^{-4}(1 + CN^{-1})$$

The $O(N^{-1})$ corrections are easily absorbed into the error term in (4.13) and we drop them in the calculations that follow, for expositional convenience.

By symmetry, we have

$$A_{11} = \langle H_{+(k+1)(k+2)} S^{(k)}\psi, H_{+(k+3)(k+4)} S^{(k)}\psi \rangle$$

$$A_{12} = 4N^{-1} \langle H_{+(k+1)(k+2)} S^{(k)}\psi, H_{+(k+2)(k+3)} S^{(k)}\psi \rangle$$

$$A_{13} = 2N^{-2} \langle H_{+(k+1)(k+2)} S^{(k)}\psi, H_{+(k+1)(k+2)} S^{(k)}\psi \rangle$$

Since

$$A_{13} \geqslant 0 \tag{4.19}$$

we can discard it. By Lemmas 4.1 and A.2, we have

$$A_{11} \geqslant \tfrac{1}{4} \langle \left(S_{k+1}^2 + S_{k+2}^2\right) S^{(k)}\psi, \left(S_{k+3}^2 + S_{k+4}^2\right) S^{(k)}\psi \rangle$$

By integration by parts and symmetry, we obtain

$$A_{11} \geqslant \|S^{(k+2)}\psi\|_{L^2}^2. \tag{4.20}$$

Plugging in the definition (4.9) of $H_{+ij}$ and expanding

$$A_{12} = A_{121} + A_{122} + A_{123} \tag{4.21}$$

where

$$A_{121} = 4N^{-1} \langle \left(S_{k+1}^2 + S_{k+2}^2\right) \left(S_{k+2}^2 + S_{k+3}^2\right) S^{(k)}\psi, S^{(k)}\psi \rangle$$

$$A_{122} = 8 \operatorname{Re} N^{-1} \langle \left(S_{k+1}^2 + S_{k+2}^2\right) (V_{N(k+2)(k+3)} + 2\alpha) S^{(k)}\psi, S^{(k)}\psi \rangle$$

$$A_{123} = 4N^{-1} \langle (V_{N(k+1)(k+2)} + 2\alpha)(V_{N(k+2)(k+3)} + 2\alpha) S^{(k)}\psi, S^{(k)}\psi \rangle$$

For $A_{121}$, we only need to keep one term:

$$A_{121} \geqslant 4N^{-1} \|S_{k+2}^2 S^{(k)}\psi\|_{L^2}^2 = 4N^{-1} \|S_1 S^{(k+1)}\psi\|_{L^2}^2. \tag{4.22}$$

[13] We allow that $C_k$ changes from one line to the next.

For $A_{122}$, integration by parts gives

$$\begin{aligned} A_{122} &= 8N^{-1}\langle (V_{N(k+2)(k+3)}+2\alpha)S^{(k+1)}\psi, S^{(k+1)}\psi\rangle \\ &\quad +8N^{-1}\langle (V_{N(k+2)(k+3)}+2\alpha)S_{k+2}S^{(k)}\psi, S_{k+2}S^{(k)}\psi\rangle \\ &\quad +8\operatorname{Re} N^{\beta-1}\langle (V')_{N(k+2)(k+3)}\, S^{(k)}\psi, S_{k+2}S^{(k)}\psi\rangle \end{aligned}$$

where we used the fact that $\partial_{x_j}$ is the only thing inside $S_j$ which needs the Leibniz's rule. Estimating,

$$\begin{aligned} |A_{122}| &\lesssim N^{-1}\left\Vert V_{N(k+2)(k+3)}\right\Vert_{L^1_{x_{k+3}}}\left\Vert S^{(k+1)}\psi\right\Vert^2_{L^2_cL^\infty_{x_{k+3}}}+N^{-1}\alpha\Vert S^{(k+1)}\psi\Vert^2_{L^2} \\ &\quad +N^{-1}\left\Vert V_{N(k+2)(k+3)}\right\Vert_{L^1_{x_{k+3}}}\left\Vert S_{k+2}S^{(k)}\psi\right\Vert^2_{L^2_cL^\infty_{x_{k+3}}}+N^{-1}\alpha\Vert S_{k+2}S^{(k)}\psi\Vert^2_{L^2} \\ &\quad +N^{\beta-1}\left\Vert (V')_{N(k+2)(k+3)}\right\Vert_{L^1_{x_{k+3}}}\left\Vert S^{(k)}\psi\right\Vert_{L^2_cL^\infty_{x_{k+3}}}\left\Vert S_{k+2}S^{(k)}\psi\right\Vert_{L^2_cL^\infty_{x_{k+3}}} \\ &\lesssim N^{\beta-1}\left(\left\Vert S^{(k+1)}\psi\right\Vert^2_{L^2_cL^\infty_{x_{k+3}}}+\Vert S^{(k+1)}\psi\Vert^2_{L^2}+\left\Vert S^{(k)}\psi\right\Vert^2_{L^2_cL^\infty_{x_{k+3}}}\right). \end{aligned}$$

By the 1D estimate (4.4), Cauchy-Schwarz, and symmetry, we have

$$\begin{aligned} |A_{122}| &\leqslant CN^{\beta-1}\left(\Vert S^{(k+2)}\psi\Vert^2_{L^2}+\Vert S^{(k+1)}\psi\Vert^2_{L^2}\right) \\ &\leqslant CN^{\beta-1}\Vert S^{(k+2)}\psi\Vert^2_{L^2}. \end{aligned} \tag{4.23}$$

For $A_{123}$,

$$\begin{aligned} |A_{123}| &\leqslant CN^{-1}\left\Vert V_{N(k+1)(k+2)}\right\Vert_{L^1_{x_{k+1}}}\left\Vert V_{N(k+2)(k+3)}\right\Vert_{L^1_{x_{k+3}}}\left\Vert S^{(k)}\psi\right\Vert^2_{L^2_cL^\infty_{x_{k+1}}L^\infty_{x_{k+3}}} \\ &\quad +CN^{-1}\left\Vert V_{N(k+1)(k+2)}\right\Vert_{L^1_{x_{k+1}}}\left\Vert S^{(k)}\psi\right\Vert^2_{L^2_cL^\infty_{x_{k+1}}}+CN^{-1}\left\Vert S^{(k)}\psi\right\Vert^2_{L^2}. \end{aligned}$$

Using (4.4) twice, we obtain

$$\begin{aligned} |A_{123}| &\leqslant CN^{-1}(\Vert S^{(k+2)}\psi\Vert^2_{L^2}+\Vert S^{(k+1)}\psi\Vert^2_{L^2}+\left\Vert S^{(k)}\psi\right\Vert^2_{L^2}) \\ &\leqslant CN^{-1}\Vert S^{(k+2)}\psi\Vert^2_{L^2}. \end{aligned} \tag{4.24}$$

By (4.21), (4.22), (4.23), and (4.24),

$$A_{12}\geqslant 4N^{-1}\Vert S_1S^{(k+1)}\psi\Vert^2_{L^2}-CN^{\beta-1}\Vert S^{(k+2)}\psi\Vert^2_{L^2}. \tag{4.25}$$

Collecting (4.18), (4.19), (4.20), and (4.25), we have the estimate for $A_1$:

$$A_1\geqslant \left(1-CN^{\beta-1}\right)\left(\Vert S^{(k+2)}\psi\Vert^2_{L^2}+4N^{-1}\Vert S_1S^{(k+1)}\psi\Vert^2_{L^2}\right) \tag{4.26}$$

The above estimate yields the positive contribution on the right-side of (4.13).

Next we turn our attention to estimating $A_2$. We will prove that

$$A_2\geqslant -CN^{\beta-1}\left(\Vert S^{(k+2)}\psi\Vert^2_{L^2}+N^{-1}\Vert S_1S^{(k+1)}\psi\Vert^2_{L^2}\right).$$

Recall that in the case $k=0$, $A_2=0$, so we can assume $k\geq 1$. We decompose

$$A_2=A_{21}+A_{22}+A_{23} \tag{4.27}$$

where

- $A_{21}$ contains those terms with $j_1 \leq k$. There are $\sim N^2$ such terms. (In the case $k=1$, there are no terms of this type, so $A_{21}=0$)
- $A_{22}$ contains those terms with $j_1 > k$, and ($j_1 = i_2$ OR $j_1 = j_2$). There are $\sim N^2$ such terms.
- $A_{23}$ contains those terms with $j_1 > k$, $j_1 \neq i_2$ and $j_1 \neq j_2$. There are $\sim N^3$ such terms.

By symmetry of $\psi$ and $H_{+ij} = H_{+ji}$,

$$A_{21} = N^{-2}\langle S^{(k)} H_{+12}\psi, S^{(k)} H_{+(k+1)(k+2)}\psi\rangle$$
$$A_{22} = N^{-2}\langle S^{(k)} H_{+1(k+1)}\psi, S^{(k)} H_{+(k+1)(k+2)}\psi\rangle$$
$$A_{23} = N^{-1}\langle S^{(k)} H_{+1(k+1)}\psi, S^{(k)} H_{+(k+2)(k+3)}\psi\rangle$$

First, we address $A_{21}$. We decompose

$$A_{21} = A_{211} + A_{212} + A_{213}, \tag{4.28}$$

where

$$A_{211} = N^{-2}\langle H_{+12} S^{(k)}\psi, H_{+(k+1)(k+2)} S^{(k)}\psi\rangle$$
$$A_{212} = N^{-2}\langle [S_1, H_{+12}] S_2 \cdots S_k\psi, H_{+(k+1)(k+2)} S^{(k)}\psi\rangle$$
$$\begin{aligned} A_{213} &= N^{-2}\langle S_1[S_2, H_{+12}] S_3 \cdots S_k\psi, H_{+(k+1)(k+2)} S^{(k)}\psi\rangle \\ &= N^{-2}\langle [S_2, H_{+12}] S_3 \cdots S_k\psi, H_{+(k+1)(k+2)} S_1 S^{(k)}\psi\rangle. \end{aligned}$$

By Lemmas 4.1 and A.2,

$$A_{211} \geqslant 0. \tag{4.29}$$

Since $[S_1, H_{+12}] = N^{\beta}(V')_{N12}$, integrating by parts half the Hermite terms in $H_{+(k+1)(k+2)}$ and using symmetry,

$$\begin{aligned} A_{212} &= 2N^{\beta-2}\langle (V')_{N12} S_2 \cdots S_k S_{k+1}\psi, S^{(k+1)}\psi\rangle \\ &\quad +2\alpha N^{\beta-2}\langle (V')_{N12} S_2 \cdots S_k\psi, S^{(k)}\psi\rangle \\ &\quad +N^{\beta-2}\langle (V)_{N(k+1)(k+2)}(V')_{N12} S_2 \cdots S_k\psi, S^{(k)}\psi\rangle \end{aligned}$$

Estimating

$$\begin{aligned} |A_{212}| &\leqslant CN^{\frac{3\beta}{2}-2}\|V'\|_{L^2_{x_1}} \|S_2 \cdots S_k S_{k+1}\psi\|_{L^2_c L^\infty_{x_1}} \left\|S^{(k+1)}\psi\right\|_{L^2} \\ &\quad +CN^{\frac{3\beta}{2}-2}\|V'\|_{L^2_{x_1}} \|S_2 \cdots S_k\psi\|_{L^2_c L^\infty_{x_1}} \left\|S^{(k)}\psi\right\|_{L^2} \\ &\quad +CN^{\frac{3\beta}{2}-2}\|V\|_{L^1_{x_{k+1}}} \|V'\|_{L^2_{x_1}} \|S_2 \cdots S_k\psi\|_{L^2_c L^\infty_{x_1} L^\infty_{x_{k+1}}} \left\|S^{(k)}\psi\right\|_{L^2_c L^\infty_{x_{k+1}}} \end{aligned}$$

Using (4.4) and symmetry,

$$\begin{aligned} |A_{212}| &\leqslant CN^{\frac{3\beta}{2}-2}\left(\left\|S^{(k+1)}\psi\right\|^2_{L^2} + \left\|S^{(k)}\psi\right\|^2_{L^2} + \left\|S^{(k+1)}\psi\right\|^2_{L^2}\right) \\ &\leqslant CN^{\frac{3\beta}{2}-2}\left\|S^{(k+1)}\psi\right\|^2_{L^2} \end{aligned} \tag{4.30}$$

For $A_{213}$, we use $[S_2, H_{+12}] = -N^{\beta}(V')_{N12}$ to get

$$A_{213} = -N^{\beta-2}\langle (V')_{N12} S_3 \cdots S_k\psi, H_{+(k+1)(k+2)} S_1 S^{(k)}\psi\rangle$$

Split up the terms of $H_{+(k+1)(k+2)}$ via integration by parts and use symmetry to obtain

$$\begin{aligned} A_{213} &= -2N^{\beta-2}\langle (V')_{N12}S_3\cdots S_kS_{k+1}\psi, S_1S^{(k+1)}\psi\rangle \\ &\quad -2\alpha N^{\beta-2}\langle (V')_{N12}S_3\cdots S_k\psi, S_1S^{(k)}\psi\rangle \\ &\quad -N^{\beta-2}\langle V_{N(k+1)(k+2)}(V')_{N12}S_3\cdots S_k\psi, S_1S^{(k)}\psi\rangle \end{aligned}$$

We now implement the same estimates used to treat $A_{212}$ but carry a factor $N^{-1/2}$ with $S_1S^{(k)}\psi$ and $S_1S^{(k+1)}\psi$

$$\begin{aligned} |A_{213}| &\leqslant CN^{\frac{3\beta}{2}-\frac{3}{2}}\|V'\|_{L^2_{x_1}}\|S_3\cdots S_kS_{k+1}\psi\|_{L^2_cL^\infty_{x_1}}N^{-1/2}\left\|S_1S^{(k+1)}\psi\right\|_{L^2} \\ &\quad +CN^{\frac{3\beta}{2}-\frac{3}{2}}\|V'\|_{L^2_{x_1}}\|S_3\cdots S_k\psi\|_{L^2_cL^\infty_{x_1}}N^{-1/2}\left\|S_1S^{(k)}\psi\right\|_{L^2} \\ &\quad +CN^{\frac{3\beta}{2}-\frac{3}{2}}\|V\|_{L^2_{x_{k+1}}}\|V'\|_{L^2_{x_1}}\|S_3\cdots S_k\psi\|_{L^2_cL^\infty_{x_1}L^\infty_{x_{k+1}}}N^{-1/2}\left\|S_1S^{(k)}\psi\right\|_{L^2_cL^\infty_{x_{k+1}}} \end{aligned}$$

Arguing as above using (4.4) and symmetry

$$\begin{aligned} |A_{213}| &\leqslant CN^{\frac{3\beta}{2}-\frac{3}{2}}\left\|S^{(k)}\psi\right\|_{L^2}N^{-1/2}\left\|S_1S^{(k+1)}\psi\right\|_{L^2} \\ &\quad +CN^{\frac{3\beta}{2}-\frac{3}{2}}\left\|S^{(k-1)}\psi\right\|_{L^2}N^{-1/2}\left\|S_1S^{(k)}\psi\right\|_{L^2} \\ &\quad +CN^{\frac{3\beta}{2}-\frac{3}{2}}\left\|S^{(k)}\psi\right\|_{L^2}N^{-1/2}\left\|S_1S^{(k+1)}\psi\right\|_{L^2} \\ &\leqslant CN^{\frac{3\beta}{2}-\frac{3}{2}}\left(\left\|S^{(k)}\psi\right\|^2_{L^2}+N^{-1}\left\|S_1S^{(k+1)}\psi\right\|^2_{L^2}\right) \end{aligned} \tag{4.31}$$

By (4.28), (4.29), (4.30), and (4.31), we obtain

$$A_{21}\geqslant -CN^{\frac{3\beta}{2}-\frac{3}{2}}\left(\left\|S^{(k+1)}\psi\right\|^2_{L^2}+N^{-1}\left\|S_1S^{(k+1)}\psi\right\|^2_{L^2}\right) \tag{4.32}$$

Next, we address $A_{22}$. Recall

$$A_{22}=N^{-2}\langle S^{(k)}H_{+1(k+1)}\psi, H_{+(k+1)(k+2)}S^{(k)}\psi\rangle$$

Decompose

$$A_{22}=A_{221}+A_{222} \tag{4.33}$$

where

$$A_{221}=N^{-2}\langle H_{+1(k+1)}S^{(k)}\psi, H_{+(k+1)(k+2)}S^{(k)}\psi\rangle$$

$$A_{222}=N^{-2}\langle [S_1,H_{+1(k+1)}]S_2...S_k\psi, H_{+(k+1)(k+2)}S^{(k)}\psi\rangle$$

For $A_{221}$, plug in the definition (4.9) of $H_{+ij}$ to obtain the decomposition

$$A_{221}=A_{2211}+A_{2212}+A_{2213}+A_{2214}$$

where

$$A_{2211}=N^{-2}\langle \left(S_1^2+S_{k+1}^2\right)S^{(k)}\psi, \left(S_{k+1}^2+S_{k+2}^2\right)S^{(k)}\psi\rangle$$

$$A_{2212}=N^{-2}\langle \left(S_1^2+S_{k+1}^2\right)S^{(k)}\psi, (V_{N(k+1)(k+2)}+2\alpha)S^{(k)}\psi\rangle$$

$$A_{2213}=N^{-2}\langle (V_{N1(k+1)}+2\alpha)S^{(k)}\psi, \left(S_{k+1}^2+S_{k+2}^2\right)S^{(k)}\psi\rangle$$

$$A_{2214}=N^{-2}\langle (V_{N1(k+1)}+2\alpha)S^{(k)}\psi, (V_{N(k+1)(k+2)}+2\alpha)S^{(k)}\psi\rangle$$

Note that $A_{2211} \geq 0$, so we can discard this term. Integrating by parts,

$$\begin{aligned} A_{2212} &= N^{-2}\langle S_1 S^{(k)}\psi, (V_{N(k+1)(k+2)} + 2\alpha) S_1 S^{(k)}\psi\rangle \\ &+N^{-2}\langle S^{(k+1)}\psi, (V_{N(k+1)(k+2)} + 2\alpha) S^{(k+1)}\psi\rangle \\ &+N^{\beta-2}\langle S^{(k+1)}\psi, (V')_{N(k+1)(k+2)}\, S^{(k)}\psi\rangle \end{aligned}$$

Putting every instance of $V$ or $V'$ in $L^\infty$, we obtain the estimate

$$\begin{aligned} |A_{2212}| &\leqslant CN^{\beta-2}\left\|S_1 S^{(k)}\psi\right\|_{L^2}^2 + CN^{\beta-2}\left\|S^{(k+1)}\psi\right\|_{L^2}^2 \\ &+CN^{2\beta-2}\left\|S^{(k+1)}\psi\right\|_{L^2}\left\|S^{(k)}\psi\right\|_{L^2} \end{aligned}$$

Using that $\max\left(N^{2\beta-2}, N^{\beta-2}\right) \leq N^{\beta-1}$,

$$|A_{2212}| \leqslant CN^{\beta-1}(N^{-1}\left\|S_1 S^{(k)}\psi\right\|_{L^2}^2 + \left\|S^{(k+1)}\psi\right\|_{L^2}^2)$$

By integration by parts,

$$\begin{aligned} A_{2213} &= N^{-2}\langle (V_{N1(k+1)} + 2\alpha) S^{(k+1)}\psi, S^{(k+1)}\psi\rangle \\ &-N^{\beta-2}\langle (V')_{N1(k+1)}\, S^{(k)}\psi, S^{(k+1)}\psi\rangle \\ &+N^{-2}\langle (V_{N1(k+1)} + 2\alpha) S_{k+2} S^{(k)}\psi, S_{k+2} S^{(k)}\psi\rangle \end{aligned}$$

Putting every instance of $V$ or $V'$ in $L^\infty$,

$$\begin{aligned} |A_{2213}| &\leqslant CN^{\beta-2}\left\|S^{(k+1)}\psi\right\|_{L^2}^2 + CN^{2\beta-2}\left\|S^{(k+1)}\psi\right\|_{L^2}\left\|S^{(k)}\psi\right\|_{L^2} \\ &+CN^{\beta-2}\left\|S^{(k+1)}\psi\right\|_{L^2}^2 \\ &\leqslant CN^{2\beta-2}\left\|S^{(k+1)}\psi\right\|_{L^2}^2 \end{aligned}$$

For $A_{2214}$, we put both $V$ terms in $L^\infty$ to obtain

$$|A_{2214}| \leqslant CN^{2\beta-2}\|S^{(k)}\psi\|_{L^2}^2$$

This completes the bound for $A_{221}$. Specifically,

$$A_{221} \geqslant -CN^{\beta-1}(\left\|S^{(k+1)}\psi\right\|_{L^2}^2 + N^{-1}\left\|S_1 S^{(k)}\psi\right\|_{L^2}^2) \tag{4.34}$$

For $A_{222}$, substitute $[S_1, H_{+1(k+1)}] = N^\beta (V')_{N1(k+1)}$ and plug in the definition (4.9) of $H_{+(k+1)(k+2)}$ to obtain

$$A_{222} = A_{2221} + A_{2222} + A_{2223}$$

where

$$A_{2221} = N^{\beta-2}\langle (V')_{N1(k+1)} S_2 ... S_k\psi, S_{k+1}^2 S^{(k)}\psi\rangle$$

$$A_{2222} = N^{\beta-2}\langle (V')_{N1(k+1)} S_2 ... S_k\psi, S_{k+2}^2 S^{(k)}\psi\rangle$$

$$A_{2223} = N^{\beta-2}\langle (V')_{N1(k+1)} S_2 ... S_k\psi, (V_{N(k+1)(k+2)} + 2\alpha) S^{(k)}\psi\rangle$$

For $A_{2221}$, we apply Hölder in $x_1$ as follows:

$$|A_{2221}| \leqslant N^{\beta-2}\|(V')_{N1(k+1)}\|_{L^2_{x_1}}\|S_2 ... S_k\psi\|_{L^2_c L^\infty_{x_1}}\|S_{k+1}^2 S^{(k)}\psi\|_{L^2}$$

By (4.4) and symmetry,

$$\begin{aligned}|A_{2221}| &\leqslant CN^{\frac{3}{2}\beta-\frac{3}{2}}\|V'\|_{L^2}\|S^{(k)}\psi\|_{L^2}N^{-1/2}\|S_1S^{(k+1)}\psi\|_{L^2}\\ &\leqslant CN^{\frac{3}{2}\beta-\frac{3}{2}}\left(\|S^{(k)}\psi\|_{L^2}^2+N^{-1}\|S_1S^{(k+1)}\psi\|_{L^2}^2\right).\end{aligned}$$

Argue the same for $A_{2222}$, we get

$$\begin{aligned}|A_{2222}| &\leqslant N^{\beta-2}\|(V')_{N1(k+1)}\|_{L^2_{x_1}}\|S_2...S_k\psi\|_{L^2_cL^\infty_{x_1}}\|S_{k+2}^2S^{(k)}\psi\|_{L^2}\\ &\leqslant CN^{\frac{3}{2}\beta-\frac{3}{2}}\|V'\|_{L^2}\|S^{(k)}\psi\|_{L^2}N^{-1/2}\|S_1S^{(k+1)}\psi\|_{L^2}\\ &\leqslant CN^{\frac{3}{2}\beta-\frac{3}{2}}\left(\|S^{(k)}\psi\|_{L^2}^2+N^{-1}\|S_1S^{(k+1)}\psi\|_{L^2}^2\right).\end{aligned}$$

For $A_{2223}$, we use Hölder in $x_{k+1}$ to obtain

$$N^{\beta-2}\langle (V')_{N1(k+1)}S_2...S_k\psi,(V_{N(k+1)(k+2)}+2\alpha)S^{(k)}\psi\rangle$$

$$\begin{aligned}|A_{2223}| &\leqslant CN^{\beta-2}\left\|(V')_{N1(k+1)}\right\|_{L^1_{x_{k+1}}}\left\|V_{N(k+1)(k+2)}+2\alpha\right\|_{L^\infty_{x_{k+1}}}\\ &\quad\times\|S_2...S_k\psi\|_{L^2_cL^\infty_{x_{k+1}}}\left\|S^{(k)}\psi\right\|_{L^2_cL^\infty_{x_{k+1}}}\\ &\leqslant CN^{2\beta-2}\left\|S^{(k)}\psi\right\|_{L^2}\left\|S^{(k+1)}\psi\right\|_{L^2}\\ &\leqslant CN^{2\beta-2}\left\|S^{(k+1)}\psi\right\|_{L^2}^2.\end{aligned}$$

This completes the estimate for $A_{222}$. Specifically, collecting the estimates for $A_{2221}\sim A_{2223}$, we obtain

$$A_{222}\geqslant -CN^{\beta-1}(\left\|S^{(k+1)}\psi\right\|_{L^2}^2+N^{-1}\|S_1S^{(k+1)}\psi\|_{L^2}^2) \tag{4.35}$$

By (4.33), (4.34) and (4.35), we complete the estimate for $A_{22}$ as

$$A_{22}\geq -CN^{\beta-1}(\left\|S^{(k+1)}\psi\right\|_{L^2}^2+N^{-1}\|S_1S^{(k+1)}\psi\|_{L^2}^2) \tag{4.36}$$

Finally, for $A_{23}$, we have

$$A_{23}=N^{-1}\langle S^{(k)}H_{+1(k+1)}\psi,S^{(k)}H_{+(k+2)(k+3)}\psi\rangle$$

$$A_{23}=A_{231}+A_{232} \tag{4.37}$$

where

$$A_{231}=N^{-1}\langle H_{+1(k+1)}S^{(k)}\psi,H_{+(k+2)(k+3)}S^{(k)}\psi\rangle$$
$$A_{232}=N^{-1}\langle [S_1,H_{+1(k+1)}]S_2...S_k\psi,H_{+(k+2)(k+3)}S^{(k)}\psi\rangle$$

By Lemmas 4.1 and A.2,

$$A_{231}\geqslant 0, \tag{4.38}$$

so we discard it. For $A_{232}$, we plug in $[S_1,H_{+1(k+1)}]=N^\beta(V')_{N1(k+1)}$, the definition (4.9) of $H_{+(k+2)(k+3)}$, integrate by parts and use symmetry to obtain

$$\begin{aligned}A_{232} &= 2N^{\beta-1}\langle (V')_{N1(k+1)}S_2...S_kS_{k+2}\psi,S_{k+2}S^{(k)}\psi\rangle\\ &\quad+2\alpha N^{\beta-1}\langle (V')_{N1(k+1)}S_2...S_k\psi,S^{(k)}\psi\rangle\\ &\quad+N^{\beta-1}\langle (V')_{N1(k+1)}S_2...S_k\psi,V_{N(k+2)(k+3)}S^{(k)}\psi\rangle\end{aligned}$$

For the first two terms, we apply Hölder in $x_{k+1}$, and for the third term, we apply Hölder in both $x_{k+1}$ and $x_{k+2}$ to obtain

$$\begin{aligned}|A_{232}| &\leqslant CN^{\beta-1}\|V'\|_{L^1_{x_{k+1}}}\|S_2...S_kS_{k+2}\psi\|_{L^2_cL^\infty_{x_{k+1}}}\left\|S_{k+2}S^{(k)}\psi\right\|_{L^2_cL^\infty_{x_{k+1}}}\\&+CN^{\beta-1}\|V'\|_{L^1_{x_{k+1}}}\|S_2...S_k\psi\|_{L^2_cL^\infty_{x_{k+1}}}\left\|S^{(k)}\psi\right\|_{L^2_cL^\infty_{x_{k+1}}}\\&+CN^{\beta-1}\|V'\|_{L^1_{x_{k+1}}}\|V\|_{L^1_{x_{k+2}}}\|S_2...S_k\psi\|_{L^2_cL^\infty_{x_{k+1}}L^\infty_{x_{k+2}}}\left\|S^{(k)}\psi\right\|_{L^2_cL^\infty_{x_{k+1}}L^\infty_{x_{k+2}}}.\end{aligned}$$

Again, use (4.4),

$$\begin{aligned}|A_{232}| &\leqslant CN^{\beta-1}\left\|S^{(k+1)}\psi\right\|_{L^2}\left\|S^{(k+2)}\psi\right\|_{L^2}\\&+CN^{\beta-1}\left\|S^{(k)}\psi\right\|_{L^2}\left\|S^{(k+1)}\psi\right\|_{L^2}\\&+CN^{\beta-1}\left\|S^{(k+1)}\psi\right\|_{L^2}\left\|S^{(k+2)}\psi\right\|_{L^2}\\&\leqslant CN^{\beta-1}\left\|S^{(k+2)}\psi\right\|^2_{L^2}\end{aligned}\tag{4.39}$$

Collecting (4.37), (4.38), and (4.39), we obtain

$$A_{23}\geq -CN^{\beta-1}\left\|S^{(k+2)}\psi\right\|^2_{L^2}.\tag{4.40}$$

By (4.27), (4.32), (4.36), and (4.40), we obtain

$$A_2\geq -CN^{\beta-1}\left(\|S^{(k+2)}\psi\|^2_{L^2}+N^{-1}\|S_1S^{(k+1)}\psi\|^2_{L^2}\right)\tag{4.41}$$

Finally, combining (4.16), (4.26), and (4.41), we complete the proof of (4.13) (assuming (4.2) for $k$). Whence, we have proved (4.2) for all $k$ and established Theorem 4.1.

## 5. Proof of Compactness and Convergence

**Theorem 5.1** (Compactness)**.** *For $T\in\left[-\frac{\tan\omega T_0}{\omega},\frac{\tan\omega T_0}{\omega}\right]$, the sequence*

$$\Gamma_N(\tau)=\left\{u_N^{(k)}\right\}_{k=1}^N\in\bigoplus_{k\geqslant 1}C\left([0,T],\mathcal{L}^1_k\right),$$

*which satisfies the 1D BBGKY hierarchy (2.1) subject to energy condition (3.1) is compact with respect to the product topology $\tau_{prod}$. For any limit point $\Gamma(t)=\left\{u^{(k)}\right\}_{k=1}^N$, $\gamma^{(k)}$ is a symmetric nonnegative trace class operator with trace bounded by* 1, *and it verifies the energy bound*

$$\sup_{\tau\in[0,T]}\operatorname{Tr}L^{(k)}u^{(k)}(\tau)L^{(k)}\leqslant C^k.\tag{5.1}$$

**Theorem 5.2** (Convergence)**.** *Let $\Gamma(\tau)=\left\{u^{(k)}\right\}_{k=1}^\infty$ be a limit point of $\Gamma_N(\tau)=\left\{u_N^{(k)}\right\}_{k=1}^N$, the sequence in Theorem 5.1, with respect to the product topology $\tau_{prod}$, then $\Gamma(\tau)$ is a solution to the focusing GP hierarchy (2.3) subject to initial data $u^{(k)}(0)=|\phi_0\rangle\langle\phi_0|^{\otimes k}$ with coupling constant $b_0=\int V(x)\,dx$, which, written in integral form, is*

$$u^{(k)}(\tau)=U^{(k)}(\tau)u^{(k)}(0)+ib_0\sum_{j=1}^k\int_0^\tau U^{(k)}(\tau-s)\operatorname{Tr}_{k+1}\left[g(s)\delta\left(y_j-y_{k+1}\right),u^{(k+1)}(s)\right]ds.\tag{5.2}$$

*Proof of Compactness.* By the standard diagonalization argument, it suffices to show the compactness of $u_N^{(k)}$ for fixed $k$ with respect to the metric $\hat{d}_k$. By the Arzelà-Ascoli theorem, this is equivalent to the equicontinuity of $u_N^{(k)}$, and by [27, Lemma 6.2], this is equivalent to the statement that for every observable $J^{(k)}$ from a dense subset of $\mathcal{K}_k$ and for every $\varepsilon > 0$, there exists $\delta(J^{(k)}, \varepsilon)$ such that for all $\tau_1, \tau_2 \in [0, T]$ with $|\tau_1 - \tau_2| \leqslant \delta$, we have

$$\sup_N \left| \operatorname{Tr} J^{(k)} u_N^{(k)}(\tau_1) - \operatorname{Tr} J^{(k)} u_N^{(k)}(\tau_2) \right| \leqslant \varepsilon .$$

We select the observables $J^{(k)} \in \mathcal{K}_k$ which satisfy

$$\left\| L_i L_j J^{(k)} L_i^{-1} L_j^{-1} \right\|_{\mathrm{op}} + \left\| L_i^{-1} L_j^{-1} J^{(k)} L_i L_j \right\|_{\mathrm{op}} < \infty,$$

where $L_j = \left(1 - \partial_j^2\right)^{\frac{1}{2}}$. Assume $0 \leqslant \tau_1 \leqslant \tau_2 \leqslant T$, hierarchy (2.1) yields

$$\begin{aligned}
& \left| \operatorname{Tr} J^{(k)} u_N^{(k)}(\tau_1) - \operatorname{Tr} J^{(k)} u_N^{(k)}(\tau_2) \right| \\
\leqslant & \sum_{j=1}^{k} \int_{\tau_1}^{\tau_2} \left| \operatorname{Tr} J^{(k)} \left[ -\partial_j^2, u_N^{(k)}(s) \right] \right| ds \\
& + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant k} \int_{\tau_1}^{\tau_2} \left| \operatorname{Tr} J^{(k)} \left[ g(s) V_{N,s}(y_i - y_j), u_N^{(k)}(s) \right] \right| ds \\
& + \frac{N-k}{N} \sum_{j=1}^{k} \int_{\tau_1}^{\tau_2} \left| \operatorname{Tr} J^{(k)} \left[ g(s) V_{N,s}(y_i - y_j), u_N^{(k+1)}(s) \right] \right| ds \\
\leqslant & \sum_{j=1}^{k} \int_{\tau_1}^{\tau_2} I ds + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant k} \int_{\tau_1}^{\tau_2} II ds + \frac{N-k}{N} \sum_{j=1}^{k} \int_{\tau_1}^{\tau_2} III ds.
\end{aligned}$$

For $I$, we have, by (3.1), that,

$$\begin{aligned}
I &= \left| \operatorname{Tr} J^{(k)} L_j^2 u_N^{(k)}(s) - \operatorname{Tr} J^{(k)} u_N^{(k)}(s) L_j^2 \right| \\
&= \left| \operatorname{Tr} L_j^{-1} J^{(k)} L_j L_j u_N^{(k)}(s) L_j - \operatorname{Tr} L_j J^{(k)} L_j^{-1} L_j u_N^{(k)}(s) L_j \right| \\
&\leqslant \left( \left\| L_j^{-1} J^{(k)} L_j \right\|_{\mathrm{op}} + \left\| L_j J^{(k)} L_j^{-1} \right\|_{\mathrm{op}} \right) \operatorname{Tr} L_j u_N^{(k)}(s) L_j \\
&\leqslant C_J.
\end{aligned}$$

Lemma A.1 and (3.1) will handle $II$ and $III$. Write

$$W_{ij} = \left( L_i^{-1} L_j^{-1} V_{N,s}(y_i - y_j) L_i^{-1} L_j^{-1} \right)$$

which, by Lemma A.1, is a bounded operator with the bound

$$\left\| W_{ij} \right\|_{op} \leqslant C \left\| V \right\|_{L^1},$$

uniformly in $s$. So then

$$\begin{aligned} II &= \left|\operatorname{Tr} J^{(k)} g(s) V_{N,s}\left(y_i-y_j\right) u_N^{(k)}(s)-\operatorname{Tr} J^{(k)} u_N^{(k)}(s) g(s) V_{N,s}\left(y_i-y_j\right)\right| \\ &= |g(s)|\,|\operatorname{Tr} L_i^{-1} L_j^{-1} J^{(k)} L_i L_j W_{ij} L_i L_j u_N^{(k)}(s) L_i L_j \\ &\quad -\operatorname{Tr} L_i L_j J^{(k)} L_i^{-1} L_j^{-1} L_i L_j u_N^{(k)}(s) L_i L_j W_{ij}| \\ &\leqslant C\left(\left\|L_i^{-1} L_j^{-1} J^{(k)} L_i L_j\right\|_{op}+\left\|L_i L_j J^{(k)} L_i^{-1} L_j^{-1}\right\|_{op}\right)\|V\|_{L^1} \operatorname{Tr} L_i L_j u_N^{(k)}(s) L_i L_j \\ &\leqslant C_J \end{aligned}$$

and

$$\begin{aligned} III &= \left|\operatorname{Tr} J^{(k)} g(s) V_{N,s}\left(y_j-y_{k+1}\right) u_N^{(k+1)}(s)-\operatorname{Tr} J^{(k)} u_N^{(k+1)}(s) g(s) V_{N,s}\left(y_j-y_{k+1}\right)\right| \\ &= |g(s)|\,|\operatorname{Tr} L_j^{-1} J^{(k)} L_j W_{j(k+1)} L_j L_{k+1} u_N^{(k+1)}(s) L_j L_{k+1} \\ &\quad -\operatorname{Tr} L_j J^{(k)} L_j^{-1} L_j L_{k+1} u_N^{(k+1)}(s) L_j L_{k+1} W_{j(k+1)}| \\ &\leqslant C\left(\left\|L_j^{-1} J^{(k)} L_j\right\|_{op}+\left\|L_j J^{(k)} L_j^{-1}\right\|_{op}\right)\|V\|_{L^1} \operatorname{Tr} L_j L_{k+1} u_N^{(k)}(s) L_j L_{k+1} \\ &\leqslant C_J . \end{aligned}$$

Putting together the estimates of $I$, $II$, and $III$, we have

$$\sup_N\left|\operatorname{Tr} J^{(k)} u_N^{(k)}(\tau_1)-\operatorname{Tr} J^{(k)} u_N^{(k)}(\tau_2)\right| \leqslant C_J^{(k)}\left|\tau_1-\tau_2\right|$$

which is enough to end the proof of Theorem 5.1. □

*Proof of Convergence.* By Theorem 5.1, passing to subsequences if necessary, we have

$$\lim_{N\to\infty} \sup_{\tau\in[0,T]} \operatorname{Tr} J^{(k)}\left(u_N^{(k)}-u^{(k)}\right)=0,\ \forall J^{(k)}\in\mathcal{K}_k. \tag{5.3}$$

We test (5.2) against the observables $J^{(k)}$ in Theorem 5.1. We prove that the limit point verifies

$$\operatorname{Tr} J^{(k)} u^{(k)}(0)=\operatorname{Tr} J^{(k)}\left|\phi_0\right\rangle\left\langle\phi_0\right|^{\otimes k}, \tag{5.4}$$

and

$$\begin{aligned} \operatorname{Tr} J^{(k)} u^{(k)} &= \operatorname{Tr} J^{(k)} U^{(k)}(\tau) u^{(k)}(0) \\ &\quad +i b_0 \sum_{j=1}^k \int_0^\tau \operatorname{Tr} J^{(k)} U^{(k)}(\tau-s)\left[g(s) \delta\left(y_j-y_{k+1}\right), u^{(k+1)}(s)\right] ds . \end{aligned} \tag{5.5}$$

We use the BBGKY hierarchy (2.1) for this purpose. Written in the form we need here, it becomes

$$\begin{aligned}
\operatorname{Tr} J^{(k)} u_N^{(k)} &= \operatorname{Tr} J^{(k)} U^{(k)}(\tau) u_N^{(k)}(0) \\
&\quad + \frac{i}{N} \sum_{1 \leqslant i<j \leqslant k} \int_0^\tau \operatorname{Tr} J^{(k)} U^{(k)}(\tau-s)\left[-g(s) V_{N,s}(y_i-y_j), u_N^{(k)}(s)\right] ds \\
&\quad + i \frac{N-k}{N} \sum_{j=1}^k \int_0^\tau \operatorname{Tr} J^{(k)} U^{(k)}(\tau-s)\left[-g(s) V_{N,s}(y_j-y_{k+1}), u_N^{(k+1)}(s)\right] ds \\
&= A + \frac{i}{N} \sum_{1 \leqslant i<j \leqslant k} B + i\left(1-\frac{k}{N}\right) \sum_{j=1}^k D.
\end{aligned}$$

We put a minus sign in front of $V_{N,s}$ so that the above takes the same form as (5.5) because $b_0 = -\int V_{N,s}(x)dx$.

First of all, (5.3) yields

$$\begin{aligned}
\lim_{N\to\infty} \operatorname{Tr} J^{(k)} u_N^{(k)} &= \operatorname{Tr} J^{(k)} u^{(k)} \\
\lim_{N\to\infty} \operatorname{Tr} J^{(k)} U^{(k)}(\tau) u_N^{(k)}(0) &= \operatorname{Tr} J^{(k)} U^{(k)}(\tau) u^{(k)}(0).
\end{aligned}$$

Since

$$u_N^{(1)}(0) = \gamma_N^{(1)}(0) \to |\phi_0\rangle\langle\phi_0| \text{ strongly as trace operators,}$$

we obtain through the argument on [40, p.64] that

$$u_N^{(k)}(0) = \gamma_N^{(k)}(0) \to |\phi_0\rangle\langle\phi_0|^{\otimes k} \text{ strongly as trace operators.}$$

So far, we have checked relation (5.4) and the left hand side and the first term on the right hand side of (5.5) for the limit point. We will prove

$$\lim_{N\to\infty} \frac{B}{N} = \lim_{N\to\infty} \frac{k}{N} D = 0,$$

$$\lim_{N\to\infty} D = \int_0^\tau g(s) \operatorname{Tr} J^{(k)} U^{(k)}(\tau-s)\left[\delta\left(y_j-y_{k+1}\right), u^{(k+1)}(s)\right] ds. \tag{5.6}$$

A computation similar to the estimate of $II$ and $III$ in the proof of Theorem 5.1 shows that $|B|$ and $|D|$ are uniformly bounded for every finite time, thus

$$\lim_{N\to\infty} \frac{B}{N} = \lim_{N\to\infty} \frac{k}{N} D = 0.$$

To acquire limit (5.6), we use Lemma A.3. Take a probability measure $\rho \in L^1(\mathbb{R})$, define $\rho_\alpha(y) = \frac{1}{\alpha}\rho\left(\frac{y}{\alpha}\right)$. Use the short notation $J_{s-\tau}^{(k)} = J^{(k)}U^{(k)}(\tau - s)$, we have

$$\begin{aligned}
&\left|\operatorname{Tr} J^{(k)}U^{(k)}(\tau - s)\left(-V_{N,s}(y_j - y_{k+1})\, u_N^{(k+1)}(s) - b_0\delta(y_j - y_{k+1})\, u^{(k+1)}(s)\right)\right| \\
&\quad\leqslant \left|\operatorname{Tr} J_{s-\tau}^{(k)}\left(-V_{N,s}(y_j - y_{k+1}) - b_0\delta(y_j - y_{k+1})\right) u_N^{(k+1)}(s)\right| \\
&\quad\quad + b_0\left|\operatorname{Tr} J_{s-\tau}^{(k)}\left(\delta(y_j - y_{k+1}) - \rho_\alpha(y_j - y_{k+1})\right) u_N^{(k+1)}(s)\right| \\
&\quad\quad + b_0\left|\operatorname{Tr} J_{s-\tau}^{(k)}\rho_\alpha(y_j - y_{k+1})\left(u_N^{(k+1)}(s) - u^{(k+1)}(s)\right)\right| \\
&\quad\quad + b_0\left|\operatorname{Tr} J_{s-\tau}^{(k)}\left(\rho_\alpha(y_j - y_{k+1}) - \delta(y_j - y_{k+1})\right) u^{(k+1)}(s)\right| \\
&\quad = E + F + G + H.
\end{aligned}$$

A direct application of Lemma A.3 and the energy condition (3.1) hands us

$$\begin{aligned}
E &\leqslant \frac{C}{N^{\kappa\beta}(g(s))^\kappa}\left(\left\|L_j^{-1}J^{(k)}L_j\right\|_{op} + \left\|L_jJ^{(k)}L_j^{-1}\right\|_{op}\right)\operatorname{Tr} L_jL_{k+1}u_N^{(k+1)}L_jL_{k+1} \\
&\leqslant \frac{C_J}{N^{\kappa\beta}} \to 0 \text{ as } N \to \infty, \text{ uniformly for } s \in [0,T] \text{ with } T < \infty.
\end{aligned}$$

Similarly, using Lemma A.3 and (3.1) and (5.1), we have

$$\begin{aligned}
F &\leqslant C_\kappa\alpha^\kappa b_0\left(\left\|L_j^{-1}J^{(k)}L_j\right\|_{op} + \left\|L_jJ^{(k)}L_j^{-1}\right\|_{op}\right)\operatorname{Tr} L_jL_{k+1}u_N^{(k+1)}L_jL_{k+1} \\
&\leqslant C_J\alpha^\kappa \to 0 \text{ as } \alpha \to 0, \\
H &\leqslant C_\kappa\alpha^\kappa b_0\left(\left\|L_j^{-1}J^{(k)}L_j\right\|_{op} + \left\|L_jJ^{(k)}L_j^{-1}\right\|_{op}\right)\operatorname{Tr} L_jL_{k+1}u^{(k+1)}L_jL_{k+1} \\
&\leqslant C_J\alpha^\kappa \to 0 \text{ as } \alpha \to 0.
\end{aligned}$$

For G,

$$\begin{aligned}
G \leqslant\ & b_0\left|\operatorname{Tr} J_{s-\tau}^{(k)}\rho_\alpha(y_j - y_{k+1})\frac{1}{1+\varepsilon L_{k+1}}\left(u_N^{(k+1)}(s) - u^{(k+1)}(s)\right)\right| \\
&+ b_0\left|\operatorname{Tr} J_{s-\tau}^{(k)}\rho_\alpha(y_j - y_{k+1})\frac{\varepsilon L_{k+1}}{1+\varepsilon L_{k+1}}\left(u_N^{(k+1)}(s) - u^{(k+1)}(s)\right)\right|.
\end{aligned}$$

The first term in the above estimate goes to zero as $N \to \infty$ for every $\varepsilon > 0$, since we have assumed (5.3) and $J_{s-\tau}^{(k)}\rho_\alpha(y_j - y_{k+1})(1+\varepsilon L_{k+1})^{-1}$ is a compact operator. Due to the energy bounds on $u_N^{(k+1)}$ and $u^{(k+1)}$, the second term tends to zero as $\varepsilon \to 0$, uniformly in $N$.

Combining the estimates for $E - H$, we have justified limit (5.6) and thus limit (5.5). Hence, we have finished proving Theorem 5.2. □

## 6. Proof of the Optimal 1D Collapsing Estimate (Theorem 1.3)

We prove the optimality in §6.1. It suffices to prove Theorem 3.1 for $k = 1$. We aim to prove that, for each $\varepsilon > 0$ and each bump function $\theta$,

$$\|\theta(\tau)R_\varepsilon^{(1)}U^{(1)}(-\tau)B_{1,2}U^{(2)}(\tau)\phi^{(2)}\|_{L_\tau^2L_{\mathbf{y},\mathbf{y}'}^2} \leq C_{\varepsilon,\theta}\|R_\varepsilon^{(2)}\phi^{(2)}\|_{L_{\mathbf{y},\mathbf{y}'}^2}$$

which is equivalent to

$$\|\theta(\tau)R_{\varepsilon}^{(1)}U^{(1)}(-\tau)B_{1,2}U^{(2)}(\tau)R_{-\varepsilon}^{(2)}\phi^{(2)}\|_{L_{\tau}^{2}L_{\mathbf{y},\mathbf{y}'}^{2}}\leq C_{\varepsilon,\theta}\|\phi^{(2)}\|_{L_{\mathbf{y},\mathbf{y}'}^{2}}$$

The space-time Fourier transform of the operator on the left side is (dropping the $x_1'$ variable)

$$\iint_{\xi_2,\xi_2'}\frac{\langle\xi_1\rangle^{\varepsilon}\hat{\theta}\left(\eta+(\xi_1-\xi_2-\xi_2')^2-\xi_1^2+\xi_2^2-(\xi_2')^2\right)}{\langle\xi_1-\xi_2-\xi_2'\rangle^{\varepsilon}\langle\xi_2\rangle^{\varepsilon}\langle\xi_2'\rangle^{\varepsilon}}\hat{\phi}(\xi_1-\xi_2-\xi_2',\xi_2,\xi_2')d\xi_2 d\xi_2'$$

where $(\eta,\xi)$ is the space-time Fourier variable. By the usual Cauchy-Schwarz procedure, it suffices to prove the boundedness (independent of $\eta$, $\xi_1$) of

$$I(\eta,\xi_1)\overset{\text{def}}{=}\iint_{\xi_2,\xi_2'}\frac{\langle\xi_1\rangle^{2\varepsilon}|\hat{\theta}\left(\eta-2\xi_1\left(\xi_2+\xi_2'\right)+\left(\xi_2+\xi_2'\right)^2+\xi_2^2-(\xi_2')^2\right)|}{\langle\xi_1-\xi_2-\xi_2'\rangle^{2\varepsilon}\langle\xi_2\rangle^{2\varepsilon}\langle\xi_2'\rangle^{2\varepsilon}}d\xi_2 d\xi_2'.$$

Changing variables $(\xi_2,\xi_2')\mapsto(u,v)$, where

$$\begin{aligned} u &= \xi_2+\xi_2' \\ v &= \xi_2-\xi_2' \end{aligned} \tag{6.1}$$

we obtain

$$I(\eta,\xi_1)=\iint_{u,v}|\hat{\theta}(\eta-2\xi_1 u+u^2+uv)|\frac{\langle\xi_1\rangle^{2\varepsilon}}{\langle\xi_1-u\rangle^{2\varepsilon}\langle u+v\rangle^{2\varepsilon}\langle u-v\rangle^{2\varepsilon}}dudv$$

Doing the $dv$ integral first and change $v\mapsto w$, where $w=\frac{\eta}{u}-2\xi_1+u+v$, we obtain

$$\begin{aligned} I(\eta,\xi_1) &= \int_u\frac{\langle\xi_1\rangle^{2\epsilon}}{\langle\xi_1-u\rangle^{2\epsilon}}H(\eta,\xi_1,u)du \\ &= \int_{|u|<1}+\int_{|u|>1} \\ &= I_1(\eta,\xi_1)+I_2(\eta,\xi_1) \end{aligned}$$

where

$$H(\eta,\xi_1,u)=\int_w\frac{|\hat{\theta}(uw)|}{\langle w-\frac{\eta}{u}+2\xi_1\rangle^{2\varepsilon}\langle w-2u-\frac{\eta}{u}+2\xi_1\rangle^{2\varepsilon}}dw \tag{6.2}$$

For convenience, we introduce

$$\sigma(\eta,\xi_1,u)\overset{\text{def}}{=}\frac{\eta}{u}-2\xi_1 \tag{6.3}$$

6.0.3. *Treating I.* We first address $I_1(\eta,\xi_1)$. For $|u|\leqslant 1$, we have by (6.2) and (6.3) that

$$\begin{aligned} H(\eta,\xi_1,u) &\leqslant C\int_w\frac{|\hat{\theta}(uw)|}{\langle w-\sigma\rangle^{4\varepsilon}}dw \\ &\leqslant C\int_w\frac{|\hat{\theta}(uw)|}{|w-\sigma|^{4\varepsilon}}dw. \end{aligned}$$

Change variables, we get

$$\begin{aligned} &= C\int_w \frac{|\hat{\theta}(w)|}{\left|\frac{w}{u}-\sigma\right|^{4\varepsilon}}\frac{dw}{|u|} \\ &= \frac{C}{|u|^{1-4\varepsilon}}\int_w \frac{|\hat{\theta}(w)|}{|w-u\sigma|^{4\varepsilon}}dw. \end{aligned}$$

Divide the integral into two pieces,

$$\begin{aligned} &= \frac{C}{|u|^{1-4\varepsilon}}\left(\int_{|w-u\sigma|\leqslant 1}\frac{|\hat{\theta}(w)|}{|w-u\sigma|^{4\varepsilon}}dw+\int_{|w-u\sigma|\geqslant 1}\frac{|\hat{\theta}(w)|}{|w-u\sigma|^{4\varepsilon}}dw\right) \\ &\leqslant \frac{C}{|u|^{1-4\varepsilon}}\left(\int_{|w-u\sigma|\leqslant 1}\frac{1}{|w-u\sigma|^{4\varepsilon}}dw+\int_{|w-u\sigma|\geqslant 1}|\hat{\theta}(w)|dw\right). \end{aligned}$$

Thus

$$H(\eta,\xi_1,u)\lesssim \frac{1}{|u|^{1-4\varepsilon}}.$$

Therefore, plugging the above into $I_1$, we have

$$I_1(\eta,\xi_1)\lesssim \int_{|u|\leq 1}\frac{\langle\xi_1\rangle^{2\varepsilon}}{\langle\xi_1-u\rangle^{2\varepsilon}|u|^{1-4\varepsilon}}\,du.$$

Since $|u|\leqslant 1$, we have $\frac{\langle\xi_1\rangle^{2\varepsilon}}{\langle u-\xi_1\rangle^{2\varepsilon}}\sim 1$ and therefore

$$I_1(\eta,\xi_1)\lesssim \int_{|u|\leqslant 1}\frac{du}{|u|^{1-4\varepsilon}}\lesssim 1.$$

6.0.4. *Treating $II$.* We turn our attention to $I_2(\eta,\xi_1)$. For $|u|\geqslant 1$, by (6.2) and (6.3),

$$H(\eta,\xi_1,u)\lesssim \frac{1}{|u|\langle\sigma\rangle^{2\varepsilon}\langle\sigma-2u\rangle^{2\varepsilon}} \tag{6.4}$$

Indeed, in this case, the integral in (6.2) is effectively constrained to the small interval $|w|\lesssim |u|^{-1}\leq 1$, and the extra factors $\langle\sigma\rangle^{2\varepsilon}\langle\sigma-2u\rangle^{2\varepsilon}$ in the denominator in (6.4) come from the factors $\langle w-\sigma\rangle^{2\varepsilon}\langle w-2u-\sigma\rangle^{2\varepsilon}$ in the denominator in (6.2). Plugging (6.4) into $I_2(\eta,\xi_1)$, we get

$$I_2(\eta,\xi_1)\lesssim \int_{|u|\geq 1}\frac{\langle\xi_1\rangle^{2\varepsilon}}{\langle\xi_1-u\rangle^{2\varepsilon}|u|\langle\sigma\rangle^{2\varepsilon}\langle\sigma-2u\rangle^{2\varepsilon}}\,du$$

If $|\xi_1|\leq 1$, then $I_2(\eta,\xi_1)\lesssim \int_{|u|\geq 1}\frac{du}{|u|^{1+2\varepsilon}}$ (by neglecting the two terms $\langle\sigma\rangle^{2\varepsilon}\langle\sigma-2u\rangle^{2\varepsilon}$ in the denominator), and this integral converges. If $|\xi_1|\geq 1$, then $\langle\xi_1\rangle\sim|\xi_1|$ and hence

$$I_2(\eta,\xi_1)\lesssim \int_{|u|\geq 1}\frac{|\xi_1|^{2\varepsilon}}{|\xi_1-u|^{2\varepsilon}|u||\sigma|^{2\varepsilon}|\sigma-2u|^{2\varepsilon}}\,du$$

Substituting (6.3),

$$\begin{aligned} I_2(\eta,\xi_1) &\lesssim \int_{|u|\geq 1}\frac{|\xi_1|^{2\varepsilon}}{|\xi_1-u|^{2\varepsilon}|u||\frac{\eta}{u}-2\xi_1|^{2\varepsilon}|u+\xi_1-\frac{\eta}{2u}|^{2\varepsilon}}\,du \\ &= \int_{|u|\geq 1}\frac{|\xi_1|^{2\varepsilon}}{|\xi_1-u|^{2\varepsilon}|u|^{1-4\varepsilon}|\eta-2\xi_1u|^{2\varepsilon}|u^2+\xi_1u-\frac{\eta}{2}|^{2\varepsilon}}\,du \end{aligned}$$

If $\frac{\eta}{2}+\frac{\xi_1^2}{4}\geq 0$, then let $a$, $b$ be the roots of the quadratic $u^2+\xi_1 u-\frac{\eta}{2}$ (which are real). If $\frac{\eta}{2}+\frac{\xi_1^2}{4}<0$, then let $a=b=-\frac{\xi_1}{2}$. Then we obtain

$$I_2(\eta,\xi_1)\lesssim\int_u\frac{du}{|u-\xi_1|^{2\varepsilon}\langle u\rangle^{1-4\varepsilon}|u-\frac{\eta}{2\xi_1}|^{2\varepsilon}|u-a|^{2\varepsilon}|u-b|^{2\varepsilon}}$$

The fact that this is bounded uniformly in $\xi_1$ and $\eta$ follows from Lemma 6.1.

**Lemma 6.1.** *Suppose that* $0<\varepsilon<\frac{1}{8}$. *Then*

$$\int\frac{du}{|u-a|^{2\varepsilon}|u-b|^{2\varepsilon}|u-c|^{2\varepsilon}|u-d|^{2\varepsilon}\langle u\rangle^{1-4\varepsilon}}$$

*is bounded independently of* $a$, $b$, $c$, $d$.

*Proof.* Call the given integral $G(a,b,c,d)$. Let

$$F(e)\overset{\text{def}}{=}\int_{u=-\infty}^{+\infty}\frac{du}{|u-e|^{8\varepsilon}\langle u\rangle^{1-4\varepsilon}} \tag{6.5}$$

We claim that

$$G(a,b,c,d)\leq F(a)+F(b)+F(c)+F(d) \tag{6.6}$$

To show (6.6), we might as well assume that

$$-\infty<a\leq b\leq c\leq d<+\infty$$

Divide the $u$-integration into the four intervals $-\infty<u\leq\frac{a+b}{2}$, $\frac{a+b}{2}\leq u\leq\frac{b+c}{2}$, $\frac{b+c}{2}\leq u\leq\frac{c+d}{2}$, $\frac{c+d}{2}\leq u<+\infty$. For $-\infty<u\leq\frac{a+b}{2}$, it is evident that the integral is bounded by $F(a)$. For $\frac{a+b}{2}\leq u\leq\frac{b+c}{2}$, the integral is bounded by $F(b)$, etc.

Hence it suffices to show that $F(e)$ is bounded independently of $e$. If $|e|\geq 1$, then we use that

$$F(e)\leq\int_u\frac{du}{|u-e|^{8\varepsilon}|u|^{1-4\varepsilon}}$$

and then change variables $u\mapsto x$ where $u=ex$ to obtain

$$F(e)\leq\frac{1}{|e|^{4\varepsilon}}\int\frac{dx}{|x-1|^{8\varepsilon}|x|^{1-4\varepsilon}}\lesssim 1$$

If $|e|\leq 1$, then dividing the integration in $u$ in (6.5) into $|u|\leq 1$ and $|u|\geq 1$ gives two integrals individually bounded independently of $e$. □

6.1. **Proof of Optimality.** We prove the failure of Theorem 1.3 for the $T=\infty$ and $\varepsilon\geqslant 0$ case and the $T<\infty$ and $\varepsilon=0$ case separately. We remark that both cases deduce to the fact that $\int_{|u|\leqslant 1}\frac{1}{|u|}du=\infty$.

6.1.1. *The $T=\infty$ and $\varepsilon\geqslant 0$ Case.* We disprove the estimate:

$$\|R_{\varepsilon}^{(1)}B_{1,2}U^{(2)}(\tau)R_{-\varepsilon}^{(2)}\phi^{(2)}\|_{L_{\tau}^{2}L_{\mathbf{y},\mathbf{y}'}^{2}}\leqslant C\|\phi^{(2)}\|_{L_{\mathbf{y},\mathbf{y}'}^{2}}. \tag{6.7}$$

By duality, it is equivalent to the estimate that

$$\left|\int_{\mathbb{R}^{1+1}}J\left(\eta,\xi_{1}\right)g(\eta,\xi_{1})d\eta d\xi_{1}\right|\leqslant C\|\phi^{(2)}\|_{L_{\mathbf{y}}^{2}}\|g\|_{L_{\eta}^{2}L_{\xi_{1}}^{2}} \tag{6.8}$$

for all $g\in L_{\eta}^{2}L_{\xi_{1}}^{2}$, where $J\left(\eta,\xi_{1}\right)$ is the space-time Fourier transform of $R_{\varepsilon}^{(1)}B_{1,2}U^{(2)}(\tau)R_{-\varepsilon}^{(2)}\phi^{(2)}$ (dropping the $x_{1}'$ variable) which is

$$J\left(\eta,\xi_{1}\right)=\int\frac{\langle\xi_{1}\rangle^{\varepsilon}\delta\left(\eta+\left(\xi_{1}-\xi_{2}-\xi_{2}'\right)^{2}+\xi_{2}^{2}-\left(\xi_{2}'\right)^{2}\right)}{\langle\xi_{1}-\xi_{2}-\xi_{2}'\rangle^{\varepsilon}\langle\xi_{2}\rangle^{\varepsilon}\langle\xi_{2}'\rangle^{\varepsilon}}\hat{\phi}(\xi_{1}-\xi_{2}-\xi_{2}',\xi_{2},\xi_{2}')d\xi_{2}d\xi_{2}'$$

Write out the left hand side of (6.8).

$$\begin{aligned}
&\int_{\mathbb{R}^{1+1}}J\left(\eta,\xi_{1}\right)g(\eta,\xi_{1})d\eta d\xi_{1}\\
=&\int d\xi_{1}d\xi_{2}d\xi_{2}'\hat{\phi}(\xi_{1},\xi_{2},\xi_{2}')\\
&\times\left(\int d\eta\frac{\langle\xi_{1}+\xi_{2}+\xi_{2}'\rangle^{\varepsilon}\delta\left(\eta+\xi_{1}^{2}+\xi_{2}^{2}-\left(\xi_{2}'\right)^{2}\right)}{\langle\xi_{1}\rangle^{\varepsilon}\langle\xi_{2}\rangle^{\varepsilon}\langle\xi_{2}'\rangle^{\varepsilon}}g(\eta,\xi_{1}+\xi_{2}+\xi_{2}')\right)
\end{aligned}$$

Thus estimate (6.8) is equivalent to the estimate

$$\int\frac{\langle\xi_{1}+\xi_{2}+\xi_{2}'\rangle^{2\epsilon}}{\langle\xi_{1}\rangle^{2\varepsilon}\langle\xi_{2}\rangle^{2\varepsilon}\langle\xi_{2}'\rangle^{2\varepsilon}}\left|g(-\xi_{1}^{2}-\xi_{2}^{2}+\left(\xi_{2}'\right)^{2},\xi_{1}+\xi_{2}+\xi_{2}')\right|^{2}d\xi_{1}d\xi_{2}d\xi_{2}'\leqslant C\left\|g\right\|_{L_{\tau,\xi_{1}}^{2}}$$

Performing the change of variables in (6.1) to the left hand side we get

$$\begin{aligned}
&\int\frac{\langle\xi_{1}+u\rangle^{2\epsilon}}{\langle\xi_{1}\rangle^{2\varepsilon}\langle u+v\rangle^{2\varepsilon}\langle u-v\rangle^{2\varepsilon}}\left|g(-\xi_{1}^{2}-uv,\xi_{1}+u)\right|^{2}d\xi_{1}dudv\\
=&\int\left(\int\frac{\langle\xi_{1}\rangle^{2\epsilon}}{\langle\xi_{1}-u\rangle^{2\varepsilon}\langle u-\frac{\eta+(\xi_{1}-u)^{2}}{u}\rangle^{2\varepsilon}\langle u+\frac{\eta+(\xi_{1}-u)^{2}}{u}\rangle^{2\varepsilon}}\frac{1}{|u|}du\right)\left|g(\eta,\xi_{1})\right|^{2}d\xi_{1}d\eta.
\end{aligned}$$

Over the region $|\eta|\lesssim 1$, $|\xi_{1}|\lesssim 1$, the $du$ integral effectively becomes

$$\int\frac{1}{\langle u\rangle^{4\varepsilon}}\frac{1}{|u|}du$$

which diverges to $\infty$. Whence, we have disproved estimate (6.7).

6.1.2. *The $T<\infty$ and $\varepsilon=0$ Case.* Here, we disprove the following estimate:

$$\|\theta(\tau)B_{1,2}U^{(2)}(\tau)\phi^{(2)}\|_{L_{\tau}^{2}L_{\mathbf{y},\mathbf{y}'}^{2}}\leqslant C\|\phi^{(2)}\|_{L_{\mathbf{y},\mathbf{y}'}^{2}}. \tag{6.9}$$

We proceed as in the $T=\infty$ and $\varepsilon>0$ case. This time

$$J\left(\eta,\xi_{1}\right)=\int\hat{\theta}\left(\eta+\left(\xi_{1}-\xi_{2}-\xi_{2}'\right)^{2}+\xi_{2}^{2}-\left(\xi_{2}'\right)^{2}\right)\hat{\phi}(\xi_{1}-\xi_{2}-\xi_{2}',\xi_{2},\xi_{2}')d\xi_{2}d\xi_{2}'$$

and hence (6.9) is equivalent to the estimate that

$$\int \left| \left( \hat{\theta} * g \right) \left( -\xi_1^2 - \xi_2^2 + \left( \xi_2' \right)^2, \xi_1 + \xi_2 + \xi_2' \right) \right|^2 d\xi_1 d\xi_2 d\xi_2' \leqslant C \left\| g \right\|_{L^2_{\tau,\xi_1}}$$

for all $g \in L^2_\eta L^2_{\xi_1}$. By the change of variables (6.1), the left side of the above estimate is

$$\begin{aligned} & \int \left| \left( \hat{\theta} * g \right) \left( -\xi_1^2 - uv, \xi_1 + u \right) \right|^2 d\xi_1 du dv \\ = & \int \left( \int \frac{1}{|u|} du \right) \left| \left( \hat{\theta} * g \right) (\eta, \xi_1) \right|^2 d\xi_1 d\eta. \end{aligned}$$

This disproves estimate (6.9). Together with the $T = \infty$ and $\varepsilon \geqslant 0$ case, we have attained the optimality of Theorem 1.3.

## Appendix A. Basic Operator Facts

**Lemma A.1** ([37, Lemma A.1])**.** *Let $x_i, x_j \in \mathbb{R}$,*

$$\left\| L_i^{-1} L_j^{-1} V(x_i - x_j) L_i^{-1} L_j^{-1} \right\|_{op} \leqslant \left\| V \right\|_{L^1}$$

**Lemma A.2.** *If $A_2 \geqslant A_1 \geqslant 0$, $B_2 \geqslant B_1 \geqslant 0$, and $[A_i, B_j] = 0$, $\forall i, j = 1, 2$, i.e. all A-B pairs commute. Then $A_2 B_2 \geqslant A_1 B_1$.*

*Proof.* We compute directly that

$$\langle u, A_1 B_1 u \rangle = \left\langle B_1^{\frac{1}{2}} u, A_1 B_1^{\frac{1}{2}} u \right\rangle \leqslant \left\langle B_1^{\frac{1}{2}} u, A_2 B_1^{\frac{1}{2}} u \right\rangle = \left\langle A_2^{\frac{1}{2}} u, B_1 A_2^{\frac{1}{2}} u \right\rangle \leqslant \langle u, A_2 B_2 u \rangle.$$

□

**Lemma A.3.** *Let $\rho \in L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} \rho(x)\, dx = 1$ and $\int_{\mathbb{R}} \langle x \rangle |\rho(x)|\, dx < \infty$, and let $\rho_\alpha(x) = \frac{1}{\alpha} \rho\left(\frac{x}{\alpha}\right)$. Then, for every $\kappa \in (0,1)$, there exists $C_\kappa > 0$ s.t.*

$$\begin{aligned} & \left| \operatorname{Tr} J^{(k)} \left( \rho_\alpha (x_j - x_{k+1}) - \delta (x_j - x_{k+1}) \right) \gamma^{(k+1)} \right| \\ \leqslant \; & C \left( \int |\rho(x)| \, |x|^\kappa \, dx \right) \alpha^\kappa \left( \left\| L_j J^{(k)} L_j^{-1} \right\|_{\mathrm{op}} + \left\| L_j^{-1} J^{(k)} L_j \right\|_{\mathrm{op}} \right) \\ & \times \operatorname{Tr} L_j L_{k+1} \gamma^{(k+1)} L_j L_{k+1} \end{aligned}$$

*for all nonnegative $\gamma^{(k+1)} \in \mathcal{L}^1_{k+1}$.*

*Proof.* Kirkpatrick, Schlein, and Staffilani stated a similar lemma ([37, Lemma A.2]) with $\rho \geqslant 0$. Their proof, slightly modified, gives Lemma A.3. For completeness, we include the details. It suffices to prove the estimate for $k = 1$. We represent $\gamma^{(2)}$ by $\gamma^{(2)} = \sum_j \lambda_j \left| \varphi_j \right\rangle \left\langle \varphi_j \right|$, where $\varphi_j \in L^2(\mathbb{R})$ and $\lambda_j \geqslant 0$. We write

$$\begin{aligned} & \operatorname{Tr} J^{(1)} \left( \rho_\alpha (x_1 - x_2) - \delta (x_1 - x_2) \right) \gamma^{(2)} \\ & = \sum_j \lambda_j \left\langle \varphi_j, J^{(1)} \left( \rho_\alpha (x_1 - x_2) - \delta (x_1 - x_2) \right) \varphi_j \right\rangle \\ & = \sum_j \lambda_j \left\langle \psi_j, \left( \rho_\alpha (x_1 - x_2) - \delta (x_1 - x_2) \right) \varphi_j \right\rangle \end{aligned}$$

where $\psi_j = \left(J^{(1)} \otimes 1\right) \varphi_j$. By Parseval, we find

$$\begin{aligned} & |\langle \psi_j, (\rho_\alpha(x_1 - x_2) - \delta(x_1 - x_2))\varphi_j\rangle| \\ = & |\int \overline{\hat{\psi}}_j(\xi_1, \xi_2)\hat{\varphi}_j(\xi_1', \xi_2') \left(\hat{\rho}(\alpha(\xi_1 - \xi_1')) - 1\right) \\ & \times \delta(\xi_1 + \xi_2 - \xi_1' - \xi_2') d\xi_1 d\xi_2 d\xi_1' d\xi_2'|. \end{aligned}$$

With $\int \rho = 1$, we rewrite

$$\begin{aligned} = & |\int \overline{\hat{\psi}}_j(\xi_1, \xi_2)\hat{\varphi}_j(\xi_1', \xi_2')\rho(x)(e^{i\alpha x\cdot(\xi_1' - \xi_1)} - 1) \\ & \times \delta(\xi_1 + \xi_2 - \xi_1' - \xi_2') dx d\xi_1 d\xi_2 d\xi_1' d\xi_2'| \\ \leqslant & \int |\hat{\psi}_j(\xi_1, \xi_2)||\hat{\varphi}_j(\xi_1', \xi_2')|\delta(\xi_1 + \xi_2 - \xi_1' - \xi_2') \\ & \times |\int \rho(x)(e^{i\alpha x\cdot(\xi_1' - \xi_1)} - 1) dx| d\xi_1 d\xi_2 d\xi_1' d\xi_2'. \end{aligned}$$

Using the inequality that $\forall \kappa \in (0,1)$

$$\begin{aligned} \left|e^{i\alpha x\cdot(\xi_1' - \xi_1)} - 1\right| & \leqslant \alpha^\kappa |x|^\kappa |\xi_1 - \xi_1'|^\kappa \\ & \leqslant \alpha^\kappa |x|^\kappa \left(|\xi_1|^\kappa + |\xi_1'|^\kappa\right), \end{aligned}$$

we get

$$\begin{aligned} & |\langle \psi_j, (\rho_\alpha(x_1 - x_2) - \delta(x_1 - x_2))\varphi_j\rangle| \\ \leqslant & \alpha^\kappa (\int |\rho(x)| |x|^\kappa dr) \\ & \times \int |\xi_1|^\kappa |\hat{\psi}_j(\xi_1, \xi_2)||\hat{\varphi}_j(\xi_1', \xi_2')|\delta(\xi_1 + \xi_2 - \xi_1' - \xi_2') d\xi_1 d\xi_2 d\xi_1' d\xi_2' \\ & + \alpha^\kappa (\int |\rho(x)| |x|^\kappa dr) \\ & \times \int |\xi_1'|^\kappa |\hat{\psi}_j(\xi_1, \xi_2)||\hat{\varphi}_j(\xi_1', \xi_2')|\delta(\xi_1 + \xi_2 - \xi_1' - \xi_2') d\xi_1 d\xi_2 d\xi_1' d\xi_2' \\ = & \alpha^\kappa (\int |\rho(x)| |x|^\kappa dr)(I + II). \end{aligned}$$

The estimates for $I$ and $II$ are similar, so we only deal with $I$ explicitly. We rewrite $I$ as

$$\begin{aligned} I = & \int \delta\left(\xi_1 + \xi_2 - \xi_1' - \xi_2'\right) \frac{\langle \xi_1\rangle \langle \xi_2\rangle}{\langle \xi_1'\rangle \langle \xi_2'\rangle} \left|\hat{\psi}_j\left(\xi_1, \xi_2\right)\right| \\ & \times \frac{\langle \xi_1'\rangle \langle \xi_2'\rangle}{\langle \xi_1\rangle^{1-\kappa} \langle \xi_2\rangle} \left|\hat{\varphi}_j\left(\xi_1', \xi_2'\right)\right| d\xi_1 d\xi_2 d\xi_1' d\xi_2'. \end{aligned}$$

Apply Cauchy-Schwarz:

$$\begin{aligned}
&\leqslant \varepsilon \int \delta\left(\xi_1+\xi_2-\xi_1'-\xi_2'\right) \frac{\langle\xi_1\rangle^2 \langle\xi_2\rangle^2}{\langle\xi_1'\rangle^2 \langle\xi_2'\rangle^2} \left|\hat{\psi}_j\left(\xi_1,\xi_2\right)\right|^2 d\xi_1 d\xi_2 d\xi_1' d\xi_2' \\
&\quad +\frac{1}{\varepsilon} \int \delta\left(\xi_1+\xi_2-\xi_1'-\xi_2'\right) \frac{\langle\xi_1'\rangle^2 \langle\xi_2'\rangle^2}{\langle\xi_1\rangle^{2(1-\kappa)} \langle\xi_2\rangle^2} \left|\hat{\varphi}_j\left(\xi_1',\xi_2'\right)\right|^2 d\xi_1 d\xi_2 d\xi_1' d\xi_2'.
\end{aligned}$$

Rearrange terms:

$$\begin{aligned}
&= \varepsilon \int \langle\xi_1\rangle^2 \langle\xi_2\rangle^2 \left|\hat{\psi}_j\left(\xi_1,\xi_2\right)\right|^2 d\xi_1 d\xi_2 \int \frac{1}{\langle\xi_1+\xi_2-\xi_2'\rangle^2 \langle\xi_2'\rangle^2} d\xi_2' \\
&\quad +\frac{1}{\varepsilon} \int \langle\xi_1'\rangle^2 \langle\xi_2'\rangle^2 \left|\hat{\varphi}_j\left(\xi_1',\xi_2'\right)\right|^2 d\xi_1' d\xi_2' \int \frac{1}{\langle\xi_1'+\xi_2'-\xi_2\rangle^{2(1-\kappa)} \langle\xi_2\rangle^2} d\xi_2 \\
&\leqslant \varepsilon \left\langle\psi_j, L_1^2 L_2^2 \psi_j\right\rangle \sup_{\xi} \int_{\mathbb{R}} \frac{1}{\langle\xi-\eta\rangle^2 \langle\eta\rangle^2} d\eta \\
&\quad +\frac{1}{\varepsilon} \left\langle\varphi_j, L_1^2 L_2^2 \varphi_j\right\rangle \sup_{\xi} \int_{\mathbb{R}} \frac{1}{\langle\xi-\eta\rangle^{2(1-\kappa)} \langle\eta\rangle^2} d\eta.
\end{aligned}$$

When $\kappa \in [0,1]$,

$$\begin{aligned}
\sup_{\xi} \int_{\mathbb{R}} \frac{1}{\langle\xi-\eta\rangle^{2(1-\kappa)} \langle\eta\rangle^2} d\eta &< \infty, \\
\sup_{\xi} \int_{\mathbb{R}} \frac{1}{\langle\xi-\eta\rangle^2 \langle\eta\rangle^2} d\eta &< \infty,
\end{aligned}$$

hence we have

$$\begin{aligned}
&\left|\operatorname{Tr} J^{(1)}\left(\rho_\alpha\left(x_1-x_2\right)-\delta\left(x_1-x_2\right)\right)\gamma^{(k+1)}\right| \\
&\leqslant C\left(\int |\rho(x)|\,|x|^\kappa\,dx\right)\alpha^\kappa \left(\varepsilon \operatorname{Tr} J^{(1)} L_1^2 L_2^2 J^{(1)} \gamma^{(2)} + \frac{1}{\varepsilon} \operatorname{Tr} L_1^2 L_2^2 \gamma^{(2)}\right) \\
&= C\left(\int |\rho(x)|\,|x|^\kappa\,dx\right)\alpha^\kappa \\
&\quad \times\left(\varepsilon \operatorname{Tr} L_1^{-1} L_2^{-1} J^{(1)} L_1 L_1 J^{(1)} L_1^{-1} L_1 L_2^2 \gamma^{(2)} L_1 L_2 + \frac{1}{\varepsilon} \operatorname{Tr} L_1^2 L_2^2 \gamma^{(2)}\right) \\
&\leqslant C\left(\int |\rho(x)|\,|x|^\kappa\,dx\right)\alpha^\kappa \left(\varepsilon \left\|L_1^{-1} J^{(1)} L_1\right\|_{\mathrm{op}} \left\|L_1 J^{(1)} L_1^{-1}\right\|_{\mathrm{op}} + \frac{1}{\varepsilon}\right) \\
&\quad \times \operatorname{Tr} L_1^2 L_2^2 \gamma^{(2)}.
\end{aligned}$$

Let $\varepsilon = \|L_1 J^{(1)} L_1^{-1}\|_{\mathrm{op}}^{-1}$, we reach

$$\begin{aligned}
&\leqslant C\left(\int |\rho(x)|\,|x|^\kappa\,dx\right)\alpha^\kappa \left(\left\|L_1^{-1} J^{(1)} L_1\right\|_{\mathrm{op}} + \left\|L_1 J^{(1)} L_1^{-1}\right\|_{\mathrm{op}}\right) \\
&\quad \times \operatorname{Tr} L_1^2 L_2^2 \gamma^{(2)}
\end{aligned}$$

as claimed. □

## Appendix B. Deducing Theorem 1.1 from Theorem 1.2

If $\psi_N(0)$ satisfies (a), (b), and (c) in Theorem 1.1, then $\psi_N(0)$ checks the requirements of Lemma B.1. Thus we can define an approximation $\psi_N^\kappa(0)$ of $\psi_N(0)$ as in (B.2). Via (i) and (iii) of Lemma B.1, $\psi_N^\kappa(0)$ verifies the hypothesis of Theorem 1.2 for small enough $\kappa > 0$. Therefore, for $\gamma_N^{\kappa,(k)}(t)$, the marginal density associated with $e^{itH_N}\psi_N^\kappa(0)$, Theorem 1.2 gives the convergence

$$\lim_{N\to\infty} \operatorname{Tr}\left|\gamma_N^{\kappa,(k)}(t)(t,\mathbf{x}_k;\mathbf{x}_k') - \prod_{j=1}^k \phi(t,x_j)\overline{\phi}(t,x_j')\right| = 0. \tag{B.1}$$

for all small enough $\kappa > 0$, all $k \geqslant 1$, and all $t \in \mathbb{R}$.

For $\gamma_N^{(k)}(t)$ in Theorem 1.1, we notice that, $\forall J^{(k)} \in \mathcal{K}_k$, $\forall t \in \mathbb{R}$, we have

$$\begin{aligned} & \left|\operatorname{Tr} J^{(k)}\left(\gamma_N^{(k)}(t) - |\phi(t)\rangle\langle\phi(t)|^{\otimes k}\right)\right| \\ \leqslant & \left|\operatorname{Tr} J^{(k)}\left(\gamma_N^{(k)}(t) - \gamma_N^{\kappa,(k)}(t)\right)\right| \\ & + \left|\operatorname{Tr} J^{(k)}\left(\gamma_N^{\kappa,(k)}(t) - |\phi(t)\rangle\langle\phi(t)|^{\otimes k}\right)\right| \\ = & \ \mathrm{I} + \mathrm{II}. \end{aligned}$$

Convergence (B.1) then takes care of II. To handle I , part (ii) of Lemma B.1 yields

$$\left\|e^{itH_N}\psi_N^\kappa(0) - e^{itH_N}\psi_N(0)\right\|_{L^2} = \|\psi_N^\kappa(0) - \psi_N(0)\|_{L^2} \leqslant C\kappa^{\frac{1}{2}}$$

which implies

$$I = \left|\operatorname{Tr} J^{(k)}\left(\gamma_N^{(k)}(t) - \gamma_N^{\kappa,(k)}(t)\right)\right| \leqslant C\left\|J^{(k)}\right\|_{op}\kappa^{\frac{1}{2}}.$$

Since $\kappa > 0$ is arbitrary, we deduce that

$$\lim_{N\to\infty}\left|\operatorname{Tr} J^{(k)}\left(\gamma_N^{(k)}(t) - |\phi(t)\rangle\langle\phi(t)|^{\otimes k}\right)\right| = 0.$$

i.e. as trace class operators

$$\gamma_N^{(k)}(t) \to |\phi(t)\rangle\langle\phi(t)|^{\otimes k} \text{ weak*}.$$

Then again, the Grümm's convergence theorem upgrades the above weak* convergence to strong. Thence, we have concluded Theorem 1.1 via Theorem 1.2.

**Lemma B.1.** *Assume $\psi_N(0)$ satisfies (a), (b), and (c) in Theorem 1.1. Let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off such that $0 \leqslant \chi \leqslant 1$, $\chi(s) = 1$ for $0 \leqslant s \leqslant 1$ and $\chi(s) = 0$ for $s \geqslant 2$. For $\kappa > 0$, we define an approximation $\psi_N^\kappa(0)$ of $\psi_N(0)$ by*

$$\psi_N^\kappa(0) = \frac{\chi(\kappa H_N/N)\psi_N(0)}{\|\chi(\kappa H_N/N)\psi_N(0)\|}. \tag{B.2}$$

*This approximation has the following properties:*

*(i) $\psi_N^\kappa(0)$ verifies the energy condition*

$$\langle\psi_N^\kappa(0), H_N^k\psi_N^\kappa(0)\rangle \leqslant \frac{2^k N^k}{\kappa^k}.$$

*(ii)*

$$\sup_N \|\psi_N^\kappa(0)-\psi_N(0)\|_{L^2} \leqslant C\kappa^{\frac{1}{2}}.$$

*(iii) For small enough $\kappa>0$, $\psi_N^\kappa(0)$ is asymptotically factorized as well*

$$\lim_{N\to\infty}\operatorname{Tr}\left|\gamma_N^{\kappa,(1)}(0,x_1;x_1')-\phi_0(x_1)\overline{\phi_0}(x_1')\right|=0,$$

*where $\gamma_N^{\kappa,(1)}(0)$ is the marginal density associated with $\psi_N^\kappa(0)$ and $\phi_0$ is the same as in assumption (b) in Theorem 1.1.*

*Proof.* Let us write $\chi(\kappa H_N/N)$ as $\chi$ and $\psi_N(0)$ as $\psi_N$. This proof of Lemma B.1 closely follows the proof of [26, Proposition 9.1 (i) and (ii)] and [25, Proposition 5.1 (iii)].

(i) is from definition. In fact, denote the characteristic function of $[0,\lambda]$ with $\mathbf{1}(s\leqslant\lambda)$. We see that $\mathbf{1}(H_N\leqslant 2N/\kappa)\chi(\kappa H_N/N)=\chi(\kappa H_N/N)$. Thus

$$\begin{aligned}\left\langle\psi_N^\kappa(0),H_N^k\psi_N^\kappa(0)\right\rangle &= \left\langle\frac{\chi\psi_N}{\|\chi\psi_N\|},\mathbf{1}(H_N\leqslant 2N/\kappa)H_N^k\frac{\chi\psi_N}{\|\chi\psi_N\|}\right\rangle\\ &\leqslant \left\|\mathbf{1}(H_N\leqslant 2N/\kappa)H_N^k\right\|_{op}\\ &\leqslant \frac{2^kN^k}{\kappa^k}.\end{aligned}$$

We prove (ii) with a slightly modified proof of [26, Proposition 9.1 (ii)]. We still have

$$\begin{aligned}\|\psi_N^\kappa-\psi_N\|_{L^2} &\leqslant \|\chi\psi_N-\psi_N\|_{L^2}+\left\|\frac{\chi\psi_N}{\|\chi\psi_N\|}-\chi\psi_N\right\|_{L^2}\\ &\leqslant \|\chi\psi_N-\psi_N\|_{L^2}+|1-\|\chi\psi_N\||\\ &\leqslant 2\|\chi\psi_N-\psi_N\|_{L^2},\end{aligned}$$

where

$$\begin{aligned}\|\chi\psi_N-\psi_N\|_{L^2}^2 &= \left\langle\psi_N,(1-\chi(\kappa H_N/N))^2\psi_N\right\rangle\\ &\leqslant \left\langle\psi_N,\mathbf{1}(\frac{\kappa H_N}{N}\geqslant 1)\psi_N\right\rangle.\end{aligned}$$

To continue estimating, we notice that if $C\geqslant 0$, then $\mathbf{1}(s\geqslant 1)\leqslant\mathbf{1}(s+C\geqslant 1)$ for all $s$. So

$$\begin{aligned}\|\chi\psi_N-\psi_N\|_{L^2}^2 &\leqslant \left\langle\psi_N,\mathbf{1}(\frac{\kappa H_N}{N}\geqslant 1)\psi_N\right\rangle\\ &\leqslant \left\langle\psi_N,\mathbf{1}(\frac{\kappa(H_N+N\alpha+N)}{N}\geqslant 1)\psi_N\right\rangle.\end{aligned}$$

With the inequality that $\mathbf{1}(s\geqslant 1)\leqslant s$ for all $s\geqslant 0$ and the fact that

$$H_N+N\alpha+N\geqslant 0$$

proved in Lemma 4.1, we arrive at

$$\begin{aligned}\|\chi\psi_N-\psi_N\|_{L^2}^2 &\leqslant \frac{\kappa}{N}\left\langle\psi_N,(H_N+N\alpha+N)\psi_N\right\rangle\\ &\leqslant \frac{\kappa}{N}\left\langle\psi_N,H_N\psi_N\right\rangle+(1+\alpha)\kappa\left\langle\psi_N,\psi_N\right\rangle,\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{N}\left\langle \psi_N, H_N\psi_N\right\rangle &= \left\langle \psi_N, \left(-\partial_{x_1}^2+\omega^2x_1^2\right)\psi_N\right\rangle \\
&\quad +\frac{1}{N^2}\sum_{1\leqslant i<j\leqslant N}\int N^\beta V(N^\beta\left(x_i-x_j\right))\left|\psi_N\right|^2 d\mathbf{x}_N \\
&\leqslant \left\langle \psi_N, \left(-\partial_{x_1}^2+\omega^2x_1^2\right)\psi_N\right\rangle \\
&\quad +C\left\|V\right\|_{L^1}\int\left(\left|\psi_N\right|^2+\left|\partial_{x_1}\psi_N\right|^2\right)d\mathbf{x}_N \\
&\leqslant C\left\langle \psi_N, \left(-\partial_{x_1}^2+\omega^2x_1^2\right)\psi_N\right\rangle + C.
\end{aligned}$$

Using (a) and (c) in the assumptions of Theorem 1.1, we deduce that

$$\left\|\chi\psi_N-\psi_N\right\|_{L^2}^2\leqslant C\kappa$$

which implies

$$\left\|\psi_N^\kappa-\psi_N\right\|_{L^2}\leqslant C\kappa^{\frac{1}{2}}.$$

(iii) does not follow from the proof of [26, Proposition 9.1 (iii)] in which the positivity of $V$ is used. (iii) follows from the proof of [25, Proposition 5.1 (iii)] which does not require $V$ to hold a definite sign.[14] Notice that we are working in one dimension, we get a $N^{\frac{\beta}{2}}$ instead of a $N^{\frac{3\beta}{2}}$ in [25, (5.20)] and hence we get a $N^{\frac{\beta}{2}-1}$ in the estimate of [25, (5.18)] which goes to zero for $\beta\in(0,1)$. □

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