ON THE RIGOROUS DERIVATION OF THE 2D CUBIC NONLINEAR SCHRÖDINGER EQUATION FROM 3D QUANTUM MANY-BODY DYNAMICS

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ABSTRACT. We consider the 3D quantum many-body dynamics describing a dilute bose gas with strong confining in one direction. We study the corresponding BBGKY hierarchy which contains a diverging coefficient as the strength of the confining potential tends to ∞. We find that this diverging coefficient is counterbalanced by the limiting structure of the density matrices and establish the convergence of the BBGKY hierarchy. Moreover, we prove that the limit is fully described by a 2D cubic NLS and obtain the exact 3D to 2D coupling constant.

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1. INTRODUCTION

It is widely believed that the cubic nonlinear Schrödinger equation (NLS)

$$i\partial_t \phi = L\phi + |\phi|^2 \phi \text{ in } \mathbb{R}^{n+1},$$

where L is the Laplacian $-\triangle$ or the Hermite operator $-\triangle + \omega^2 |x|^2$, describes the physical phenomenon of Bose-Einstein condensation (BEC). This belief is one of the main motivations for studying the cubic NLS. BEC is the phenomenon that particles of integer spin (bosons)

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occupy a macroscopic quantum state. This unusual state of matter was first predicted theoretically by Einstein for non-interacting particles. The first experimental observation of BEC in an interacting atomic gas did not occur until 1995 using laser cooling techniques [4, 20]. E. A. Cornell, W. Ketterle, and C. E. Wieman were awarded the 2001 Nobel Prize in physics for observing BEC. Many similar successful experiments [8, 36, 52] were performed later.

Let $t \in \mathbb{R}$ be the time variable and $\mathbf{r}_N = (r_1, r_2, ..., r_N) \in \mathbb{R}^{nN}$ be the position vector of N particles in $\mathbb{R}^n$. Then BEC naively means that the N-body wave function $\psi_N(t, \mathbf{r}_N)$ satisfies

$$\psi_N(t, \mathbf{r}_N) \sim \prod_{j=1}^{N} \varphi(t, r_j)$$

up to a phase factor solely depending on t, for some one particle state φ. In other words, every particle is in the same quantum state. Equivalently, there is the Penrose-Onsager formulation [46] of BEC: if we define $\gamma_N^{(k)}$ to be the k-particle marginal densities associated with ψ_N by

$$\gamma_N^{(k)}(t, \mathbf{r}_k; \mathbf{r}'_k) = \int \psi_N(t, \mathbf{r}_k, \mathbf{r}_{N-k}) \overline{\psi_N}(t, \mathbf{r}'_k, \mathbf{r}_{N-k}) d\mathbf{r}_{N-k}, \quad \mathbf{r}_k, \mathbf{r}'_k \in \mathbb{R}^{nk} \tag{1.1}$$

then, equivalently, BEC means

$$\gamma_N^{(k)}(t, \mathbf{r}_k; \mathbf{r}'_k) \sim \prod_{j=1}^{k} \varphi(t, r_j) \bar{\varphi}(t, r'_j). \tag{1.2}$$

Gross [33, 34] and Pitaevskii [47] proposed that the many-body effect should be model by a strong on-site interaction and hence the one-particle state φ should be modeled by the a cubic NLS. In a series of works [44, 42, 21, 23, 24, 25, 26, 27, 11, 17] , it has been proven rigorously that, under suitable assumptions on the interaction potential, relation (1.2) holds in 3D and the one-particle state φ satisfies the 3D cubic NLS.

It is then natural to believe that the 2D cubic NLS describes the 2D BEC as well. However, there is no BEC in 2D unless the temperature is absolute zero (see p. 69 of [43] and the references within). In other words, 2D BEC is physically impossible due to the third law of thermodynamics. In a physically realistic setting, 2D NLS can only arise from a 3D BEC with strong confining in one direction (which we take to be the z-direction). Such an effective 3D to 2D phenomenon has been experimentally observed [28, 53, 19, 35, 18]. (See [6] for a review.) It is then natural to consider the derivation of the 2D NLS from a 3D N-body quantum dynamic. Combining [1, 2, 17] suggests a route of getting the 2D NLS from 3D. First, a special case of Theorem 2 in [17] establishes the 3D cubic NLS

$$i\partial_t \varphi = -\triangle_x \varphi + \left(-\partial_z^2 + \omega^2 z^2\right) \varphi + |\varphi|^2 \varphi, \;\; (x, z) \in \mathbb{R}^{2+1} \tag{1.3}$$

from the 3D N-body quantum dynamic as a $N \to \infty$ limit. Then the result in [1, 2] shows that the 2D cubic NLS arises from equation (1.3) as a $\omega \to \infty$ limit. This path corresponds to the iterated limit ($\lim_{\omega\to\infty} \lim_{N\to\infty}$) of the N-body dynamic, thus the 2D cubic NLS coming from such a path approximates the 3D N-body dynamic when ω is large and N is infinity. In experiments, it is fully possible to have N and ω comparable to each other. In fact, N

is about 10^4 and ω is about 10^3 in [28, 53, 35, 18]. In this paper, we derive rigorously the 2D cubic NLS as the double limit ($\lim_{N,\omega\to\infty}$) of a 3D quantum N-body dynamic directly, without passing through any 3D cubic NLS. It is elementary mathematical analysis that $\lim_{\omega\to\infty}\lim_{N\to\infty}$ and $\lim_{N,\omega\to\infty}$ are topologically different and one does not imply each other. Let us adopt the notation

$$r_i = (x_i, z_i) \in \mathbb{R}^{2+1}$$

and investigate the procedure of laboratory experiments of BEC according to [28, 53, 19, 35, 18].

Step A. Confine a large number of bosons inside a trap with strong confining in the z-direction. Cool it down so that the many-body system reaches its ground state. It is expected that this ground state is a BEC state / factorized state. To formulate the problem mathematically, we use the quadratic potential $|\cdot|^2$ to represent the trap and

$$V_a(r) = \frac{1}{a^{3\beta}} V\left(\frac{r}{a^\beta}\right),\ \beta > 0$$

to represent the interaction potential. We use the quadratic potential to represent the trap because this simplified yet reasonably general model is expected to capture the salient features of the actual trap: on the one hand the quadratic potential varies slowly, on the other hand it tends to ∞ as $|x| \to \infty$. In the physics literature, Lieb, Seiringer and Yngvason remarked in [44] that the confining potential is typically $\sim |x|^2$ in the available experiments. The review [6] on [28, 53, 19, 35, 18] also mentioned that the trap is harmonic. We use $V_a(r)$ to represent the interaction potential to match the Gross-Pitaevskii description [33, 34, 47] that the many-body effect should be modeled by an on-site self interaction because V_a is an approximation of the identity as $a \to 0$. This step then corresponds to the following mathematical problem:

Problem 1. *Show that, for large N and large $\omega \gg \omega_0$, the ground state of the N-body Hamiltonian*

$$\sum_{j=1}^{N}\left(-\triangle_{r_j} + \omega_0^2|x_j|^2 + \omega^2 z_j^2\right) + \sum_{1\leqslant i<j\leqslant N} \frac{1}{a^{3\beta-1}} V\left(\frac{r_i - r_j}{a^\beta}\right) \tag{1.4}$$

is a factorized state under proper assumptions on a and V.

Step B. Switch the trap in order to enable measurement or direct observation. It is assumed that such a shift of the confining potential is instant and does not destroy the BEC obtained from Step A. To be more precise about the word "switch": in [19, 18], the trap in the x-spatial directions are tuned very loose to generate a 2D Bose gas. For mathematical convenience, we can assume ω_0 becomes 0. The system is then time dependent. Therefore, the factorized structure obtained in Step A must be preserved in time for the observation of BEC. Mathematically, this step stands for the following problem.

Problem 2. *Take the BEC state obtained in Step A as initial datum, show that, for large N and ω, the solution to the $N-$body Schrödinger equation*

$$i\partial_t\psi_{N,\omega}=\sum_{j=1}^{N}\left(-\frac{1}{2}\triangle_{r_j}+\frac{\omega^2}{2}z_j^2\right)\psi_{N,\omega}+\sum_{1\leqslant i<j\leqslant N}\frac{1}{a^{3\beta-1}}V\left(\frac{r_i-r_j}{a^\beta}\right)\psi_{N,\omega} \tag{1.5}$$

is a BEC state / factorized state under the same assumptions of the interaction potential V in Problem 1.

We first remark that neither of the problems listed above admits a factorized state solution. It is also unrealistic to solve the equations in Problems 1 and 2 for large N. Moreover, both problems are linear so that it is not clear how the 2D cubic NLS arises from either problem. Therefore, in order to justify the statement that the 2D cubic NLS depicts the 3D to 2D BEC, we have to show mathematically that, in an appropriate sense, for some 3D one particle state φ fully described by the 2D cubic NLS

$$\gamma_{N,\omega}^{(k)}(t,\mathbf{r}_k;\mathbf{r}_k')\sim\prod_{j=1}^{k}\varphi(t,r_j)\bar{\varphi}(t,r_j')\text{ as }N,\omega\to\infty$$

where $\gamma_{N,\omega}^{(k)}$ are the k-marginal densities associated with $\psi_{N,\omega}$.

For Problem 1 (Step A), a satisfying answer has been found by Schnee and Yngvason. Let $\operatorname{scat}(W)$ denote the 3D scattering length of the potential W. By [24, Lemma A.1], for $0<\beta\leq 1$ and $a\ll 1$, we have

$$\operatorname{scat}\left(a\cdot\frac{1}{a^{3\beta}}V\left(\frac{r}{a^\beta}\right)\right)\sim\begin{cases}a\displaystyle\int_{\mathbb{R}^3}V & \text{if }0\leq\beta<1\\ a\operatorname{scat}(V) & \text{if }\beta=1\end{cases}$$

Consider $\phi_{\omega_0,Ng}$, the minimizer to the 2D NLS energy functional

$$E_{\omega_0,Ng}=\int_{\mathbb{R}^2}\left(|\nabla\phi(x)|^2+\omega_0^2\left|x\right|^2|\phi(x)|^2+4\pi Ng|\phi(x)|^4\right)dx \tag{1.6}$$

subject to the constraint $\|\phi\|_{L^2(\mathbb{R}^2)}=1$. The existence of this nonlinear ground state stems from the presence of the confining potential $\omega_0^2\left|x\right|^2$; otherwise the nonlinear term is defocusing (as it is called in the NLS literature).

Given parameters ω_0,ω,N,a, Schnee-Yngvason [49] define $g=g(\omega_0,\omega,N,a)$ and $\bar\rho=\bar\rho(\omega_0,\omega,N,a)$ by the two simultaneous equations (see (1.15) and (1.18) in [49])

$$g\stackrel{\text{def}}{=}\left|-\log(\frac{\bar\rho}{\omega})+\frac{1}{\sqrt{\omega}a\int_{\mathbb{R}}h_1^4}\right|^{-1},\qquad\bar\rho=N\int|\phi_{\omega_0,Ng}|^4.$$

They argue that this definition for g makes the 2D NLS Hamiltonian (1.6) relevant to the analysis of the limiting behavior of the ground state of (1.4) describing a dilute interacting Bose gas in a 3D trap that is strongly confining in the z-direction. (See also [54] for the case with rotation)

The Gross-Pitaevskii limit means $Ng\sim 1$. We have liberty to fix the value of ω_0 by scaling, so we take $\omega_0=1$. Then the minimizer $\phi_{\omega_0,Ng}$ is fixed and hence $\bar\rho\sim N$.

In this paper, we consider Problem 2 (Step B) and offer a rigorous derivation of the 2D cubic NLS from the 3D quantum many-body dynamic. For the scaling of the interaction potential, we consider the case (called Region I in [49]) in which the term $(\sqrt{\omega}a)^{-1}$ dominates in the definition of g. Then

$$1 \sim Ng \sim Na\sqrt{\omega} \iff a \sim \frac{1}{N\sqrt{\omega}}$$

This then implies that

$$\frac{1}{\sqrt{\omega}a} \sim N \gg \log\frac{N}{\omega} \sim \log\frac{\bar{\rho}}{\omega}$$

so that our assumption that the term $(\sqrt{\omega}a)^{-1}$ dominates in the definition of g is self-consistent.

We will take for mathematical convenience $a = (N\sqrt{\omega})^{-1}$ for Problem 2 (Step B). The Hamiltonian (1.4) then becomes

$$H_{N,\omega} = \sum_{j=1}^{N} \left(-\triangle_{r_j} + \omega^2 z_j^2\right) + \frac{1}{N\sqrt{\omega}} \sum_{1\leqslant i<j\leqslant N} \left(N\sqrt{\omega}\right)^{3\beta} V\left(\left(N\sqrt{\omega}\right)^{\beta}(r_i - r_j)\right) \tag{1.7}$$

Let $h(z) = \pi^{-1}e^{-z^2/2}$ so that h is the normalized ground state eigenfunction of $-\partial_z^2 + z^2$, i.e. it solves $(-1 - \partial_z^2 + z^2)h = 0$. Then the normalized ground state eigenfunction $h_\omega(z)$ of $-\partial_z^2 + \omega^2 z^2$ is given by $h_\omega(z) = \omega^{1/4}h(\omega^{1/2}z)$, i.e. it solves $(-\omega - \partial_z^2 + \omega^2 z^2)h_\omega = 0$. In particular, $h_1 = h$.

We consider initial data that is asymptotically (as $N \to \infty, \omega \to \infty$) factorized in the x-direction and in the ground state in the z-direction; in particular we could take

$$\psi_{N,\omega}(0, \mathbf{r}_N) = \prod_{j=1}^{N} \phi_0(x_j)h_\omega(z_j)\,, \qquad \|\phi_0\|_{L^2(\mathbb{R}^2)} = 1.$$

Let

$$\psi_{N,\omega}(t, \cdot) = e^{itH_{N,\omega}}\psi_{N,\omega}(0, \cdot) \tag{1.8}$$

denote the evolution of this initial data according to the Hamiltonian (1.7). We prove that in a certain sense, as $N \to \infty, \omega \to \infty$,

$$\psi_{N,\omega}(t, \mathbf{r}_N) \sim \prod_{j=1}^{N} \phi(t, x_j)h_\omega(z_j) \tag{1.9}$$

where $\phi(t)$ solves a 2D cubic NLS with initial data $\phi_0(x)$. To make this statement more precise, we introduce the rescaled solution

$$\tilde{\psi}_{N,\omega}(t, \mathbf{r}_N) \stackrel{\text{def}}{=} \frac{1}{\omega^{N/4}}\psi_{N,\omega}(t, \mathbf{x}_N, \frac{\mathbf{z}_N}{\sqrt{\omega}}) \tag{1.10}$$

and the rescaled Hamiltonian

$$\tilde{H}_{N,\omega} = \sum_{j=1}^{N}(-\Delta_{x_j} + \omega(-\partial_{z_j}^2 + z_j^2)) + \frac{1}{N}\sum_{1\leq i<j\leq N} V_{N,\omega}(r_i - r_j) \tag{1.11}$$

where

$$V_{N,\omega}(r) = N^{3\beta}\left(\sqrt{\omega}\right)^{3\beta-1} V\left(\left(N\sqrt{\omega}\right)^{\beta} x, \frac{\left(N\sqrt{\omega}\right)^{\beta}}{\sqrt{\omega}} z\right), \tag{1.12}$$

Then

$$(\tilde{H}_{N,\omega}\tilde{\psi}_{N,\omega})(t,\mathbf{x}_N,\mathbf{z}_N) = \frac{1}{\omega^{N/4}}(H_{N,\omega}\psi_{N,\omega})(t,\mathbf{x}_N,\frac{\mathbf{z}_N}{\sqrt{\omega}})$$

and hence when $\psi_{N,\omega}(t)$ is given by (1.8) and $\tilde{\psi}_{N,\omega}$ is defined by (1.10), we have

$$\tilde{\psi}_{N,\omega}(t,\mathbf{r}_N) = e^{it\tilde{H}_{N,\omega}}\tilde{\psi}(0,\mathbf{r}_N)$$

The informal statement of convergence given by (1.9) becomes the informal statement

$$\tilde{\psi}(t,\mathbf{r}_N) \sim \prod_{j=1}^{N}\phi(t,x_j)h(z_j) \tag{1.13}$$

where $\phi(t)$ solves 2D NLS with initial data $\phi_0(x)$. In fact, the convergence we prove is stated in terms of the associated density operators with kernels

$$\tilde{\gamma}_{N,\omega}(t,\mathbf{r}_N,\mathbf{r}_N') = \tilde{\psi}(t,\mathbf{r}_N)\overline{\tilde{\psi}(t,\mathbf{r}_N')} \tag{1.14}$$

The version of (1.13) that we prove is the convergence

$$\tilde{\gamma}_{N,\omega}^{(k)}(t,\mathbf{r}_k,\mathbf{r}_k') \to \prod_{j=1}^{k}\phi(x_j)h(z_j)\overline{\phi(x_j')h(z_j')}$$

in trace class, for each $k \geq 0$.

We define

$$v(\beta) = \max\left(\frac{1-\beta}{2\beta}, \frac{\frac{5}{4}\beta - \frac{1}{12}}{1-\frac{5}{2}\beta}, \frac{\frac{1}{2}\beta + \frac{5}{6}}{1-\beta}, \frac{\beta+\frac{1}{3}}{1-2\beta}\right) \tag{1.15}$$

(see Fig. 1)

Our main theorem is the following:

Theorem 1.1 (main theorem)**.** *Assume the pair interaction V is a nonnegative Schwartz class function. Let $\{\tilde{\gamma}_{N,\omega}^{(k)}(t,\mathbf{r}_k;\mathbf{r}_k')\}$ be the family of marginal densities associated with the 3D rescaled Hamiltonian evolution $\tilde{\psi}_{N,\omega}(t) = e^{it\tilde{H}_{N,\omega}}\tilde{\psi}_{N,\omega}(0)$ for some $\beta \in (0,2/5)$, (see* (1.1), (1.11), (1.14)*). Suppose the initial datum $\tilde{\psi}_{N,\omega}(0)$ satisfies the following:*

(a) *$\tilde{\psi}_{N,\omega}(0)$ is normalized, that is, $\|\tilde{\psi}_{N,\omega}(0)\|_{L^2} = 1$,*

(b) *$\tilde{\psi}_{N,\omega}(0)$ is asymptotically factorized in the sense that*

$$\lim_{N,\omega\to\infty}\operatorname{Tr}\left|\tilde{\gamma}_{N,\omega}^{(1)}(0,x_1,z_1;x_1',z_1') - \phi_0(x_1)\overline{\phi_0}(x_1')h(z_1)h(z_1')\right| = 0,$$

for some one particle state $\phi_0 \in H^1(\mathbb{R}^2)$,

(c) *Away from the z-directional ground state energy, $\tilde{\psi}_{N,\omega}(0)$ has finite energy per particle:*

$$\sup_{\omega,N}\frac{1}{N}\langle\tilde{\psi}_{N,\omega}(0),(\tilde{H}_{N,\omega}-N\omega)\tilde{\psi}_{N,\omega}(0)\rangle \leqslant C,$$

Then $\forall k \geqslant 1, t \geqslant 0$, and $\varepsilon > 0$, we have the convergence in trace norm (propagation of chaos) that

$$\lim_{\substack{N,\omega\to\infty \\ N\geqslant\omega^{v(\beta)+\varepsilon}}} \operatorname{Tr}\left|\tilde{\gamma}_{N,\omega}^{(k)}(t,\mathbf{x}_k,\mathbf{z}_k;\mathbf{x}_k',\mathbf{z}_k') - \prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')h_1(z_j)h_1(z_j')\right| = 0,$$

where $v(\beta)$ is given by (1.15) *and $\phi(t,x)$ solves the 2D cubic NLS with coupling constant $b_0\left(\int |h_1(z)|^4\, dz\right)$ that is*

$$i\partial_t\phi = -\triangle_x\phi + b_0\left(\int |h_1(z)|^4\, dz\right)|\phi|^2\phi \quad \text{in } \mathbb{R}^{2+1} \tag{1.16}$$

with initial condition $\phi(0,x) = \phi_0(x)$ and $b_0 = \int V(r)\, dr$.

Theorem 1.1 is equivalent to the following theorem.

Theorem 1.2 (main theorem). *Assume the pair interaction V is a nonnegative Schwartz class function. Let $\{\tilde{\gamma}_{N,\omega}^{(k)}(t,\mathbf{r}_k;\mathbf{r}_k')\}$ be the family of marginal densities associated with the 3D rescaled Hamiltonian evolution $\tilde{\psi}_{N,\omega}(t) = e^{it\tilde{H}_{N,\omega}}\tilde{\psi}_{N,\omega}(0)$ for some $\beta \in (0,2/5)$, (see* (1.1), (1.11), (1.14)*). Suppose the initial datum $\tilde{\psi}_{N,\omega}(0)$ is normalized, asymptotically factorized and satisfies the energy condition that*

(c′) *there is a $C > 0$ such that*

$$\langle\tilde{\psi}_{N,\omega}(0),(\tilde{H}_{N,\omega}-N\omega)^k\tilde{\psi}_{N,\omega}(0)\rangle \leqslant C^kN^k,\ \forall k\geqslant 1, \tag{1.17}$$

Then $\forall k \geqslant 1, t \geqslant 0$, and $\varepsilon > 0$, we have the convergence in trace norm (propagation of chaos) that

$$\lim_{\substack{N,\omega\to\infty \\ N\geqslant\omega^{v(\beta)+\varepsilon}}} \operatorname{Tr}\left|\tilde{\gamma}_{N,\omega}^{(k)}(t,\mathbf{x}_k,\mathbf{z}_k;\mathbf{x}_k',\mathbf{z}_k') - \prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')h_1(z_j)h_1(z_j')\right| = 0,$$

where $v(\beta)$ is given by (1.15) *and $\phi(t,x)$ solves the 2D cubic NLS* (1.16).

We remark that assumptions (a), (b), and (c) in Theorem 1.1 are reasonable assumptions on the initial datum coming from Step A. In fact, if we assume further that ϕ_0 minimizes the 2D Gross-Pitaevskii functional (1.6), then (a), (b) and (c) are the conclusion of [49, Theorem 1.1, 1.3]. The limit in Theorem 1.1, which is taken as $N,\omega\to\infty$ within the subregion $N \geqslant \omega^{v(\beta)+\varepsilon}$ is optimal in the sense that if $N \leqslant \omega^{\frac{1}{2\beta}-\frac{1}{2}}$, then the limit of $V_{N,\omega}$ defined by (1.12) is not a delta function.

The equivalence of Theorems 1.1 and 1.2 for asymptotically factorized initial data is well-known. In the main part of this paper, we prove Theorem 1.2 in full detail. For completeness, we discuss briefly how to deduce Theorem 1.1 from Theorem 1.2 in Appendix B.

The main tool used to prove Theorem 1.2 is the analysis of the BBGKY hierarchy of $\left\{\tilde{\gamma}_{N,\omega}^{(k)}\right\}_{k=1}^{N}$ as $N,\omega\to\infty$. With our definition, the sequence of the marginal densities $\left\{\tilde{\gamma}_{N,\omega}^{(k)}\right\}_{k=1}^{N}$ associated with $\tilde{\psi}_{N,\omega}$ satisfies the BBGKY hierarchy

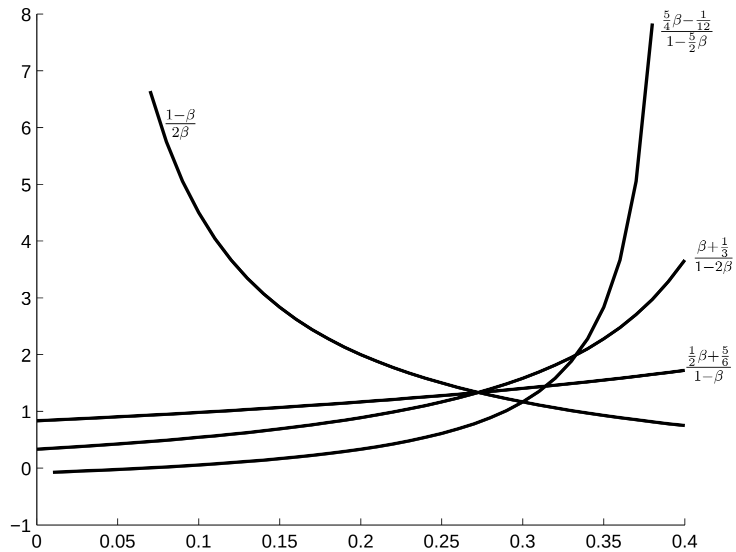

FIGURE 1. A graph of the various rational functions of β appearing in (1.15). In Theorems 1.1, 1.2, the limit $(N,\omega)\to\infty$ is taken with $N\geq\omega^{v(\beta)+\epsilon}$. As shown here, there are values of β for which $v(\beta)\sim 1$, which allows $N\sim\omega$, as in the experimental paper [28, 53, 35, 18]. We conjecture that Theorems 1.1, 1.2 hold with (1.15) replaced by the weaker constraint $v(\beta)=\frac{1-\beta}{2\beta}$ for all $0<\beta<1$.

(1.18)

$$i\partial_t\tilde{\gamma}_{N,\omega}^{(k)}=\sum_{j=1}^{k}\left[-\triangle_{x_j},\tilde{\gamma}_{N,\omega}^{(k)}\right]+\sum_{j=1}^{k}\omega\left[-\partial_{z_j}^2+z_j^2,\tilde{\gamma}_{N,\omega}^{(k)}\right]+\frac{1}{N}\sum_{i<j}^{k}\left[V_{N,\omega}\left(r_i-r_j\right),\tilde{\gamma}_{N,\omega}^{(k)}\right]$$
$$+\frac{N-k}{N}\operatorname{Tr}_{r_{k+1}}\sum_{j=1}^{k}\left[V_{N,\omega}\left(r_j-r_{k+1}\right),\tilde{\gamma}_{N,\omega}^{(k+1)}\right]$$

In the classical setting, deriving mean-field type equations by studying the limit of the BBGKY hierarchy was proposed by Kac and demonstrated by Landford's work [41] on the Boltzmann equation. In the quantum setting, the usage of the BBGKY hierarchy was suggested by Spohn [51] and has been proven to be successful by Elgart, Erdös, Schlein, and Yau in their fundamental papers [21, 23, 24, 25, 26, 27] which rigorously derives the 3D cubic NLS from a 3D quantum many-body dynamic without a trap. The Elgart-Erdös-Schlein-Yau program consists of two principal parts: in one part, they consider the sequence of the marginal densities $\left\{\gamma_N^{(k)}\right\}$ associated with the Hamiltonian evolution $e^{itH_N}\psi_N(0)$ where

$$H_N=\sum_{j=1}^{N}-\triangle_{r_j}+\frac{1}{N}\sum_{1\leqslant i<j\leqslant N}N^{3\beta}V(N^{\beta}\left(r_i-r_j\right))$$

and prove that an appropriate limit of as $N \to \infty$ solves the 3D Gross-Pitaevskii hierarchy

$$i\partial_t \gamma^{(k)} + \sum_{j=1}^{k} \left[\triangle_{r_k}, \gamma^{(k)}\right] = b_0 \sum_{j=1}^{k} \mathrm{Tr}_{r_{k+1}}[\delta(r_j - r_{k+1}), \gamma^{(k+1)}], \text{ for all } k \geq 1\,. \tag{1.19}$$

In another part, they show that hierarchy (1.19) has a unique solution which is therefore a completely factorized state. However, the uniqueness theory for hierarchy (1.19) is surprisingly delicate due to the fact that it is a system of infinitely many coupled equations over an unbounded number of variables. In [39], by imposing a space-time bound on the limit of $\left\{\gamma_N^{(k)}\right\}$, Klainerman and Machedon gave another proof of the uniqueness in [24] through a collapsing estimate originating from the ordinary multilinear Strichartz estimates in their null form paper [38] and a board game argument inspired by the Feynman graph argument in [24].

Later, the method in Klainerman and Machedon [39] was taken up by Kirkpatrick, Schlein, and Staffilani [37], who derived the 2D cubic NLS from the 2D quantum many-body dynamic; by Chen and Pavlović [9, 10], who considered the 1D and 2D 3-body interaction problem and the general existence theory of hierarchy (1.19); and by X.C. [16], who investigated the trapping problem in 2D and 3D. In [12, 13], Chen, Pavlović and Tzirakis worked out the virial and Morawetz identities for hierarchy (1.19). In 2011, for the 3D case without traps, Chen and Pavlović [11] proved that, for $\beta \in (0, 1/4)$, the limit of $\left\{\gamma_N^{(k)}\right\}$ actually satisfies the space-time bound assumed by Klainerman and Machedon [39] as $N \to \infty$. This has been a well-known open problem in the field. In 2012, X.C. [17] extended and simplified their method to study the 3D trapping problem for $\beta \in (0, 2/7]$.

The $\beta = 0$ case has been studied by many authors as well [22, 7, 40, 45, 48].

Away from the usage of the BBGKY hierarchy, there has been work by X.C., Grillakis, Machedon and Margetis [31, 32, 15, 30] using the second order correction which can deal with $e^{itH_N}\psi_N$ directly.

To our knowledge, this is the first direct rigorous treatment of the 3D to 2D dynamic problem. We now compare our theorem with the known work which derives nD cubic NLS from the nD quantum many-body dynamic. It is easy to tell that Theorem 1.2 deals with a different limit than the known work [3, 21, 23, 24, 25, 26, 27, 37, 10, 16, 11, 17] which derives nD NLS from nD dynamics. On the one hand, Theorem 1.2 deals with a 3D to 2D effect. Such a phenomenon is described by the limit equation (1.16) and the coupling constant $\int |h_1(z)|^4\, dz$. The limit in Theorem 1.2 is with the scaling

$$\lim_{\substack{N,\omega\to\infty \\ N\geqslant \omega^{v(\beta)+\varepsilon}}} N\sqrt{\omega}\,\mathrm{scat}\left(\frac{V_{N,\omega}}{N}\right) = \text{constant},$$

instead of the scaling

$$\lim_{N\to\infty} N\,\mathrm{scat}(N^{n\beta-1}V(N^{\beta}\cdot)) = \text{constant},$$

in the known nD to nD work.

The main idea of the proof of Theorem 1.2 is to investigate the limit of hierarchy (1.18) which at a glance is similar to the nD to nD work. However, in contrast with the nD to nD case, even the formal limit of hierarchy (1.18) is not known.

Heuristically, according to the uncertainty principle, in 3D, as the z-component of the particles' position becomes more and more determined to be 0, the z-component of the momentum and thus the energy must blow up. Hence the energy of the system is dominated by its z-directional part which is in fact infinity as $N,\omega\to\infty$. This renders the energy and thus the analysis of the $x-$component intractable.

Technically, it is not clear whether the term

$$\omega\left[-\partial_{z_j}^2+z_j^2,\tilde{\gamma}_{N,\omega}^{(k)}\right]$$

tends to a limit as $N,\omega\to\infty$. Since $\tilde{\gamma}_{N,\omega}^{(k)}$ is not a factorized state for $t>0$, one cannot expect the commutator to be zero. Thus we formally have an $\infty-\infty$ in hierarchy (1.18) as $N,\omega\to\infty$. This is the main difficulty we need to circumvent in the proof of Theorem 1.2.

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2. Outline of the proof of Theorem 1.2

We begin by setting down some notation that will be used in the remainder of the paper. We will always assume $\omega\geq 1$. Note that, as an operator, we have the positivity:

$$-1-\partial_{z_j}^2+z_j^2\geq 0$$

Define

$$\tilde{S}_j\stackrel{\text{def}}{=}(1-\Delta_{x_j}+\omega(-1-\partial_{z_j}^2+z_j^2))^{1/2} \tag{2.1}$$

We have $\tilde{S}_j^2(\phi(x_j)h(z_j))=(1-\Delta_{x_j})\phi(x_j)\,h(z_j)$ and thus the diverging ω parameter has no consequence when the operator is applied to a tensor product function $\phi(x_j)h(z_j)$ for which the z_j-component rests in the ground state.

Let P_0 denote the orthogonal projection onto the ground state of $-\partial_z^2+z^2$ and P_1 denote the orthogonal projection onto all higher energy modes, so $I=P_0+P_1$, where $I:L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$. Let P_0^j and P_1^j be the corresponding operators acting on $L^2(\mathbb{R}^{3N})$ in the z_j component, $1\leq j\leq N$. Then

$$I=\prod_{j=1}^k(P_0^j+P_1^j)\,,\quad\text{where}\quad I:L^2(\mathbb{R}^{3N})\to L^2(\mathbb{R}^{3N}) \tag{2.2}$$

For a k-tuple $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)$ with $\alpha_j\in\{0,1\}$, let $P_{\boldsymbol{\alpha}}=P_{\alpha_1}^1\cdots P_{\alpha_k}^k$. Adopt the notation

$$|\boldsymbol{\alpha}|=\alpha_1+\cdots+\alpha_k$$

This leads to the coercivity (operator lower bounds) given in Lemma A.5.

We next introduce an appropriate topology on the density matrices as was previously done in [21, 22, 23, 24, 25, 26, 27, 37, 10, 16, 17]. Denote the spaces of compact operators and trace class operators on $L^2\left(\mathbb{R}^{3k}\right)$ as $\mathcal{K}_k$ and $\mathcal{L}_k^1$, respectively. Then $(\mathcal{K}_k)' = \mathcal{L}_k^1$. By the fact that $\mathcal{K}_k$ is separable, we select a dense countable subset $\{J_i^{(k)}\}_{i\geqslant 1} \subset \mathcal{K}_k$ in the unit ball of $\mathcal{K}_k$ (so $\|J_i^{(k)}\|_{\text{op}} \leqslant 1$ where $\|\cdot\|_{\text{op}}$ is the operator norm). For $\gamma^{(k)}, \tilde{\gamma}^{(k)} \in \mathcal{L}_k^1$, we then define a metric d_k on $\mathcal{L}_k^1$ by

$$d_k(\gamma^{(k)}, \tilde{\gamma}^{(k)}) = \sum_{i=1}^{\infty} 2^{-i} \left| \operatorname{Tr} J_i^{(k)} \left(\gamma^{(k)} - \tilde{\gamma}^{(k)}\right)\right|.$$

A uniformly bounded sequence $\tilde{\gamma}_{N,\omega}^{(k)} \in \mathcal{L}_k^1$ converges to $\tilde{\gamma}^{(k)} \in \mathcal{L}_k^1$ with respect to the weak* topology if and only if

$$\lim_{N,\omega\to\infty} d_k(\tilde{\gamma}_{N,\omega}^{(k)}, \tilde{\gamma}^{(k)}) = 0.$$

For fixed $T > 0$, let $C\left([0,T], \mathcal{L}_k^1\right)$ be the space of functions of $t \in [0,T]$ with values in $\mathcal{L}_k^1$ which are continuous with respect to the metric d_k. On $C\left([0,T], \mathcal{L}_k^1\right)$, we define the metric

$$\hat{d}_k(\gamma^{(k)}(\cdot), \tilde{\gamma}^{(k)}(\cdot)) = \sup_{t\in[0,T]} d_k(\gamma^{(k)}(t), \tilde{\gamma}^{(k)}(t)),$$

and denote by τ_{prod} the topology on the space $\oplus_{k\geqslant 1} C\left([0,T], \mathcal{L}_k^1\right)$ given by the product of topologies generated by the metrics $\hat{d}_k$ on $C\left([0,T], \mathcal{L}_k^1\right)$.

With the above topology on the space of marginal densities, we now outline the proof of Theorem 1.2. We divide the proof into five steps.

Step I (Energy estimate). We transform, through Theorem 3.1, the energy condition (1.17) into an "easier to use" H^1 type energy bound in which the interaction V is not involved. Since the quantity on the left-hand side of energy condition (1.17) is conserved by the evolution, we deduce the *a priori* bounds on the scaled marginal densities

$$\sup_t \operatorname{Tr} \prod_{j=1}^{k} \left(1 - \triangle_{x_j} + \omega\left(-1 - \partial_{z_j}^2 + z_j^2\right)\right) \tilde{\gamma}_{N,\omega}^{(k)} \leqslant C^k$$

$$\sup_t \operatorname{Tr} \prod_{j=1}^{k} \left(1 - \triangle_{r_j}\right) \tilde{\gamma}_{N,\omega}^{(k)} \leqslant C^k$$

$$\sup_t \operatorname{Tr} P_{\boldsymbol{\alpha}} \tilde{\gamma}_{N,\omega}^{(k)} P_{\boldsymbol{\beta}} \leq C^k \omega^{-\frac{1}{2}|\boldsymbol{\alpha}| - \frac{1}{2}|\boldsymbol{\beta}|}$$

via Corollary 3.1. We remark that, in contrast to the nD to nD work, the quantity

$$\operatorname{Tr}\left(1 - \triangle_{r_1}\right) \tilde{\gamma}_{N,\omega}^{(1)}$$

is not the one particle kinetic energy of the system; the one particle kinetic energy of the system is $\operatorname{Tr}\left(1 - \triangle_{x_1} - \omega\partial_{z_1}^2\right) \tilde{\gamma}_{N,\omega}^{(1)}$ and grows like ω.

Step II (Compactness of BBGKY). We fix $T > 0$ and work in the time-interval $t \in [0,T]$. In Theorem 4.1, we establish the compactness of the sequence $\Gamma_{N,\omega}(t) = \left\{\tilde{\gamma}_{N,\omega}^{(k)}\right\}_{k=1}^{N} \in \oplus_{k\geqslant 1} C\left([0,T], \mathcal{L}_k^1\right)$ with respect to the product topology τ_{prod} even though there is an $\infty - \infty$

in hierarchy (1.18). Moreover, in Corollary 4.1, we prove that, to be compatible with the energy bound obtained in Step I, every limit point $\Gamma(t) = \left\{\tilde{\gamma}^{(k)}\right\}_{k=1}^{N}$ must take the form

$$\tilde{\gamma}^{(k)}\left(t,(\mathbf{x}_k,\mathbf{z}_k);(\mathbf{x}_k',\mathbf{z}_k')\right) = \tilde{\gamma}_x^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\prod_{j=1}^{k} h_1\left(z_j\right)h_1\left(z_j'\right),$$

where $\tilde{\gamma}_x^{(k)} = \operatorname{Tr}_z \tilde{\gamma}^{(k)}$ is the x-component of $\tilde{\gamma}^{(k)}$.

Step III (Limit points of BBGKY satisfy GP). In Theorem 5.1, we prove that if $\Gamma(t) = \left\{\tilde{\gamma}^{(k)}\right\}_{k=1}^{\infty}$ is a $N \geqslant \omega^{v(\beta)+\varepsilon}$ limit point of $\Gamma_{N,\omega}(t) = \left\{\tilde{\gamma}_{N,\omega}^{(k)}\right\}_{k=1}^{N}$ with respect to the product topology τ_{prod}, then $\left\{\tilde{\gamma}_x^{(k)} = \operatorname{Tr}_z \tilde{\gamma}^{(k)}\right\}_{k=1}^{\infty}$ is a solution to the coupled Gross-Pitaevskii (GP) hierarchy subject to initial data $\tilde{\gamma}_x^{(k)}(0) = \left|\phi_0\right\rangle\left\langle\phi_0\right|^{\otimes k}$ with coupling constant $b_0 = \int V(r)\,dr$, which written in differential form, is

$$i\partial_t\tilde{\gamma}_x^{(k)} = \sum_{j=1}^{k}\left[-\triangle_{x_j},\tilde{\gamma}_x^{(k)}\right] + b_0\sum_{j=1}^{k}\operatorname{Tr}_{x_{k+1}}\operatorname{Tr}_z\left[\delta\left(r_j - r_{k+1}\right),\tilde{\gamma}^{(k+1)}\right].$$

Together with Corollary 4.1, we then deduce that $\left\{\tilde{\gamma}_x^{(k)} = \operatorname{Tr}_z \tilde{\gamma}^{(k)}\right\}_{k=1}^{\infty}$ is a solution to the well-known 2D GP hierarchy subject to initial data $\tilde{\gamma}_x^{(k)}(0) = \left|\phi_0\right\rangle\left\langle\phi_0\right|^{\otimes k}$ with coupling constant $b_0\left(\int\left|h_1(z)\right|^4 dz\right)$, which, written in differential form, is

$$i\partial_t\tilde{\gamma}_x^{(k)} = \sum_{j=1}^{k}\left[-\triangle_{x_j},\tilde{\gamma}_x^{(k)}\right] + b_0\left(\int\left|h_1(z)\right|^4 dz\right)\sum_{j=1}^{k}\operatorname{Tr}_{x_{k+1}}\left[\delta\left(x_j - x_{k+1}\right),\tilde{\gamma}_x^{(k+1)}\right]. \tag{2.3}$$

Step IV (GP has a unique solution). When $\tilde{\gamma}_x^{(k)}(0) = \left|\phi_0\right\rangle\left\langle\phi_0\right|^{\otimes k}$, we know one solution to the 2D Gross-Pitaevskii hierarchy (2.3), namely $\left|\phi\right\rangle\left\langle\phi\right|^{\otimes k}$, where ϕ solves equation (1.16). Since we have the *a priori* bound

$$\sup_t \operatorname{Tr}\prod_{j=1}^{k}\left(1-\triangle_{x_j}\right)\tilde{\gamma}_x^{(k)} \leqslant C^k,$$

the uniqueness theorem (Theorem 6.3) then gives that $\tilde{\gamma}_x^{(k)} = \left|\phi\right\rangle\left\langle\phi\right|^{\otimes k}$. Thus the compact sequence $\Gamma_{N,\omega}(t) = \left\{\tilde{\gamma}_{N,\omega}^{(k)}\right\}_{k=1}^{N}$ has only one $N \geqslant \omega^{v(\beta)+\varepsilon}$ limit point, namely

$$\tilde{\gamma}^{(k)} = \prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')h_1\left(z_j\right)h_1(z_j').$$

By the definition of the topology, we know, as trace class operators

$$\tilde{\gamma}_{N,\omega}^{(k)} \to \prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')h_1\left(z_j\right)h_1(z_j') \text{ weak*}.$$

Remark 1. *This is in fact the very first time that the Klainerman-Machedon theory applies to a 3D many-body system with $\beta \geqslant 1/3$. The previous best is $\beta \in (0,2/7)$ in* [17] *after the $\beta \in (0,1/4)$ work* [11]*. Of course, we are not actually using any 3D Gross-Pitaevskii hierarchies here.*

Step V (Weak convergence upgraded to strong). We use the argument in the bottom of p. 296 of [27] to conclude that the weak* convergence obtained in Step IV is in fact strong. We include this argument for completeness. We test the sequence obtained in Step IV against the compact observable

$$J^{(k)}=\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')h_1\left(z_j\right)h_1(z_j'),$$

and notice the fact that $\left(\tilde{\gamma}_{N,\omega}^{(k)}\right)^2\leqslant\tilde{\gamma}_{N,\omega}^{(k)}$ since the initial data is normalized, we see that as Hilbert-Schmidt operators

$$\tilde{\gamma}_{N,\omega}^{(k)}\to\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')h_1\left(z_j\right)h_1(z_j')\text{ strongly.}$$

Since $\operatorname{Tr}\tilde{\gamma}_{N,\omega}^{(k)}=\operatorname{Tr}\tilde{\gamma}^{(k)}$, we deduce the strong convergence

$$\lim_{\substack{N,\omega\to\infty\\N\geqslant\omega^{v(\beta)+\varepsilon}}}\operatorname{Tr}\left|\tilde{\gamma}_{N,\omega}^{(k)}(t,\mathbf{x}_k,\mathbf{z}_k;\mathbf{x}_k',\mathbf{z}_k')-\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')h_1\left(z_j\right)h_1(z_j')\right|=0,$$

via the Grümm's convergence theorem [50, Theorem 2.19]

3. Energy estimate

We find it more convenient to prove the energy estimate for $\psi_{N,\omega}$ and then convert it by scaling to an estimate for $\tilde{\psi}_{N,\omega}$ (see (1.10)). Note that, as an operator, we have the positivity:

$$-\omega-\partial_{z_j}^2+\omega^2z_j^2\geq 0$$

Define

$$S_j\overset{\text{def}}{=}(1-\Delta_{x_j}-\omega-\partial_{z_j}^2+\omega^2z_j^2)^{1/2}=(1-\omega-\Delta_{r_j}+\omega^2z_j^2)^{1/2}$$

Theorem 3.1. *Let the Hamiltonian be defined as in* (1.7) *with* $\beta\in(0,2/5)$*. Then for all* $\varepsilon>0$*, there exists a constant* $C>0$*, and for all* $\omega,k\geqslant 0$*, there exists* $N_0(k,\omega)$ *such that*

$$\left\langle\psi_{N,\omega},\left(N+H_{N,\omega}-N\omega\right)^k\psi_{N,\omega}\right\rangle\geqslant C^kN^k\left\|\prod_{j=1}^{k}S_j\psi_{N,\omega}\right\|_{L^2(\mathbb{R}^{3N})}^2\tag{3.1}$$

for all $N\geqslant\omega^{v(\beta)+\epsilon}$*, and all* $\psi\in L_s^2\left(\mathbb{R}^{3N}\right)\cap\mathcal{D}(H_{N,\omega}^k)$.

Proof. We adapt the proof of [21, Prop. 3.1] to accommodate the operator $-\omega-\partial_{z_j}^2+\omega^2z_j^2$ in place of $-\partial_{z_j}^2$. The case $k=0$ is trivial and the case $k=1$ follows from the positivity of V and symmetry of ψ. We proceed by induction. Suppose that the result holds for $k=n$, and we will prove it for $k=n+2$. By the induction hypothesis,

$$\begin{aligned}&\langle\psi,(N-N\omega+H_{N,\omega})^{n+2}\psi\rangle\\&\geq C^nN^n\langle\psi,(N-N\omega+H_{N,\omega})\prod_{j=1}^{n}S_j^2(N-N\omega+H_{N,\omega})\psi\rangle\end{aligned}\tag{3.2}$$

For convenience, let

$$\tilde{V}(r) = (N\sqrt{\omega})^{3\beta-1}V((N\sqrt{\omega})^{\beta}r)$$

Expand

$$N - N\omega + H_{N,\omega} = \sum_{\ell=n+1}^{N} S_\ell^2 + \left(\sum_{\ell=1}^{n} S_\ell^2 + H_{N,\omega}^I\right)$$

and substitute in both occurrences of the operator $N - N\omega + H_{N,\omega}$ in the right side of (3.2) to obtain four terms. We ignore the last (positive) one of these terms to obtain

$$\langle\psi, (N - N\omega + H_{N,\omega})^{n+2}\psi\rangle \geq C^n N^n(\mathrm{I} + \mathrm{II} + \mathrm{III}) \tag{3.3}$$

We have

$$\mathrm{I} = \sum_{\ell_1,\ell_2=n+1}^{N} \langle\psi, S_{\ell_1}^2 S_{\ell_2}^2 \prod_{j=1}^{n} S_j^2\psi\rangle$$

In this double sum, there are $(N-n)(N-n-1)$ terms where $\ell_1 \neq \ell_2$ that are all the same by symmetry, and there are $(N-n)$ terms where $\ell_1 = \ell_2$ that are all the same by symmetry. We have

$$\mathrm{I} = (N-n)(N-n-1)\langle\psi, \prod_{j=1}^{n+2} S_j^2\psi\rangle + (N-n)\langle\psi, S_1^2 \prod_{j=1}^{n+1} S_j^2\psi\rangle \tag{3.4}$$

the first of which will ultimately fulfill the induction claim. In (3.3), we also have

$$\begin{aligned}\mathrm{II} + \mathrm{III} = \; & 2\sum_{\ell_1=n+1}^{N}\sum_{\ell_2=1}^{n}\langle\psi, S_{\ell_1}^2 \prod_{j=1}^{n} S_j^2 S_{\ell_2}^2\psi\rangle + \sum_{\ell=n+1}^{N}\langle\psi, S_\ell^2 \prod_{j=1}^{n} S_j^2 H_{N,\omega}^I\psi\rangle \\ & + \sum_{\ell=n+1}^{N}\langle\psi, H_{N,\omega}^I \prod_{j=1}^{n} S_j^2 S_\ell^2\psi\rangle\end{aligned}$$

Exploiting symmetry this becomes

$$\mathrm{II} + \mathrm{III} = 2(N-n)n\langle\psi, S_1^2 \prod_{j=1}^{n+1} S_j^2\psi\rangle + 2(N-n)\,\mathrm{Re}\langle\psi, \prod_{j=1}^{n+1} S_j^2 H_{N,\omega}^I\psi\rangle \tag{3.5}$$

In the first term, we have applied the permutation that swaps ℓ_1 and $n+1$ and ℓ_2 and 1. In the second and third terms, we have applied the permutation σ that swaps ℓ and $n+1$. Strictly speaking, this permutation maps $H_{N,\omega}^I$ to $H_{N,\omega,\sigma}^I$ where

$$H_{N,\omega,\sigma}^I \stackrel{\text{def}}{=} \frac{1}{N\omega^{1/2}} \sum_{1\leq i<j\leq N} (N\omega^{1/2})^{3\beta} V((\pm 1)(N\omega^{1/2})^{\beta}(r_i - r_j))$$

where ± 1 is chosen according to the affect of the permutation on the pair (i,j). The distinction between $H_{N,\omega}^I$ and $H_{N,\omega,\sigma}^I$ is inconsequential for the remainder of the analysis (and in fact $H_{N,\omega}^I = H_{N,\omega,\sigma}^I$ if V is even), so we have ignored it in (3.5). The first of the terms in (3.5) is positive – it is the second term that requires attention; in particular, we have to manage commutators.

Assuming $N \geq 2n+2$, we substitute (3.4), (3.5) into (3.3) to obtain

$$\langle \psi, (N - N\omega + H_{N,\omega})^{n+2}\psi\rangle \geq \tfrac{1}{4}C^n N^{n+2}\langle \psi, \prod_{j=1}^{n+2} S_j^2\psi\rangle + C^n N^{n+1}\langle \psi, S_1^2 \prod_{j=1}^{n+1} S_j^2\psi\rangle \tag{3.6}$$
$$+2C^n N^n (N-n)\operatorname{Re}\langle \psi, \prod_{j=1}^{n+1} S_j^2 H_{N,\omega}^I \psi\rangle =: D + E + F$$

The first two terms, D and E, in (3.6) are positive. The third term F will be decomposed into components, some of which are positive and others that can be bounded in terms of the first two terms appearing in (3.6). In the expression for $H_{N,\omega}^I$, there are

- $\frac{1}{2}(n+1)n$ terms of the form $\tilde{V}(r_i - r_j)$ for $1 \leq i < j \leq n+1$.
- $(n+1)(N-n-1)$ terms of the form $\tilde{V}(r_i - r_j)$ for $1 \leq i \leq n+1$ and $n+2 \leq j \leq N$.
- $\frac{1}{2}(N-n-1)(N-n-2)$ terms of the form $\tilde{V}(r_i - r_j)$ for $n+2 \leq i < j \leq N$.

For convenience, let

$$V_{ij} \stackrel{\text{def}}{=} (N\omega^{1/2})^{3\beta-1} V((N\omega^{1/2})^{\beta}(r_i - r_j))$$

Using symmetry, we obtain

$$\begin{aligned} F =\ & 2C^n N^n (N-n)(n+1)n \operatorname{Re}\langle \psi, \prod_{j=1}^{n+1} S_j^2 V_{12}\psi\rangle \\ & + 2C^n N^n (N-n)(n+1)(N-n-1)\operatorname{Re}\langle \psi, \prod_{j=1}^{n+1} S_j^2 V_{1(n+2)}\psi\rangle \\ & + C^n N^n (N-n)(N-n-1)(N-n-2)\operatorname{Re}\langle \psi, \prod_{j=1}^{n+1} S_j^2 V_{(n+2)(n+3)}\psi\rangle \\ =:\ & F_1 + F_2 + F_3 \end{aligned}$$

The last term F_3 is positive since each S_j for $1 \leq j \leq n+1$ commutes with $V_{(n+2)(n+3)}$. We will show $F_1 \geq -\frac{1}{2}E$ and $F_2 \geq -\frac{1}{2}D$ provided $N \geq N_0(n)$, which together with (3.6) will complete the induction argument. We have

$$\begin{aligned} F_1 &= 2C^n N^n (N-n)(n+1)n \operatorname{Re}\langle \psi, \prod_{j=1}^{n+1} S_j^2 V_{12}\psi\rangle \\ &= 2C^n N^n (N-n)(n+1)n \operatorname{Re}\int_{r_3,\dots,r_N} \underbrace{\langle f, S_1^2 S_2^2 V_{12} f\rangle_{r_1,r_2}}_{=:\tilde{F}_1} dr_3 \cdots dr_N \end{aligned}$$

where $f = \prod_{j=3}^{n+1} S_j \psi$. We can regard $r_3, \dots, r_N$ as frozen in the following computation, so to prove $|F_1| \leq \frac{1}{2}E$, it will suffice to show that

$$|\tilde{F}_1| \leq \tfrac{1}{4} n^{-2} \|S_1^2 S_2 f\|^2_{L^2_{r_1} L^2_{r_2}} \tag{3.7}$$

Toward this end, we have

$$\begin{aligned}|\tilde{F}_1| &= |\langle S_1^2 f, V_{12} S_2^2 f\rangle + 2\langle S_1^2 f, \nabla_{r_2} V_{12}\cdot \nabla_{r_2} f\rangle + \langle S_1^2 f, (\Delta_{r_2} V_{12})\, f\rangle| \\ &\lesssim \|S_1^2 f\|_{L^2_{r_1}L^6_{r_2}} \|V_{12}\|_{L^\infty_{r_1}L^3_{r_2}} \|S_2^2 f\|_{L^2_{r_1}L^2_{r_2}} + \|S_1^2 f\|_{L^2_{r_1}L^6_{r_2}} \|\nabla_{r_2} V_{12}\|_{L^\infty_{r_1}L^{3/2}_{r_2}} \|\nabla_{r_2} f\|_{L^2_{r_1}L^6_{r_2}} \\ &\qquad + \|S_1^2 f\|_{L^2_{r_1}L^6_{r_2}} \|\Delta_{r_2} V_{12}\|_{L^\infty_{r_1}L^{6/5}_{r_2}} \|f\|_{L^2_{r_1}L^\infty_{r_2}}\end{aligned}$$

By evaluation of

$$\|V_{12}\|_{L^3_{r_2}} \sim (N\omega^{1/2})^{2\beta-1}, \quad \|\nabla_{r_2} V_{12}\|_{L^{3/2}_{r_2}} \sim (N\omega^{1/2})^{2\beta-1}, \quad \|\Delta_{r_2} V_{12}\|_{L^{6/5}_{r_2}} \sim (N\omega^{1/2})^{\frac{5}{2}\beta-1}$$

the above estimate reduces to

$$\begin{aligned}|\tilde{F}_1| &\lesssim (N\omega^{1/2})^{2\beta-1} \|S_1^2 f\|_{L^2_{r_1}L^6_{r_2}} \|S_2^2 f\|_{L^2_{r_1}L^2_{r_2}} + (N\omega^{1/2})^{2\beta-1} \|S_1^2 f\|_{L^2_{r_1}L^6_{r_2}} \|\nabla_{r_2} f\|_{L^2_{r_1}L^6_{r_2}} \\ &\qquad + (N\omega^{1/2})^{\frac{5}{2}\beta-1} \|S_1^2 f\|_{L^2_{r_1}L^6_{r_2}} \|f\|_{L^2_{r_1}L^\infty_{r_2}}\end{aligned}$$

Applying Lemma A.4, this reduces further to

$$\begin{aligned}|\tilde{F}_1| &\lesssim (N\omega^{1/2})^{2\beta-1}\omega^{1/6} \|S_1^2 S_2 f\|_{L^2_{r_1}L^2_{r_2}} \|S_2^2 f\|_{L^2_{r_1}L^2_{r_2}} \\ &\quad + (N\omega^{1/2})^{2\beta-1}\omega^{1/6}\omega^{2/3} \|S_1^2 S_2 f\|_{L^2_{r_1}L^2_{r_2}} \|S_2^2 f\|_{L^2_{r_1}L^2_{r_2}} \\ &\quad + (N\omega^{1/2})^{\frac{5}{2}\beta-1}\omega^{1/6}\omega^{1/4} \|S_1^2 S_2 f\|_{L^2_{r_1}L^2_{r_2}} \|S_2^2 f\|_{L^2_{r_1}L^2_{r_2}}\end{aligned}$$

Hence we need $\beta < \frac{2}{5}$ and conditions (3.13), (3.11) below to achieve (3.7).

Let us now establish $F_2 \geq -\frac{1}{2}D$. We have

$$\begin{aligned}F_2 &= 2C^n N^n (N-n)(n+1)(N-n-1)\,\mathrm{Re}\langle \psi, \prod_{j=1}^{n+1} S_j^2 V_{1(n+2)}\psi\rangle \\ &= 2C^n N^n (N-n)(n+1)(N-n-1) \int \underbrace{\langle f, S_1^2 V_{1(n+2)} f\rangle_{r_1,r_{n+2}}}_{=:\tilde{F}_2} dr_2\cdots dr_{n+1} dr_{n+3}\cdots dr_N\end{aligned}$$

where $f = \prod_{j=2}^{n+1} S_j \psi$. Now

$$\begin{aligned}\tilde{F}_2 &= \langle f, (-\omega - \partial_{z_1}^2 + \omega^2 z_1^2) V_{1(n+2)} f\rangle_{r_1,r_{n+2}} \\ &= -\omega\langle f, V_{1(n+2)} f\rangle_{r_1 r_{n+2}} + \langle \partial_{z_1} f, (\partial_{z_1} V_{1(n+2)}) f\rangle_{r_1 r_{n+2}} \\ &\qquad + \langle \partial_{z_1} f, V_{1(n+2)} \partial_{z_1} f\rangle_{r_1 r_{n+2}} + \langle f, \omega^2 z_1^2 f\rangle_{r_1 r_{n+2}} \\ &=: \tilde{F}_{2,1} + \tilde{F}_{2,2} + \tilde{F}_{2,3} + \tilde{F}_{2,4}\end{aligned}$$

Note that $\tilde{F}_{2,3}$ and $\tilde{F}_{2,4}$ are positive and can thus be disregarded. To prove $F_2 \geq -\frac{1}{2}D$, it suffices to prove

$$|\tilde{F}_{2,1}| + |\tilde{F}_{2,2}| \leq \tfrac{1}{16} n^{-1} \|S_1 S_{n+2} f\|^2_{L^2_{r_1}L^2_{r_{n+2}}} \tag{3.8}$$

But

$$|\tilde{F}_{2,1}| \lesssim \omega \|f\|_{L^2_{r_1}L^6_{r_{n+2}}} \|V_{1(n+2)}\|_{L^\infty_{r_1}L^{3/2}_{r_{n+2}}} \|f\|_{L^2_{r_1}L^6_{r_{n+2}}}$$

By Lemma A.4 and $\|V_{1(n+2)}\|_{L^\infty_{r_1}L^{3/2}_{r_{n+2}}} \sim (N\omega^{1/2})^{\beta-1}$, we obtain

$$|\tilde{F}_{2,1}| \lesssim \omega^{4/3} (N\omega^{1/2})^{\beta-1} \|S_{n+2} f\|^2_{L^2_{r_1}L^2_{r_{n+2}}} \tag{3.9}$$

The upper bound in (3.8) will be achieved provided (3.12) below holds. Also,

$$|\tilde{F}_{2,2}| \lesssim \|\partial_{z_1} f\|_{L^2_{r_1} L^6_{r_{n+2}}} \|\partial_{z_1} V_{1(n+2)}\|_{L^\infty_{r_1} L^{3/2}_{r_{n+2}}} \|f\|_{L^2_{r_1} L^6_{r_{n+2}}}$$

Note that $\|\partial_{z_1} V_{1(n+2)}\|_{L^\infty_{r_1} L^{3/2}_{r_{n+2}}} \sim (N\omega^{1/2})^{2\beta-1}$. By Lemma A.4,

$$\|\partial_{z_1} f\|_{L^2_{r_1} L^6_{r_{n+2}}} \lesssim \omega^{1/6} \|S_{n+2} \partial_{z_1} f\|_{L^2_{r_1} L^2_{r_{n+2}}} \lesssim \omega^{2/3} \|S_1 S_{n+2} f\|_{L^2_{r_1} L^2_{r_{n+2}}}$$

and $\|f\|_{L^2_{r_1} L^6_{r_{n+2}}} \lesssim \omega^{1/6} \|S_{n+2} f\|_{L^2_{r_1} L^2_{r_{n+2}}}$. From this, it follows that

$$|\tilde{F}_{2,2}| \lesssim \omega^{5/6} (N\omega^{1/2})^{2\beta-1} \|S_1 S_{n+2} f\|_{L^2_{r_1} L^2_{r_{n+2}}} \|S_{n+2} f\|_{L^2_{r_1} L^2_{r_{n+2}}} \tag{3.10}$$

The upper bound in (3.8) will be achieved provided (3.13) holds. By (3.9), (3.10), we obtain (3.8), completing the proof. Let us collect the conditions on N and ω. We have

$$(N\omega^{1/2})^{\frac{5}{2}\beta-1} \omega^{5/12} \ll n^{-2} \qquad \iff \quad N \gg \omega^{\frac{\frac{5}{4}\beta - \frac{1}{12}}{1-\frac{5}{2}\beta}} n^{\frac{2}{1-\frac{5}{2}\beta}} \tag{3.11}$$

$$(N\omega^{1/2})^{\beta-1} \omega^{4/3} \ll n^{-1} \qquad \iff \quad N \gg \omega^{\frac{\frac{1}{2}\beta + \frac{5}{6}}{1-\beta}} n^{\frac{1}{1-\beta}} \tag{3.12}$$

$$(N\omega^{1/2})^{2\beta-1} \omega^{5/6} \ll n^{-1} \qquad \iff \quad N \gg \omega^{\frac{\beta + \frac{1}{3}}{1-2\beta}} n^{\frac{1}{1-2\beta}} \tag{3.13}$$

The requirement that (3.11), (3.12), and (3.13) hold is imposed in the definition (1.15) of $v(\beta)$. □

Now consider the rescaled operator (2.1) so that

$$(S_j \psi)(t, \mathbf{x}_N, \mathbf{z}_N) = \omega^{N/4} (\tilde{S}_j \tilde{\psi})(t, \mathbf{x}_N, \sqrt{\omega} \mathbf{z}_N) \,.$$

We will convert the conclusions of Theorem 3.1 into statements about $\tilde{\psi}$, $\tilde{S}_j$, and $\tilde{\gamma}^{(k)}_{N,\omega}$ that we will then apply in the remainder of the paper.

Corollary 3.1. *Let $\tilde{\psi}_{N,\omega}(t) = e^{it\tilde{H}_{N,\omega}} \tilde{\psi}_{N,\omega}(0)$ and $\{\tilde{\gamma}^{(k)}_{N,\omega}(t)\}$ be the marginal densities associated with it, then for all $\omega \geq 1$, $k \geq 0$, $N \geq \omega^{v(\beta)+\epsilon}$, we have the uniform-in-time bound*

$$\operatorname{Tr} \prod_{j=1}^k \tilde{S}_j^2 \tilde{\gamma}^{(k)}_{N,\omega} = \left\| \prod_{j=1}^k \tilde{S}_j \tilde{\psi}_{N,\omega}(t) \right\|^2_{L^2(\mathbb{R}^{3N})} \leq C^k \tag{3.14}$$

Consequently,

$$\operatorname{Tr} \prod_{j=1}^k (1 - \Delta_{r_j}) \tilde{\gamma}^{(k)}_{N,\omega} = \left\| \prod_{j=1}^k (1 - \Delta_{r_j})^{1/2} \tilde{\psi}_{N,\omega}(t) \right\|^2_{L^2(\mathbb{R}^{3N})} \leq C^k \tag{3.15}$$

and

$$\|P_{\boldsymbol{\alpha}} \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \leq C^k \omega^{-|\boldsymbol{\alpha}|/2} \,, \qquad \operatorname{Tr} P_{\boldsymbol{\alpha}} \tilde{\gamma}^{(k)}_{N,\omega} P_{\boldsymbol{\beta}} \leq C^k \omega^{-\frac{1}{2}|\boldsymbol{\alpha}| - \frac{1}{2}|\boldsymbol{\beta}|} \tag{3.16}$$

Proof. Substituting (1.10) into (3.1) of Theorem 3.1 and rescaling, we obtain

$$\langle \tilde{\psi}_{N,\omega}, (N - \tilde{H}_{N,\omega} - N\omega)^k \tilde{\psi}_{N,\omega} \rangle \geq C^k N^k \left\| \prod_{j=1}^k \tilde{S}_j \tilde{\psi}_{N,\omega} \right\|^2_{L^2(\mathbb{R}^{3N})} \tag{3.17}$$

Since $N-\tilde{H}_{N,\omega}-N\omega$ is self-adjoint and $[\tilde{H}_{N,\omega},N-\tilde{H}_{N,\omega}-N\omega]=0$,

$$\partial_t\langle\tilde{\psi}_{N,\omega},(N-\tilde{H}_{N,\omega}-N\omega)^k\tilde{\psi}_{N,\omega}\rangle=0$$

Hence by (3.17),

$$\begin{aligned}C^kN^k\left\|\prod_{j=1}^{k}\tilde{S}_j\tilde{\psi}_{N,\omega}(t)\right\|_{L^2(\mathbb{R}^{3N})}^2 &\leq\langle\tilde{\psi}_{N,\omega}(t),(N-\tilde{H}_{N,\omega}-N\omega)^k\tilde{\psi}_{N,\omega}(t)\rangle\\ &=\langle\tilde{\psi}_{N,\omega}(0),(N-\tilde{H}_{N,\omega}-N\omega)^k\tilde{\psi}_{N,\omega}(0)\rangle\leq(C')^kN^k\end{aligned}$$

where the last estimate follows from the hypothesis (1.17) of Theorem 1.2.

The inequality (3.15) follows from (3.14) and (A.27). The inequality on the left of (3.16) follows from (A.29) and (3.14). By Lemma A.6, $\operatorname{Tr}P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}}=\langle P_{\boldsymbol{\alpha}}\tilde{\psi}_{N,\omega},P_{\boldsymbol{\beta}}\tilde{\psi}_{N,\omega}\rangle$, so the inequality on the right of (3.16) follows by Cauchy-Schwarz. □

4. Compactness of the BBGKY sequence

Theorem 4.1. *The sequence*

$$\Gamma_{N,\omega}(t)=\left\{\tilde{\gamma}_{N,\omega}^{(k)}\right\}_{k=1}^{N}\in\bigoplus_{k\geqslant1}C\left([0,T],\mathcal{L}_k^1\right)$$

which satisfies the $\infty-\infty$ *BBGKY hierarchy* (1.18)*, is compact with respect to the product topology* τ_{prod}*. For any limit point* $\Gamma(t)=\left\{\tilde{\gamma}^{(k)}\right\}_{k=1}^{N}$, $\tilde{\gamma}^{(k)}$ *is a symmetric nonnegative trace class operator with trace bounded by* 1*.*

We establish Theorem 4.1 at the end of this section. With Theorem 4.1, we can start talking about the limit points of $\Gamma_{N,\omega}(t)=\{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^{N}$.

Corollary 4.1. *Let* $\Gamma(t)=\{\tilde{\gamma}^{(k)}\}_{k=1}^{\infty}$ *be a limit point of* $\Gamma_{N,\omega}(t)=\{\tilde{\gamma}_{N,\omega}^{(k)}\}_{k=1}^{N}$ *with respect to the product topology* τ_{prod}*, then* $\tilde{\gamma}^{(k)}$ *satisfies*

$$\operatorname{Tr}\prod_{j=1}^{k}\left(1-\triangle_{r_j}\right)\tilde{\gamma}^{(k)}\leqslant C^k \tag{4.1}$$

$$\tilde{\gamma}^{(k)}\left(t,(\mathbf{x}_k,\mathbf{z}_k);(\mathbf{x}_k',\mathbf{z}_k')\right)=\tilde{\gamma}_x^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\prod_{j=1}^{k}h_1\left(z_j\right)h_1\left(z_j'\right) \tag{4.2}$$

Proof. The estimate (4.1) is a direct consequence of (3.15) in Corollary 3.1 and Theorem 4.1. The formula (4.2) is equivalent to the statement that if either $\boldsymbol{\alpha}\neq0$ or $\boldsymbol{\beta}\neq0$, then $P_{\boldsymbol{\alpha}}\tilde{\gamma}^{(k)}P_{\boldsymbol{\beta}}=0$. This is equivalent to the statement that for any $J^{(k)}\in\mathcal{K}_k$, $\operatorname{Tr}J^{(k)}P_{\boldsymbol{\alpha}}\tilde{\gamma}^{(k)}P_{\boldsymbol{\beta}}=0$. However,

$$\operatorname{Tr}J^{(k)}P_{\boldsymbol{\alpha}}\tilde{\gamma}^{(k)}P_{\boldsymbol{\beta}}=\lim_{(N,\omega)\to\infty}\operatorname{Tr}J^{(k)}P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}} \tag{4.3}$$

By Lemma A.6,

$$\operatorname{Tr}J^{(k)}P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}}=\langle J^{(k)}P_{\boldsymbol{\alpha}}\tilde{\psi}_{N,\omega},P_{\boldsymbol{\beta}}\tilde{\psi}_{N,\omega}\rangle_{\mathbf{r}_k}$$

and by Cauchy-Schwarz and (3.16),

$$|\operatorname{Tr} J^{(k)} P_{\boldsymbol{\alpha}} \tilde{\gamma}^{(k)}_{N,\omega} P_{\boldsymbol{\beta}}| \leq \|J^{(k)}\|_{\mathrm{op}} \|P_{\boldsymbol{\alpha}} \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \|P_{\boldsymbol{\beta}} \tilde{\psi}_{N,\omega}\|_{L^2(\mathbb{R}^{3N})} \leq C^k \omega^{-\frac{1}{2}|\boldsymbol{\alpha}|-\frac{1}{2}|\boldsymbol{\beta}|}$$

Hence the right side of (4.3) is 0. ☐

Proof of Theorem 4.1. By the standard diagonalization argument, it suffices to show the compactness of $\tilde{\gamma}^{(k)}_{N,\omega}$ for fixed k with respect to the metric $\hat{d}_k$. By the Arzelà-Ascoli theorem, this is equivalent to the equicontinuity of $\tilde{\gamma}^{(k)}_{N,\omega}$, and by [27, Lemma 6.2], this is equivalent to the statement that for every observable $J^{(k)}$ from a dense subset of $\mathcal{K}(L^2(\mathbb{R}^{3k}))$ and for every $\varepsilon > 0$, there exists $\delta(J^{(k)}, \varepsilon)$ such that for all $t_1, t_2 \in [0, T]$ with $|t_1 - t_2| \leqslant \delta$, we have

$$\sup_{N,\omega} \left| \operatorname{Tr} J^{(k)} \tilde{\gamma}^{(k)}_{N,\omega}(t_1) - \operatorname{Tr} J^{(k)} \tilde{\gamma}^{(k)}_{N,\omega}(t_2) \right| \leqslant \varepsilon \,. \tag{4.4}$$

We assume that our compact operators $J^{(k)}$ have been cutoff as in Lemma A.7. Assume $t_1 \leq t_2$. Inserting the decomposition (2.2) on the left and right side of $\gamma^{(k)}_{N,\omega}$, we obtain

$$\tilde{\gamma}^{(k)}_{N,\omega} = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}} P_{\boldsymbol{\alpha}} \tilde{\gamma}^{(k)}_{N,\omega} P_{\boldsymbol{\beta}}$$

where the sum is taken over all k-tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of the type described above.

To establish (4.4) it suffices to establish, for each $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

$$\sup_{N,\omega} \left| \operatorname{Tr} J^{(k)} P_{\boldsymbol{\alpha}} \tilde{\gamma}^{(k)}_{N,\omega} P_{\boldsymbol{\beta}}(t_1) - \operatorname{Tr} J^{(k)} P_{\boldsymbol{\alpha}} \tilde{\gamma}^{(k)}_{N,\omega} P_{\boldsymbol{\beta}}(t_2) \right| \leqslant \varepsilon \,. \tag{4.5}$$

Below, we establish the estimate

$$\begin{aligned} &|\operatorname{Tr} J^{(k)} P_{\boldsymbol{\alpha}} \tilde{\gamma}^{(k)}_{N,\omega} P_{\boldsymbol{\beta}}(t_2) - \operatorname{Tr} J^{(k)} P_{\boldsymbol{\alpha}} \tilde{\gamma}^{(k)}_{N,\omega} P_{\boldsymbol{\beta}}(t_1)| \\ &\lesssim |t_2 - t_1| \begin{cases} 1 & \text{if both } \boldsymbol{\alpha} = 0 \text{ and } \boldsymbol{\beta} = 0 \\ \max(1, \omega^{1-\frac{1}{2}|\boldsymbol{\alpha}|-\frac{1}{2}|\boldsymbol{\beta}|}) & \text{otherwise} \end{cases} \end{aligned} \tag{4.6}$$

Estimate (4.6) suffices to prove (4.5) except when $|\boldsymbol{\alpha}| = 0$ and $|\boldsymbol{\beta}| = 1$ or vice versa, in which case it yields the upper bound $\omega^{1/2}|t_2 - t_1|$ with the adverse factor $\omega^{1/2}$. On the other hand, we can also prove the (comparatively simpler) bound

$$|\operatorname{Tr} J^{(k)} P_{\boldsymbol{\alpha}} \tilde{\gamma}^{(k)}_{N,\omega} P_{\boldsymbol{\beta}}(t_2) - \operatorname{Tr} J^{(k)} P_{\boldsymbol{\alpha}} \tilde{\gamma}^{(k)}_{N,\omega} P_{\boldsymbol{\beta}}(t_1)| \lesssim \omega^{-\frac{1}{2}|\boldsymbol{\alpha}|-\frac{1}{2}|\boldsymbol{\beta}|} \tag{4.7}$$

that provides no gain as $t_2 \to t_1$, but a better power of ω. By averaging (4.6) and (4.7) in the case $|\boldsymbol{\alpha}| = 0$ and $|\boldsymbol{\beta}| = 1$ (or vice versa), we obtain

$$|\operatorname{Tr} J^{(k)} P_{\boldsymbol{\alpha}} \tilde{\gamma}^{(k)}_{N,\omega} P_{\boldsymbol{\beta}}(t_2) - \operatorname{Tr} J^{(k)} P_{\boldsymbol{\alpha}} \tilde{\gamma}^{(k)}_{N,\omega} P_{\boldsymbol{\beta}}(t_1)| \lesssim |t_2 - t_1|^{1/2}$$

which suffices to establish (4.5).

Thus, it remains to prove both (4.6) and (4.7), and we begin with (4.6). Hierarchy (1.18) yields

$$i\partial_t \, P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}} = \sum_{j=1}^{k}\left[-\triangle_{x_j},\ P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}}\right]+\sum_{j=1}^{k}\omega\left[-\partial_{z_j}^2+z_j^2,\ P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}}\right] \tag{4.8}$$
$$+\frac{1}{N}\sum_{i<j}^{k}P_{\boldsymbol{\alpha}}\left[V_{N,\omega}\left(r_i-r_j\right),\ \tilde{\gamma}_{N,\omega}^{(k)}\right]P_{\boldsymbol{\beta}}$$
$$+\frac{N-k}{N}\operatorname{Tr}_{r_{k+1}}\sum_{j=1}^{k}P_{\boldsymbol{\alpha}}\left[V_{N,\omega}\left(r_j-r_{k+1}\right),\ \tilde{\gamma}_{N,\omega}^{(k+1)}\right]P_{\boldsymbol{\beta}}$$

Let

$$\mathrm{I}=-i\sum_{j=1}^{k}\operatorname{Tr}J^{(k)}[-\Delta_{x_j},P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}}]$$

$$\mathrm{II}=-\omega i\sum_{j=1}^{k}\operatorname{Tr}J^{(k)}[-\partial_{z_j}^2+z_j^2,P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}}] \tag{4.9}$$

$$\mathrm{III}=-iN^{-1}\sum_{1\leq i<j\leq k}\operatorname{Tr}J^{(k)}P_{\boldsymbol{\alpha}}[V_{N,\omega}(r_i-r_j),\tilde{\gamma}_{N,\omega}^{(k)}]P_{\boldsymbol{\beta}}$$

$$\mathrm{IV}=-i\frac{N-k}{N}\sum_{j=1}^{k}\operatorname{Tr}J^{(k)}P_{\boldsymbol{\alpha}}[V_{N,\omega}(r_j-r_{k+1}),\tilde{\gamma}_{N,\omega}^{(k+1)}]P_{\boldsymbol{\beta}}$$

Then it follows from (4.8) that

$$\partial_t\operatorname{Tr}J^{(k)}P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}}=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV} \tag{4.10}$$

First, consider I. Applying Lemma A.6 and then integration by parts, we obtain

$$\mathrm{I}=i\sum_{j=1}^{k}\left(\langle J^{(k)}\Delta_{x_j}P_{\boldsymbol{\alpha}}\psi,P_{\boldsymbol{\beta}}\psi\rangle_{\mathbf{r}_k}-\langle J^{(k)}P_{\boldsymbol{\alpha}}\psi,P_{\boldsymbol{\beta}}\Delta_{x_j}\psi\rangle_{\mathbf{r}_k}\right)$$
$$=i\sum_{j=1}^{k}\left(\langle J^{(k)}\Delta_{x_j}P_{\boldsymbol{\alpha}}\psi,P_{\boldsymbol{\beta}}\psi\rangle_{\mathbf{r}_k}-\langle\Delta_{x_j}J^{(k)}P_{\boldsymbol{\alpha}}\psi,P_{\boldsymbol{\beta}}\psi\rangle_{\mathbf{r}_k}\right)$$

Hence

$$|\mathrm{I}|\leq\sum_{j=1}^{k}(\|J^{(k)}\Delta_{x_j}\|_{\mathrm{op}}+\|\Delta_{x_j}J^{(k)}\|_{\mathrm{op}})\|P_{\boldsymbol{\alpha}}\psi\|_{L^2(\mathbb{R}^{3N})}\|P_{\boldsymbol{\beta}}\psi\|_{L^2(\mathbb{R}^{3N})}\leq C_{k,J^{(k)}} \tag{4.11}$$

where in the last step we applied the energy estimate.

Now, consider II. When $\boldsymbol{\alpha}=0$ and $\boldsymbol{\beta}=0$, we use that

$$\mathrm{II}=-\omega i\sum_{j=1}^{k}\operatorname{Tr}J^{(k)}[1-\partial_{z_j}^2+z_j^2,P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}}]=0$$

Otherwise, we proceed directly from (4.9), applying Lemma A.6 and integration by parts to obtain ($H_j = -\partial_{z_j}^2 + z_j^2$)

$$\begin{aligned} \mathrm{II} &= \omega i \sum_{j=1}^{k} \langle J^{(k)} H_j P_{\boldsymbol{\alpha}}\psi, P_{\boldsymbol{\beta}}\psi\rangle - \langle J^{(k)} P_{\boldsymbol{\alpha}}\psi, H_j P_{\boldsymbol{\beta}}\psi\rangle \\ &= \omega i \sum_{j=1}^{k} \langle J^{(k)} H_j P_{\boldsymbol{\alpha}}\psi, P_{\boldsymbol{\beta}}\psi\rangle - \langle H_j J^{(k)} P_{\boldsymbol{\alpha}}\psi, P_{\boldsymbol{\beta}}\psi\rangle \end{aligned}$$

Hence

$$|\mathrm{II}| \lesssim \omega \sum_{j=1}^{k} (\|J^{(k)} H_j\|_{\mathrm{op}} + \|H_j J^{(k)}\|_{\mathrm{op}}) \|P_{\boldsymbol{\alpha}}\psi\|_{L^2(\mathbb{R}^{3N})} \|P_{\boldsymbol{\beta}}\psi\|_{L^2(\mathbb{R}^{3N})}$$

By the energy estimates,

$$\mathrm{II} \begin{cases} = 0 & \text{if } \boldsymbol{\alpha} = 0 \text{ and } \boldsymbol{\beta} = 0 \\ \lesssim C_{k,J^{(k)}} \ \omega^{1-\frac{1}{2}|\boldsymbol{\alpha}| - \frac{1}{2}|\boldsymbol{\beta}|} & \text{otherwise} \end{cases} \tag{4.12}$$

Now, consider III.

$$\begin{aligned} \mathrm{III} &= -iN^{-1} \sum_{1\le i<j\le k} \langle J^{(k)} P_{\boldsymbol{\alpha}} V_{N,\omega}(r_i - r_j)\psi, P_{\boldsymbol{\beta}}\psi\rangle - \langle J^{(k)} P_{\boldsymbol{\alpha}}\psi, P_{\boldsymbol{\beta}} V_{N,\omega}(r_i - r_j)\psi\rangle \\ &= -iN^{-1} \sum_{1\le i<j\le k} \langle J^{(k)} P_{\boldsymbol{\alpha}} V_{N,\omega}(r_i - r_j)\psi, P_{\boldsymbol{\beta}}\psi\rangle - \langle P_{\boldsymbol{\alpha}}\psi, J^{(k)} P_{\boldsymbol{\beta}} V_{N,\omega}(r_i - r_j)\psi\rangle \end{aligned}$$

Let $L_i = (1 - \Delta_{r_i})^{1/2}$ and

$$W_{ij} = L_i^{-1} L_j^{-1} V_{N,\omega}(r_i - r_j) L_i^{-1} L_j^{-1} .$$

Then

$$\mathrm{III} = -iN^{-1} \sum_{1\le i<j\le k} \langle J^{(k)} P_{\boldsymbol{\alpha}} L_i L_j W_{ij} L_i L_j \psi, P_{\boldsymbol{\beta}}\psi\rangle - \langle P_{\boldsymbol{\alpha}}\psi, J^{(k)} P_{\boldsymbol{\beta}} L_i L_j W_{ij} L_i L_j \psi\rangle$$

Hence

$$\begin{aligned} |\mathrm{III}| \lesssim\ & N^{-1} \|J^{(k)} L_i L_j\|_{\mathrm{op}} \|W_{ij}\|_{\mathrm{op}} \|L_i L_j \psi\|_{L^2(\mathbb{R}^{3N})} \|P_{\boldsymbol{\beta}}\psi\|_{L^2(\mathbb{R}^{3N})} \\ & + N^{-1} \|P_{\boldsymbol{\alpha}}\psi\|_{L^2(\mathbb{R}^{3N})} \|J^{(k)} L_i L_j\|_{\mathrm{op}} \|W_{ij}\|_{\mathrm{op}} \|L_i L_j \psi\|_{L^2(\mathbb{R}^{3N})} \end{aligned}$$

By Lemma A.1, $\|W_{ij}\|_{\mathrm{op}} \lesssim \|V_{N,\omega}\|_{L^1} = \|V\|_{L^1}$ (independent of N, ω), and hence the energy estimates imply that

$$|\mathrm{III}| \lesssim C_{k,J^{(k)}} \ N^{-1} \tag{4.13}$$

Now consider IV.

$$\mathrm{IV} = -i\frac{N-k}{N} \sum_{j=1}^{k} \big(\langle J^{(k)} P_{\boldsymbol{\alpha}} V_{N,\omega}(r_j - r_{k+1})\psi, P_{\boldsymbol{\beta}}\psi\rangle - \langle J^{(k)} P_{\boldsymbol{\alpha}}\psi, P_{\boldsymbol{\beta}} V_{N,\omega}(r_j - r_{k+1})\psi\rangle \big)$$

Then, since $J^{(k)}L_{k+1}=L_{k+1}J^{(k)}$,

$$\begin{aligned}\mathrm{IV}= & -i\frac{N-k}{N}\sum_{j=1}^{k}\langle J^{(k)}L_jP_{\boldsymbol{\alpha}}W_{j(k+1)}L_jL_{k+1}\psi,P_{\boldsymbol{\beta}}L_{k+1}\psi\rangle \\ & -i\frac{N-k}{N}\sum_{j=1}^{k}\langle L_jJ^{(k)}P_{\boldsymbol{\alpha}}L_{k+1}\psi,P_{\boldsymbol{\beta}}W_{j(k+1)}L_jL_{k+1}\psi\rangle\end{aligned}$$

Estimating yields

$$|\mathrm{IV}|\lesssim\sum_{j=1}^{k}(\|J^{(k)}L_j\|_{\mathrm{op}}+\|L_jJ^{(k)}\|_{\mathrm{op}})\|W_{j(k+1)}\|_{\mathrm{op}}\|L_jL_{k+1}\psi\|_{L^2(\mathbb{R}^{3N})}\|L_{k+1}\psi\|_{L^2(\mathbb{R}^{3N})}$$

By (3.15),

$$|\mathrm{IV}|\lesssim C_{k,J^{(k)}} \tag{4.14}$$

Integrating (4.10) from t_1 to t_2 and applying the bounds obtained in (4.11), (4.12), (4.13), and (4.14), we obtain (4.6).

Finally, we proceed to prove (4.7). We have, by Lemma A.1,

$$\begin{aligned}&|\operatorname{Tr}J^{(k)}P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}}(t_2)-\operatorname{Tr}J^{(k)}P_{\boldsymbol{\alpha}}\tilde{\gamma}_{N,\omega}^{(k)}P_{\boldsymbol{\beta}}(t_1)| \\ &\quad\leq 2\sup_t|\langle J^{(k)}P_{\boldsymbol{\alpha}}\tilde{\psi}_{N,\omega}(t),P_{\boldsymbol{\beta}}\tilde{\psi}_{N,\omega}(t)\rangle_{\mathbf{r}_k}| \\ &\quad\lesssim\|J^{(k)}\|_{\mathrm{op}}\|P_{\boldsymbol{\alpha}}\tilde{\psi}_{N,\omega}(t)\|_{L^2(\mathbb{R}^{3N})}\|P_{\boldsymbol{\beta}}\tilde{\psi}_{N,\omega}(t)\|_{L^2(\mathbb{R}^{3N})} \\ &\quad\lesssim\omega^{-\frac{1}{2}|\boldsymbol{\alpha}|-\frac{1}{2}|\boldsymbol{\beta}|}\end{aligned}$$

where in the last step we applied (3.16).

□

According to Corollary 4.1, the study of the limit point of $\Gamma_{N,\omega}(t)=\left\{\tilde{\gamma}_{N,\omega}^{(k)}\right\}_{k=1}^{N}$ is directly related to the sequence $\Gamma_{x,N,\omega}(t)=\left\{\tilde{\gamma}_{x,N,\omega}^{(k)}=\operatorname{Tr}_z\tilde{\gamma}_{N,\omega}^{(k)}\right\}_{k=1}^{N}\in\oplus_{k\geqslant 1}C\left([0,T],\mathcal{L}_k^1\left(\mathbb{R}^{2k}\right)\right)$. We will do so in Section 5. We end this section on compactness by proving that $\Gamma_{x,N,\omega}(t)$ is compact with respect to the two dimensional version of the product topology τ_{prod} used in Theorem 4.1. This proof is not as delicate as the proof of Theorem 4.1 because we do not need to deal with $\infty-\infty$ here.

Theorem 4.2. *The sequence*

$$\Gamma_{x,N,\omega}(t)=\left\{\tilde{\gamma}_{x,N,\omega}^{(k)}=\operatorname{Tr}_z\tilde{\gamma}_{N,\omega}^{(k)}\right\}_{k=1}^{N}\in\bigoplus_{k\geqslant 1}C\left([0,T],\mathcal{L}_k^1\left(\mathbb{R}^{2k}\right)\right).$$

is compact with respect to the two dimensional version of the product topology τ_{prod} used in Theorem 4.1.

Proof. Similar to Theorem 4.1, we show that for every observable $J_x^{(k)}$ from a dense subset of $\mathcal{K}\left(L^2\left(\mathbb{R}^{2k}\right)\right)$ and for every $\varepsilon>0$, $\exists\delta(J_x^{(k)},\varepsilon)$ s.t. $\forall t_1,t_2\in[0,T]$ with $|t_1-t_2|\leqslant\delta$, we have

$$\sup_{N,\omega}\left|\operatorname{Tr} J_x^{(k)}\left(\tilde{\gamma}_{x,N,\omega}^{(k)}(t_1)-\tilde{\gamma}_{x,N,\omega}^{(k)}(t_2)\right)\right|\leqslant\varepsilon.$$

We utilize the observables $J_x^{(k)}\in\mathcal{K}\left(L^2\left(\mathbb{R}^{2k}\right)\right)$ which satisfy

$$\left\|\left\langle\nabla_{x_i}\right\rangle\left\langle\nabla_{x_j}\right\rangle J_x^{(k)}\left\langle\nabla_{x_i}\right\rangle^{-1}\left\langle\nabla_{x_j}\right\rangle^{-1}\right\|_{\mathrm{op}}+\left\|\left\langle\nabla_{x_i}\right\rangle^{-1}\left\langle\nabla_{x_j}\right\rangle^{-1}J_x^{(k)}\left\langle\nabla_{x_i}\right\rangle\left\langle\nabla_{x_j}\right\rangle\right\|_{\mathrm{op}}<\infty.$$

Here we choose similar but different observables from the proof of Theorem 4.1 since $\tilde{\gamma}_{x,N,\omega}^{(k)}$ acts on $L^2\left(\mathbb{R}^{2k}\right)$ instead of $L^2\left(\mathbb{R}^{3k}\right)$. This seems to make a difference when we deal with the terms involving $\tilde{\gamma}_{N,\omega}^{(k)}$ or $\tilde{\gamma}^{(k)}$. But $J_x^{(k)}$ does nothing on the z variable, hence

$$\begin{aligned}\left\|L_jJ_x^{(k)}L_j^{-1}\right\|_{\mathrm{op}} &\sim \left\|\left(\left\langle\nabla_{x_j}\right\rangle+\partial_{z_j}\right)J_x^{(k)}\frac{1}{\left(\left\langle\nabla_{x_j}\right\rangle+\partial_{z_j}\right)}\right\|_{\mathrm{op}}\\ &\leqslant \left\|\left\langle\nabla_{x_j}\right\rangle J_x^{(k)}\frac{1}{\left(\left\langle\nabla_{x_j}\right\rangle+\partial_{z_j}\right)}\right\|_{\mathrm{op}}+\left\|J_x^{(k)}\frac{\partial_{z_j}}{\left(\left\langle\nabla_{x_j}\right\rangle+\partial_{z_j}\right)}\right\|_{\mathrm{op}}\\ &\leqslant \left\|\left\langle\nabla_{x_j}\right\rangle J_x^{(k)}\left\langle\nabla_{x_j}\right\rangle^{-1}\right\|_{\mathrm{op}}+\left\|J_x^{(k)}\right\|_{\mathrm{op}},\end{aligned}$$

i.e. $\|L_jJ_x^{(k)}L_j^{-1}\|_{\mathrm{op}}, \|L_j^{-1}J_x^{(k)}L_j\|_{\mathrm{op}}, \|L_iL_jJ_x^{(k)}L_i^{-1}L_j^{-1}\|_{\mathrm{op}}$ and $\|L_i^{-1}L_j^{-1}J_x^{(k)}L_iL_j\|_{\mathrm{op}}$ are all finite. It is true that $J_x^{(k)}$ and the related operators listed are only in $\mathcal{L}^\infty\left(L^2\left(\mathbb{R}^{3k}\right)\right)$, but this is good enough for our purpose here.

Taking Tr_z on both sides of hierarchy (1.18), we have that $\tilde{\gamma}_{x,N,\omega}^{(k)}$ satisfies the coupled BBGKY hierarchy:

$$\begin{aligned}i\partial_t\tilde{\gamma}_{x,N,\omega}^{(k)} &= \sum_{j=1}^k\left[-\triangle_{x_j},\tilde{\gamma}_{x,N,\omega}^{(k)}\right]+\frac{1}{N}\sum_{i<j}^k\operatorname{Tr}_z\left[V_{N,\omega}\left(r_i-r_j\right),\tilde{\gamma}_{N,\omega}^{(k)}\right]\\ &\quad+\frac{N-k}{N}\sum_{j=1}^k\operatorname{Tr}_{x_{k+1}}\operatorname{Tr}_z\left[V_{N,\omega}\left(r_j-r_{k+1}\right),\tilde{\gamma}_{N,\omega}^{(k+1)}\right].\end{aligned}\tag{4.15}$$

Assume $t_1\leqslant t_2$, the above hierarchy yields

$$\begin{aligned}&\left|\operatorname{Tr} J_x^{(k)}\left(\tilde{\gamma}_{x,N,\omega}^{(k)}(t_1)-\tilde{\gamma}_{x,N,\omega}^{(k)}(t_2)\right)\right|\\ \leqslant&\sum_{j=1}^k\int_{t_1}^{t_2}\left|\operatorname{Tr} J_x^{(k)}\left[-\triangle_{x_j},\tilde{\gamma}_{x,N,\omega}^{(k)}\right]\right|dt+\frac{1}{N}\sum_{i<j}^k\int_{t_1}^{t_2}\left|\operatorname{Tr} J_x^{(k)}\left[V_{N,\omega}\left(r_i-r_j\right),\tilde{\gamma}_{N,\omega}^{(k)}\right]\right|dt\\ &+\frac{N-k}{N}\sum_{j=1}^k\int_{t_1}^{t_2}\left|\operatorname{Tr} J_x^{(k)}\left[V_{N,\omega}\left(r_j-r_{k+1}\right),\tilde{\gamma}_{N,\omega}^{(k+1)}\right]\right|dt.\\ =&\sum_{j=1}^k\int_{t_1}^{t_2}\mathrm{I}(t)\,dt+\frac{1}{N}\sum_{i<j}^k\int_{t_1}^{t_2}\mathrm{II}(t)\,dt+\frac{N-k}{N}\sum_{j=1}^k\int_{t_1}^{t_2}\mathrm{III}(t)\,dt.\end{aligned}$$

For I, we have

$$
\begin{aligned}
&\left|\operatorname{Tr} J_x^{(k)}\left[-\triangle_{x_j}, \tilde{\gamma}_{x,N,\omega}^{(k)}\right]\right| \\
&= \left|\operatorname{Tr} J_x^{(k)}\left[\left\langle\nabla_{x_j}\right\rangle^2, \tilde{\gamma}_{x,N,\omega}^{(k)}\right]\right| \text{ (1 commutes with everything)} \\
&= \left|\operatorname{Tr}\left\langle\nabla_{x_j}\right\rangle^{-1} J_x^{(k)}\left\langle\nabla_{x_j}\right\rangle^2 \tilde{\gamma}_{x,N,\omega}^{(k)}\left\langle\nabla_{x_j}\right\rangle-\operatorname{Tr}\left\langle\nabla_{x_j}\right\rangle J_x^{(k)}\left\langle\nabla_{x_j}\right\rangle^{-1}\left\langle\nabla_{x_j}\right\rangle \tilde{\gamma}_{x,N,\omega}^{(k)}\left\langle\nabla_{x_j}\right\rangle\right| \\
&\leqslant \left(\left\|\left\langle\nabla_{x_j}\right\rangle^{-1} J_x^{(k)}\left\langle\nabla_{x_j}\right\rangle\right\|_{\mathrm{op}}+\left\|\left\langle\nabla_{x_j}\right\rangle J_x^{(k)}\left\langle\nabla_{x_j}\right\rangle^{-1}\right\|_{\mathrm{op}}\right) \operatorname{Tr}\left\langle\nabla_{x_j}\right\rangle \tilde{\gamma}_{x,N,\omega}^{(k)}\left\langle\nabla_{x_j}\right\rangle \\
&\leqslant C_J \operatorname{Tr}\left\langle\nabla_{x_j}\right\rangle^2 \tilde{\gamma}_{N,\omega}^{(k)} \\
&\leqslant C_J \text{ (Corollary 3.1).}
\end{aligned}
$$

for II and III, we have

$$
\begin{aligned}
\mathrm{II} &= \left|\operatorname{Tr} J_x^{(k)}\left[V_{N,\omega}\left(r_i-r_j\right), \tilde{\gamma}_{N,\omega}^{(k)}\right]\right| \\
&= \left|\operatorname{Tr} L_i^{-1} L_j^{-1} J_x^{(k)} L_i L_j W_{ij} L_i L_j \tilde{\gamma}_{N,\omega}^{(k)} L_i L_j-\operatorname{Tr} L_i L_j J_x^{(k)} L_i^{-1} L_j^{-1} L_i L_j \tilde{\gamma}_{N,\omega}^{(k)} L_i L_j W_{ij}\right| \\
&\leqslant \left(\left\|L_i^{-1} L_j^{-1} J_x^{(k)} L_i L_j\right\|_{\mathrm{op}}+\left\|L_i L_j J_x^{(k)} L_i^{-1} L_j^{-1}\right\|_{\mathrm{op}}\right)\left\|W_{ij}\right\|_{\mathrm{op}} \operatorname{Tr} L_i L_j \tilde{\gamma}_{N,\omega}^{(k)} L_i L_j \\
&\leqslant C_J,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
\mathrm{III} &= \left|\operatorname{Tr} J_x^{(k)}\left[V_{N,\omega}\left(r_j-r_{k+1}\right), \tilde{\gamma}_{N,\omega}^{(k+1)}\right]\right| \\
&= \left|\operatorname{Tr} L_j^{-1} L_{k+1}^{-1} J_x^{(k)} L_j L_{k+1} W_{j(k+1)} L_j L_{k+1} \tilde{\gamma}_{N,\omega}^{(k+1)} L_j L_{k+1}\right. \\
&\quad \left.-\operatorname{Tr} L_j L_{k+1} J_x^{(k)} L_j^{-1} L_{k+1}^{-1} L_j L_{k+1} \tilde{\gamma}_{N,\omega}^{(k+1)} L_j L_{k+1} W_{j(k+1)}\right| \\
&\leqslant \left(\left\|L_j^{-1} J_x^{(k)} L_j\right\|_{\mathrm{op}}+\left\|L_j J_x^{(k)} L_j^{-1}\right\|_{\mathrm{op}}\right)\left\|W_{j(k+1)}\right\|_{\mathrm{op}} \operatorname{Tr} L_j L_{k+1} \tilde{\gamma}_{N,\omega}^{(k+1)} L_j L_{k+1} \\
&\leqslant C_J .
\end{aligned}
$$

Up to this point, we have proven uniform in time bounds for I - III, thus we conclude the compactness of the sequence $\Gamma_{x,N,\omega}(t)=\left\{\tilde{\gamma}_{x,N,\omega}^{(k)}\right\}_{k=1}^N$. □

5. Limit points satisfy GP hierarchy

Theorem 5.1. *Let $\Gamma(t)=\left\{\tilde{\gamma}^{(k)}\right\}_{k=1}^{\infty}$ be a $N \geqslant \omega^{v(\beta)+\varepsilon}$ limit point of $\Gamma_{N,\omega}(t)=\left\{\tilde{\gamma}_{N,\omega}^{(k)}\right\}_{k=1}^N$ with respect to the product topology τ_{prod}, then $\left\{\tilde{\gamma}_x^{(k)}=\operatorname{Tr}_z \tilde{\gamma}^{(k)}\right\}_{k=1}^{\infty}$ is a solution to the coupled Gross-Pitaevskii hierarchy subject to initial data $\tilde{\gamma}_x^{(k)}(0)=\left|\phi_0\right\rangle\left\langle\phi_0\right|^{\otimes k}$ with coupling constant $b_0=\int V(r)\, dr$, which, written in integral form, is*

$$
\tilde{\gamma}_x^{(k)}=U^{(k)}(t) \tilde{\gamma}_x^{(k)}(0)-i b_0 \sum_{j=1}^k \int_0^t U^{(k)}(t-s) \operatorname{Tr}_{x_{k+1}} \operatorname{Tr}_z\left[\delta\left(r_j-r_{k+1}\right), \tilde{\gamma}^{(k+1)}(s)\right] ds, \tag{5.1}
$$

where

$$U^{(k)}=\prod_{j=1}^{k}e^{it\triangle_{x_j}}e^{-it\triangle_{x_j'}}.$$

We prove Theorem 5.1 below. Combining Corollary 4.1 and Theorem 5.1, we see that $\tilde{\gamma}_x^{(k)}$ in fact solves the 2D Gross-Pitaevskii hierarchy with the desired coupling constant $b_0\left(\int |h_1(z)|^4\,dz\right)$.

Corollary 5.1. *Let* $\Gamma(t)=\left\{\tilde{\gamma}^{(k)}\right\}_{k=1}^{\infty}$ *be a* $N\geqslant\omega^{v(\beta)+\varepsilon}$ *limit point of* $\Gamma_{N,\omega}(t)=\left\{\tilde{\gamma}_{N,\omega}^{(k)}\right\}_{k=1}^{N}$ *with respect to the product topology* τ_{prod}*, then* $\left\{\tilde{\gamma}_x^{(k)}=\operatorname{Tr}_z\tilde{\gamma}^{(k)}\right\}_{k=1}^{\infty}$ *is a solution to the 2D Gross-Pitaevskii hierarchy subject to initial data* $\tilde{\gamma}_x^{(k)}(0)=\left|\phi_0\right\rangle\left\langle\phi_0\right|^{\otimes k}$ *with coupling constant* $b_0\left(\int|h_1(z)|^4\,dz\right)$*, which, written in integral form, is*

(5.2)

$$\tilde{\gamma}_x^{(k)}=U^{(k)}(t)\tilde{\gamma}_x^{(k)}(0)-ib_0\left(\int|h_1(z)|^4\,dz\right)\sum_{j=1}^{k}\int_0^t U^{(k)}(t-s)\operatorname{Tr}_{x_{k+1}}\left[\delta\left(x_j-x_{k+1}\right),\tilde{\gamma}_x^{(k+1)}(s)\right]ds.$$

Proof. We compute the $k=1$ case explicitly here. Written in kernels, the inhomogeneous term in hierarchy (5.1) is

$$\begin{aligned}&ib_0\int U^{(1)}(t-s)ds\int\delta(z_1-z_1')dz_1dz_1'\int\delta(r_1-r_2)\tilde{\gamma}^{(2)}(r_1,r_2,r_1',r_2)dr_2\\&-ib_0\int U^{(1)}(t-s)ds\int\delta(z_1-z_1')dz_1dz_1'\int\delta(r_1'-r_2)\tilde{\gamma}^{(2)}(r_1,r_2,r_1',r_2)dr_2\end{aligned}$$

which, by Corollary 4.1, is

$$\begin{aligned}=&ib_0\int U^{(1)}(t-s)ds\int\delta(z_1-z_1')\delta(r_1-r_2)\tilde{\gamma}_x^{(2)}(x_1,x_2,x_1',x_2)\\&\qquad\times h_1(z_1)h_1(z_2)h_1(z_1')h_1(z_2)dr_2dz_1dz_1'\\&-ib_0\int U^{(1)}(t-s)ds\int\delta(z_1-z_1')\delta(r_1'-r_2)\tilde{\gamma}_x^{(2)}(x_1,x_2,x_1',x_2)\\&\qquad\times h_1(z_1)h_1(z_2)h_1(z_1')h_1(z_2)dr_2dz_1dz_1'\end{aligned}$$

Further simplifications lead to

$$\begin{aligned}=&ib_0\int U^{(1)}(t-s)ds\int\delta(x_1-x_2)\tilde{\gamma}_x^{(2)}(x_1,x_2,x_1',x_2)|h_1(z_1)|^4dx_2dz_1\\&-ib_0\int U^{(1)}(t-s)ds\int\delta(x_1'-x_2)\tilde{\gamma}_x^{(2)}(x_1,x_2,x_1',x_2)|h_1(z_1')|^4dx_2dz_1'.\end{aligned}$$

In summary, we have

$$\begin{aligned}&ib_0\int U^{(1)}(t-s)\operatorname{Tr}_{x_2}\operatorname{Tr}_z\left[\delta\left(r_1-r_2\right),\tilde{\gamma}^{(2)}(s)\right]ds\\&=ib_0\left(\int|h_1(z)|^4\,dz\right)\int U^{(2)}(t-s)\operatorname{Tr}_{x_2}\left[\delta\left(x_1-x_2\right),\tilde{\gamma}_x^{(2)}(s)\right]ds.\end{aligned}$$

□

Proof of Theorem 5.1. By Theorems 4.1, 4.2, passing to subsequences if necessary, we have

$$
\begin{aligned}
\lim_{\substack{N,\omega\to\infty \\ N\geqslant\omega^{v(\beta)+\varepsilon}}} \sup_t \operatorname{Tr} J^{(k)}\left(\tilde{\gamma}_{N,\omega}^{(k)}(t)-\tilde{\gamma}^{(k)}(t)\right) &= 0, \quad \forall\, J^{(k)}\in\mathcal{K}\left(L^2\left(\mathbb{R}^{3k}\right)\right), \\
\lim_{\substack{N,\omega\to\infty \\ N\geqslant\omega^{v(\beta)+\varepsilon}}} \sup_t \operatorname{Tr} J_x^{(k)}\left(\tilde{\gamma}_{x,N,\omega}^{(k)}(t)-\tilde{\gamma}_x^{(k)}(t)\right) &= 0, \quad \forall\, J_x^{(k)}\in\mathcal{K}\left(L^2\left(\mathbb{R}^{2k}\right)\right).
\end{aligned}
\tag{5.3}
$$

We establish (5.1) by testing the limit point against the observables $J_x^{(k)}\in\mathcal{K}\left(L^2\left(\mathbb{R}^{2k}\right)\right)$ as in the proof of Theorem 4.2. We will prove that the limit point satisfies

$$
\operatorname{Tr} J_x^{(k)}\tilde{\gamma}_x^{(k)}(0)=\operatorname{Tr} J_x^{(k)}\left|\phi_0\right\rangle\left\langle\phi_0\right|^{\otimes k}
\tag{5.4}
$$

and

$$
\begin{aligned}
\operatorname{Tr} J_x^{(k)}\tilde{\gamma}_x^{(k)}(t) &= \operatorname{Tr} J_x^{(k)}U^{(k)}(t)\,\tilde{\gamma}_x^{(k)}(0) \\
&\quad - ib_0\sum_{j=1}^{k}\int_0^t \operatorname{Tr} J_x^{(k)}U^{(k)}(t-s)\left[\delta\left(r_j-r_{k+1}\right),\tilde{\gamma}^{(k+1)}(s)\right]ds.
\end{aligned}
\tag{5.5}
$$

To this end, we use the coupled BBGKY hierarchy (4.15) satisfied by $\tilde{\gamma}_{x,N,\omega}^{(k)}$, which, written in the form needed here, is

$$
\begin{aligned}
\operatorname{Tr} J_x^{(k)}\tilde{\gamma}_{x,N,\omega}^{(k)}(t) &= \operatorname{Tr} J_x^{(k)}U^{(k)}(t)\,\tilde{\gamma}_{x,N,\omega}^{(k)}(0) \\
&\quad -\frac{i}{N}\sum_{i<j}^{k}\int_0^t \operatorname{Tr} J_x^{(k)}U^{(k)}(t-s)\left[V_{N,\omega}\left(r_i-r_j\right),\tilde{\gamma}_{N,\omega}^{(k)}(s)\right]ds \\
&\quad -i\left(\frac{N-k}{N}\right)\sum_{j=1}^{k}\int_0^t \operatorname{Tr} J_x^{(k)}U^{(k)}(t-s)\left[V_{N,\omega}\left(r_j-r_{k+1}\right),\tilde{\gamma}_{N,\omega}^{(k+1)}(s)\right]ds \\
&= A-\frac{i}{N}\sum_{i<j}^{k}B-i\left(1-\frac{k}{N}\right)\sum_{j=1}^{k}D.
\end{aligned}
$$

By (5.3), we know

$$
\begin{aligned}
\lim_{\substack{N,\omega\to\infty \\ N\geqslant\omega^{v(\beta)+\varepsilon}}} \operatorname{Tr} J_x^{(k)}\tilde{\gamma}_{x,N,\omega}^{(k)}(t) &= \operatorname{Tr} J_x^{(k)}\tilde{\gamma}_x^{(k)}(t), \\
\lim_{\substack{N,\omega\to\infty \\ N\geqslant\omega^{v(\beta)+\varepsilon}}} \operatorname{Tr} J_x^{(k)}U^{(k)}(t)\,\tilde{\gamma}_{x,N,\omega}^{(k)}(0) &= \operatorname{Tr} J_x^{(k)}U^{(k)}(t)\,\tilde{\gamma}_x^{(k)}(0).
\end{aligned}
$$

By the argument that appears between Theorem 1 and Corollary 1 in [42], we know that assumption (b) in Theorem 1.1,

$$
\tilde{\gamma}_{N,\omega}^{(1)}(0)\to\left|\phi_0\otimes h_1\right\rangle\left\langle\phi_0\otimes h_1\right|, \quad \text{strongly in trace norm},
$$

in fact implies

$$
\tilde{\gamma}_{N,\omega}^{(k)}(0)\to\left|\phi_0\otimes h_1\right\rangle\left\langle\phi_0\otimes h_1\right|^{\otimes k}, \quad \text{strongly in trace norm}.
$$

Thus we have tested relation (5.4), the left-hand side of (5.5), and the first term on the right-hand side of (5.5) for the limit point. We are left to prove that

$$\begin{aligned}
\lim_{\substack{N,\omega\to\infty\\ N\geqslant\omega^{v(\beta)+\varepsilon}}}\frac{B}{N} &= 0,\\
\lim_{\substack{N,\omega\to\infty\\ N\geqslant\omega^{v(\beta)+\varepsilon}}}\left(1-\frac{k}{N}\right)D &= b_0\int_0^t \operatorname{Tr} J_x^{(k)}U^{(k)}(t-s)\left[\delta\left(r_j-r_{k+1}\right),\tilde{\gamma}^{(k+1)}\left(s\right)\right]ds.
\end{aligned}$$

First of all, we can use an argument similar to the estimate of III and IV in the proof of Theorem 4.1 to show the boundedness of $|B|$ and $|D|$ for every finite time t. In fact, noticing that $U^{(k)}$ commutes with Fourier multipliers, we have

$$\begin{aligned}
|B| &\leqslant \int_0^t \left|\operatorname{Tr} J_x^{(k)}U^{(k)}\left(t-s\right)\left[V_{N,\omega}\left(r_i-r_j\right),\tilde{\gamma}_{N,\omega}^{(k)}\left(s\right)\right]\right|ds\\
&= \int_0^t ds|\operatorname{Tr} L_i^{-1}L_j^{-1}J_x^{(k)}L_iL_jU^{(k)}\left(t-s\right)W_{ij}L_iL_j\tilde{\gamma}_{N,\omega}^{(k)}\left(s\right)L_iL_j\\
&\quad -\operatorname{Tr} L_iL_jJ_x^{(k)}L_i^{-1}L_j^{-1}U^{(k)}\left(t-s\right)L_iL_j\tilde{\gamma}_{N,\omega}^{(k)}\left(s\right)L_iL_jW_{ij}|\\
&\leqslant \int_0^t ds\left\|L_i^{-1}L_j^{-1}J_x^{(k)}L_iL_j\right\|_{\mathrm{op}}\left\|U^{(k)}\right\|_{\mathrm{op}}\left\|W_{ij}\right\|\operatorname{Tr} L_i^2L_j^2\tilde{\gamma}_{N,\omega}^{(k)}\left(s\right)\\
&\quad +\int_0^t ds\left\|L_iL_jJ_x^{(k)}L_i^{-1}L_j^{-1}\right\|_{\mathrm{op}}\left\|U^{(k)}\right\|_{\mathrm{op}}\left\|W_{ij}\right\|\operatorname{Tr} L_i^2L_j^2\tilde{\gamma}_{N,\omega}^{(k)}\left(s\right)\\
&\leqslant C_Jt.
\end{aligned}$$

Hence

$$\lim_{\substack{N,\omega\to\infty\\ N\geqslant\omega^{v(\beta)+\varepsilon}}}\frac{B}{N}=\lim_{\substack{N,\omega\to\infty\\ N\geqslant\omega^{v(\beta)+\varepsilon}}}\frac{kD}{N}=0.$$

To prove

$$\lim_{\substack{N,\omega\to\infty\\ N\geqslant\omega^{v(\beta)+\varepsilon}}} D=\int_0^t \operatorname{Tr} J_x^{(k)}U^{(k)}(t-s)\left[\delta\left(r_j-r_{k+1}\right),\tilde{\gamma}^{(k+1)}\left(s\right)\right]ds, \tag{5.6}$$

we need Lemma A.2 (stated and proved in Appendix A) which compares the $\delta-$function and its approximation. We choose a probability measure $\rho\in L^1\left(\mathbb{R}^3\right)$ and define $\rho_\alpha\left(r\right)=\alpha^{-3}\rho\left(\frac{r}{\alpha}\right)$. In fact, ρ can be the square of any 3D Hermite function. Write $J_{s-t}^{(k)}=J_x^{(k)}U^{(k)}\left(t-s\right)$, we then have

$$
\begin{aligned}
&\left|\operatorname{Tr} J_x^{(k)} U^{(k)}(t-s)\left(V_{N,\omega}(r_j-r_{k+1})\tilde{\gamma}_{N,\omega}^{(k+1)}(s)-b_0\delta(r_j-r_{k+1})\tilde{\gamma}^{(k+1)}(s)\right)\right| \\
&\quad\leqslant\left|\operatorname{Tr} J_{s-t}^{(k)}\left(V_{N,\omega}(r_j-r_{k+1})-b_0\delta(r_j-r_{k+1})\right)\tilde{\gamma}_{N,\omega}^{(k+1)}(s)\right| \\
&\qquad+b_0\left|\operatorname{Tr} J_{s-t}^{(k)}\left(\delta(r_j-r_{k+1})-\rho_\alpha(r_j-r_{k+1})\right)\tilde{\gamma}_{N,\omega}^{(k+1)}(s)\right| \\
&\qquad+b_0\left|\operatorname{Tr} J_{s-t}^{(k)}\rho_\alpha(r_j-r_{k+1})\left(\tilde{\gamma}_{N,\omega}^{(k+1)}(s)-\tilde{\gamma}^{(k+1)}(s)\right)\right| \\
&\qquad+b_0\left|\operatorname{Tr} J_{s-t}^{(k)}\left(\rho_\alpha(r_j-r_{k+1})-\delta(r_j-r_{k+1})\right)\tilde{\gamma}^{(k+1)}(s)\right| \\
&\quad=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}
\end{aligned}
$$

We take care of I first because it is a term which requires $N>\omega^{\frac{1}{2\beta}-\frac{1}{2}}$. Write $V_\omega(r)=\frac{1}{\sqrt{\omega}}V(x,\frac{z}{\sqrt{\omega}})$, we have $V_{N,\omega}=(N\sqrt{\omega})^{3\beta}V_\omega((N\sqrt{\omega})^\beta r)$, Lemma A.2 then yields

$$
\begin{aligned}
\mathrm{I} &\leqslant \frac{Cb_0}{(N\sqrt{\omega})^{\beta\kappa}}\left(\int V_\omega(r)\,|r|^\kappa\,dr\right) \\
&\quad\times\left(\left\|L_jJ_x^{(k)}L_j^{-1}\right\|_{\mathrm{op}}+\left\|L_j^{-1}J_x^{(k)}L_j\right\|_{\mathrm{op}}\right)\operatorname{Tr} L_jL_{k+1}\tilde{\gamma}_{N,\omega}^{(k+1)}(s)\,L_jL_{k+1} \\
&= C_J\frac{\left(\int V_\omega(r)\,|r|^\kappa\,dr\right)}{(N\sqrt{\omega})^{\beta\kappa}}.
\end{aligned}
$$

Notice that $\left(\int V_\omega(r)\,|r|^\kappa\,dr\right)$ grows like $(\sqrt{\omega})^\kappa$, so $I\leqslant C_J\left(\frac{(\sqrt{\omega})^{1-\beta}}{N^\beta}\right)^\kappa$ which converges to zero as $N,\omega\to\infty$ in the way that $N\geqslant\omega^{\frac{1}{2\beta}-\frac{1}{2}+\varepsilon}$. More precisely,

$$
\lim_{\substack{N,\omega\to\infty\\ N\geqslant\omega^{v(\beta)+\varepsilon}}} I=0.
$$

So we have handled I.

For II and IV, we have

$$
\begin{aligned}
\mathrm{II} &\leqslant Cb_0\alpha^\kappa\left(\left\|L_jJ_x^{(k)}L_j^{-1}\right\|_{\mathrm{op}}+\left\|L_j^{-1}J_x^{(k)}L_j\right\|_{\mathrm{op}}\right)\operatorname{Tr} L_jL_{k+1}\tilde{\gamma}_{N,\omega}^{(k+1)}(s)\,L_jL_{k+1}\ \text{(Lemma A.2)} \\
&\leqslant C_J\alpha^\kappa\ \text{(Corollary 3.1)} \\
\mathrm{IV} &\leqslant Cb_0\alpha^\kappa\left(\left\|L_jJ_x^{(k)}L_j^{-1}\right\|_{\mathrm{op}}+\left\|L_j^{-1}J_x^{(k)}L_j\right\|_{\mathrm{op}}\right)\operatorname{Tr} L_jL_{k+1}\tilde{\gamma}^{(k+1)}(s)\,L_jL_{k+1}\ \text{(Lemma A.2)} \\
&\leqslant C_J\alpha^\kappa\ \text{(Corollary 4.1)}
\end{aligned}
$$

which converges to 0 as $\alpha\to 0$, uniformly in N,ω.

For III,

$$
\begin{aligned}
\mathrm{III} &\leqslant b_0\left|\operatorname{Tr} J_{s-t}^{(k)}\rho_\alpha(r_j-r_{k+1})\frac{1}{1+\varepsilon L_{k+1}}\left(\tilde{\gamma}_{N,\omega}^{(k+1)}(s)-\tilde{\gamma}^{(k+1)}(s)\right)\right| \\
&\quad+b_0\left|\operatorname{Tr} J_{s-t}^{(k)}\rho_\alpha(r_j-r_{k+1})\frac{\varepsilon L_{k+1}}{1+\varepsilon L_{k+1}}\left(\tilde{\gamma}_{N,\omega}^{(k+1)}(s)-\tilde{\gamma}^{(k+1)}(s)\right)\right|.
\end{aligned}
$$

The first term in the above estimate goes to zero as $N, \omega \to \infty$ for every $\varepsilon > 0$, since we have assumed condition (5.3) and $J_{s-t}^{(k)} \rho_\alpha (r_j - r_{k+1}) (1 + \varepsilon L_{k+1})^{-1}$ is a compact operator. Due to the energy bounds on $\tilde{\gamma}_{N,\omega}^{(k+1)}$ and $\tilde{\gamma}^{(k+1)}$, the second term tends to zero as $\varepsilon \to 0$, uniformly in N.

Combining the estimates for I-IV, we have justified limit (5.6). Hence, we have obtained Theorem 5.1.

□

6. Uniqueness of the 2D GP hierarchy

For completeness, we discuss the uniqueness theory of the 2D Gross-Pitaevskii hierarchy. To be specific, we have the following theorem.

Theorem 6.1 ([16, Theorem 3]). *Define the collision operator $B_{j,k+1}$ by*

$$B_{j,k+1} \gamma_x^{(k+1)} = \mathrm{Tr}_{k+1} \left[\delta (x_j - x_{k+1}), \gamma_x^{(k+1)} \right].$$

Suppose that $\left\{ \gamma_x^{(k)} \right\}_{k=1}^{\infty}$ solves the 2D constant coefficient Gross-Pitaevskii hierarchy

$$i \partial_t \gamma_x^{(k)} + \sum_{j=1}^{k} \left[-\triangle_{x_j}, \gamma_x^{(k)} \right] = c_0 \sum_{j=1}^{k} B_{j,k+1} \left(\gamma_x^{(k+1)} \right), \tag{6.1}$$

subject to zero initial data and the space-time bound

$$\int_0^T \left\| \prod_{j=1}^{k} \left(\left| \nabla_{x_j} \right|^{\frac{1}{2}} \left| \nabla_{x_j'} \right|^{\frac{1}{2}} \right) B_{j,k+1} \gamma_x^{(k+1)}(t, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{2k} \times \mathbb{R}^{2k})} dt \leqslant C^k \tag{6.2}$$

for some $C > 0$ and all $1 \leqslant j \leqslant k$. Then $\forall k, t \in [0, T]$,

$$\left\| \prod_{j=1}^{k} \left(\left| \nabla_{x_j} \right|^{\frac{1}{2}} \left| \nabla_{x_j'} \right|^{\frac{1}{2}} \right) \gamma_x^{(k)}(t, \cdot; \cdot) \right\|_{L^2(\mathbb{R}^{2k} \times \mathbb{R}^{2k})} = 0.$$

Proof. This is the constant coefficient version of [16, Theorem 3]. W. Beckner obtained the key estimate of this theorem independently in [5]. Some other estimates of this type can be found in [14, 29]. K. Kirpatrick, G. Staffilani and B. Schlein are the first to obtain uniqueness theorems for 2D Gross-Pitaevskii hierarchies. One will find their Theorem 7.1 in [37] by replacing $|\nabla|^{\frac{1}{2}}$ by $\langle \nabla \rangle^{\frac{1}{2}+\varepsilon}$ in the statement of the above theorem. □

To apply Theorem 6.1 to our problem here, it is necessary to prove that both the known solution to the 2D Gross-Pitaevskii hierarchy (namely $|\phi\rangle \langle \phi|^{\otimes k}$, where ϕ solves the 2D cubic NLS) and the limit obtained from the coupled BBGKY hierarchy (4.15), satisfy the space-time bound (6.2). It is easy to see that $|\phi\rangle \langle \phi|^{\otimes k}$ verifies the space-time bound (6.2) because it is part of the standard procedure of proving well-posedness of the 2D cubic NLS. We use the following trace theorem to prove the space-time bound (6.2) for the limit.

Theorem 6.2 ([37, Theorem 5.2])**.** *For every $\alpha < 1$, there is a $C_\alpha > 0$ such that*

$$\left\| \prod_{j=1}^{k} \left(\left\langle \nabla_{x_j} \right\rangle^{\alpha} \left\langle \nabla_{x_j'} \right\rangle^{\alpha} \right) B_{j,k+1} \gamma_x^{(k+1)} \right\|_{L^2(\mathbb{R}^{2k} \times \mathbb{R}^{2k})} \leqslant C_\alpha \operatorname{Tr} \left(\prod_{j=1}^{k+1} \left(1 - \triangle_{x_j} \right) \right) \gamma_x^{(k+1)}$$

for all nonnegative $\gamma_x^{(k+1)} \in \mathcal{L}^1 \left(L^2 \left(\mathbb{R}^{2k} \right) \right)$.

We can combine the above theorems so that it is easy to see how they apply to our problem.

Theorem 6.3. *There is at most one nonnegative operator sequence*

$$\left\{ \gamma_x^{(k)} \right\}_{k=1}^{\infty} \in \bigoplus_{k \geqslant 1} C \left([0, T], \mathcal{L}_k^1 \left(\mathbb{R}^{2k} \right) \right)$$

that solves the 2D Gross-Pitaevskii hierarchy (6.1) *subject to the energy condition*

$$\operatorname{Tr} \left(\prod_{j=1}^{k} \left(1 - \triangle_{x_j} \right) \right) \gamma_x^{(k)} \leqslant C^k.$$

7. Conclusion

In this paper, by proving the limit of a BBGKY hierarchy whose limit is not even formally known since it contains $(\infty - \infty)$, we have rigorously derived the 2D cubic nonlinear Schrödinger equation from a 3D quantum many-body dynamic and we have accurately described the 3D to 2D phenomenon by establishing the exact emergence of the coupling constant $\left(\int |h_1(z)|^4 \, dz \right)$. This is the first direct rigorous treatment of the 3D to 2D dynamic problem in the literature.

Appendix A. Basic operator facts and Sobolev-type lemmas

Lemma A.1 ([24, Lemma A.3])**.** *Let $L_j = \left(1 - \triangle_{r_j} \right)^{\frac{1}{2}}$. Then we have*

$$\left\| L_i^{-1} L_j^{-1} V \left(r_i - r_j \right) L_i^{-1} L_j^{-1} \right\|_{\mathrm{op}} \leqslant C \left\| V \right\|_{L^1}.$$

Lemma A.2. *Let $\rho \in L^1 \left(\mathbb{R}^3 \right)$ be a probability measure such that $\int_{\mathbb{R}^3} \langle r \rangle^{\frac{1}{2}} \rho(r) \, dr < \infty$ and let $\rho_\alpha(r) = \alpha^{-3} \rho \left(\frac{r}{\alpha} \right)$. Then, for every $\kappa \in (0, 1/2)$, there exists $C > 0$ s.t.*

$$\begin{aligned} &\left| \operatorname{Tr} J^{(k)} \left(\rho_\alpha \left(r_j - r_{k+1} \right) - \delta \left(r_j - r_{k+1} \right) \right) \gamma^{(k+1)} \right| \\ &\quad \leqslant C \left(\int \rho(r) \left| r \right|^{\kappa} dr \right) \alpha^{\kappa} \left(\left\| L_j J^{(k)} L_j^{-1} \right\|_{\mathrm{op}} + \left\| L_j^{-1} J^{(k)} L_j \right\|_{\mathrm{op}} \right) \operatorname{Tr} L_j L_{k+1} \gamma^{(k+1)} L_j L_{k+1} \end{aligned}$$

for all nonnegative $\gamma^{(k+1)} \in \mathcal{L}^1 \left(L^2 \left(\mathbb{R}^{3k+3} \right) \right)$.

Proof. We give a proof by modifying the proof of [37, Lemma A.2]. We remark that the range of κ is smaller here because we are working in 3D. It suffices to prove the estimate for $k=1$. We represent $\gamma^{(2)}$ by $\gamma^{(2)}=\sum_j \lambda_j \left|\varphi_j\right\rangle\left\langle\varphi_j\right|$, where $\varphi_j \in L^2\left(\mathbb{R}^6\right)$ and $\lambda_j \geqslant 0$. We write

$$\begin{aligned}
&\operatorname{Tr} J^{(1)}\left(\rho_\alpha\left(r_1-r_2\right)-\delta\left(r_1-r_2\right)\right) \gamma^{(2)} \\
&\quad=\sum_j \lambda_j\left\langle\varphi_j, J^{(1)}\left(\rho_\alpha\left(r_1-r_2\right)-\delta\left(r_1-r_2\right)\right) \varphi_j\right\rangle \\
&\quad=\sum_j \lambda_j\left\langle\psi_j,\left(\rho_\alpha\left(r_1-r_2\right)-\delta\left(r_1-r_2\right)\right) \varphi_j\right\rangle
\end{aligned}$$

where $\psi_j=\left(J^{(1)} \otimes 1\right) \varphi_j$. By Parseval, we find

$$\begin{aligned}
&|\langle\psi_j,(\rho_\alpha(r_1-r_2)-\delta(r_1-r_2))\varphi_j\rangle| \\
&\quad=|\int \overline{\hat{\psi}}_j(\xi_1,\xi_2)\hat{\varphi}_j(\xi_1',\xi_2')\rho(r)(e^{i\alpha r\cdot(\xi_1-\xi_1')}-1)\delta(\xi_1+\xi_2-\xi_1'-\xi_2')dr d\xi_1 d\xi_2 d\xi_1' d\xi_2'| \\
&\quad\leqslant \int |\hat{\psi}_j(\xi_1,\xi_2)||\hat{\varphi}_j(\xi_1',\xi_2')|\delta(\xi_1+\xi_2-\xi_1'-\xi_2')|\int \rho(r)(e^{i\alpha r\cdot(\xi_1-\xi_1')}-1)dr| d\xi_1 d\xi_2 d\xi_1' d\xi_2'.
\end{aligned}$$

Using the inequality that $\forall \kappa \in (0,1)$

$$\begin{aligned}
\left|e^{i\alpha r\cdot(\xi_1-\xi_1')}-1\right| &\leqslant \alpha^\kappa |r|^\kappa \left|\xi_1-\xi_1'\right|^\kappa \\
&\leqslant \alpha^\kappa |r|^\kappa \left(|\xi_1|^\kappa+|\xi_1'|^\kappa\right),
\end{aligned}$$

we get

$$\begin{aligned}
&|\langle\psi_j,(\rho_\alpha(r_1-r_2)-\delta(r_1-r_2))\varphi_j\rangle| \\
&\quad\leqslant \alpha^\kappa(\int \rho(r)|r|^\kappa dr)\int |\xi_1|^\kappa |\hat{\psi}_j(\xi_1,\xi_2)||\hat{\varphi}_j(\xi_1',\xi_2')|\delta(\xi_1+\xi_2-\xi_1'-\xi_2')d\xi_1 d\xi_2 d\xi_1' d\xi_2' \\
&\qquad+\alpha^\kappa(\int \rho(r)|r|^\kappa dr)\int |\xi_1'|^\kappa |\hat{\psi}_j(\xi_1,\xi_2)||\hat{\varphi}_j(\xi_1',\xi_2')|\delta(\xi_1+\xi_2-\xi_1'-\xi_2')d\xi_1 d\xi_2 d\xi_1' d\xi_2' \\
&\quad=\alpha^\kappa(\int \rho(r)|r|^\kappa dr)(\mathrm{I}+\mathrm{II}).
\end{aligned}$$

The estimate for I and II are similar, so we only deal with I explicitly.

$$\begin{aligned}
\mathrm{I} &\leqslant \int \delta\left(\xi_1+\xi_2-\xi_1'-\xi_2'\right) \frac{\langle\xi_1\rangle\langle\xi_2\rangle}{\langle\xi_1'\rangle\langle\xi_2'\rangle}\left|\hat{\psi}_j\left(\xi_1,\xi_2\right)\right| \frac{\langle\xi_1'\rangle\langle\xi_2'\rangle}{\langle\xi_1\rangle^{1-\kappa}\langle\xi_2\rangle}\left|\hat{\varphi}_j\left(\xi_1',\xi_2'\right)\right| d\xi_1 d\xi_2 d\xi_1' d\xi_2' \\
&\leqslant \varepsilon \int \delta\left(\xi_1+\xi_2-\xi_1'-\xi_2'\right) \frac{\langle\xi_1\rangle^2\langle\xi_2\rangle^2}{\langle\xi_1'\rangle^2\langle\xi_2'\rangle^2}\left|\hat{\psi}_j\left(\xi_1,\xi_2\right)\right|^2 d\xi_1 d\xi_2 d\xi_1' d\xi_2' \\
&\quad +\frac{1}{\varepsilon}\int \delta\left(\xi_1+\xi_2-\xi_1'-\xi_2'\right) \frac{\langle\xi_1'\rangle^2\langle\xi_2'\rangle^2}{\langle\xi_1\rangle^{2(1-\kappa)}\langle\xi_2\rangle^2}\left|\hat{\varphi}_j\left(\xi_1',\xi_2'\right)\right|^2 d\xi_1 d\xi_2 d\xi_1' d\xi_2' \\
&= \varepsilon \int \langle\xi_1\rangle^2\langle\xi_2\rangle^2\left|\hat{\psi}_j\left(\xi_1,\xi_2\right)\right|^2 d\xi_1 d\xi_2 \int \frac{1}{\langle\xi_1+\xi_2-\xi_2'\rangle^2\langle\xi_2'\rangle^2} d\xi_2' \\
&\quad \frac{1}{\varepsilon}\int \langle\xi_1'\rangle^2\langle\xi_2'\rangle^2\left|\hat{\varphi}_j\left(\xi_1',\xi_2'\right)\right|^2 d\xi_1' d\xi_2' \int \frac{1}{\langle\xi_1'+\xi_2'-\xi_2\rangle^{2(1-\kappa)}\langle\xi_2\rangle^2} d\xi_2 \\
&\leqslant \varepsilon\left\langle\psi_j, L_1^2L_2^2\psi_j\right\rangle \sup_{\xi}\int_{\mathbb{R}^3}\frac{1}{\langle\xi-\eta\rangle^2\langle\eta\rangle^2}d\eta + \frac{1}{\varepsilon}\left\langle\varphi_j, L_1^2L_2^2\varphi_j\right\rangle \sup_{\xi}\int_{\mathbb{R}^3}\frac{1}{\langle\xi-\eta\rangle^{2(1-\kappa)}\langle\eta\rangle^2}d\eta.
\end{aligned}$$

When $\kappa\in[0,1/2)$,

$$\begin{aligned}
\sup_{\xi}\int_{\mathbb{R}^3}\frac{1}{\langle\xi-\eta\rangle^{2(1-\kappa)}\langle\eta\rangle^2}d\eta &< \infty, \\
\sup_{\xi}\int_{\mathbb{R}^3}\frac{1}{\langle\xi-\eta\rangle^2\langle\eta\rangle^2}d\eta &< \infty,
\end{aligned}$$

and hence we have (with $\varepsilon=\|L_1J^{(1)}L_1^{-1}\|_{\mathrm{op}}^{-1}$),

$$\begin{aligned}
&\left|\operatorname{Tr} J^{(1)}\left(\rho_\alpha\left(r_1-r_2\right)-\delta\left(r_1-r_2\right)\right)\gamma^{(k+1)}\right| \\
&\leqslant C\left(\int\rho(r)\,|r|^\kappa\,dr\right)\alpha^\kappa\left(\varepsilon\operatorname{Tr} J^{(1)}L_1^2L_2^2J^{(1)}\gamma^{(2)}+\frac{1}{\varepsilon}\operatorname{Tr} L_1^2L_2^2\gamma^{(2)}\right) \\
&= C\left(\int\rho(r)\,|r|^\kappa\,dr\right)\alpha^\kappa\left(\varepsilon\operatorname{Tr} L_1^{-1}L_2^{-1}J^{(1)}L_1L_1J^{(1)}L_1^{-1}L_1L_2^2\gamma^{(2)}L_1L_2+\frac{1}{\varepsilon}\operatorname{Tr} L_1^2L_2^2\gamma^{(2)}\right) \\
&\leqslant C\left(\int\rho(r)\,|r|^\kappa\,dr\right)\alpha^\kappa\left(\varepsilon\left\|L_1^{-1}J^{(1)}L_1\right\|_{\mathrm{op}}\left\|L_1J^{(1)}L_1^{-1}\right\|_{\mathrm{op}}+\frac{1}{\varepsilon}\right)\operatorname{Tr} L_1^2L_2^2\gamma^{(2)} \\
&\leqslant C\left(\int\rho(r)\,|r|^\kappa\,dr\right)\alpha^\kappa\left(\left\|L_1^{-1}J^{(1)}L_1\right\|_{\mathrm{op}}+\left\|L_1J^{(1)}L_1^{-1}\right\|_{\mathrm{op}}\right)\operatorname{Tr} L_1^2L_2^2\gamma^{(2)}
\end{aligned}$$

□

Lemma A.3 (some standard operator inequalities)**.**

(1) *Suppose that* $A\geq 0$, $P_j=P_j^*$, *and* $I=P_0+P_1$. *Then* $A\leq 2P_0AP_0+2P_1AP_1$.
(2) *If* $A\geq B\geq 0$, *and* $AB=BA$, *then* $A^\alpha\geq B^\alpha$ *for any* $\alpha\geq 0$.
(3) *If* $A_1\geq A_2\geq 0$, $B_1\geq B_2\geq 0$ *and* $A_iB_j=B_jA_i$ *for all* $1\leq i,j\leq 2$, *then* $A_1B_1\geq A_2B_2$.
(4) *If* $A\geq 0$ *and* $AB=BA$, *then* $A^{1/2}B=BA^{1/2}$.

Proof. For (1), $\|A^{1/2}f\|^2=\|A^{1/2}(P_0+P_1)f\|^2\leq 2\|A^{1/2}P_0f\|^2+2\|A^{1/2}P_1f\|^2$. The rest are standard facts in operator theory. □

Recall that

$$S^2 = 1 - \Delta_x - \omega - \partial_z^2 + \omega^2 z^2$$

Lemma A.4 (Estimates with ω-loss)**.** *Suppose $f = f(x,z)$. Then*

$$\|\nabla_r f\|_{L^2_r} \lesssim \omega^{1/2} \|Sf\|_{L^2_r} \tag{A.1}$$

$$\|f\|_{L^6_r} \lesssim \omega^{1/6} \|Sf\|_{L^2_r} \tag{A.2}$$

$$\|\nabla_r f\|_{L^6_r} \lesssim \omega^{2/3} \|S^2 f\|_{L^2_r} \tag{A.3}$$

$$\|f\|_{L^\infty_r} \lesssim \omega^{1/4} \|S^2 f\|_{L^2_r} \tag{A.4}$$

The factors of ω appearing here are seen to be optimal by taking $f(x,z) = g(x)h_\omega(z)$, where $g(x)$ is a smooth bump function. Then $S^2 f = (1-\Delta_x)g(x)h_\omega(z)$ and hence

$$\|Sf\|^2_{L^2_r} = \langle S^2 f, f\rangle = \langle (1-\Delta_x)gh_\omega, h_\omega\rangle = (\|g\|^2_{L^2_x} + \|\nabla_x g\|^2_{L^2_x})\|h_\omega\|^2_{L^2_z}$$

which is ω-independent. Also, $\|S^2 f\|_{L^2_r} = \|(1-\Delta_x)g\|_{L^2_x}$ is ω-independent. On the other hand, it is apparent that $\|\nabla_r f\|_{L^2_r} = \omega^{1/2}$, $\|f\|_{L^6_r} = \omega^{1/6}$, $\|\nabla_r f\|_{L^6_r} = \omega^{2/3}$ and $\|f\|_{L^\infty_r} = \omega^{1/4}$, which demonstrates sharpness of the estimates.

Proof. Recall $I = P_0 + P_1$. First, we establish

$$\|\nabla_r P_1 f\|_{L^2_r} \lesssim \|Sf\|_{L^2_r} \tag{A.5}$$

$$\|P_1 f\|_{L^6_r} \lesssim \|Sf\|_{L^2_r} \tag{A.6}$$

$$\|\nabla_r P_1 f\|_{L^6_r} \lesssim \|S^2 f\|_{L^2_r} \tag{A.7}$$

$$\|P_1 f\|_{L^\infty_r} \lesssim \|S^2 f\|_{L^2_r} \tag{A.8}$$

Note that $P_j S^2 = S^2 P_j$. By the definition of S,

$$P_1(1 - \Delta_r + \omega^2 z^2)P_1 = S^2 P_1 + \omega P_1$$

By spectral considerations $2\omega P_1 \leq P_1 S^2$, and hence

$$\underbrace{P_1(1 - \Delta_r + \omega^2 z^2)P_1}_{\text{all terms positive}} \lesssim S^2 P_1 \tag{A.9}$$

Since $[P_1(1-\Delta_x)P_1, S^2 P_1] = 0$ and $P_1(1-\Delta_x)P_1 \leq S^2 P_1$ (from (A.9)), we have by Lemma A.3(3)

$$P_1(1-\Delta_x)^2 P_1 \lesssim S^4 P_1 \tag{A.10}$$

Since $[P_1(-\partial_z^2 + \omega^2 z^2)P_1, S^2 P_1] = 0$ and $P_1(-\partial_z^2 + \omega^2 z^2)P_1 \lesssim S^2 P_1$ (from (A.9)), we have by Lemma A.3(3)

$$P_1(-\partial_z^2 + \omega^2 z^2)^2 P_1 \lesssim S^4 P_1 \tag{A.11}$$

Expanding and "integrating by parts"

$$(-\partial_z^2 + \omega^2 z^2)^2 = \underbrace{\partial_z^4 - 2\omega^2 \partial_z z^2 \partial_z + \omega^4 z^4}_{\text{terms all positive}} + B + B^* \tag{A.12}$$

where $B \stackrel{\text{def}}{=} -2\omega^2 \partial_z z$. We claim

$$P_1(B+B^*)P_1 \lesssim S^4 P_1 \tag{A.13}$$

Since $\|\partial_z P_1 f\|_{L^2_r} \lesssim \|SP_1 f\|_{L^2_r}$ and $\omega\|zP_1 f\|_{L^2_r} \lesssim \|SP_1 f\|_{L^2_r}$, it follows by Cauchy-Schwarz that

$$\omega^2 |\operatorname{Re}\langle \partial_z P_1 f, zP_1 f\rangle| \lesssim \omega \|SP_1 f\|_{L^2_r}^2 \lesssim \|S^2 P_1 f\|_{L^2_r}^2$$

which is equivalent to (A.13). By (A.11), (A.12), (A.13), we obtain

$$P_1(\partial_z^4)P_1 \lesssim S^4 P_1 \tag{A.14}$$

Now, (A.10), (A.14) imply

$$P_1(1-\Delta_r)^2 P_1 \lesssim S^4 P_1 \tag{A.15}$$

Then (A.5), (A.6), (A.7), (A.8) follow from Sobolev embedding and (A.9), (A.15). For example, to prove (A.7), we apply 3D Sobolev embedding and (A.15) to obtain

$$\|\nabla_r P_1 f\|_{L^6_r} \lesssim \|\Delta_r P_1 f\|_{L^2_r} \lesssim \|S^2 P_1 f\|_{L^2_r} \lesssim \|S^2 f\|_{L^2_r}\,.$$

Next we prove

$$\|\nabla_r P_0 f\|_{L^2_r} \lesssim \omega^{1/2} \|Sf\|_{L^2_r} \tag{A.16}$$

$$\|P_0 f\|_{L^6_r} \lesssim \omega^{1/6} \|Sf\|_{L^2_r} \tag{A.17}$$

$$\|\nabla_r P_0 f\|_{L^6_r} \lesssim \omega^{2/3} \|S^2 f\|_{L^2_r} \tag{A.18}$$

$$\|P_0 f\|_{L^\infty_r} \lesssim \omega^{1/4} \|S^2 f\|_{L^2_r} \tag{A.19}$$

Recall that

$$P_0 f(x,z) = \int_{z'} f(x,z') h_\omega(z')\, dz'\ h_\omega(z) = \langle f(x,\cdot), h_\omega\rangle_{z'}\ h_\omega(z) \tag{A.20}$$

We have

$$\nabla_x P_0 f(x,z) = \langle \nabla_x f(x,\cdot), h_\omega\rangle\ h_\omega(z) \tag{A.21}$$

By Cauchy-Schwarz,

$$\|\nabla_x P_0 f\|_{L^2_r} \lesssim \|\nabla_x f\|_{L^2_r} \lesssim \|Sf\|_{L^2_r} \tag{A.22}$$

Also,

$$\partial_z P_0 f(x,z) = \langle f(x,\cdot), h_\omega\rangle\ \partial_z h_\omega(z) \tag{A.23}$$

and hence by Cauchy-Schwarz,

$$\|\partial_z P_0 f\|_{L^2_r} \lesssim \omega^{1/2} \|f\|_{L^2_r} \tag{A.24}$$

(A.22) and (A.24) together imply (A.16). By Cauchy-Schwarz, Minkowski, and 2D Sobolev,

$$\begin{aligned}
\|P_0 f\|_{L^6_r} &\le \|\langle f(x,\cdot), h_\omega\rangle_{z'}\|_{L^6_x} \|h_\omega\|_{L^6_z} \\
&\lesssim \|f\|_{L^6_x L^2_z} \|h_\omega\|_{L^2_z} \|h_\omega\|_{L^6_z} \\
&\lesssim \|(1-\Delta_x)^{1/2} f\|_{L^2_r}\ \omega^{1/6}
\end{aligned}$$

Since $(1-\Delta_x) \leq S^2$, we obtain (A.17) as a consequence of the previous estimate. Next, we prove (A.18). By (A.21), Cauchy-Schwarz, Minkowski, and 2D Sobolev,

$$\begin{aligned}\|\nabla_x P_0 f\|_{L^6_r} &\lesssim \|\langle \nabla_x f(x,\cdot), h_\omega\rangle\|_{L^6_x} \|h_\omega\|_{L^6_z} \\ &\lesssim \|\nabla_x f\|_{L^6_x L^2_z}\, \omega^{1/6} \\ &\lesssim \|(-\Delta_x)^{5/6} f\|_{L^2_r}\, \omega^{1/6}\end{aligned}$$

Since $(-\Delta_x)^{5/3} \leq (1-\Delta_x)^2 \leq S^4$, we obtain

$$\|\nabla_x P_0 f\|_{L^6_r} \lesssim \omega^{1/6} \|S^2 f\|_{L^2_r} \tag{A.25}$$

By (A.23), Cauchy-Schwarz, Minkowski, and 2D Sobolev,

$$\begin{aligned}\|\partial_z P_0 f\|_{L^6_r} &\lesssim \|\langle f(x,\cdot), h_\omega\rangle\|_{L^6_x} \|\partial_z h_\omega\|_{L^6_z} \\ &\lesssim \|f\|_{L^6_x L^2_z}\, \omega^{2/3} \\ &\lesssim \|(1-\Delta_x)^{1/3} f\|_{L^2_r}\, \omega^{2/3}\end{aligned}$$

Since $(1-\Delta_x)^{2/3} \leq (1-\Delta_x)^2 \leq S^4$, we obtain

$$\|\partial_z P_0 f\|_{L^6_r} \lesssim \|S^2 f\|_{L^2_r}\, \omega^{2/3} \tag{A.26}$$

Combining (A.25) and (A.26), we obtain (A.18). Next, we prove (A.19). By (A.20) and 2D Sobolev,

$$\begin{aligned}\|P_0 f\|_{L^\infty_r} &\lesssim \|\langle f(x,\cdot), h_\omega\rangle\|_{L^\infty_x} \|h_\omega\|_{L^\infty_z} \\ &\lesssim \|f\|_{L^\infty_x L^2_z}\, \omega^{1/4} \\ &\lesssim \|(1-\Delta_x)^{\frac{1}{2}+\epsilon} f\|_{L^2_r}\, \omega^{1/4}\end{aligned}$$

Since $(1-\Delta_x)^{1+2\epsilon} \leq (1-\Delta_x)^2 \leq S^4$, we obtain (A.19) as a consequence of the previous estimate.

Note that combining (A.5)–(A.8) and (A.16)–(A.19) yeilds (A.1)–(A.4). ☐

Let

$$\tilde{S} = (1-\Delta_x + \omega(-1-\partial_z^2+z^2))^{1/2}$$

Lemma A.5.

$$\tilde{S}^2 \gtrsim 1-\Delta_r \tag{A.27}$$

$$\tilde{S}^2 P_1 \geq P_1(1-\Delta_x-\omega\partial_z^2+\omega z^2)P_1 \tag{A.28}$$

$$\tilde{S}^2 P_1 \geq \omega P_1 \tag{A.29}$$

Proof. Directly from the definition of $\tilde{S}$, we have

$$\underbrace{P_1(1-\Delta_x-\omega\partial_z^2+\omega z^2)P_1}_{\text{all terms positive}} \leq \omega P_1 + \tilde{S}^2 P_1 \tag{A.30}$$

By spectral considerations

$$2\omega P_1 \leq \omega(-1-\partial_z^2+z^2)P_1 \leq \tilde{S}^2 P_1 \tag{A.31}$$

Combining (A.30) and (A.31) yields (A.28). Also, (A.29) follows from (A.31). Next, we establish (A.27) using (A.28). It is immediate that

$$\tilde{S}^2 \geq (1 - \Delta_x) \tag{A.32}$$

On the other hand, since P_0 is just projection onto the smooth function e^{-z^2},

$$P_0(-\partial_z^2)P_0 \lesssim 1 \leq \tilde{S}^2 \tag{A.33}$$

By (A.28),

$$P_1(-\partial_z^2)P_1 \leq \tilde{S}^2 P_1 \leq \tilde{S}^2 \tag{A.34}$$

By Lemma A.3(1), (A.33), (A.34),

$$-\partial_z^2 \lesssim \tilde{S}^2 \tag{A.35}$$

The claimed inequality (A.27) follows from (A.32) and (A.35). □

Lemma A.6. *Suppose $\sigma : L^2(\mathbb{R}^{3k}) \to L^2(\mathbb{R}^{3k})$ has kernel*

$$\sigma(\mathbf{r}_k, \mathbf{r}'_k) = \int \psi(\mathbf{r}_k, \mathbf{r}_{N-k})\overline{\psi}(\mathbf{r}'_k, \mathbf{r}_{N-k})\, d\mathbf{r}_{N-k}\,,$$

for some $\psi \in L^2(\mathbb{R}^{3N})$, and let $A, B : L^2(\mathbb{R}^{3k}) \to L^2(\mathbb{R}^{3k})$. Then the composition $A\sigma B$ has kernel

$$(A\sigma B)(\mathbf{r}_k, \mathbf{r}'_k) = \int (A\psi)(\mathbf{r}_k, \mathbf{r}_{N-k})(\overline{B^*\psi})(\mathbf{r}'_k, \mathbf{r}_{N-k})\, d\mathbf{r}_{N-k}$$

It follows that

$$\operatorname{Tr} A\sigma B = \langle A\psi, B^*\psi \rangle\,.$$

Let $\mathcal{K}_k$ denote the class of compact operators on $L^2(\mathbb{R}^{3k})$, $\mathcal{L}_k^1$ denote the trace class operators on $L^2(\mathbb{R}^{3k})$, and $\mathcal{L}_k^2$ denote the Hilbert-Schmidt operators on $L^2(\mathbb{R}^{3k})$. We have

$$\mathcal{L}_k^1 \subset \mathcal{L}_k^2 \subset \mathcal{K}_k$$

For an operator J on $L^2(\mathbb{R}^{3k})$, let $|J| = (J^*J)^{1/2}$ and denote by $J(\mathbf{r}_k, \mathbf{r}'_k)$ the kernel of J and $|J|(\mathbf{r}_k, \mathbf{r}'_k)$ the kernel of $|J|$, which satisfies $|J|(\mathbf{r}_k, \mathbf{r}'_k) \geq 0$. Let

$$\mu_1 \geq \mu_2 \geq \cdots \geq 0$$

be the eigenvalues of $|J|$ repeated according to multiplicity (the *singular values* of J). Then

$$\|J\|_{\mathcal{K}_k} = \|\mu_n\|_{\ell_n^\infty} = \mu_1 = \|\,|J|\,\|_{\text{op}} = \|J\|_{\text{op}}$$

$$\|J\|_{\mathcal{L}_k^2} = \|\mu_n\|_{\ell_n^2} = \|J(\mathbf{r}_k, \mathbf{r}'_k)\|_{L^2(\mathbf{r}_k, \mathbf{r}'_k)} = (\operatorname{Tr} J^*J)^{1/2}$$

$$\|J\|_{\mathcal{L}_k^1} = \|\mu_n\|_{\ell_n^1} = \||J|(\mathbf{r}_k, \mathbf{r}_k)\|_{L^1(\mathbf{r}_k)} = \operatorname{Tr}|J|$$

The topology on $\mathcal{K}_k$ coincides with the operator topology, and $\mathcal{K}_k$ is a closed subspace of the space of bounded operators on $L^2(\mathbb{R}^{3k})$.

Lemma A.7. *Let χ be a smooth function on $\mathbb{R}^3$ such that $\chi(\xi) = 1$ for $|\xi| \leq 1$ and $\chi(\xi) = 0$ for $|\xi| \geq 2$. Let*

$$(Q_M f)(\mathbf{r}_k) = \int e^{i\mathbf{r}_k\cdot\boldsymbol{\xi}_k} \prod_{j=1}^{k} \chi(M^{-1}\xi_j) \hat{f}(\boldsymbol{\xi}_k)\, d\boldsymbol{\xi}_k$$

With respect to the spectral decomposition of $L^2(\mathbb{R})$ corresponding to the operator $H_j = -\partial_{z_j}^2 + z_j^2$, let Z_M^j be the orthogonal projection onto the sum of the first M eigenspaces (in the z_j variable only). Let

$$R_M = \prod_{j=1}^{k} Z_M^j$$

1. *Suppose that J is a compact operator. Then $J_M \stackrel{\text{def}}{=} R_M Q_M J Q_M R_M \to J$ in the operator norm.*
2. *$H_j J_M$, $J_M H_j$, $\Delta_{r_j} J_M$ and $J_M \Delta_{r_j}$ are all bounded.*
3. *There exists a countable dense subset $\{T_i\}$ of the closed unit ball in the space of bounded operators on $L^2(\mathbb{R}^{3k})$ such that each T_i is compact and in fact for each i there exists M (depending on i) such that $T_i = R_M Q_M T_i Q_M R_M$.*

Proof. (1) If $S_n \to S$ strongly and $J \in \mathcal{K}_k$, then $S_n J \to SJ$ in the operator norm and $JS_n \to JS$ in the operator norm. (2) is straightforward. For (3), start with a subset $\{Y_n\}$ of the closed unit ball in the space of bounded operators on $L^2(\mathbb{R}^{3k})$ such that each Y_n is compact. Then let $\{T_i\}$ be an enumeration of the set $R_M Q_M Y_n Q_M R_M$ where M ranges over the dyadic integers. By (1) this collection will still be dense. □

Appendix B. Deducing Theorem 1.1 from Theorem 1.2

The argument presented here which deduces Theorem 1.1 from Theorem 1.2 has been used in all the nD to nD work. We refer the readers to them for more details. We first give the following proposition.

Proposition B.1. *Assume $\tilde{\psi}_{N,\omega}(0)$ satisfies (a), (b) and (c) in Theorem 1.1. Let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off such that $0 \leqslant \chi \leqslant 1$, $\chi(s) = 1$ for $0 \leqslant s \leqslant 1$ and $\chi(s) = 0$ for $s \geqslant 2$. For $\kappa > 0$, we define an approximation of $\tilde{\psi}_{N,\omega}(0)$ by*

$$\tilde{\psi}_{N,\omega}^{\kappa}(0) = \frac{\chi\left(\kappa\left(\tilde{H}_{N,\omega} - N\omega\right)/N\right)\tilde{\psi}_{N,\omega}(0)}{\left\|\chi\left(\kappa\left(\tilde{H}_{N,\omega} - N\omega\right)/N\right)\tilde{\psi}_{N,\omega}(0)\right\|}.$$

This approximation has the following properties:

(i) $\tilde{\psi}_{N,\omega}^{\kappa}(0)$ verifies the energy condition

$$\langle \tilde{\psi}_{N,\omega}^{\kappa}(0), (\tilde{H}_{N,\omega} - N\omega)^k \tilde{\psi}_{N,\omega}^{\kappa}(0)\rangle \leqslant \frac{2^k N^k}{\kappa^k}.$$

(ii)

$$\sup_{N,\omega}\left\|\tilde{\psi}_{N,\omega}(0) - \tilde{\psi}_{N,\omega}^{\kappa}(0)\right\|_{L^2} \leqslant C\kappa^{\frac{1}{2}}.$$

(iii) For small enough $\kappa > 0$, $\tilde{\psi}_{N,\omega}^{\kappa}(0)$ is asymptotically factorized as well

$$\lim_{N,\omega\to\infty} \operatorname{Tr}\left|\tilde{\gamma}_{N,\omega}^{\kappa,(1)}(0,x_1,z_1;x_1',z_1')-\phi_0(x_1)\overline{\phi_0}(x_1')h(z_1)h(z_1')\right|=0,$$

where $\tilde{\gamma}_{N,\omega}^{\kappa,(1)}(0)$ is the marginal density associated with $\tilde{\psi}_{N,\omega}^{\kappa}(0)$, and ϕ_0 is the same as in assumption (b) in Theorem 1.1.

Proof. Proposition B.1 follows the same proof as [26, Proposition 9.1] if one replaces H_N by $(\tilde{H}_{N,\omega}-N\omega)$ and $\hat{H}_N$ by

$$\sum_{j=2}^{N}(-\Delta_{x_j}+\omega(-1+-\partial_{z_j}^2+z_j^2))+\frac{1}{N}\sum_{1<i<j\leq N}V_{N,\omega}(r_i-r_j).$$

□

Via (i) and (iii) of Proposition 1.2, $\tilde{\psi}_{N,\omega}^{\kappa}(0)$ verifies the hypothesis of Theorem 1.2 for small enough $\kappa > 0$. Therefore, for $\tilde{\gamma}_{N,\omega}^{\kappa,(1)}(t)$, the marginal density associated with $e^{it\tilde{H}_{N,\omega}}\tilde{\psi}_{N,\omega}^{\kappa}(0)$, Theorem 1.2 gives the convergence

$$\lim_{\substack{N,\omega\to\infty\\ N\geqslant\omega^{v(\beta)+\varepsilon}}} \operatorname{Tr}\left|\tilde{\gamma}_{N,\omega}^{\kappa,(k)}(t,\mathbf{x}_k,\mathbf{z}_k;\mathbf{x}_k',\mathbf{z}_k')-\prod_{j=1}^{k}\phi(t,x_j)\overline{\phi}(t,x_j')h_1(z_j)h_1(z_j')\right|=0. \tag{B.1}$$

for all small enough $\kappa > 0$, all $k \geqslant 1$, and all $t\in\mathbb{R}$.

For $\tilde{\gamma}_{N,\omega}^{(k)}(t)$ in Theorem 1.1, we notice that, $\forall J^{(k)}\in\mathcal{K}_k$, $\forall t\in\mathbb{R}$, we have

$$\begin{aligned}
&\left|\operatorname{Tr} J^{(k)}\left(\tilde{\gamma}_{N,\omega}^{(k)}(t)-\left|\phi(t)\otimes h_1\right\rangle\left\langle\phi(t)\otimes h_1\right|^{\otimes k}\right)\right|\\
\leqslant\;&\left|\operatorname{Tr} J^{(k)}\left(\tilde{\gamma}_{N,\omega}^{(k)}(t)-\tilde{\gamma}_{N,\omega}^{\kappa,(k)}(t)\right)\right|+\left|\operatorname{Tr} J^{(k)}\left(\tilde{\gamma}_{N,\omega}^{\kappa,(k)}(t)-\left|\phi(t)\otimes h_1\right\rangle\left\langle\phi(t)\otimes h_1\right|^{\otimes k}\right)\right|\\
=\;&\mathrm{I}+\mathrm{II}.
\end{aligned}$$

Convergence (B.1) then takes care of II. To handle I , part (ii) of Proposition 1.2 yields

$$\left\|e^{it\tilde{H}_{N,\omega}}\tilde{\psi}_{N,\omega}(0)-e^{it\tilde{H}_{N,\omega}}\tilde{\psi}_{N,\omega}^{\kappa}(0)\right\|_{L^2}=\left\|\tilde{\psi}_{N,\omega}(0)-\tilde{\psi}_{N,\omega}^{\kappa}(0)\right\|_{L^2}\leqslant C\kappa^{\frac{1}{2}}$$

which implies

$$I=\left|\operatorname{Tr} J^{(k)}\left(\tilde{\gamma}_{N,\omega}^{(k)}(t)-\tilde{\gamma}_{N,\omega}^{\kappa,(k)}(t)\right)\right|\leqslant C\left\|J^{(k)}\right\|_{op}\kappa^{\frac{1}{2}}.$$

Since $\kappa > 0$ is arbitrary, we deduce that

$$\lim_{\substack{N,\omega\to\infty\\ N\geqslant\omega^{v(\beta)+\varepsilon}}}\left|\operatorname{Tr} J^{(k)}\left(\tilde{\gamma}_{N,\omega}^{(k)}(t)-\left|\phi(t)\otimes h_1\right\rangle\left\langle\phi(t)\otimes h_1\right|^{\otimes k}\right)\right|=0.$$

i.e. as trace class operators

$$\tilde{\gamma}_{N,\omega}^{(k)}(t)\to\left|\phi(t)\otimes h_1\right\rangle\left\langle\phi(t)\otimes h_1\right|^{\otimes k}\ \text{ weak*}.$$

Then again, the Grümm's convergence theorem upgrades the above weak* convergence to strong. Thence, we have concluded Theorem 1.1 via Theorem 1.2 and Proposition B.1.

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