ON THE KLAINERMAN-MACHEDON CONJECTURE OF THE QUANTUM BBGKY HIERARCHY WITH SELF-INTERACTION

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ABSTRACT. We consider the 3D quantum BBGKY hierarchy which corresponds to the N-particle Schrödinger equation. We assume the pair interaction is $N^{3\beta-1}V(N^{\beta}\bullet)$. For interaction parameter $\beta \in (0, \frac{2}{3})$, we prove that, as $N \to \infty$, the limit points of the solutions to the BBGKY hierarchy satisfy the space-time bound conjectured by Klainerman-Machedon [37] in 2008. This allows for the application of the Klainerman-Machedon uniqueness theorem, and hence implies that the limit is uniquely determined as a tensor product of solutions to the Gross-Pitaevski equation when the N-body initial data is factorized. The first result in this direction in 3D was obtained by T. Chen and N. Pavlović [11] for $\beta \in (0, \frac{1}{4})$ and subsequently by X. Chen [15] for $\beta \in (0, \frac{2}{7}]$. We build upon the approach of X. Chen but apply frequency localized Klainerman-Machedon collapsing estimates and the endpoint Strichartz estimate to extend the range to $\beta \in (0, \frac{2}{3})$. Overall, this provides an alternative approach to the mean-field program by Erdös-Schlein-Yau [23], whose uniqueness proof is based upon Feynman diagram combinatorics.

CONTENTS

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1. Introduction

The quantum BBGKY hierarchy refers to a sequence of trace class operator kernels $\left\{\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\right\}_{k=1}^{N}$, where $t\in\mathbb{R}$, $\mathbf{x}_k=(x_1,x_2,\cdots,x_k)\in\mathbb{R}^{3k}$, $\mathbf{x}_k'=(x_1',x_2',\cdots,x_k')\in\mathbb{R}^{3k}$, which are symmetric, in the sense that

$$\gamma_N^{(k)}(t,\mathbf{x}_k,\mathbf{x}_k')=\overline{\gamma_N^{(k)}(t,\mathbf{x}_k',\mathbf{x}_k)},$$

and

$$\gamma_N^{(k)}(t,x_{\sigma(1)},\cdots x_{\sigma(k)},x'_{\sigma(1)},\cdots x'_{\sigma(k)})=\gamma_N^{(k)}(t,x_1,\cdots x_k,x_1',\cdots x_k'), \tag{1.1}$$

for any permutation σ, and satisfy the quantum BBGKY linear hierarchy of equations which written in operator form is

$$\begin{aligned} i\partial_t\gamma_N^{(k)}+\left[\triangle_{\mathbf{x}_k},\gamma_N^{(k)}\right] &= \frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\left[V_N\left(x_i-x_j\right),\gamma_N^{(k)}\right] \\ &\quad+\frac{N-k}{N}\sum_{j=1}^{k}\operatorname{Tr}_{k+1}\left[V_N\left(x_j-x_{k+1}\right),\gamma_N^{(k+1)}\right] \end{aligned} \tag{1.2}$$

with prescribed initial conditions

$$\gamma_N^{(k)}(0,\mathbf{x}_k,\mathbf{x}_k')=\gamma_{N,0}^{(k)}(\mathbf{x}_k,\mathbf{x}_k').$$

Here $\triangle_{\mathbf{x}_k}$ denotes the standard Laplacian with respect to the variables $\mathbf{x}_k\in\mathbb{R}^{3k}$, the operator $V_N(x)$ represents multiplication by the function $V_N(x)$, where

$$V_N(x)=N^{3\beta}V(N^\beta x) \tag{1.3}$$

is an approximation to the Dirac δ function, and Tr_{k+1} means taking the $k+1$ trace, for example,

$$\operatorname{Tr}_{k+1}V_N(x_j-x_{k+1})\,\gamma_N^{(k+1)}=\int V_N(x_j-x_{k+1})\,\gamma_N^{(k+1)}(t,\mathbf{x}_k,x_{k+1};\mathbf{x}_k',x_{k+1})dx_{k+1}.$$

We devote this paper to proving the following theorem.

Theorem 1.1 (Main theorem)**.** *Assume the interaction parameter $\beta\in(0,2/3)$ and the pair interaction $V\in L^1\cap W^{2,\frac{6}{5}+}$. Suppose that the sequence $\left\{\gamma_N^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')\right\}_{k=1}^{N}$ is a solution to the quantum BBGKY hierarchy (1.2) subject to the energy condition: there is a C (independent of N and k) such that for any $k\geqslant 0$, there is a $N_0(k)$ such that*

$$\forall\, N\geqslant N_0(k)\,,\qquad \sup_{t\in\mathbb{R}}\left\|S^{(k)}\gamma_N^{(k)}\right\|_{L^2_{\mathbf{x},\mathbf{x}'}}\leqslant C^k \tag{1.4}$$

where $S^{(k)}=\prod_{j=1}^{k}\left(\left\langle\nabla_{x_j}\right\rangle\left\langle\nabla_{x_j'}\right\rangle\right)$. Then, for every finite time T, every limit point $\Gamma=\left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$ of $\left\{\Gamma_N\right\}_{N=1}^{\infty}=\left\{\left\{\gamma_N^{(k)}\right\}_{k=1}^{N}\right\}_{N=1}^{\infty}$ in $\bigoplus_{k\geqslant1}C\left([0,T],\mathcal{L}_k^1\right)$ with respect to the product *topology τ_{prod} (defined in Appendix A) satisfies the Klainerman-Machedon space-time bound: there is a C independent of j,k such that*

$$\int_0^T \left\| R^{(k)} B_{j,k+1} \gamma^{(k+1)}(t) \right\|_{L^2_{\mathbf{x},\mathbf{x}'}} dt \leqslant C^k, \tag{1.5}$$

where $\mathcal{L}_k^1$ is the space of trace class operators on $L^2(\mathbb{R}^{3k})$, $R^{(k)} = \prod_{j=1}^{k} \left(\left|\nabla_{x_j}\right| \left|\nabla_{x_j'}\right| \right)$, and

$$B_{j,k+1} = \operatorname{Tr}_{k+1} \left[\delta\left(x_j - x_{k+1}\right), \gamma^{(k+1)} \right].$$

In particular, this theorem establishes a positive answer to Conjecture 1 by Klainerman and Machedon in 2008 for $\beta \in (0, 2/3)$.

Conjecture 1 (Klainerman-Machedon [37])**.** *Under condition (1.4), for $\beta \in (0,1]$, every limit point $\Gamma = \left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$ of $\left\{\Gamma_N\right\}_{N=1}^{\infty}$ satisfies space-time bound (1.5).*

The quantum BBGKY hierarchy (1.2) is generated from the N-body Hamiltonian evolution $\psi_N(t) = e^{itH_N}\psi_N(0)$ with the N-body Hamiltonian

$$H_N = -\triangle_{\mathbf{x}_N} + \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} N^{3\beta} V(N^\beta (x_i - x_j)) \tag{1.6}$$

where the factor $1/N$ is to make sure that the interactions are proportional to the number of particles, and the pair interaction $N^{3\beta}V(N^\beta(x_i - x_j))$ is an approximation to the Dirac δ function which matches the Gross-Pitaevskii description of Bose-Einstein condensation that the many-body effect should be modeled by a strong on-site self-interaction. Since $\psi_N \overline{\psi_N}$ is a probability density, we define the marginal densities $\left\{\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k')\right\}_{k=1}^{N}$ by

$$\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k') = \int \psi_N(t, \mathbf{x}_k, \mathbf{x}_{N-k}) \overline{\psi_N}(t, \mathbf{x}_k', \mathbf{x}_{N-k}) d\mathbf{x}_{N-k}, \ \ \mathbf{x}_k, \mathbf{x}_k' \in \mathbb{R}^{3k}.$$

Then we have that $\left\{\gamma_N^{(k)}(t, \mathbf{x}_k; \mathbf{x}_k')\right\}_{k=1}^{N}$ satisfies the the quantum BBGKY hierarchy (1.2) if we do not distinguish $\gamma_N^{(k)}$ as a kernel and the operator it defines.[1]

Establishing the $N \to \infty$ limit of hierarchy (1.2) justifies the mean-field limit in the Gross-Pitaevskii theory. Such an approach was first proposed by Spohn [43] and can be regarded as a quantum version of Kac's program. We see that, as $N \to \infty$, hierarchy (1.2) formally converges to the infinite Gross-Pitaevskii hierarchy

$$i\partial_t \gamma^{(k)} + \left[\triangle_{\mathbf{x}_k}, \gamma^{(k)}\right] = \left(\int V(x)dx\right) \sum_{j=1}^{k} \operatorname{Tr}_{k+1}\left[\delta\left(x_j - x_{k+1}\right), \gamma^{(k+1)}\right]. \tag{1.7}$$

When the initial data is factorized

$$\gamma^{(k)}(0, \mathbf{x}_k; \mathbf{x}_k') = \prod_{j=1}^{k} \phi_0(x_j) \bar{\phi}_0(x_j),$$

[1] From here on out, we consider only the $\beta > 0$ case. For $\beta = 0$, see [21, 38, 40, 42, 30, 31, 13, 6] .

hierarchy (1.7) has a special solution

$$\gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}_k')=\prod_{j=1}^{k}\phi(t,x_j)\bar{\phi}(t,x_j), \tag{1.8}$$

if ϕ solves the cubic NLS

$$i\partial_t\phi=-\triangle_x\phi+\left(\int V(x)dx\right)|\phi|^2\phi. \tag{1.9}$$

Thus such a limit process shows that, in an appropriate sense,

$$\lim_{N\to\infty}\gamma_N^{(k)}=\prod_{j=1}^{k}\phi(t,x_j)\bar{\phi}(t,x_j),$$

hence justifies the mean-field limit.

Such a limit in 3D was first proved in a series of important papers [20, 22, 23, 24, 25] by Elgart, Erdös, Schlein, and Yau.[2] Briefly, the Elgart-Erdös-Schlein-Yau approach[3] can be described as the following:

Step A. Prove that, with respect to the topology τ_{prod} defined in Appendix A, the sequence $\{\Gamma_N\}_{N=1}^{\infty}$ is compact in the space $\bigoplus_{k\geqslant 1}C\left([0,T],\mathcal{L}^1\left(\mathbb{R}^{3k}\right)\right)$.

Step B. Prove that every limit point $\Gamma=\left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$ of $\{\Gamma_N\}_{N=1}^{\infty}$ must verify hierarchy (1.7).

Step C. Prove that, in the space in which the limit points from Step B lie, there is a unique solution to hierarchy (1.7). Thus $\{\Gamma_N\}_{N=1}^{\infty}$ is a compact sequence with only one limit point. Hence $\Gamma_N\to\Gamma$ as $N\to\infty$.

In 2007, Erdös, Schlein, and Yau obtained the first uniqueness theorem of solutions [23, Theorem 9.1] to the hierarchy (1.7). The proof is surprisingly delicate – it spans 63 pages and uses complicated Feynman diagram techniques. The main difficulty is that hierarchy (1.7) is a system of infinitely coupled equations. Briefly, [23, Theorem 9.1] is the following:

Theorem 1.2 (Erdös-Schlein-Yau uniqueness [23, Theorem 9.1])**.** *There is at most one nonnegative symmetric operator sequence* $\left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$ *that solves hierarchy* (1.7) *subject to the energy condition*

$$\sup_{t\in[0,T]}\operatorname{Tr}\left(\prod_{j=1}^{k}\left(1-\triangle_{x_j}\right)\right)\gamma^{(k)}\leqslant C^k. \tag{1.10}$$

In [37], based on their null form paper [36], Klainerman and Machedon gave a different proof of the uniqueness of hierarchy (1.7) in a space different from that used in [23, Theorem 9.1]. The proof is shorter (13 pages) than the proof of [23, Theorem 9.1]. Briefly, [37, Theorem 1.1] is the following:

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[2]Around the same time, there was the 1D work [1].

[3]See [5, 29, 41] for different approaches.

Theorem 1.3 (Klainerman-Machedon uniqueness [37, Theorem 1.1]). *There is at most one symmetric operator sequence* $\left\{\gamma^{(k)}\right\}_{k=1}^{\infty}$ *that solves hierarchy* (1.7) *subject to the space-time bound* (1.5).

For special cases like (1.8), condition (1.10) is actually

$$\sup_{t\in[0,T]} \left\Vert \left\langle \nabla_x\right\rangle \phi\right\Vert_{L^2} \leqslant C, \tag{1.11}$$

while condition (1.5) means

$$\int_0^T \left\Vert \left|\nabla_x\right| \left(\left|\phi\right|^2 \phi\right)\right\Vert_{L^2} dt \leqslant C. \tag{1.12}$$

When ϕ satisfies NLS (1.9), both are known. In fact, due to the Strichartz estimate [33], (1.11) implies (1.12), that is, condition (1.5) seems to be a bit weaker than condition (1.10). The proof of [37, Theorem 1.1] (13 pages) is also considerably shorter than the proof of [23, Theorem 9.1] (63 pages). It is then natural to wonder whether [37, Theorem 1.1] simplifies Step C. To answer such a question it is necessary to know whether the limit points in Step B satisfy condition (1.10), that is, whether Conjecture 1 holds.

Away from curiosity, there are realistic reasons to study Conjecture 1. While [23, Theorem 9.1] is a powerful theorem, it is very difficult to adapt such an argument to various other interesting and colorful settings: a different spatial dimension, a three-body interaction instead of a pair interaction, or the Hermite operator instead of the Laplacian. The last situation mentioned is physically important. On the one hand, all the known experiments of BEC use harmonic trapping to stabilize the condensate [2, 19, 7, 34, 44]. On the other hand, different trapping strength produces quantum behaviors which do not exist in the Boltzmann limit of classical particles nor in the quantum case when the trapping is missing and have been experimentally observed [26, 45, 18, 32, 17]. The Klainerman-Machedon approach applies easily in these meaningful situations ([35, 9, 14, 15, 16, 27]). Thus proving Conjecture 1 actually helps to advance the study of quantum many-body dynamic and the mean-field approximation in the sense that it provides a flexible and powerful tool in 3D.

The well-posedness theory of the Gross-Pitaevskii hierarchy (1.7) subject to general initial datum also requires that the limits of the BBGKY hierarchy (1.2) lie in the space in which the space-time bound (1.5) holds. See [8, 10, 11].

As pointed out in [20], the study of the Hamiltonian (1.6) is of particular interest when $\beta \in (1/3, 1]$. The reason is the following. In physics, the initial datum $\psi_N(0)$ of the Hamiltonian evolution $e^{itH_N}\psi_N(0)$ is usually assumed to be close to the ground state of the Hamiltonian

$$H_{N,0} = -\triangle_{\mathbf{x}_N} + \omega_0^2 \left|\mathbf{x}_N\right|^2 + \frac{1}{N}\sum_{1\leqslant i<j\leqslant N} N^{3\beta}V(N^{\beta}(x_i - x_j)).$$

The preparation of the available experiments and the mathematical work [39] by Lieb, Seiringer, Solovej and Yngvason confirm this assumption. Such an initial datum $\psi_N(0)$ is localized in space. We can assume all N particles are in a box of length 1. Let the effective radius of the pair interaction V be a, then the effective radius of V_N is about

a/N^{β}. Thus every particle in the box interacts with $\left(a/N^{\beta}\right)^3 \times N$ other particles. Thus, for $\beta > 1/3$ and large N, every particle interacts with only itself. This exactly matches the Gross-Pitaevskii theory that the many-body effect should be modeled by a strong on-site self-interaction. Therefore, for the mathematical justification of the Gross-Pitaevskii theory, it is of particular interest to prove Conjecture 1 for self-interaction ($\beta > 1/3$) as well.

To the best of our knowledge, the main theorem (Theorem 1.1) in the current paper is the first result in proving Conjecture 1 for self-interaction ($\beta > 1/3$). For $\beta \leqslant 1/3$, the first progress of Conjecture 1 is the $\beta \in (0, 1/4)$ work [11] by T. Chen and N. Pavlović and then the $\beta \in (0, 2/7]$ work [15] by X.C. As a matter of fact, the main theorem (Theorem 1.1) in the current paper has already fulfilled the original intent of [37], namely, simplifying the uniqueness argument of [23], because [23] deals with $\beta \in (0, 3/5)$. Conjecture 1 for $\beta \in [2/3, 1]$ is still open.

1.1. **Organization of the paper.** In §2, we outline the proof of Theorem 1.1. The overall pattern follows that introduced by X.C.[15], who obtained Theorem 1.1 for $\beta \in (0, \frac{2}{7}]$. Let $P_{\leq M}^{(k)}$ be the Littlewood-Paley projection defined in (2.1). Theorem 1.1 will follow once it is established that for all $M \geq 1$, there exists N_0 depending on M such that for all $N \geq N_0$, there holds

$$\|P_{\leq M}^{(k)} R^{(k)} B_{N,j,k+1} \gamma_N^{(k+1)}(t)\|_{L_T^1 L^2_{\mathbf{x},\mathbf{x}'}} \leqslant C^k \tag{1.13}$$

where $B_{N,j,k+1}$ is defined by (2.3). Substituting the Duhamel-Born expansion, carried out to coupling level K, of the BBGKY hierarchy, this is reduced to proving analogous bounds on the free part, potential part, and interaction part, defined in §2. Each part is reduced via the Klainerman-Machedon board game. Estimates for the free part and interaction part were previously obtained by X.C. [15]. For the estimate of the interaction part, one takes $K = \ln N$, the utility of which was first observed by T. Chen and N. Pavlović [11].

The main new achievement of our paper is the improved estimates on the potential part, which are discussed in §3. We make use of the endpoint Strichartz estimate in place of the Sobolev inequality employed by X.C [15]. The Strichartz estimate is phrased in terms of X_b norms. We also introduce frequency localized versions of the Klainerman-Machedon collapsing estimates, allowing us to exploit the frequency localization in (1.13). Specifically, the operator $P_{\leq M}^{(k)}$ does not commute with $B_{N,j,k+1}$, however, the composition $P_{\leq M_k}^{(k)} B_{N,j,k+1} P_{\sim M_{k+1}}^{(k+1)}$ enjoys better bounds if $M_{k+1} \gg M_k$. We prove the Strichartz estimate and the frequency localized Klainerman-Machedon collapsing estimates in §4. Frequency localized space-time techniques of this type were introduced by Bourgain [4, Chapter IV, §3] into the study of the well-posedness for nonlinear Schrödinger equations and other nonlinear dispersive PDE.

In X.C. [15], (1.13) is obtained without the frequency localization $P_{\leq M}^{(k)}$ for $\beta \in (0, \frac{2}{7}]$. In Theorem 3.2, we prove that this estimate still holds without frequency localization for $\beta \in (0, \frac{2}{5})$ by using the Strichartz estimate alone. This already surpasses the self-interaction threshold $\beta = \frac{1}{3}$. For the purpose of proving Conjecture 1, the frequency localized estimate (1.13) is equally good, but allows us to achieve higher β.

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2. Proof of the Main Theorem

We establish Theorem 1.1 in this section. For simplicity of notation, we denote $\|\cdot\|_{L^p[0,T]L^2_{\mathbf{x},\mathbf{x}'}}$ by $\|\cdot\|_{L^p_T L^2_{\mathbf{x},\mathbf{x}'}}$ and denote $\|\cdot\|_{L^p_t(\mathbb{R})L^2_{\mathbf{x},\mathbf{x}'}}$ by $\|\cdot\|_{L^p_t L^2_{\mathbf{x},\mathbf{x}'}}$. Let us begin by introducing some notation for Littlewood-Paley theory. Let $P^i_{\leq M}$ be the projection onto frequencies $\leq M$ and P^i_M the analogous projections onto frequencies $\sim M$, acting on functions of $x_i \in \mathbb{R}^3$ (the ith coordinate). We take M to be a dyadic frequency range $2^\ell \geq 1$. Similarly, we define $P^{i'}_{\leq M}$ and $P^{i'}_M$, which act on the variable x'_i. Let

$$P^{(k)}_{\leq M} = \prod_{i=1}^{k} P^i_{\leq M} P^{i'}_{\leq M}. \tag{2.1}$$

To establish Theorem 1.1, it suffices to prove the following theorem.

Theorem 2.1. *Under the assumptions of Theorem 1.1, there exists a C (independent of k, M, N) such that for each $M \geq 1$ there exists N_0 (depending on M) such that for $N \geqslant N_0$, there holds*

$$\|P^{(k)}_{\leq M} R^{(k)} B_{N,j,k+1} \gamma^{(k+1)}_N(t)\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \leqslant C^k \tag{2.2}$$

where

$$B_{N,j,k+1}\gamma^{(k+1)}_N = \mathrm{Tr}_{k+1}\left[V_N\left(x_j - x_{k+1}\right), \gamma^{(k+1)}_N\right]. \tag{2.3}$$

We first explain how, assuming Theorem 2.1, we can prove Theorem 1.1. Passing to the weak* limit $\gamma^{(k)}_N \to \gamma^{(k)}$ as $N \to \infty$, we obtain

$$\|P^{(k)}_{\leq M} R^{(k)} B_{j,k+1} \gamma^{(k+1)}\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \leqslant C^k$$

Since this holds uniformly in M, we can send $M \to \infty$ and, by the monotone convergence theorem, we obtain

$$\|R^{(k)} B_{j,k+1} \gamma^{(k+1)}\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \leqslant C^k$$

which is exactly the Klainerman-Machedon space-time bound (1.5). This completes the proof Theorem 1.1, assuming Theorem 2.1.

The rest of this paper is devoted to proving Theorem 2.1. Without loss of generality, we prove estimate (2.2) for $k = 1$, that is

$$\|P^{(1)}_{\leq M} R^{(1)} B_{N,1,2} \gamma^{(2)}_N\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \leqslant C \tag{2.4}$$

for $N \geqslant N_0(M)$. We are going to establish estimate (2.4) for a sufficiently small T which depends on the controlling constant in condition (1.4) and is independent of N and M, then a bootstrap argument together with condition (1.4) give estimate (2.4) for every finite time at the price of a larger constant C.

We start by rewriting hierarchy (1.2) as

$$\begin{aligned} \gamma_N^{(k)}(t_k) &= U^{(k)}(t_k)\gamma_{N,0}^{(k)} + \int_0^{t_k} U^{(k)}(t_k - t_{k+1})V_N^{(k)}\gamma_N^{(k)}(t_{k+1})dt_{k+1} \\ &\quad + \frac{N-k}{N}\int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}\gamma_N^{(k+1)}(t_{k+1})dt_{k+1} \end{aligned} \tag{2.5}$$

with the short-hand notation:

$$\begin{aligned} U^{(k)} &= e^{it\triangle_{\mathbf{x}_k}}e^{-it\triangle_{\mathbf{x}'_k}}, \\ V_N^{(k)}\gamma_N^{(k)} &= \frac{1}{N}\sum_{1\leqslant i<j\leqslant k}\left[V_N(x_i - x_j), \gamma_N^{(k)}\right] \\ B_N^{(k+1)}\gamma_N^{(k+1)} &= \sum_{j=1}^{k} B_{N,j,k+1}\gamma_N^{(k+1)}. \end{aligned}$$

We omit the i in front of the potential term and the interaction term so that we do not need to keep track of its exact power.

Writing out the kth Duhamel-Born series of $\gamma_N^{(2)}$ by iterating hierarchy (2.5) k times, we have

$$\begin{aligned} \gamma_N^{(2)}(t_2) &= U^{(2)}(t_2)\gamma_{N,0}^{(2)} + \int_0^{t_2} U^{(2)}(t_2 - t_3)V_N^{(2)}\gamma_N^{(2)}(t_3)dt_3 \\ &\quad + \frac{N-2}{N}\int_0^{t_2} U^{(2)}(t_2 - t_3)B_N^{(3)}\gamma_N^{(3)}(t_3)dt_3 \\ &= U^{(2)}(t_2)\gamma_{N,0}^{(2)} + \frac{N-2}{N}\int_0^{t_2} U^{(2)}(t_2 - t_3)B_N^{(3)}U^{(3)}(t_3)\gamma_{N,0}^{(3)}dt_3 \\ &\quad + \int_0^{t_2} U^{(2)}(t_2 - t_3)V_N^{(2)}\gamma_N^{(2)}(t_3)dt_3 \\ &\quad + \frac{N-2}{N}\int_0^{t_2} U^{(2)}(t_2 - t_3)B_N^{(3)}\int_0^{t_3} U^{(3)}(t_3 - t_4)V_N^{(3)}\gamma_N^{(3)}(t_4)dt_4dt_3 \\ &\quad + \frac{N-2}{N}\frac{N-3}{N}\int_0^{t_2} U^{(2)}(t_2 - t_3)B_N^{(3)}\int_0^{t_3} U^{(3)}(t_3 - t_4)B_N^{(4)}\gamma_N^{(4)}(t_4)dt_4dt_3 \\ &= \ldots \end{aligned}$$

After k iterations[4]

$$\gamma_N^{(2)}(t_2) = FP^{(k)}(t_2) + PP^{(k)}(t_2) + IP^{(k)}(t_2) \tag{2.6}$$

[4]Henceforth, the k's appearing in our formulas are the coupling level which is distinct from the k in the statement of Theorem 2.1 (which has been fixed at $k = 1$).

where the *free part* at coupling level k is given by

$$\begin{aligned} FP^{(k)} &= U^{(2)}(t_2)\gamma_{N,0}^{(2)} + \\ &\sum_{j=3}^{k}\left(\prod_{l=3}^{j}\frac{N+1-l}{N}\right)\int_0^{t_2}\cdots\int_0^{t_{j-1}} U^{(2)}(t_2-t_3)B_N^{(3)}\cdots U^{(j-1)}(t_{j-1}-t_j)B_N^{(j)} \\ &\times\left(U^{(j)}(t_j)\gamma_{N,0}^{(j)}\right)dt_3\cdots dt_j, \end{aligned}$$

the *potential part* is given by

$$\begin{aligned} PP^{(k)} &= \int_0^{t_2} U^{(2)}(t_2-t_3)V_N^{(2)}\gamma_N^{(2)}(t_3)dt_3 + \sum_{j=3}^{k}\left(\prod_{l=3}^{j}\frac{N+1-l}{N}\right) \\ &\times\int_0^{t_2}\cdots\int_0^{t_{j-1}} U^{(2)}(t_2-t_3)B_N^{(3)}\cdots U^{(j-1)}(t_{j-1}-t_j)B_N^{(j)} \\ &\times\left(\int_0^{t_j} U^{(j)}(t_j-t_{j+1})V_N^{(j)}\gamma_N^{(j)}(t_{j+1})dt_{j+1}\right)dt_3\cdots dt_j, \end{aligned}$$

and the *interaction part* is given by

$$\begin{aligned} IP^{(k)} &= \left(\prod_{l=3}^{k}\frac{N+1-l}{N}\right)\int_0^{t_2}\cdots\int_0^{t_k} U^{(2)}(t_2-t_3)B_N^{(3)}\cdots \\ &\cdots U^{(k)}(t_k-t_{k+1})B_N^{(k+1)}\left(\gamma_N^{(k+1)}(t_{k+1})\right)dt_3\cdots dt_{k+1}. \end{aligned}$$

By (2.6), to establish (2.4), it suffices to prove

$$\left\|P_{\leq M}^{(1)}R^{(1)}B_{N,1,2}FP^{(k)}\right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C \tag{2.7}$$

$$\left\|P_{\leq M}^{(1)}R^{(1)}B_{N,1,2}PP^{(k)}\right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C \tag{2.8}$$

$$\left\|P_{\leq M}^{(1)}R^{(1)}B_{N,1,2}IP^{(k)}\right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C \tag{2.9}$$

for some C and a sufficiently small T determined by the controlling constant in condition (1.4) and independent of N and M. We observe that $B_N^{(j)}$ has $2j$ terms inside so that each summand of $\gamma_N^{(2)}(t_2)$ contains factorially many terms ($\sim k!$). We use the Klainerman-Machedon board game to combine them and hence reduce the number of terms that need to be treated. Define

$$J_N(\underline{t}_{j+1})(f^{(j+1)}) = U^{(2)}(t_2-t_3)B_N^{(3)}\cdots U^{(j)}(t_j-t_{j+1})B_N^{(j+1)}f^{(j+1)},$$

where $\underline{t}_{j+1}$ means $(t_3,\ldots,t_{j+1})$, then the Klainerman-Machedon board game implies the lemma.

Lemma 2.1 (Klainerman-Machedon board game). *One can express*

$$\int_0^{t_2}\cdots\int_0^{t_j} J_N(\underline{t}_{j+1})(f^{(j+1)})d\underline{t}_{j+1}$$

as a sum of at most 4^{j-1} *terms of the form*

$$\int_D J_N(\underline{t}_{j+1},\mu_m)(f^{(j+1)})d\underline{t}_{j+1},$$

or in other words,

$$\int_0^{t_2}\cdots\int_0^{t_j} J_N(\underline{t}_{j+1})(f^{(j+1)})d\underline{t}_{j+1}=\sum_m\int_D J_N(\underline{t}_{j+1},\mu_m)(f^{(j+1)})d\underline{t}_{j+1}.$$

Here $D\subset[0,t_2]^{j-1}$*,* μ_m *are a set of maps from* $\{3,\ldots,j+1\}$ *to* $\{2,\ldots,j\}$ *satisfying* $\mu_m(3)=2$ *and* $\mu_m(l)<l$ *for all* l*, and*

$$\begin{aligned}J_N(\underline{t}_{j+1},\mu_m)(f^{(j+1)}) &= U^{(2)}(t_2-t_3)B_{N,2,3}U^{(3)}(t_3-t_4)B_{N,\mu_m(4),4}\cdots\\ &\quad\cdots U^{(j)}(t_j-t_{j+1})B_{N,\mu_m(j+1),j+1}(f^{(j+1)}).\end{aligned}$$

Proof. Lemma 2.1 follows the exact same proof as [37, Theorem 3.4], the Klainerman-Machedon board game, if one replaces $B_{j,k+1}$ by $B_{N,j,k+1}$ and notices that $B_{N,j,k+1}$ still commutes with $e^{it\triangle_{x_i}}e^{-it\triangle_{x_i}}$ whenever $i\neq j$. This argument reduces the number of terms by combining them. ∎

In the rest of this paper, we establish estimate (2.8) only. The reason is the following. On the one hand, the proof of estimate (2.8) is exactly the place that relies on the restriction $\beta\in(0,2/3)$ in this paper. On the other hand, X.C. has already proven estimates (2.7) and (2.9) as estimates (6.3) and (6.5) in [15] without using any frequency localization. For completeness, we include a proof of estimates (2.7) and (2.9) in Appendix B. Before we delve into the proof of estimate (2.8), we remark that the proof of estimates (2.7) and (2.9) is independent of the coupling level k and we will take the coupling level k to be $\ln N$ for estimate (2.9).[5]

3. Estimate of the Potential Part

In this section, we prove estimate (2.8). To be specific, we establish the following theorem.

Theorem 3.1. *Under the assumptions of Theorem 1.1, there exists a* C *(independent of* k,M_1,N*) such that for each* $M_1\geqslant 1$ *there exists* N_0 *(depending on* M_1*) such that for* $N\geqslant N_0$*, there holds*

$$\left\Vert P_{\leq M_1}^{(1)}R^{(1)}B_{N,1,2}PP^{(k)}\right\Vert_{L_T^1L^2_{\mathbf{x},\mathbf{x}'}}\leqslant C$$

where $PP^{(k)}$ *is given by (2.7).*

In this section, we will employ the estimates stated and proved in Section 4. Due to the technicality of the proof of Theorem 3.1 involving Littlewood-Paley theory, we prove a simpler $\beta\in(0,\frac{2}{5})$ version first to illustrate the basic steps in establishing Theorem 3.1. We then prove Theorem 3.1 in Section 3.2.

[5]The technique of taking $k=\ln N$ for estimate (2.9) was first observed by T.Chen and N.Pavlović [11].

3.1. The simpler $\beta \in (0, \frac{2}{5})$ case.

Theorem 3.2. *For $\beta \in (0, \frac{2}{5})$, we have the estimate*

$$\left\| R^{(1)} B_{N,1,2} PP^{(k)} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \leqslant C$$

for some C and a sufficiently small T determined by the controlling constant in condition (1.4) and independent of N.

Proof. The proof is divided into four steps. We will reproduce every step for Theorem 3.1 in Section 3.2.

Step I. By Lemma 2.1, we know that

$$\begin{aligned} PP^{(k)} &= \int_0^{t_2} U^{(2)}(t_2 - t_3) V_N^{(2)} \gamma_N^{(2)}(t_3) dt_3 \\ &+ \sum_{j=3}^{k} \left(\prod_{l=3}^{j} \frac{N+1-l}{N} \right) \\ &\qquad \times \left(\sum_m \int_D J_N(\underline{t}_j, \mu_m) \left(\int_0^{t_j} U^{(j)}(t_j - t_{j+1}) V_N^{(j)} \gamma_N^{(j)}(t_{j+1}) dt_{j+1} \right) d\underline{t}_j \right) \end{aligned} \tag{3.1}$$

where $\sum_m$ has at most 4^{j-3} terms inside.

For the second term, we iterate Lemma 4.2 to prove the following estimate[6]:

$$\begin{aligned} &\left\| R^{(1)} B_{N,1,2} \int_D J_N(\underline{t}_{j+1}, \mu_m)(f^{(j+1)}) d\underline{t}_{j+1} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \\ &= \int_0^T \left\| \int_D R^{(1)} B_{N,1,2} U^{(2)}(t_2 - t_3) B_{N,2,3} \cdots dt_3 \ldots dt_{j+1} \right\|_{L^2_{\mathbf{x},\mathbf{x}'}} dt_2 \\ &\leqslant \int_{[0,T]^j} \left\| R^{(1)} B_{N,1,2} U^{(2)}(t_2 - t_3) B_{N,2,3} \cdots \right\|_{L^2} dt_2 dt_3 ... dt_{j+1} \\ &\leqslant T^{\frac{1}{2}} \int_{[0,T]^{j-1}} \left(\int \left\| R^{(1)} B_{N,1,2} U^{(2)}(t_2 - t_3) B_{N,2,3} ... \right\|_{L^2}^2 dt_2 \right)^{\frac{1}{2}} dt_3 ... dt_{j+1} \\ &\quad \text{(Cauchy-Schwarz)} \\ &\leqslant CT^{\frac{1}{2}} \int_{[0,T]^{j-1}} \left\| R^{(2)} B_{N,2,3} U^{(3)}(t_3 - t_4) ... \right\| dt_3 ... dt_{j+1} \quad \text{(Lemma 4.2)} \\ &\quad \text{(Iterate } j-2 \text{ times)} \\ &\quad ... \\ &\leqslant (CT^{\frac{1}{2}})^{j-1} \left\| R^{(j)} B_{N,\mu_m(j+1),j+1} f^{(j+1)} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \end{aligned} \tag{3.2}$$

[6]This also helps in proving estimates (2.7) and (2.9)–see Appendix B

Applying relation (3.2), we have

$$\begin{aligned}
&\left\|R^{(1)}B_{N,1,2}PP^{(k)}\right\|_{L_T^1L^2_{\mathbf{x},\mathbf{x}'}}\\
\leqslant\ &\left\|R^{(1)}B_{N,1,2}\int_0^{t_2}U^{(2)}(t_2-t_3)V_N^{(2)}\gamma_N^{(2)}(t_3)dt_3\right\|_{L_T^1L^2_{\mathbf{x},\mathbf{x}'}}\\
&+\sum_{j=3}^k 4^{j-3}(CT^{\frac{1}{2}})^{j-2}\left\|R^{(j-1)}B_{N,\mu_m(j),j}\left(\int_0^{t_j}U^{(j)}(t_j-t_{j+1})V_N^{(j)}\gamma_N^{(j)}(t_{j+1})dt_{j+1}\right)\right\|_{L_T^1L^2_{\mathbf{x},\mathbf{x}'}}\\
\leqslant\ &\left\|R^{(1)}B_{N,1,2}\int_0^{t_2}U^{(2)}(t_2-t_3)V_N^{(2)}\gamma_N^{(2)}(t_3)dt_3\right\|_{L_T^1L^2_{\mathbf{x},\mathbf{x}'}}\\
&+\sum_{j=3}^k (CT^{\frac{1}{2}})^{j-2}\left\|R^{(j-1)}B_{N,\mu_m(j),j}\left(\int_0^{t_j}U^{(j)}(t_j-t_{j+1})V_N^{(j)}\gamma_N^{(j)}(t_{j+1})dt_{j+1}\right)\right\|_{L_T^1L^2_{\mathbf{x},\mathbf{x}'}}.
\end{aligned}$$

Inserting a smooth cut-off $\theta(t)$ with $\theta(t)=1$ for $t\in[-T,T]$ and $\theta(t)=0$ for $t\in[-2T,2T]^c$ into the above estimate, we get

$$\begin{aligned}
&\left\|R^{(1)}B_{N,1,2}PP^{(k)}\right\|_{L_T^1L^2_{\mathbf{x},\mathbf{x}'}}\\
\leqslant\ &\left\|R^{(1)}B_{N,1,2}\theta(t_2)\int_0^{t_2}U^{(2)}(t_2-t_3)\theta(t_3)V_N^{(2)}\gamma_N^{(2)}(t_3)dt_3\right\|_{L_T^1{L^2}_{\mathbf{x},\mathbf{x}'}}\\
&+\sum_{j=3}^k (CT^{\frac{1}{2}})^{j-2}\left\|R^{(j-1)}B_{N,\mu_m(j),j}\theta(t_j)\left(\int_0^{t_j}U^{(j)}(t_j-t_{j+1})\theta(t_{j+1})V_N^{(j)}\gamma_N^{(j)}(t_{j+1})dt_{j+1}\right)\right\|_{L_T^1L^2_{\mathbf{x},\mathbf{x}'}}
\end{aligned}$$

Step II. The X_b space version of Lemma 4.2, Lemma 4.3, then turns the last step into

$$\begin{aligned}
&\left\|R^{(1)}B_{N,1,2}PP^{(k)}\right\|_{L_T^1L^2_{\mathbf{x},\mathbf{x}'}}\\
\leqslant\ &C\|\theta(t_2)\int_0^{t_2}U^{(2)}(t_2-t_3)R^{(2)}\left(\theta(t_3)V_N^{(2)}\gamma_N^{(2)}(t_3)\right)dt_3\|_{X^{(2)}_{\frac{1}{2}+}}\\
&+C\sum_{j=3}^k (CT^{\frac{1}{2}})^{j-2}\|\theta(t_j)\int_0^{t_j}U^{(j)}(t_j-t_{j+1})R^{(j)}\left(\theta(t_{j+1})V_N^{(j)}\gamma_N^{(j)}(t_{j+1})\right)dt_{j+1}\|_{X^{(j)}_{\frac{1}{2}+}}
\end{aligned}$$

Step III. We then proceed with Lemma 4.1 to get

$$\begin{aligned}
&\left\|R^{(1)}B_{N,1,2}PP^{(k)}\right\|_{L_T^1L^2_{\mathbf{x},\mathbf{x}'}}\\
\leqslant\ &C\|R^{(2)}\left(\theta(t_3)V_N^{(2)}\gamma_N^{(2)}(t_3)\right)\|_{X^{(2)}_{-\frac{1}{2}+}}+C\sum_{j=3}^k (CT^{\frac{1}{2}})^{j-2}\|R^{(j)}\left(\theta(t_{j+1})V_N^{(j)}\gamma_N^{(j)}(t_{j+1})\right)\|_{X^{(j)}_{-\frac{1}{2}+}}.
\end{aligned}$$

Step IV. Now we would like to utilize Lemma 4.6. We first analyse a typical term to demonstrated the effect of Lemma 4.6. To be specific, we have

$$\begin{aligned}
& \|R^{(2)}\left(\theta(t_3)V_N^{(2)}\gamma_N^{(2)}(t_3)\right)\|_{X^{(2)}_{-\frac{1}{2}+}} \\
\leqslant\ & \frac{C}{N}\|V_N(x_1-x_2)\theta(t_3)R^{(2)}\gamma_N^{(2)}(t_3)\|_{X^{(2)}_{-\frac{1}{2}+}} \\
& +\frac{C}{N}\|\left(V_N\right)'(x_1-x_2)\theta(t_3)\left|\nabla_{x_2}\right|\left|\nabla_{x_1'}\right|\left|\nabla_{x_2'}\right|\gamma_N^{(2)}(t_3)\|_{X^{(2)}_{-\frac{1}{2}+}} \\
& +\frac{C}{N}\|\left(V_N\right)''(x_1-x_2)\theta(t_3)\left|\nabla_{x_1'}\right|\left|\nabla_{x_2'}\right|\gamma_N^{(2)}(t_3)\|_{X^{(2)}_{-\frac{1}{2}+}} \\
\leqslant\ & \frac{C}{N}\left\|V_N\right\|_{L^{3+}}\|\theta(t_3)R^{(2)}\gamma_N^{(2)}\|_{L^2_{t_3}L^2_{x,x'}} \\
& +\frac{C}{N}\left\|V_N'\right\|_{L^{2+}}\|\theta(t_3)\left\langle\nabla_{x_1}\right\rangle^{\frac{1}{2}}\left\langle\nabla_{x_2}\right\rangle\left|\nabla_{x_1'}\right|\left|\nabla_{x_2'}\right|\gamma_N^{(2)}\|_{L^2_{t_3}L^2_{x,x'}} \\
& +\frac{C}{N}\left\|V_N''\right\|_{L^{\frac{6}{5}+}}\|\theta(t_3)S^{(2)}\gamma_N^{(2)}\|_{L^2_{t_3}L^2_{x,x'}} \\
\leqslant\ & C\|S^{(2)}\gamma_N^{(2)}\|_{L^2_{2T}L^2_{\mathbf{x},\mathbf{x}'}}
\end{aligned}$$

since $\|V_N/N\|_{L^{3+}}$ $\|V_N'/N\|_{L^{2+}}$, and $\|V_N''/N\|_{L^{\frac{6}{5}+}}$ are uniformly bounded in N for $\beta\in(0,\frac{2}{5})$. In fact,

$$\begin{aligned}
\|V_N/N\|_{L^{3+}} &\leqslant N^{2\beta-1}\left\|V\right\|_{L^{3+}} \\
\|V_N'/N\|_{L^{2+}} &\leqslant N^{\frac{5\beta}{2}-1}\left\|V'\right\|_{L^{2+}} \\
\|V_N''/N\|_{L^{\frac{6}{5}+}} &\leqslant N^{\frac{5\beta}{2}-1}\left\|V''\right\|_{L^{\frac{6}{5}+}}
\end{aligned}$$

where by Sobolev, $V\in W^{2,\frac{6}{5}+}$ implies $V\in L^{\frac{6}{5}+}\cap L^{6+}$ and $V'\in L^{2+}$.

Using the same idea for all the terms, we end up with

$$\begin{aligned}
& \left\|R^{(1)}B_{N,1,2}PP^{(k)}\right\|_{L^1_TL^2_{\mathbf{x},\mathbf{x}'}} \\
\leqslant\ & CT^{\frac{1}{2}}\|S^{(2)}\gamma_N^{(2)}\|_{L^\infty_{2T}L^2_{\mathbf{x},\mathbf{x}'}}+CT^{\frac{1}{2}}\sum_{j=3}^{k}(CT^{\frac{1}{2}})^{j-2}j^2\|S^{(j)}\gamma_N^{(j)}\|_{L^\infty_{2T}L^2_{\mathbf{x},\mathbf{x}'}} \\
& (j^2\text{ terms inside }V_N^{(j)}) \\
\leqslant\ & CT^{\frac{1}{2}}C^2+CT^{\frac{1}{2}}\sum_{j=3}^{\infty}(CT^{\frac{1}{2}})^{j-2}C^j\text{ (Condition (1.4))} \\
\leqslant\ & C<\infty.
\end{aligned}$$

This concludes the proof of Theorem 3.2. ∎

3.2. **Proof of Theorem 3.1.** To make formulas shorter, let us write

$$R^{(k)}_{\leqslant M_k}=P^{(k)}_{\leqslant M_k}R^{(k)},$$

since $P^{(k)}_{\leqslant M_k}$ and $R^{(k)}$ are usually bundled together.

3.2.1. *Step I.* By (3.1),

$$\left\|R_{\leqslant M_1}^{(1)}B_{N,1,2}PP^{(k)}\right\|_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\leqslant\left\|R_{\leqslant M_1}^{(1)}B_{N,1,2}\int_0^{t_2}U^{(2)}(t_2-t_3)V_N^{(2)}\gamma_N^{(2)}(t_3)dt_3\right\|_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}+\sum_{j=3}^{k}\sum_{m}\left\|R_{\leqslant M_1}^{(1)}B_{N,1,2}\int_D J_N(\underline{t}_j,\mu_m)(f^{(j)})d\underline{t}_j\right\|_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}$$

where

$$f^{(j)}=\int_0^{t_j}U^{(j)}(t_j-t_{j+1})V_N^{(j)}\gamma_N^{(j)}(t_{j+1})dt_{j+1}$$

where $\sum_m$ has at most 4^{j-3} terms inside. By Minkowski's integral inequality,

$$\begin{aligned}&\left\|R_{\leqslant M_1}^{(1)}B_{N,1,2}\int_D J_N(\underline{t}_j,\mu_m)(f^{(j)})d\underline{t}_j\right\|_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\\&=\int_0^T\left\|\int_D R_{\leqslant M_1}^{(1)}B_{N,1,2}U^{(2)}(t_2-t_3)B_{N,2,3}\cdots dt_3\dots dt_j\right\|_{L_{\mathbf{x},\mathbf{x}'}^2}dt_2\\&\leqslant\int_{[0,T]^{j-1}}\left\|R_{\leqslant M_1}^{(1)}B_{N,1,2}U^{(2)}(t_2-t_3)B_{N,2,3}\cdots\right\|_{L^2}dt_2dt_3\dots dt_j\end{aligned}$$

By Cauchy-Schwarz in the t_2 integration,

$$\leqslant T^{\frac{1}{2}}\int_{[0,T]^{j-1}}\left(\int\left\|R_{\leqslant M_1}^{(1)}B_{N,1,2}U^{(2)}(t_2-t_3)B_{N,2,3}\cdots\right\|_{L^2}^2dt_2\right)^{\frac{1}{2}}dt_3\dots dt_j$$

By Lemma 4.4,

$$\leqslant C_\varepsilon T^{\frac{1}{2}}\sum_{M_2\geqslant M_1}\left(\frac{M_1}{M_2}\right)^{1-\varepsilon}\int_{[0,T]^{j-1}}\left\|R_{\leqslant M_2}^{(2)}B_{N,2,3}U^{(3)}(t_3-t_4)\cdots\right\|dt_3\dots dt_j$$

Iterating the previous step $(j-3)$ times,

$$\begin{aligned}&\leqslant(C_\varepsilon T^{\frac{1}{2}})^{j-2}\sum_{M_{j-1}\geqslant\cdots\geqslant M_2\geqslant M_1}\left(\frac{M_1}{M_2}\frac{M_2}{M_3}\cdots\frac{M_{j-2}}{M_{j-1}}\right)^{1-\varepsilon}\left\|R_{\leqslant M_{j-1}}^{(j-1)}B_{N,\mu_m(j),j}f^{(j)}\right\|_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\\&=(C_\varepsilon T^{\frac{1}{2}})^{j-2}\sum_{M_{j-1}\geqslant\cdots\geqslant M_2\geqslant M_1}\left(\frac{M_1}{M_{j-1}}\right)^{1-\varepsilon}\left\|R_{\leqslant M_{j-1}}^{(j-1)}B_{N,\mu_m(j),j}f^{(j)}\right\|_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\end{aligned}$$

where the sum is over all $M_2,\dots,M_{j-1}$ dyadic such that $M_{j-1}\geqslant\cdots\geqslant M_2\geqslant M_1$.

Hence

$$\begin{aligned}&\left\|R_{\leqslant M_1}^{(1)}B_{N,1,2}PP^{(k)}\right\|_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\\\leqslant&\left\|R_{\leqslant M_1}^{(1)}B_{N,1,2}\left(\int_0^{t_2}U^{(2)}(t_2-t_3)V_N^{(2)}\gamma_N^{(2)}(t_3)dt_3\right)\right\|_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\\&+\sum_{j=3}^{k}(C_\varepsilon T^{\frac{1}{2}})^{j-2}\sum_{M_{j-1}\geqslant\cdots\geqslant M_1}\frac{M_1^{1-\varepsilon}}{M_{j-1}^{1-\varepsilon}}\left\|R_{\leqslant M_{j-1}}^{(j-1)}B_{N,\mu_m(j),j}f^{(j)}\right\|_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}.\end{aligned}$$

We then insert a smooth cut-off $\theta(t)$ with $\theta(t)=1$ for $t\in[-T,T]$ and $\theta(t)=0$ for $t\in[-2T,2T]^c$ into the above estimate to get

$$\begin{aligned}&\left\|R^{(1)}_{\leqslant M_1}B_{N,1,2}PP^{(k)}\right\|_{L^1_TL^2_{\mathbf{x},\mathbf{x}'}}\\ \leqslant\ &\left\|R^{(1)}_{\leqslant M_1}B_{N,1,2}\theta(t_2)\left(\int_0^{t_2}U^{(2)}(t_2-t_3)\theta(t_3)V^{(2)}_N\gamma^{(2)}_N(t_3)dt_3\right)\right\|_{L^1_TL^2_{\mathbf{x},\mathbf{x}'}}\\ &+\sum_{j=3}^k(C_\varepsilon T^{\frac12})^{j-2}\sum_{M_{j-1}\geqslant\cdots\geqslant M_1}\frac{M_1^{1-\varepsilon}}{M_{j-1}^{1-\varepsilon}}\left\|R^{(j-1)}_{\leqslant M_{j-1}}B_{N,\mu_m(j),j}\theta(t_j)\tilde f^{(j)}\right\|_{L^1_TL^2_{\mathbf{x},\mathbf{x}'}},\end{aligned}$$

where the sum is over all $M_2,\ldots,M_{j-1}$ dyadic such that $M_{j-1}\geqslant\cdots\geqslant M_2\geqslant M_1$, and

$$\tilde f^{(j)}=\int_0^{t_j}U^{(j)}(t_j-t_{j+1})\left(\theta(t_{j+1})V^{(j)}_N\gamma^{(j)}_N(t_{j+1})\right)dt_{j+1}$$

3.2.2. *Step II.* Using Lemma 4.5, the X_b space version of Lemma 4.4, we turn Step I into

$$\begin{aligned}&\left\|R^{(1)}_{\leqslant M_1}B_{N,1,2}PP^{(k)}\right\|_{L^1_TL^2_{\mathbf{x},\mathbf{x}'}}\\ \leqslant\ &\sum_{M_2\geqslant M_1}\frac{M_1^{1-\varepsilon}}{M_2^{1-\varepsilon}}\left\|\theta(t_2)\left(\int_0^{t_2}U^{(2)}(t_2-t_3)\left(R^{(2)}_{\leqslant M_2}\theta(t_3)V^{(2)}_N\gamma^{(2)}_N(t_3)\right)dt_3\right)\right\|_{X^{(2)}_{\frac12+}}\\ &+\sum_{j=3}^k(C_\varepsilon T^{\frac12})^{j-2}\sum_{M_j\geqslant M_{j-1}\geqslant\cdots\geqslant M_1}\frac{M_1^{1-\varepsilon}}{M_j^{1-\varepsilon}}\left\|\theta(t_j)R^{(j)}_{\leqslant M_j}\tilde f^{(j)}\right\|_{X^{(j)}_{\frac12+}}\end{aligned}$$

3.2.3. *Step III.* Lemma 4.1 gives us

$$\left\|R^{(1)}_{\leqslant M_1}B_{N,1,2}PP^{(k)}\right\|_{L^1_TL^2_{\mathbf{x},\mathbf{x}'}}\leq A+B$$

where

$$A=\sum_{M_2\geqslant M_1}\frac{M_1^{1-\varepsilon}}{M_2^{1-\varepsilon}}\left\|R^{(2)}_{\leqslant M_2}\left(\theta(t_3)V^{(2)}_N\gamma^{(2)}_N(t_3)\right)\right\|_{X^{(2)}_{-\frac12+}}$$

and

$$B=\sum_{j=3}^k(C_\varepsilon T^{\frac12})^{j-2}\sum_{M_j\geqslant M_{j-1}\geqslant\cdots\geqslant M_1}\frac{M_1^{1-\varepsilon}}{M_j^{1-\varepsilon}}\left\|R^{(j)}_{\leqslant M_j}\left(\theta(t_{j+1})V^{(j)}_N\gamma^{(j)}_N(t_{j+1})\right)\right\|_{X^{(j)}_{-\frac12+}}$$

3.2.4. *Step IV.* We focus for a moment on B. Applying Lemmas 3.1 and 3.2, we can carry out the sum in $M_2\leq\cdots\leq M_{j-1}$ at the expense of a factor $\left(\frac{M_j}{M_1}\right)^\epsilon$. Hence

$$B\lesssim\sum_{j=3}^k(C_\varepsilon T^{\frac12})^{j-2}\sum_{M_j\geqslant M_1}\left(\frac{M_1}{M_j}\right)^{1-2\epsilon}\left\|R^{(j)}_{\leqslant M_j}\left(\theta(t_{j+1})V^{(j)}_N\gamma^{(j)}_N(t_{j+1})\right)\right\|_{X^{(j)}_{-\frac12+}}$$

where the sum is over dyadic M_j such that $M_j\geq M_1$. Applying (4.25),

$$B\lesssim\sum_{j=3}^k(C_\varepsilon T^{\frac12})^{j-2}j^2\sum_{M_j\geqslant M_1}\left(\frac{M_1}{M_j}\right)^{1-2\epsilon}\min(M_j^2,N^{2\beta})N^{\frac12\beta-1}\|\theta(t_{j+1})S^{(j)}\gamma^{(j)}_N\,(t_{j+1})\,\|_{L^2_{t_{j+1}}L^2_{\mathbf{x},\mathbf{x}'}}$$

Rearranging terms

$$B \lesssim \sum_{j=3}^{k} (C_\varepsilon T^{\frac{1}{2}})^{j-2} j^2 \|\theta(t_{j+1}) S^{(j)} \gamma_N^{(j)}(t_{j+1})\|_{L^2_{t_{j+1}} L^2_{\mathbf{x}\mathbf{x}'}} M_1^{1-2\epsilon} N^{\frac{1}{2}\beta-1} \sum_{M_j \geqslant M_1} (\cdots)$$

where

$$\sum_{M_j \geqslant M_1} (\cdots) = \sum_{M_j \geqslant M_1} \min(M_j^{1+2\epsilon}, M_j^{-1+2\epsilon} N^{2\beta}).$$

We carry out the sum in M_j by dividing into $M_j \leqslant N^\beta$ (for which $\min(M_j^{1+2\epsilon}, M_j^{-1+2\epsilon}N^{2\beta}) = M_j^{1+2\epsilon}$) and $M_j \geqslant N^\beta$ (for which $\min(M_j^{1+2\epsilon}, M_j^{-1+2\epsilon}N^{2\beta}) = M_j^{-1+2\epsilon}N^{2\beta}$). This yields

$$\begin{aligned}
&\sum_{M_j \geqslant M_1} \min(M_j^{1+2\epsilon}, M_j^{-1+2\epsilon} N^{2\beta}) \\
\lesssim\ & \left(\sum_{N^\beta \geqslant M_j \geqslant M_1} + \sum_{M_j \geqslant M_1, M_j \geqslant N^\beta} \right) (...) \\
\lesssim\ & \sum_{N^\beta \geqslant M_j \geqslant 1} M_j^{1+2\epsilon} + \sum_{M_j \geqslant N^\beta} M_j^{-1+2\epsilon} N^{2\beta} \\
\lesssim\ & N^{\beta+2\epsilon}.
\end{aligned}$$

Hence

$$\begin{aligned}
B &\lesssim \sum_{j=3}^{k} (C_\varepsilon T^{\frac{1}{2}})^{j-2} j^2 \|\theta(t_{j+1}) S^{(j)} \gamma_N^{(j)}(t_{j+1})\|_{L^2_{t_{j+1}} L^2_{\mathbf{x},\mathbf{x}'}} M_1^{1-2\epsilon} N^{\frac{3}{2}\beta-1+2\epsilon} \\
&\lesssim M_1^{1-2\epsilon} N^{\frac{3}{2}\beta-1+2\epsilon} \sum_{j=3}^{k} (C_\varepsilon T^{\frac{1}{2}})^{j-2} T^{\frac{1}{2}} j^2 \|S^{(j)} \gamma_N^{(j)}\|_{L^\infty_t L^2_{\mathbf{x},\mathbf{x}'}} \\
&\lesssim M_1^{1-2\epsilon} N^{\frac{3}{2}\beta-1+2\epsilon} \sum_{j=3}^{k} (C_\varepsilon T^{\frac{1}{2}})^{j-2} j^2 T^{\frac{1}{2}} C^j \qquad \text{by Condition (1.4)} \\
&\lesssim C M_1^{1-2\epsilon} N^{\frac{3}{2}\beta-1+2\epsilon}, \text{ for } T \text{ small enough.}
\end{aligned}$$

Therefore, for $\beta < 2/3$, there is a C independent of M_1 and N s.t. given a M_1, there is $N_0(M_1)$ which makes

$$B \leqslant C, \text{ for all } N \geqslant N_0.$$

This completes the treatment of term B for $\beta < 2/3$. Term A is treated similarly (without the need to appeal to Lemmas 3.1, 3.2 below). Whence we have completed the proof of Theorem 3.1 and thence Theorem 2.1.

Lemma 3.1.

$$\left(\sum_{M_1 \leq M_2 \leq \cdots \leq M_{j-2} \leq M_{j-1}} 1 \right) \leq \frac{(\log_2 \frac{M_{j-1}}{M_1} + j - 3)^{j-3}}{(j-3)!},$$

where the sum is in $M_2, \ldots, M_{j-2}$ *over dyads, such that* $M_1 \leq M_2 \leq M_3 \leq \cdots \leq M_{j-2} \leq M_{j-1}$.

Proof. This is equivalent to

$$\left(\sum_{i_1 \leq i_2 \leq \cdots \leq i_{j-2} \leq i_{j-1}} 1 \right) \leq \frac{(i_{j-1} - i_1 + j - 3)^{j-3}}{(j-3)!},$$

where the sum is taken over integers $i_2, \dots, i_{j-2}$ such that $i_1 \leq i_2 \leq \cdots \leq i_{j-2} \leq i_{j-1}$. We use the estimate (for $p \geq 0$, $\ell \geq 0$)

$$\sum_{i=0}^{q} (i+\ell)^p \leq \frac{(q+\ell+1)^{p+1}}{p+1},$$

which just follows by estimating the sum by an integral.

First, carry out the sum in i_2 from i_1 to i_3 to obtain

$$= \sum_{i_1 \leq i_3 \leq \cdots \leq i_{j-1}} \left(\sum_{i_2 = i_1}^{i_3} 1 \right) \leq \sum_{i_1 \leq i_3 \leq \cdots \leq i_{j-1}} (i_3 - i_1 + 1).$$

Next, carry out the sum in i_3 from i_1 to i_4,

$$\begin{aligned} &\leq \sum_{i_1 \leq i_4 \leq \cdots \leq i_{j-1}} \left(\sum_{i_3 = i_1}^{i_4} (i_3 - i_1 + 1) \right) \\ &\leq \sum_{i_1 \leq i_4 \leq \cdots \leq i_{j-1}} \left(\sum_{i_3 = 0}^{i_4 - i_1} (i_3 + 1) \right) \\ &\leq \sum_{i_1 \leq i_4 \leq \cdots \leq i_{j-1}} \frac{(i_4 - i_1 + 2)^2}{2}. \end{aligned}$$

Continue in this manner to obtain the claimed bound. ∎

Lemma 3.2. *For each $\alpha > 0$ (possibly large) and each $\epsilon > 0$ (arbitrarily small), there exists $t > 0$ (independent of M) sufficiently small such that*

$$\forall\, j \geq 1,\ \forall\, M, \quad \textit{we have} \quad \frac{t^j (\alpha \log M + j)^j}{j!} \leq M^\epsilon$$

Proof. We use the following fact: for each $\sigma > 0$ (arbitrarily small) there exists $t > 0$ sufficiently small such that

$$\forall\, x > 0, \qquad t^x \left(\frac{1}{x} + 1 \right)^x \leq e^\sigma \tag{3.3}$$

To apply this fact to prove the lemma, use Stirling's formula to obtain

$$\frac{t^j (\alpha \log M + j)^j}{j!} \leq (et)^j \left(\frac{\alpha \log M + j}{j} \right)^j$$

Define x in terms of j by the formula $j = \alpha (\log M) x$. Then

$$= \left[(et)^x \left(\frac{1}{x} + 1 \right)^x \right]^{\alpha \log M}$$

Applying (3.3),

$$\leq e^{\sigma\alpha \log M} = M^{\sigma\alpha}$$

∎

4. The X_b Norms and a Few Strichartz Estimates

Define the norm

$$\|\alpha^{(k)}\|_{X_b^{(k)}} = \left(\int \langle \tau + |\boldsymbol{\xi}_k|^2 - |\boldsymbol{\xi}_k'|^2 \rangle^{2b} \left| \hat{\alpha}^{(k)}(\tau, \boldsymbol{\xi}_k, \boldsymbol{\xi}_k') \right|^2 d\tau\, d\boldsymbol{\xi}_k\, d\boldsymbol{\xi}_k' \right)^{1/2}$$

We will use the case $b = \frac{1}{2}+$ of the following lemma.

Lemma 4.1. *Let $\frac{1}{2} < b < 1$ and $\theta(t)$ be a smooth cutoff. Then*

$$\left\| \theta(t) \int_0^t U^{(k)}(t-s) \beta^{(k)}(s)\, ds \right\|_{X_b^{(k)}} \lesssim \|\beta^{(k)}\|_{X_{b-1}^{(k)}} \tag{4.1}$$

Proof. The estimate reduces to the space-independent estimate

$$\left\| \theta(t) \int_0^t h(t')\, dt' \right\|_{H_t^b} \lesssim \|h\|_{H_t^{b-1}}, \qquad \text{for } \tfrac{1}{2} < b \leq 1 \tag{4.2}$$

Indeed, taking $h(t) = h_{\boldsymbol{x}_k \boldsymbol{x}_k'}(t) \stackrel{\text{def}}{=} U^{(k)}(-t)\beta^{(k)}(t, \boldsymbol{x}_k, \boldsymbol{x}_k')$, applying the estimate (4.2) for fixed $\mathbf{x}_k, \mathbf{x}_k'$, and then applying the $L^2_{\mathbf{x}_k \mathbf{x}_k'}$ norm to both sides, yields (4.1). Now we prove estimate (4.2). Let $P_{\leq 1}$ and $P_{\geq 1}$ denote Littlewood-Paley projections onto frequencies $|\tau| \lesssim 1$ and $|\tau| \gtrsim 1$ respectively. Decompose $h = P_{\leq 1}h + P_{\geq 1}h$ and use that $\int_0^t P_{\geq 1}h(t') = \frac{1}{2}\int (\operatorname{sgn}(t - t') + \operatorname{sgn}(t'))P_{\geq 1}h(t')\, dt'$ to obtain the decomposition

$$\theta(t) \int_0^t h(t')\, dt' = H_1(t) + H_2(t) + H_3(t),$$

where

$$\begin{aligned} H_1(t) &= \theta(t) \int_0^t P_{\leq 1} h(t')\, dt' \\ H_2(t) &= \tfrac{1}{2}\theta(t) [\operatorname{sgn} * P_{\geq 1} h](t)\, dt' \\ H_3(t) &= \tfrac{1}{2}\theta(t) \int_{-\infty}^{+\infty} \operatorname{sgn}(t') P_{\geq 1} h(t')\, dt'. \end{aligned}$$

We begin by addressing term H_1. By Sobolev embedding (recall $\frac{1}{2} < b \leq 1$) and the $L^p \to L^p$ boundedness of the Hilbert transform for $1 < p < \infty$,

$$\|H_1\|_{H_t^b} \lesssim \|H_1\|_{L_t^2} + \|\partial_t H_1\|_{L_t^{2/(3-2b)}}.$$

Using that $\|P_{\leq 1} h\|_{L_t^\infty} \lesssim \|h\|_{H_t^{b-1}}$, we thus conclude

$$\|H_1\|_{H_t^b} \lesssim (\|\theta\|_{L_t^2} + \|\theta\|_{L_t^{2/(3-2b)}} + \|\theta'\|_{L_t^{2/3-2b}}) \|h\|_{H_t^{b-1}}.$$

Next we address the term H_2. By the fractional Leibniz rule,

$$\|H_2\|_{H_t^b} \lesssim \|\langle D_t \rangle^b \theta\|_{L_t^2} \|\operatorname{sgn} * P_{\geq 1} h\|_{L_t^\infty} + \|\theta\|_{L_t^\infty} \|\langle D_t \rangle^b (\operatorname{sgn} * P_{\geq 1} h)\|_{L_t^2}.$$

However,

$$\| \operatorname{sgn} * P_{\geq 1} h\|_{L_t^\infty} \lesssim \|\langle \tau\rangle^{-1}\hat{h}(\tau)\|_{L_\tau^1} \lesssim \|h\|_{H_t^{b-1}} .$$

On the other hand,

$$\|\langle D_t\rangle^b \operatorname{sgn} * P_{\geq 1} h\|_{L_t^2} \lesssim \|\langle \tau\rangle^b \langle \tau\rangle^{-1}\hat{h}(\tau)\|_{L_\tau^2} \lesssim \|h\|_{H_t^{b-1}} .$$

Consequently,

$$\|H_2\|_{H_t^b} \lesssim (\|\langle D_t\rangle^b \theta\|_{L_t^2} + \|\theta\|_{L_t^\infty})\|h\|_{H_t^{b-1}} .$$

For term H_3, we have

$$\|H_3\|_{H_t^b} \lesssim \|\theta\|_{H_t^b} \left\| \int_{-\infty}^{+\infty} \operatorname{sgn}(t') P_{\geq 1} h(t')\, dt' \right\|_{L_t^\infty} .$$

However, the second term is handled via Parseval's identity

$$\int_{t'} \operatorname{sgn}(t') P_{\geq 1} h(t')\, dt' = \int_{|\tau|\geq 1} \tau^{-1}\hat{h}(\tau)\, d\tau ,$$

from which the appropriate bounds follow again by Cauchy-Schwarz. Collecting our estimates for H_1, H_2, and H_3, we have

$$\left\| \theta(t) \int_0^t h(t')\, dt' \right\|_{H_t^b} \lesssim C_\theta \|h\|_{H_t^{b-1}},$$

where

$$C_\theta = \|\theta\|_{L_t^2} + \|\theta'\|_{L_t^{2/(3-2b)}} + \|\langle D_t\rangle^b \theta\|_{L_t^2} + \|\theta\|_{L_t^{2/(3-2b)}} + \|\theta\|_{L_t^\infty}$$

∎

4.1. **Various Forms of Collapsing Estimates.**

Lemma 4.2. *There is a C independent of j, k, and N such that, (for $f^{(k+1)}(\mathbf{x}_{k+1}, \mathbf{x}_{k+1})$ independent of t)*

$$\left\| R^{(k)} B_{N,j,k+1} U^{(k+1)}(t) f^{(k+1)} \right\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C \left\| V \right\|_{L^1} \left\| R^{(k+1)} f^{(k+1)} \right\|_{L_{\mathbf{x},\mathbf{x}'}^2} .$$

Proof. One can find this estimate as estimate (A.18) in [11] or a special case of Theorem 7 of [15]. For more estimates of this type, see [35, 28, 12, 14, 3, 27]. ∎

We have the following consequence of Lemma 4.2.

Lemma 4.3. *There is a C independent of j, k, and N such that (for $\alpha^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}_{k+1})$ dependent on t)*

$$\|R^{(k)} B_{N,j,k+1} \alpha^{(k+1)}\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2} \leqslant C \|R^{(k+1)} \alpha^{(k+1)}\|_{X_{\frac{1}{2}+}^{(k+1)}}$$

Proof. Let

$$f_\tau^{(k+1)}(\mathbf{x}_{k+1}, \mathbf{x}'_{k+1}) = \mathcal{F}_{t\mapsto\tau}(U^{(k+1)}(-t)\alpha^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1}))$$

where $\mathcal{F}_{t\mapsto\tau}$ denotes the Fourier transform in $t \mapsto \tau$. Then

$$\alpha^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1}) = \int_\tau e^{it\tau} U^{(k+1)}(t) f^{(k+1)}(\mathbf{x}_{k+1}, \mathbf{x}'_{k+1})\, d\tau$$

By Minkowski's inequality

$$\|R^{(k)}B_{N,j,k+1}\alpha^{(k+1)}\|_{L^2_tL^2_{\mathbf{x},\mathbf{x}'}} \leq \int_\tau \|R^{(k)}B_{N,j,k+1}U^{(k+1)}(t)f^{(k+1)}\|_{L^2_tL^2_{\mathbf{x},\mathbf{x}'}}\,d\tau$$

By Lemma 4.2,

$$\leq \int_\tau \|R^{(k+1)}f^{(k+1)}\|_{L^2_{\mathbf{x},\mathbf{x}'}}\,d\tau$$

For any $b>\frac{1}{2}$, we write $1=\langle\tau\rangle^{-b}\langle\tau\rangle^{b}$ and apply Cauchy-Schwarz in τ to obtain

$$\leq \|\langle\tau\rangle^b R^{(k+1)}f^{(k+1)}\|_{L^2_{\tau,\mathbf{x},\mathbf{x}'}} = \|R^{(k+1)}\alpha^{(k+1)}\|_{X_b^{(k+1)}}$$

∎

Lemma 4.4. *For each $\varepsilon>0$, there is a C_ε independent of M_k, j, k, and N such that*

$$\|R^{(k)}P^{(k)}_{\leqslant M_k}B_{N,j,k+1}U^{(k+1)}(t)f^{(k+1)}\|_{L^2_tL^2_{\mathbf{x},\mathbf{x}'}} \leqslant C_\varepsilon\|V\|_{L^1}\sum_{M_{k+1}\geqslant M_k}\left(\frac{M_k}{M_{k+1}}\right)^{1-\varepsilon}\left\|R^{(k+1)}P^{(k+1)}_{\leqslant M_{k+1}}f^{(k+1)}\right\|_{L^2_{\mathbf{x},\mathbf{x}'}}$$

where the sum on the right is in M_{k+1}, over dyads such that $M_{k+1}\geqslant M_k$. In particular, we have Lemma 4.2 back if we carry out the sum and let $M_k\to\infty$.

Proof. It suffices to take $k=1$ and prove

$$\|R^{(1)}P_{\leq M_1}B_{N,1,2}(R^{(2)})^{-1}U^{(2)}(t)f^{(2)}\|_{L^2_tL^2_{\mathbf{x}_1\mathbf{x}_1'}} \leq C_\varepsilon\|V\|_{L^1}\sum_{M_2\geqslant M_1}\left(\frac{M_1}{M_2}\right)^{1-\varepsilon}\|P^{(2)}_{\leq M_2}f^{(2)}\|_{L^2_tL^2_{\mathbf{x}_2\mathbf{x}_2'}}$$

where the sum is over dyadic M_2 such that $M_2\geqslant M_1$. For convenience, we take only "half" of the operator $B_{N,1,2}$: For $\alpha^{(2)}(t,x_1,x_2,x_1',x_2')$, define

$$(\tilde{B}_{N,1,2}\alpha^{(2)})(t,x_1,x_1') \overset{\text{def}}{=} \int_{x_2} V_N(x_1-x_2)\alpha^{(2)}(t,x_1,x_2,x_1',x_2)\,dx_2$$

Note that

$$\Big(R^{(1)}\tilde{B}_{N,1,2}(R^{(2)})^{-1}U^{(2)}(t)f^{(2)}\Big)^{\widehat{}}(\tau,\xi_1,\xi_1') = \iint_{\xi_2,\xi_2'}\delta(\cdots)\frac{\widehat{V_N}(\xi_2+\xi_2')|\xi_1|}{|\xi_1-\xi_2-\xi_2'||\xi_2||\xi_2'|}\widehat{f^{(2)}}(\xi_1-\xi_2-\xi_2',\xi_2,\xi_1',\xi_2')\,d\xi_2\,d\xi_2'$$

where

$$\delta(\cdots)=\delta(\tau+|\xi_1-\xi_2-\xi_2'|^2+|\xi_2|^2-|\xi_1'|^2-|\xi_2'|^2)$$

Divide this integration into two pieces:

$$=\iint_{|\xi_2|\leq|\xi_2'|}(\cdots)\,d\xi_2\,d\xi_2' + \iint_{|\xi_2'|\leq|\xi_2|}(\cdots)\,d\xi_2\,d\xi_2'$$

In the first term, decompose the ξ_2' integration into dyadic intervals, and in the second term, decompose the ξ_2 integration into dyadic intervals:

$$=\sum_{M_2\geq M_1}\iint_{|\xi_2|\leq|\xi_2'|}P^{2'}_{M_2}(\cdots)\,d\xi_2\,d\xi_2' + \sum_{M_2\geq M_1}\iint_{|\xi_2'|\leq|\xi_2|}P^{2}_{M_2}(\cdots)\,d\xi_2\,d\xi_2'$$

Observe that, in the first integration, we can insert for free the projection $P^1_{\le 3M_2}P^{1'}_{\le M_1}P^2_{\le M_2}$ and in the second integration, we can insert $P^1_{\le 3M_2}P^{1'}_{\le M_1}P^{2'}_{\le M_2}$.

$$= \sum_{M_2\ge M_1}\iint_{|\xi_2|\le|\xi_2'|} P^1_{\le 3M_2}P^{1'}_{\le M_1}P^2_{\le M_2}P^{2'}_{M_2}(\cdots)\,d\xi_2\,d\xi_2' + \sum_{M_2\ge M_1}\iint_{|\xi_2'|\le|\xi_2|} P^1_{\le 3M_2}P^{1'}_{\le M_1}P^{2'}_{\le M_2}P^{2}_{M_2}(\cdots)\,d\xi_2\,d\xi_2'$$

Then for each piece, we proceed as in Klainerman-Machedon [37], performing Cauchy-Schwarz with respect to measures supported on hypersurfaces and applying the $L^2_{\tau\xi_1\xi_1'}$ norm to both sides of the resulting inequality.[7] In this manner, it suffices to prove the following estimates, uniform in $\tau'=\tau-|\xi_1'|^2$:

$$\iint_{\substack{|\xi_2'|\sim M_2,\\ |\xi_2|\le M_2}} \delta(\cdots)\frac{|\xi_1|^2}{|\xi_1-\xi_2-\xi_2'|^2\,|\xi_2|^2\,|\xi_2'|^2}\,d\xi_2\,d\xi_2' \le C_\varepsilon\left(\frac{M_1}{M_2}\right)^{2(1-\varepsilon)}, \tag{4.3}$$

(recall that $|\xi_1|\lesssim M_1\ll M_2$) and also

$$\iint_{\substack{|\xi_2|\sim M_2,\\ |\xi_2'|\le M_2}} \delta(\cdots)\frac{|\xi_1|^2}{|\xi_1-\xi_2-\xi_2'|^2\,|\xi_2|^2\,|\xi_2'|^2}\,d\xi_2\,d\xi_2' \le C_\varepsilon\left(\frac{M_1}{M_2}\right)^{2(1-\varepsilon)}. \tag{4.4}$$

In both (4.3) and (4.4),

$$\delta(\cdots)=\delta(\tau'+|\xi_1-\xi_2-\xi_2'|^2+|\xi_2|^2-|\xi_2'|^2).$$

By rescaling $\xi_2\mapsto M_2\xi_2$ and $\xi_2'\mapsto M_2\xi_2'$, (4.3) and (4.4) reduce to, respectively, the following.

(4.5)

$$\text{for } |\xi_1|\ll 1,\quad I(\tau',\xi_1)\overset{\text{def}}{=}\iint_{\substack{|\xi_2'|\sim 1,\\ |\xi_2|\le 2}} \delta(\cdots)\frac{|\xi_1|^2}{|\xi_1-\xi_2-\xi_2'|^2\,|\xi_2|^2\,|\xi_2'|^2}\,d\xi_2\,d\xi_2' \le C_\varepsilon|\xi_1|^{2(1-\varepsilon)},$$

(4.6)

$$\text{for } |\xi_1|\ll 1,\quad I'(\tau',\xi_1)\overset{\text{def}}{=}\iint_{\substack{|\xi_2|\sim 1,\\ |\xi_2'|\le 2}} \delta(\cdots)\frac{|\xi_1|^2}{|\xi_1-\xi_2-\xi_2'|^2\,|\xi_2|^2\,|\xi_2'|^2}\,d\xi_2\,d\xi_2' \le C_\varepsilon|\xi_1|^{2(1-\varepsilon)}.$$

To be precise, the ξ_1 in estimates (4.5) and (4.6) is ξ_1/M_2 in estimates (4.3) and (4.4). We shall obtain the upper bound $|\xi_1|^2\log|\xi_1|^{-1}$ for both (4.5), (4.6).

First, we prove (4.6). Begin by carrying out the ξ_2' integral to obtain

$$I'(\tau',\xi_1)=\frac{1}{2}|\xi_1|^2\int_{\frac12\le|\xi_2|\le 2}\frac{H'(\tau',\xi_1,\xi_2)}{|\xi_1-\xi_2||\xi_2|^2}\,d\xi_2$$

where $H'(\tau',\xi_1,\xi_2)$ is defined as follows. Let P' be the truncated plane defined by

$$P'(\tau',\xi_1,\xi_2)=\left\{\xi_2'\in\mathbb{R}^3 \mid (\xi_2'-\lambda\omega)\cdot\omega=0\,,\ |\xi_2'|\le 2\right\}$$

where

$$\omega=\frac{\xi_1-\xi_2}{|\xi_1-\xi_2|}\,,\quad \lambda=\frac{\tau'+|\xi_1-\xi_2|^2+|\xi_2|^2}{2|\xi_1-\xi_2|}$$

[7]Notice that $\left\|\widehat{V_N}\right\|_{L^\infty}\leqslant\|V_N\|_{L^1}=\|V\|_{L^1}$ i.e. $\widehat{V_N}$ is a dummy factor.

Now let

$$H'(\tau',\xi_1,\xi_2)=\int_{\xi_2'\in P'(\tau',\xi_1,\xi_2)}\frac{d\sigma(\xi_2')}{|\xi_1-\xi_2-\xi_2'|^2\,|\xi_2'|^2} \tag{4.7}$$

where the integral is computed with respect to the surface measure on P'.

Since $|\xi_1-\xi_2|\sim 1$, $|\xi_2|\sim 1$, we have the following reduction

$$I'(\tau',\xi_1)\lesssim |\xi_1|^2\int_{\frac12\leq|\xi_2|\leq 2}H'(\tau',\xi_1,\xi_2)\,d\xi_2$$

We now evaluate $H'(\tau',\xi_1,\xi_2)$. Introduce polar coordinates (ρ,θ) on the plane P' with respect to the "center" $\lambda\omega$, and note that

$$\begin{aligned}|\xi_1-\xi_2-\xi_2'|^2&=||\xi_1-\xi_2|\omega-\xi_2'|^2\\&=|(|\xi_1-\xi_2|-\lambda)\omega-(\xi_2'-\lambda\omega)|^2\\&=(|\xi_1-\xi_2|-\lambda)^2+|\xi_2'-\lambda\omega|^2\\&=(|\xi_1-\xi_2|-\lambda)^2+\rho^2\\&=\alpha^2+\rho^2\end{aligned} \tag{4.8}$$

where

$$\alpha=|\xi_1-\xi_2|-\lambda=\frac{|\xi_1|^2-2\xi_1\cdot\xi_2-\tau'}{2|\xi_1-\xi_2|}$$

Also,

$$|\xi_2'|^2=|(\xi_2'-\lambda\omega)+\lambda\omega|^2=|\xi_2'-\lambda\omega|^2+\lambda^2=\rho^2+\lambda^2 \tag{4.9}$$

Using (4.8) and (4.9) in (4.7),

$$H'(\tau',\xi_1,\xi_2)=\int_0^{\sqrt{4-\lambda^2}}\frac{2\pi\rho\,d\rho}{(\rho^2+\alpha^2)(\rho^2+\lambda^2)}$$

The restriction to $0\leq\rho\leq\sqrt{4-\lambda^2}$ arises from the fact that the plane P' must sit within the ball $|\xi_2'|\leq 2$. In particular, $H'(\tau,\xi_1,\xi_2)=0$ if $|\lambda|\geq 2$ since then the plane P' is located entirely outside the ball $|\xi_2'|\leq 2$. Since $|\lambda|\leq 2$, we have $|\alpha|\leq 3$ and $|\tau'|\leq 10$.

We consider the three cases: (A) $|\lambda|\leq\frac14$ (which implies $|\alpha|\geq\frac14$), (B) $|\alpha|\leq\frac14$ (which implies $|\lambda|\geq\frac14$), and (C) $|\lambda|\geq\frac14$ and $|\alpha|\geq\frac14$. Case (C) is the easiest since clearly $|H'(\tau',\xi_1,\xi_2)|\leq C$.

Let us consider case (B). Then

$$H'(\tau,\xi_1,\xi_2)\lesssim\int_{\rho=0}^2\frac{\rho\,d\rho}{\rho^2+\alpha^2}=\int_{\nu=0}^{\sqrt2}\frac{d\nu}{\nu+\alpha^2}=\log\left(1+\frac{\sqrt2}{\alpha^2}\right)$$

Substituting back into I',

$$I'(\tau',\xi_1)\lesssim|\xi_1|^2\int_{|\xi_2|\leq 2}\log\left(1+\frac{\sqrt2}{\alpha^2}\right)\,d\xi_2$$

Since $|\alpha| \le \sqrt{3}$, it follows that[8]

$$\begin{aligned}\log(1+\frac{\sqrt{2}}{\alpha^2}) &\le c + |\log|\alpha|| \\ &\le c + |\log|(|\xi_1|^2 - 2\xi_1\cdot\xi_2 - \tau')|| \\ &= c + |\log 2|\xi_1\cdot(\xi_2 - \frac{1}{2}\xi_1 + \frac{\tau'\xi_1}{2|\xi_1|^2})| \\ &= c + |\log|\xi_1\cdot(\xi_2 - \frac{1}{2}\xi_1 + \frac{\tau'\xi_1}{2|\xi_1|^2})|\end{aligned}$$

Hence

$$I'(\tau',\xi_1) \lesssim |\xi_1|^2\left(1 + \int_{|\xi_2|\le 2} |\log|\xi_1\cdot(\xi_2 - \frac{1}{2}\xi_1 + \frac{\tau'\xi_1}{2|\xi_1|^2})|\,d\xi_2\right)$$

Denoting by $B(\mu,r)$ the ball of center μ and radius r, the substitution $\xi_2 \mapsto \xi_2 + \frac{1}{2}\xi_1 - \frac{\tau'\xi_1}{2|\xi_1|^2}$ yields, with $\mu = \frac{1}{2}\xi_1 - \frac{\tau'\xi_1}{2|\xi_1|^2}$,

$$\begin{aligned}I'(\tau',\xi_1) &\lesssim |\xi_1|^2\left(1 + \int_{B(\mu,2)} |\log|\xi_1\cdot\xi_2||\,d\xi_2\right) \\ &\lesssim |\xi_1|^2\left(\log|\xi_1|^{-1} + \int_{B(\mu,2)} |\log|\frac{\xi_1}{|\xi_1|}\cdot\xi_2||\,d\xi_2\right)\end{aligned}$$

By rotating coordinates so that $\frac{\xi}{\xi_1} = (1,0,0)$, and letting μ' denote the corresponding rotation of μ,

$$I'(\tau',\xi_1) \lesssim |\xi_1|^2\left(\log|\xi_1|^{-1} + \int_{B(\mu',2)} |\log|(\xi_2)_1|\,d\xi_2\right)$$

where $(\xi_2)_1$ denotes the first coordinate of the vector ξ_2. Since $|\tau'| \le 10$, it follows that $|\mu'| \lesssim |\xi_1|^{-1}$ and we finally obtain

$$I'(\tau',\xi_1) \lesssim |\xi_1|^2 \log|\xi_1|^{-1}$$

as claimed, completing Case (B).

Case (A) is similar except that we begin with the bound

$$H'(\tau',\xi_1,\xi_2) \lesssim \int_{\rho=0}^{2} \frac{2\pi\rho\,d\rho}{\rho^2+\lambda^2}$$

This completes the proof of (4.6).

Next, we prove (4.5). In the integral defining $I(\tau',\xi_1)$, we have the restriction $\frac{1}{2} \le |\xi_2'| \le 2$ and $|\xi_2| \le 2$. Note that if $\frac{1}{4} \le |\xi_2| \le 2$, then the argument above that provided the bound for $I'(\tau',\xi_1)$ applies. Hence it suffices to restrict to $|\xi_2| \le \frac{1}{4}$, from which it follows that $|\xi_1 - \xi_2 - \xi_2'| \sim 1$.

Begin by carrying out the ξ_2' integral to obtain

$$I(\tau',\xi_1) = \frac{1}{2}|\xi_1|^2 \int_{|\xi_2|\le 2} \frac{H(\tau',\xi_1,\xi_2)}{|\xi_1-\xi_2||\xi_2|^2}\,d\xi_2 \tag{4.10}$$

[8]The first step is simply: if $x \ge \delta > 0$, then $\log(1+x) \le \log x + \log(1+\frac{1}{\delta})$. The second step uses that $|\xi_1 - \xi_2| \sim 1$, which follows since $|\xi_1| \ll 1$ and $|\xi_2| \sim 1$.

where $H(\tau',\xi_1,\xi_2)$ is defined as follows. Let P be the truncated plane defined by

$$P(\tau',\xi_1,\xi_2) = \{ \ \xi_2' \in \mathbb{R}^3 \mid (\xi_2' - \lambda\omega)\cdot\omega = 0\,, \ \frac{1}{2} \leq |\xi_2'| \leq 2 \ \}$$

where

$$\omega = \frac{\xi_1 - \xi_2}{|\xi_1-\xi_2|}\,, \qquad \lambda = \frac{\tau' + |\xi_1-\xi_2|^2 + |\xi_2|^2}{2|\xi_1-\xi_2|}$$

Now let

$$H(\tau',\xi_1,\xi_2) = \int_{\xi_2'\in P(\tau',\xi_1,\xi_2)} \frac{d\sigma(\xi_2')}{|\xi_1-\xi_2-\xi_2'|^2\,|\xi_2'|^2}$$

where the integral is computed with respect to the surface measure on P. Since $|\xi_1-\xi_2-\xi_2'| \sim 1$ and $|\xi_2'| \sim 1$, we obtain $H(\tau',\xi_1,\xi_2) \leq C$. Substituting into (4.10), we obtain

$$\begin{aligned} I(\tau',\xi_1) &\lesssim |\xi_1|^2 \int_{|\xi_2|\leq\frac14} \frac{d\xi_2}{|\xi_1-\xi_2||\xi_2|^2} \\ &\lesssim |\xi_1|^2 \left(\int_{|\xi_2|\leq 2|\xi_1|} \frac{d\xi_2}{|\xi_1-\xi_2||\xi_2|^2} + \int_{2|\xi_1|\leq|\xi_2|\leq\frac14} \frac{d\xi_2}{|\xi_1-\xi_2||\xi_2|^2} \right) \end{aligned}$$

In the first integral, we change variables $\xi_2 = |\xi_1|\eta$, and in the second integral, we use the bound $|\xi_1-\xi_2|^{-1} \leq 2|\xi_2|^{-1}$ to obtain

$$\lesssim |\xi_1|^2 \left(\int_{|\eta|\leq 2} \frac{d\eta}{|\frac{\xi_1}{|\xi_1|} - \eta||\eta|^2} + \int_{2|\xi_1|\leq|\xi_2|\leq\frac14} \frac{d\xi_2}{|\xi_2|^3} \right) \lesssim |\xi_1|^2 \log|\xi_1|^{-1}$$

This completes the proof of (4.5). ∎

Lemma 4.5. *For each $\varepsilon > 0$, there is a C_ε independent of $M_k, j, k,$ and N such that*

$$\|R^{(k)}P^{(k)}_{\leqslant M_k} B_{N,j,k+1}\alpha^{(k+1)}\|_{L^2_t L^2_{\mathbf{x},\mathbf{x}'}} \leqslant C_\varepsilon \sum_{M_{k+1}\geqslant M_k} \left(\frac{M_k}{M_{k+1}}\right)^{1-\varepsilon} \left\| R^{(k+1)}P^{(k+1)}_{\leqslant M_{k+1}}\alpha^{(k+1)} \right\|_{X^{(k)}_{\frac12+}}.$$

where the sum on the right is in M_{k+1}, over dyads such that $M_{k+1} \geqslant M_k$.

Proof. The proof is exactly the same as deducing Lemma 4.3 from Lemma 4.2. We include the proof for completeness. Let

$$f^{(k+1)}_\tau(\mathbf{x}_{k+1},\mathbf{x}'_{k+1}) = \mathcal{F}_{t\mapsto\tau}(U^{(k+1)}(-t)\alpha^{(k+1)}(t,\mathbf{x}_{k+1},\mathbf{x}'_{k+1}))$$

where $\mathcal{F}_{t\mapsto\tau}$ denotes the Fourier transform in $t\mapsto\tau$. Then

$$\alpha^{(k+1)}(t,\mathbf{x}_{k+1},\mathbf{x}'_{k+1}) = \int_\tau e^{it\tau}U^{(k+1)}(t)f^{(k+1)}(\mathbf{x}_{k+1},\mathbf{x}'_{k+1})\,d\tau$$

By Minkowski's inequality

$$\|R^{(k)}P^{(k)}_{\leqslant M_k} B_{N,j,k+1}\alpha^{(k+1)}\|_{L^2_t L^2_{\mathbf{x},\mathbf{x}'}} \leq \int_\tau \|R^{(k)}P^{(k)}_{\leqslant M_k} B_{N,j,k+1}U^{(k+1)}(t)f^{(k+1)}\|_{L^2_t L^2_{\mathbf{x},\mathbf{x}'}}\,d\tau$$

By Lemma 4.4,

$$\leq C_\varepsilon \sum_{M_{k+1}\geqslant M_k} \left(\frac{M_k}{M_{k+1}}\right)^{1-\varepsilon} \int_\tau \|R^{(k+1)}P^{(k+1)}_{\leqslant M_{k+1}}f^{(k+1)}\|_{L^2_{\mathbf{x},\mathbf{x}'}}\,d\tau$$

For any $b > \frac{1}{2}$, we write $1 = \langle\tau\rangle^{-b}\langle\tau\rangle^{b}$ and apply Cauchy-Schwarz in τ to obtain

$$\begin{aligned} &\leq C_\varepsilon \sum_{M_{k+1}\geqslant M_k} \left(\frac{M_k}{M_{k+1}}\right)^{1-\varepsilon} \|\langle\tau\rangle^b R^{(k+1)} P^{(k+1)}_{\leqslant M_{k+1}} f^{(k+1)}\|_{L^2_{\tau,\mathbf{x},\mathbf{x}'}} \\ &= C_\varepsilon \sum_{M_{k+1}\geqslant M_k} \left(\frac{M_k}{M_{k+1}}\right)^{1-\varepsilon} \left\|R^{(k+1)} P^{(k+1)}_{\leqslant M_{k+1}} \alpha^{(k+1)}\right\|_{X^{(k)}_{\frac{1}{2}+}}. \end{aligned}$$

∎

4.2. A Strichartz Estimate for $PP^{(k)}$.

Lemma 4.6. *Assume $\gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}'_k)$ satisfies the symmetric condition* (1.1). *Let*

$$\beta^{(k)}(t,\mathbf{x}_k;\mathbf{x}'_k) = V(x_i - x_j)\gamma^{(k)}(t,\mathbf{x}_k;\mathbf{x}'_k) \tag{4.11}$$

Then we have the estimates:

$$\|\beta^{(k)}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim \|V\|_{L_x^{\frac{6}{5}+}} \|\langle\nabla_{x_i}\rangle\langle\nabla_{x_j}\rangle\gamma^{(k)}\|_{L^2_t L^2_{x,x'}}, \tag{4.12}$$

$$\|\beta^{(k)}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim \|V\|_{L_x^{3+}} \|\gamma^{(k)}\|_{L^2_t L^2_{x,x'}}, \tag{4.13}$$

$$\|\beta^{(k)}\|_{X^{(k)}_{-\frac{1}{2}+}} \lesssim \|V\|_{L_x^{2+}} \|\langle\nabla_{x_i}\rangle^{\frac{1}{2}}\gamma^{(k)}\|_{L^2_t L^2_{x,x'}}. \tag{4.14}$$

Proof. It suffices to prove Lemma 4.6 for $k = 2$. Since we will need to deal with Fourier transforms in only selected coordinates, we introduce the following notation: $\mathcal{F}_0$ denotes Fourier transform in t, $\mathcal{F}_j$ denotes Fourier transform in x_j, and $\mathcal{F}_{j'}$ denotes Fourier transform in x'_j. Fourier transforms in multiple coordinates will denoted as combined subscripts – for example, $\mathcal{F}_{01'} = \mathcal{F}_0\mathcal{F}_{1'}$ denotes the Fourier transform in t and x'_1.[9] Let T denote the translation operator

$$(Tf)(x_1,x_2) = f(x_1 + x_2, x_2)$$

Suppressing the x'_1, x'_2 dependence, we have

$$(\mathcal{F}_{12}T\beta^{(2)})(t,\xi_1,\xi_2) = (\mathcal{F}_{12}\beta^{(2)})(t,\xi_1,\xi_2-\xi_1) \tag{4.15}$$

Also

$$e^{-2it\xi_1\cdot\xi_2}(\mathcal{F}_{12}T\beta^{(2)})(t,\xi_1,\xi_2) = \mathcal{F}_1\big[(\mathcal{F}_2T\beta^{(2)})(t,x_1-2t\xi_2,\xi_2)\big](\xi_1) \tag{4.16}$$

Now

$$\begin{aligned} &(\mathcal{F}_{012}\beta^{(2)})(\tau - |\xi_2|^2 + 2\xi_1\cdot\xi_2, \xi_1, \xi_2-\xi_1) \\ &\quad= (\mathcal{F}_{012}T\beta^{(2)})(\tau - |\xi_2|^2 + 2\xi_1\cdot\xi_2, \xi_1, \xi_2) && \text{by (4.15)} \\ &\quad= \mathcal{F}_0\big[e^{it|\xi_2|^2}e^{-2it\xi_1\cdot\xi_2}(\mathcal{F}_{12}T\beta^{(2)})(t,\xi_1,\xi_2)\big](\tau) \\ &\quad= \mathcal{F}_0\big[e^{it|\xi_2|^2}\mathcal{F}_1\big[(\mathcal{F}_2T\beta^{(2)})(t,x_1-2t\xi_2,\xi_2)\big](\xi_1)\big](\tau) && \text{by (4.16)} \\ &\quad= \mathcal{F}_{01}\big[e^{it|\xi_2|^2}(\mathcal{F}_2T\beta^{(2)})(t,x_1-2t\xi_2,\xi_2)\big](\tau,\xi_1) \end{aligned} \tag{4.17}$$

[9] We are going to apply the endpoint Strichartz estimate on the non-transformed coordinates. We do not know currently the origin of such an technique. The only other place we know about it is [13, Lemma 6].

By changing variables $\xi_2 \mapsto \xi_2 - \xi_1$ and then changing $\tau \mapsto \tau - |\xi_2|^2 + 2\xi_1 \cdot \xi_2$, we obtain

$$\begin{aligned}
&\|\beta^{(2)}\|_{X^{(2)}_{-\frac{1}{2}+}} \\
&= \|\hat{\beta}^{(2)}(\tau,\xi_1,\xi_2,\xi_1',\xi_2')\langle \tau + |\xi_1|^2 + |\xi_2|^2 - |\xi_1'|^2 - |\xi_2'|^2\rangle^{-\frac{1}{2}+}\|_{L^2_{\tau\xi_1\xi_2\xi_1'\xi_2'}} \\
&= \|\hat{\beta}^{(2)}(\tau - |\xi_2|^2 + 2\xi_1\cdot\xi_2,\xi_1,\xi_2-\xi_1,\xi_1',\xi_2')\langle \tau + 2\,|\xi_1|^2 - |\xi_1'|^2 - |\xi_2'|^2\rangle^{-\frac{1}{2}+}\|_{L^2_{\tau\xi_1\xi_2\xi_1'\xi_2'}}
\end{aligned}$$

Applying the the dual Strichartz (see (4.19) below), the above is bounded by

$$\lesssim \|\mathcal{F}_{01}^{-1}\Big[(\mathcal{F}_{012}\beta^{(2)})(\tau - |\xi_2|^2 + 2\xi_1\cdot\xi_2,\xi_1,\xi_2-\xi_1)\Big](t,x_1)\|_{L^2_{\xi_2}L^2_tL^{\frac{6}{5}+}_{x_1}L^2_{x_1'x_2'}}$$

Utilizing (4.17), the above is equal to

$$= \|(\mathcal{F}_2 T\beta^{(2)})(t,x_1-2t\xi_2,\xi_2)\|_{L^2_tL^2_{\xi_2}L^{\frac{6}{5}+}_{x_1}L^2_{x_1'x_2'}}$$

Change variable in $x_1 \mapsto x_1 + 2t\xi_2$ to obtain

$$= \|(\mathcal{F}_2 T\beta^{(2)})(t,x_1,\xi_2)\|_{L^2_tL^2_{\xi_2}L^{\frac{6}{5}+}_{x_1}L^2_{x_1'x_2'^2}}$$

Now note that from (4.11), we have

$$(\mathcal{F}_2 T\beta^{(2)})(t,x_1,\xi_2) = V(x_1)(\mathcal{F}_2 T\gamma^{(2)})(t,x_1,\xi_2)$$

It follows that

$$\begin{aligned}
&\|(\mathcal{F}_2 T\beta^{(2)})(t,x_1,\xi_2)\|_{L^2_tL^2_{\xi_2}L^{\frac{6}{5}+}_{x_1}L^2_{x_1'x_2'}} \\
&= \left\| V(x_1)\Big(\|(\mathcal{F}_2 T\gamma^{(2)})(t,x_1,\xi_2)\|_{L^2_{x_1'x_2'}}\Big)\right\|_{L^2_tL^2_{\xi_2}L^{\frac{6}{5}+}_{x_1}} \\
&\le \|V\|_{L^{\frac{6}{5}+}}\|(\mathcal{F}_2 T\gamma^{(2)})(t,x_1,\xi_2)\|_{L^2_tL^2_{\xi_2}L^\infty_{x_1}L^2_{x_1'x_2'}} \qquad (4.18) \\
&\le \|V\|_{L^{\frac{6}{5}+}}\|(\mathcal{F}_2 T\gamma^{(2)})(t,x_1,\xi_2)\|_{L^2_tL^2_{\xi_2x_1'x_2'}L^\infty_{x_1}}
\end{aligned}$$

and continue with Sobolev, Plancherel, etc. We also need to remark that we split $\gamma^{(2)}$ into the piece where $|\xi_1| \ge |\xi_2|$ and the piece where $|\xi_2| \ge |\xi_1|$. The above represents the treatment of the case $|\xi_1| \le |\xi_2|$. This proves estimate (4.12). Using Hölder exponents $(3+,2,\frac{6}{5}+)$ and $(2+,3,\frac{6}{5}+)$ in (4.18) yields estimates (4.13) and (4.14).

It remains to prove the following dual Strichartz estimate (here $\sigma^{(2)}(t,x_1,x_1',x_2')$, note that the x_2 coordinate is missing):

$$\|\langle \tau + 2\,|\xi_1|^2 - |\xi_1'|^2 - |\xi_2'|^2\rangle^{-\frac{1}{2}+}\hat{\sigma}^{(2)}(\tau,\xi_1,\xi_1',\xi_2')\|_{L^2_\tau L^2_{\xi_1\xi_1'\xi_2}} \lesssim \|\sigma^{(2)}\|_{L^2_tL^{\frac{6}{5}+}_{x_1}L^2_{x_1'x_2'}} \qquad (4.19)$$

The estimate (4.19) is dual to the equivalent estimate

$$\|\sigma^{(2)}\|_{L^2_tL^{6-}_{x_1}L^2_{x_1'x_2'}} \lesssim \|\langle \tau + 2\,|\xi_1|^2 - |\xi_1'|^2 - |\xi_2'|^2\rangle^{\frac{1}{2}-}\hat{\sigma}^{(2)}(\tau,\xi_1,\xi_1',\xi_2')\|_{L^2_\tau L^2_{\xi_1\xi_1'\xi_2'}} \qquad (4.20)$$

To prove (4.20), we prove

$$\|\sigma^{(2)}\|_{L_t^2 L_{x_1}^6 L_{x_1' x_2'}^2} \lesssim \|\langle\tau + 2|\xi_1|^2 - |\xi_1'|^2 - |\xi_2'|^2\rangle^{\frac{1}{2}+}\hat{\sigma}^{(2)}(\tau, \xi_1, \xi_1', \xi_2')\|_{L_\tau^2 L_{\xi_1\xi_1'\xi_2'}^2} \tag{4.21}$$

The estimate (4.20) follows from the interpolation of (4.21) and the trivial equality

$$\|\sigma^{(2)}\|_{L_t^2 L_{x_1}^2 L_{x_1' x_2'}^2} = \|\langle\tau + 2|\xi_1|^2 - |\xi_1'|^2 - |\xi_2'|^2\rangle^{0}\hat{\sigma}^{(2)}(\tau, \xi_1, \xi_1', \xi_2')\|_{L_\tau^2 L_{\xi_1\xi_1'\xi_2'}^2}$$

Thus proving (4.19) is reduced to proving (4.21), which we do now. Let

$$\phi_\tau(x_1, x_1', x_2) \stackrel{\text{def}}{=} \mathcal{F}_0[U^1(-2t)U^{1'}(-t)U^{2'}(-t)\sigma^{(2)}(t, x_1, x_1', x_2')](\tau) \tag{4.22}$$

Then note ϕ_τ is independent of t and

$$\sigma^{(2)}(t, x_1, x_1', x_2') = \int e^{it\tau}U^1(2t)U^{1'}(t)U^{2'}(t)\phi_\tau(x_1, x_1', x_2')d\tau$$

Thus

$$\begin{aligned}\|\sigma^{(2)}\|_{L_t^2 L_{x_1}^6 L_{x_1' x_2'}^2} &\lesssim \int_\tau \|U^{1'}(t)U^{2'}(t)U^1(2t)\phi_\tau(x_1, x_1', x_2')\|_{L_t^2 L_{x_1}^6 L_{x_1' x_2'}^2}\, d\tau \\ &\lesssim \int_\tau \|U^1(2t)\phi_\tau(x_1, x_1', x_2')\|_{L_t^2 L_{x_1}^6 L_{x_1' x_2'}^2}\, d\tau \\ &\lesssim \int_\tau \|U^1(2t)\phi_\tau(x_1, x_1', x_2')\|_{L_{x_1' x_2'}^2 L_t^2 L_{x_1}^6}\, d\tau\end{aligned}$$

Now apply Keel-Tao [33] endpoint Strichartz estimate to obtain

$$\begin{aligned}&\lesssim \int_\tau \|\phi_\tau(x_1, x_1', x_2')\|_{L_{x_1' x_2'}^2 L_{x_1}^2}\, d\tau \\ &\lesssim \|\langle\tau\rangle^{\frac{1}{2}+}\phi_\tau(x_1, x_1', x_2')\|_{L_\tau^2 L_{x_1 x_1' x_2'}^2}\end{aligned}$$

It follows from (4.22) that

$$= \|\langle\tau + 2|\xi_1|^2 - |\xi_1'|^2 - |\xi_2'|^2\rangle^{\frac{1}{2}+}\hat{\sigma}^{(2)}(\tau, \xi_1, \xi_1', \xi_2')\|_{L_{\tau\xi_1\xi_1'\xi_2'}^2}$$

which completes the proof of (4.21). ∎

Corollary 4.1. *Let*

$$\beta^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k') = N^{3\beta-1}V(N^\beta(x_i - x_j))\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k')$$

Then for $N \geq 1$*, we have*

$$\|\,|\nabla_{x_i}|\,|\nabla_{x_j}|\,\beta^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}} \lesssim N^{\frac{5}{2}\beta-1}\|\langle\nabla_{x_i}\rangle\langle\nabla_{x_j}\rangle\gamma^{(k)}\|_{L_t^2 L_{\mathbf{x}\mathbf{x}'}^2} \tag{4.23}$$

and

$$\|\beta^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}} \lesssim N^{\frac{1}{2}\beta-1}\|\langle\nabla_{x_i}\rangle\langle\nabla_{x_j}\rangle\gamma^{(k)}\|_{L_t^2 L_{\mathbf{x}\mathbf{x}'}^2} \tag{4.24}$$

Consequently, $(R_{\leq M}^{(k)} = P_{\leq M}^{(k)}R^{(k)})$

$$\|R_{\leq M}^{(k)}\beta^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}} \lesssim N^{\frac{1}{2}\beta-1}\min(M^2, N^{2\beta})\|S^{(k)}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{x}\mathbf{x}'}^2} \tag{4.25}$$

Proof. Estimate (4.23) follows by applying either (4.12), (4.13), or (4.14) according to whether two derivatives, no derivatives, or one derivative, respectively, lands on $N^{3\beta-1}V(N^\beta(x_i-x_j))$.

Estimate (4.24) follows by applying (4.12).

Finally, (4.25) follows from (4.23) and (4.24), as follows. Let

$$Q = \prod_{\substack{1\leq \ell \leq k \\ \ell \neq i,j}} |\nabla_{x_\ell}|$$

Then

$$\|R_{\leq M}^{(k)}\beta^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}} \leq M^2 \|Q\beta^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}}$$

The Q operator passes directly onto $\gamma^{(k)}$, and one applies (4.24) to obtain

$$\|R_{\leq M}^{(k)}\beta^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}} \lesssim N^{\frac{1}{2}\beta-1}M^2\|S^{(k)}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{x}\mathbf{x}'}^2} \tag{4.26}$$

On the other hand,

$$\|R_{\leq M}^{(k)}\beta^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}} \leq \|Q\,|\nabla_{x_i}|\,\left|\nabla_{x_j}\right|\beta^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}}$$

The Q operator passes directly on $\gamma^{(k)}$, and one applies (4.23) to obtain

$$\|R_{\leq M}^{(k)}\beta^{(k)}\|_{X_{-\frac{1}{2}+}^{(k)}} \lesssim N^{\frac{5}{2}\beta-1}\|S^{(k)}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{x}\mathbf{x}'}^2} \tag{4.27}$$

Combining (4.26) and (4.27), we obtain (4.25). ∎

5. Conclusion

In this paper, we have established a positive answer to Conjecture 1 by Klainerman and Machedon [37] in 2008 for $\beta \in (0, 2/3)$. This is the first progress in proving Conjecture 1 for self-interaction ($\beta > 1/3$). Moreover, our main theorem (Theorem 1.1) has already fulfilled the original intent of [37], namely, simplifying the uniqueness argument of [23] which deals with $\beta \in (0, 3/5)$. Conjecture 1 for $\beta \in [2/3, 1]$ is still open.

Appendix A. The Topology on the Density Matrices

In this appendix, we define a topology τ_{prod} on the density matrices as was previously done in [20, 21, 22, 23, 24, 25, 35, 9, 14, 15]

Denote the spaces of compact operators and trace class operators on $L^2\left(\mathbb{R}^{3k}\right)$ as $\mathcal{K}_k$ and $\mathcal{L}_k^1$, respectively. Then $(\mathcal{K}_k)' = \mathcal{L}_k^1$. By the fact that $\mathcal{K}_k$ is separable, we select a dense countable subset $\{J_i^{(k)}\}_{i\geqslant 1} \subset \mathcal{K}_k$ in the unit ball of $\mathcal{K}_k$ (so $\|J_i^{(k)}\|_{\text{op}} \leqslant 1$ where $\|\cdot\|_{\text{op}}$ is the operator norm). For $\gamma^{(k)}, \tilde{\gamma}^{(k)} \in \mathcal{L}_k^1$, we then define a metric d_k on $\mathcal{L}_k^1$ by

$$d_k(\gamma^{(k)}, \tilde{\gamma}^{(k)}) = \sum_{i=1}^{\infty} 2^{-i} \left| \operatorname{Tr} J_i^{(k)} \left(\gamma^{(k)} - \tilde{\gamma}^{(k)}\right)\right|.$$

A uniformly bounded sequence $\gamma_N^{(k)} \in \mathcal{L}_k^1$ converges to $\gamma^{(k)} \in \mathcal{L}_k^1$ with respect to the weak* topology if and only if

$$\lim_N d_k(\gamma_N^{(k)}, \gamma^{(k)}) = 0.$$

For fixed $T>0$, let $C\left([0,T],\mathcal{L}_k^1\right)$ be the space of functions of $t\in[0,T]$ with values in $\mathcal{L}_k^1$ which are continuous with respect to the metric d_k. On $C\left([0,T],\mathcal{L}_k^1\right)$, we define the metric

$$\hat{d}_k(\gamma^{(k)}(\cdot),\tilde{\gamma}^{(k)}(\cdot))=\sup_{t\in[0,T]}d_k(\gamma^{(k)}(t),\tilde{\gamma}^{(k)}(t)).$$

We can then define a topology τ_{prod} on the space $\oplus_{k\geqslant 1}C\left([0,T],\mathcal{L}_k^1\right)$ by the product of topologies generated by the metrics $\hat{d}_k$ on $C\left([0,T],\mathcal{L}_k^1\right)$.

Appendix B. Proof of Estimates (2.7) and (2.9)

Proof of Estimate (2.7). Utilizing Lemma 2.1 and estimate (3.2) to the free part of $\gamma_N^{(2)}$, we obtain

$$\begin{aligned}
&\left\Vert P_{\leq M}^{(1)}R^{(1)}B_{N,1,2}FP^{(k)}(t_2)\right\Vert_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\\
\leqslant\ &\left\Vert R^{(1)}B_{N,1,2}FP^{(k)}(t_2)\right\Vert_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\\
\leqslant\ &CT^{\frac{1}{2}}\left\Vert R^{(2)}\gamma_{N,0}^{(2)}\right\Vert_{L_{\mathbf{x},\mathbf{x}'}^2}+\sum_{j=3}^{k}\sum_m\left\Vert R^{(1)}B_{N,1,2}\int_D J_N(\underline{t}_j,\mu_m)(U^{(j)}(t_j)\gamma_{N,0}^{(j)})d\underline{t}_j\right\Vert_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\\
\leqslant\ &CT^{\frac{1}{2}}\left\Vert R^{(2)}\gamma_{N,0}^{(2)}\right\Vert_{L_{\mathbf{x},\mathbf{x}'}^2}+\sum_{j=3}^{k}\sum_m C(CT^{\frac{1}{2}})^{j-2}\left\Vert R^{(j-1)}B_{N,\mu_m(j+1),j+1}U^{(j)}(t_j)\gamma_{N,0}^{(j)}\right\Vert_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\\
\leqslant\ &CT^{\frac{1}{2}}\left\Vert R^{(2)}\gamma_{N,0}^{(2)}\right\Vert_{L_{\mathbf{x},\mathbf{x}'}^2}+C\sum_{j=3}^{\infty}4^{j-2}(CT^{\frac{1}{2}})^{j-1}\left\Vert R^{(j)}\gamma_{N,0}^{(j)}\right\Vert_{L_{\mathbf{x},\mathbf{x}'}^2}.
\end{aligned}$$

Via condition (1.4), we can choose a T small enough such that the series in the above estimate converge. Whence, we have shown estimate (2.7). ∎

Proof of Estimate (2.9). We proceed like the proof of estimate (2.7) and end up with

$$\begin{aligned}
&\left\Vert P_{\leq M}^{(1)}R^{(1)}B_{N,1,2}IP^{(k)}\right\Vert_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\\
\leqslant\ &\left\Vert R^{(1)}B_{N,1,2}IP^{(k)}\right\Vert_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\\
\leqslant\ &\sum_m\left\Vert R^{(1)}B_{N,1,2}\int_D J_N(\underline{t}_{k+1},\mu_m)(\gamma_N^{(k+1)}(t_{k+1}))d\underline{t}_{k+1}\right\Vert_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}\\
\leqslant\ &\sum_m C(CT^{\frac{1}{2}})^{k-1}\left\Vert R^{(k)}B_{N,\mu_m(k+1),k+1}\gamma_N^{(k+1)}(t_{k+1})\right\Vert_{L_T^1L^2{}_{\mathbf{x},\mathbf{x}'}}.
\end{aligned}$$

We then investigate

$$\left\Vert R^{(k)}B_{N,\mu_m(k+1),k+1}\gamma_N^{(k+1)}\right\Vert_{L_T^1L_{\mathbf{x},\mathbf{x}'}^2}.$$

Set $\mu_m(k+1) = 1$ for simplicity and look at $B^1_{N,1,k+1}$, we have

$$\begin{aligned}
&\int \left| R^{(k)} B^1_{N,1,k+1} \gamma_N^{(k+1)}(t) \right|^2 d\mathbf{x}_k d\mathbf{x}'_k \\
= &\int \left| R^{(k)} \int V_N(x_1 - x_{k+1}) \gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) dx_{k+1} \right|^2 d\mathbf{x}_k d\mathbf{x}'_k \\
\leqslant\ &C \int \left| \int V'_N(x_1 - x_{k+1}) \left(\prod_{j=2}^k |\nabla_{x_j}| \right) \left(\prod_{j=1}^k \left|\nabla_{x'_j}\right| \right) \gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) dx_{k+1} \right|^2 d\mathbf{x}_k d\mathbf{x}'_k \\
&+C \int \left| \int V_N(x_1 - x_{k+1}) R^{(k)} \gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) dx_{k+1} \right|^2 d\mathbf{x}_k d\mathbf{x}'_k \\
= &C(I + II).
\end{aligned}$$

We estimate I and II as following:

$$\begin{aligned}
&I \\
= &\int \left| \int V'_N(x_1 - x_{k+1}) \left(\prod_{j=2}^k |\nabla_{x_j}| \right) \left(\prod_{j=1}^k \left|\nabla_{x'_j}\right| \right) \gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) dx_{k+1} \right|^2 d\mathbf{x}_k d\mathbf{x}'_k \\
\leqslant &\int d\mathbf{x}_k d\mathbf{x}'_k \left(\int |V'_N(x_1 - x_{k+1})|^2 dx_{k+1} \right) \\
&\times \left(\int \left| \left(\prod_{j=2}^k |\nabla_{x_j}| \right) \left(\prod_{j=1}^k \left|\nabla_{x'_j}\right| \right) \gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) \right|^2 dx_{k+1} \right) \quad \text{(Cauchy-Schwarz)} \\
\leqslant &N^{5\beta} \|V'\|^2_{L^2} \int \left(\int \left| \left(\prod_{j=2}^k |\nabla_{x_j}| \right) \left(\prod_{j=1}^k \left|\nabla_{x'_j}\right| \right) \gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) \right|^2 dx_{k+1} \right) d\mathbf{x}_k d\mathbf{x}'_k \\
\leqslant &CN^{5\beta} \|V'\|^2_{L^2} \int d\mathbf{x}_k d\mathbf{x}'_k \\
&\times \left(\int \left| \left\langle \nabla_{x_{k+1}} \right\rangle \left\langle \nabla_{x'_{k+1}} \right\rangle \left(\prod_{j=2}^k |\nabla_{x_j}| \right) \left(\prod_{j=1}^k \left|\nabla_{x'_j}\right| \right) \gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x'_{k+1}) \right|^2 dx_{k+1} dx'_{k+1} \right) \\
&\text{(Trace Theorem)} \\
\leqslant &CN^{5\beta} \|V'\|^2_{L^2} \left\| S^{(k+1)} \gamma^{(k+1)} \right\|^2_{L^\infty_T L^2_{\mathbf{x},\mathbf{x}'}}
\end{aligned}$$

and

$$\begin{aligned}
II &= \int \left| \int V_N(x_1 - x_{k+1}) R^{(k)} \gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}; \mathbf{x}'_k, x_{k+1}) dx_{k+1} \right|^2 d\mathbf{x}_k d\mathbf{x}'_k \\
&\leqslant CN^{3\beta} \|V\|^2_{L^2} \left\| S^{(k+1)} \gamma^{(k+1)} \right\|^2_{L^\infty_T L^2_{\mathbf{x},\mathbf{x}'}} \quad \text{(Same method as } I\text{)},
\end{aligned}$$

where $V \in W^{2,\frac{6}{5}+}$ implies that $V \in H^1$ by Sobolev. Accordingly,

$$\int \left| R^{(k)} B^1_{N,1,k+1} \gamma_N^{(k+1)}(t) \right|^2 d\mathbf{x}_k d\mathbf{x}'_k \leqslant CN^{5\beta} \|V\|^2_{H^1} \left\| S^{(k+1)} \gamma^{(k+1)} \right\|^2_{L^\infty_T L^2_{\mathbf{x},\mathbf{x}'}} .$$

Thence

$$\begin{aligned}
&\left\|P_{\leq M}^{(1)}R^{(1)}B_{N,1,2}IP^{(k)}\right\|_{L_T^1 L^2_{\mathbf{x},\mathbf{x}'}} \\
\leqslant\ & \sum_m C(CT^{\frac{1}{2}})^{k-1}\left\|R^{(k)}B_{N,\mu_m(k+1),k+1}\gamma_N^{(k+1)}(t_{k+1})\right\|_{L_T^1 L^2_{\mathbf{x},\mathbf{x}'}} \\
\leqslant\ & 4^{k-1}C(CT^{\frac{1}{2}})^{k-1}T\left(CN^{\frac{5\beta}{2}}\left\|V\right\|_{H^1}\left\|S^{(k+1)}\gamma^{(k+1)}\right\|^2_{L_T^\infty L^2_{\mathbf{x},\mathbf{x}'}}\right) \\
\leqslant\ & C\left\|V\right\|_{H^1}(T^{\frac{1}{2}})^{k+2}N^{\frac{5\beta}{2}}C^k. \text{ (Condition (1.4))}
\end{aligned}$$

As in [11, 15], take the coupling level $k = \ln N$, we have

$$\left\|P_{\leq M}^{(1)}R^{(1)}B_{N,1,2}IP^{(k)}\right\|_{L_T^1 L^2_{\mathbf{x},\mathbf{x}'}} \leqslant C\left\|V\right\|_{H^1}(T^{\frac{1}{2}})^{2+\ln N}N^{\frac{5\beta}{2}}N^c.$$

Selecting T such that

$$T \leqslant e^{-(5\beta+2C)},$$

ensures that

$$(T^{\frac{1}{2}})^{\ln N}N^{\frac{5\beta}{2}}N^c \leqslant 1,$$

and thence

$$\left\|P_{\leq M}^{(1)}R^{(1)}B_{N,1,2}IP^{(k)}\right\|_{L_T^1 L^2_{\mathbf{x},\mathbf{x}'}} \leqslant C,$$

where C is independent of N and M. Whence, we have finished the proof of estimate (2.9). ∎

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