EFFECTIVE DYNAMICS OF DOUBLE SOLITONS FOR PERTURBED MKDV

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ABSTRACT. We consider the perturbed mKdV equation $\partial_t u = -\partial_x(\partial_x^2 u + 2u^3 - b(x,t)u)$ where the potential $b(x,t) = b_0(hx, ht)$, $0 < h \ll 1$, is slowly varying with a double soliton initial data. On a dynamically interesting time scale the solution is $\mathcal{O}(h^2)$ close in H^2 to a double soliton whose position and scale parameters follow an effective dynamics, a simple system of ordinary differential equations. These equations are formally obtained as Hamilton's equations for the restriction of the mKdV Hamiltonian to the submanifold of solitons. The interplay between algebraic aspects of complete integrability of the unperturbed equation and the analytic ideas related to soliton stability is central in the proof.

1. Introduction

We consider 2-soliton solutions to the modified Korteweg-de Vries (mKdV) equation with a slowly varying external potential

$$\begin{gathered} \partial_t u = \partial_x(-\partial_x^2 u + bu - 2u^3)\,, \\ b = b(x,t) = b_0(hx, ht)\,, \quad 0 < h \ll 1\,, \quad \partial^\alpha b_0 \in L^\infty(\mathbb{R}^2)\,. \end{gathered} \tag{1.1}$$

The purpose of the paper is to find minimal *exact* effective dynamics valid for a long time in the semiclassical sense and describing non-perturbative 2-soliton interaction. The semiclassical parameter, h, quantifies the slowly varying nature of the potential.

The equation (1.1) is chosen as a mathematically and numerically simpler model than the nonlinear Schrödinger equation (NLS) with a slowly varying potential

$$\partial_t u = -i(-\partial_x^2 u + bu - 2|u|^2 u)\,. \tag{1.2}$$

Like (1.1), the equation (1.2) is a perturbation of a completely integrable nonlinear equation which possesses multisoliton solutions. The dynamics of such multisoliton solutions in the presence of an external potential is in fact physically relevant to experiments involving Bose-Einstein condensation – see [32].

For (1.2), one can draw an analogy to the semiclassical dynamics of coherent states – well-localized solutions of the linear Schrödinger equation with slowly varying potential – see for instance [7] or the introduction to [30]. In this one-particle quantum mechanical setting, the natural long time for which the semiclassical approximation is valid is the Ehrenfest time, $\log(1/h)/h$. For the nonlinear equation (1.2), Fröhlich-Gustafson-Jonsson-Sigal [13] obtained effective dynamics of 1-soliton propagation valid up to the Ehrenfest time. However, unlike the corresponding linear case, the exact effective dynamics of 1-soliton solutions to the nonlinear equation (1.2) valid for such a long time requires h^2-size corrections[†]. Those corrections appeared as unspecified $\mathcal{O}(h^2)$ additions to Newton's equations (which give the usual semiclassical approximation) in the aforementioned work [13]. That paper and its symplectic point of view were the starting point for our recent work [18, 19] in which the exact dynamics for 1-soliton solutions to (1.2) were obtained.

Following the 1-soliton analysis of [18, 19] for (1.2), the semiclassical dynamics for 2-solitons considered here for (1.1) is obtained by restricting the Hamiltonian to the symplectic manifold of 2-solitons and considering the finite dimensional dynamics there. The numerical experiments [17] show a remarkable agreement with the theorem below. However, they also reveal an interesting scenario not covered by our theorem: the velocities of the solitons can almost cross within an exponentially small width in h and the effective dynamics remains valid. Any long time analysis involving multiple interactions of solitons has to explain this avoided crossing which perhaps could be replaced by a direct crossing in a different parametrization. This seems the most immediate open problem of phenomenological interest.

In a recent numerical study Potter [30] showed that the same effective dynamics applies very well to N-solitons both in the case of perturbed mKdV (1.1) and perturbed NLS (1.2). The soliton matter-wave trains created for Bose-Einstein condensates [32] were a good testing ground and our effective dynamics gives an alternative explanation of the observed phenomena. At the moment it is not clear how to analytically obtain rigorous exact effective dynamics for the perturbed NLS (1.2), while a result yielding inexact equations in the spirit of [13] seems comparatively accessible.

Our method of analysis follows a long tradition of the use of modulation parameters in problems of orbital and asymptotic stability of single solitons, and the pertubative interaction of multiple solitons for nonintegrable equations – see for instance [34, 8, 24, 25, 29, 31] and the numerous references given there. For nonlinear dispersive equations with non-constant coefficients one can consult, in addition, [13, 3, 14, 15], and references given there. Here we avoid generality and, as described above, the aims are more modest: for an equation with an underlying completely integrable structure, using classical methods, we can give a remarkably accurate and phenomenologically relevant description of 2-soliton interaction with an external field, while allowing for nonperturbative self-interaction of the 2-soliton. The Lyapunov functional needed in our proof was previously employed by Maddocks-Sach [23] to prove the stability of KdV multisolitons and relies on the existence of higher-order conserved energies stemming from the completely integrable structure.

[†]A compensation for that comes however at having the semiclassical propagation accurate for larger values of h.

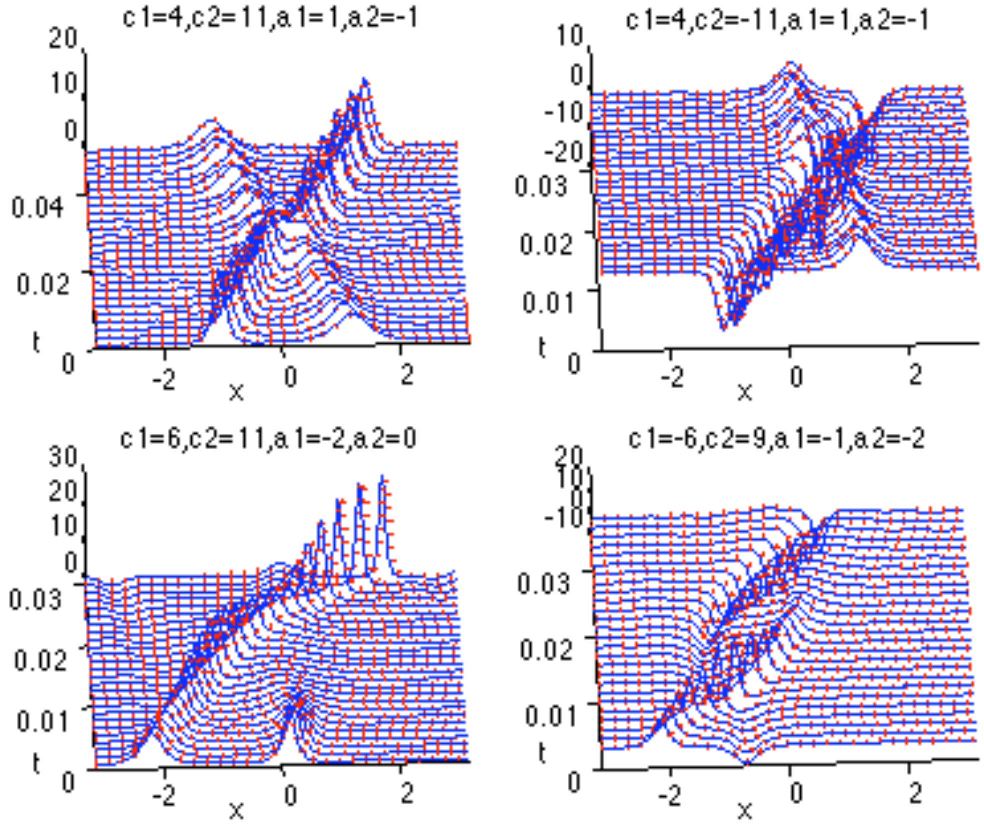

FIGURE 1. A gallery of numerical experiments showing agreement with the results of the main theorem (clockwise from the left hand corner) for the external fields listed in (1.16) with the indicated initial data. The continuous lines are the numerically computed solutions and the dotted lines follow the evolution given by (1.5). The main theorem does not apply to the bottom two figures on the whole interval of time due to the crossing of c_j's – see Fig.3. In the first figure in the second line, (1.5) still apply directy, but in the second one further modification is needed to account for the signs.

To state the exact result we recall that the unperturbed case of (1.1), $b \equiv 0$, is completely integrable and has N-soliton solutions expressed in terms of profiles $q_N(x, a, c)$ dependent upon a positional parameter $a \in \mathbb{R}^N$ and a scale parameter $c \in \mathbb{R}^N$ – see §1.1 and §3 below for detailed discussion. For $N = 2$ we obtain

Theorem. *Let* $\delta_0 > 0$ *and* $\bar{a}, \bar{c} \in \mathbb{R}^n$. *Suppose that* $u(x,t)$ *solves* (1.1) *with*

$$u(x,0) = q_2(x, \bar{a}, \bar{c}) \,, \quad |\bar{c}_1 \pm \bar{c}_2| > 2\delta_0 > 0 \,, \quad 2\delta_0 < |\bar{c}_j| < (2\delta_0)^{-1} \,. \tag{1.3}$$

Then, for $t < T(h)/h$*,*

$$\|u(\cdot,t) - q_2(\cdot,a(t),c(t))\|_{H^2} \leq Ch^2 e^{Cht}\,, \quad C = C(\delta_0, b_0) > 0\,, \tag{1.4}$$

where $a(t)$ *and* $c(t)$ *evolve according to the* effective *equations of motion,*

$$\begin{gathered} \dot{a}_j = c_j^2 - \operatorname{sgn}(c_j)\partial_{c_j} B(a,c,t)\,, \quad \dot{c}_j = \operatorname{sgn}(c_j)\partial_{a_j} B(a,c,t) \\ B(a,c,t) \stackrel{\text{def}}{=} \frac{1}{2}\int b(x,t) q_2(x,a,c)^2\,dx\,. \end{gathered} \tag{1.5}$$

The upper bound $T(h)/h$ *for the validity of* (1.4) *is given in terms of*

$$T(h) = \min(\delta \log(1/h), T_0(h))\,, \qquad \delta = \delta(\delta_0, b_0) > 0 \tag{1.6}$$

where for $t < T_0(h)/h$*,* $|c_1(t) \pm c_2(t)| > \delta_0 > 0$ *and* $\delta_0 < |c_j(t)| < \delta_0^{-1}$*. Under the assumption* (1.3) *on* $\bar{c}$*,* $T_0(h) > \delta_2$*, where* $\delta_2 = \delta_2(\delta_0, b_0) > 0$ *is independent of* h *– see* (1.13)*.*

Remarks.

1. As shown by the top two plots in Fig.1 the agreement of the approximations given by (1.5) and numerical solutions of (1.1) is remarkable. The codes are available at [17], see also §1.4.

2. For external potentials with nondegenerate maxima, the limiting Ehrenfest time $\log(1/h)/h$ appears to be optimal as the errors behave like $\mathcal{O}(h^2 \exp(Cht))$ – see [30, §4.3] – provided we insist on the agreement with classical equations of motion (1.5). We expect that the solution is close to a soliton profile $q_2(x,a,c)$ for much longer times ($h^{-\infty}$?) but with a modified evolution for the parameters. One difficulty is the lack of a good description of the long time behaviour of time dependent linearized evolution with b present – see §8. However, the modified equations would lack the transparency of (1.5) and would be harder to implement. The numerical study [30] suggests that for the *minimal exact* dynamics the error bound $\mathcal{O}(h^2)$ in (1.4) is optimal.

3. The condition that $|c_1(t) \pm c_2(t)| > \delta_1$, that is, that the perturbed effective dynamics avoids the lines shown in Fig.2, could most likely be relaxed. Allowing that provides more interesting dynamics as then the solitons can interact multiple times. As discussed in §1.2 and Appendix B, we expect avoided crossing after $\pm c_j(t)$'s get within $\exp(-c/h)$ of each other – see Fig.3. Examples of such evolution, and the comparisons with effective dynamics, are shown in the lower two plots in Fig.1. On closer inspection the agreement between the solutions and solitons moving according to effective dynamics is not as dramatic as in the case when $\pm c_j$'s stay away from each other but for smaller values of h the result should still hold. We concentrated on the simpler case at this early stage.

4. Studies of single solitons for perturbed KdV, mKdV, and their generalizations were conducted by Dejak-Jonsson [10] and Dejak-Sigal [11]. The mKdV results of [10] are improved by following [19]. For KdV one does not expect the same behaviour

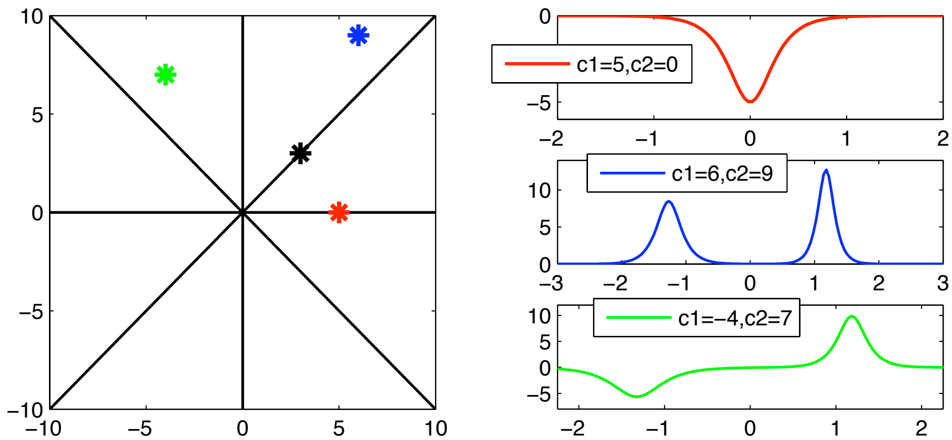

FIGURE 2. On the left we show $\mathbb{R}^2 \setminus \mathcal{C}$ and on the right examples of double solitons corresponding to (c_1, c_2) indicated on the left (with $a_1 = a_2 = 0$ in the first figure and $a_2 = -a_1 = 1$, in the other two). At the coordinate axes the double soliton degenerates into a single soliton. As one approaches the lines $c_1 = \pm c_2$ the solitons escape to infinities in opposite direction.

as for mKdV and the $\mathcal{O}(h^2)$-approximation similar to (1.4) is not valid – see the recent work by Muñoz [27] and the first author [16] for finer analysis of that case.
5. We expect the same result to be true for N-solitons for all N with the function space H^2 replaced by H^N. For $N = 1$ it follows directly from the arguments of [19]. That case is also implicit in this paper: single soliton dynamics describes the propagation away from the interaction region.
6. The condition that $u(x, 0) = q_2(x, \bar{a}, \bar{c})$ can be relaxed by allowing a small perturbation in H^2 – see [9] for the adaptation of [19] to that case. Similar statements are possible here but we prefer the simpler formulation both in the statement of the theorem and in the proofs.
7. The equation (1.1) is globally well-posed in H^k, $k \geq 1$ under even milder regularity hypotheses on b. This can be shown by modifying the techniques of Kenig-Ponce-Vega [21] – see Appendix A. Although for $k \geq 2$ more classical methods are available, we opt for a self-contained treatment dealing with all H^k's at once.

In the remainder of the introduction we will explain the origins of the effective dynamics (1.5), outline the proof, and comment on numerical experiments.

1.1. **Double solitons for mKdV.** The *single soliton* solutions to mKdV, (1.1) with $b \equiv 0$, are described in terms of the profile $\eta(x, a, c)$ as follows. Let $\eta(x) = \operatorname{sech} x$ so that $-\eta + \eta'' + 2\eta^3 = 0$, and let $\eta(x, c, a) = c\eta(c(x - a))$ for $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus 0$. Then a single soliton defined by

$$u(x,t) = \eta(x, a + c^2 t, c)$$

is easily verified to be an exact solution to mKdV. Such solitary wave solutions are available for many nonlinear evolution equations. However, mKdV has richer structure – it is completely integrable and can be studied using the inverse scattering method (Miura [26], Wadati [34]). One of the consequences is the availability of larger families of explicit solutions. In the case of mKdV, we have N-*solitons* and *breathers*. In this paper we confine our attention to the 2-soliton (or double soliton), which is described by the profile $q_2(x, a, c)$ defined in (3.2) below. The four real parameters, $a \in \mathbb{R}^2$, and $c \in \mathbb{R}^2 \setminus \mathcal{C}$,

$$\mathcal{C} \stackrel{\text{def}}{=} \{(c_1, c_2) \ : \ c_1 = \pm c_2\} \cup \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} \,,$$

describe the position (a) and scale (c) of the double soliton. At the diagonal lines the parametrization degenerates: for $c_1 = \pm c_2$, $q_2 \equiv 0$. At the coordinate axes in the c space, we recover single solitons:

$$q_2(x, a, (c_1, 0)) = -c_1 \eta(x, a_1, c_1) \,, \quad q_2(x, a, (0, c_2)) = c_2 \eta(x, a_2, c_2) \,.$$

Fig.2 shows a few examples.

Solving mKdV with $u(x, 0) = q_2(x, a, c)$ gives the solution

$$u(x,t) = q_2(x, a_1 + tc_1^2, a_2 + tc_2^2, c) \,,$$

that is, the double soliton solution.

If, say, $0 < c_1 < c_2$, then for $|a_1 - a_2|$ large,

$$q(x, a, c) \approx \eta(x, a_1 + \alpha_1, c_1) + \eta(x, a_2 + \alpha_2, c_2)$$

where α_j are shifts defined in terms of c, see Lemma 3.2 for the precise statement. This means that for large positive and negative times the evolving double soliton is effectively a sum of single solitons. The decomposition can be made exact preserving the particle-like nature of single solitons even during the interaction – see (3.11) and Fig.4.

We consider the set of 2-solitons as a submanifold of $H^2(\mathbb{R}; \mathbb{R})$ with 8 open components corresponding to the components of $\mathbb{R}^2 \setminus \mathcal{C}$:

$$M = \{ \, q(\cdot, a, c) \, | \, a = (a_1, a_2) \in \mathbb{R}^2 \,, c = (c_1, c_2) \in \mathbb{R}^2 \setminus \mathcal{C} \, \} \,. \tag{1.7}$$

As in the case of single solitons this submanifold is symplectic with respect to the natural structure recalled in the next subsection.

1.2. **Dynamical structure and effective equations of motion.** The equation (1.1) is a Hamiltonian equation of evolution for

$$H_b(u) = \frac{1}{2}\int (u_x^2 - u^4 + bu^2)dx\,, \tag{1.8}$$

on the Schwartz space, $\mathcal{S}(\mathbb{R};\mathbb{R})$ equipped with the symplectic form

$$\omega(u,v) = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{x}(u(x)v(y) - u(y)v(x))dydx\,. \tag{1.9}$$

In other words, (1.1) is equivalent to

$$u_t = \partial_x H_b'(u)\,, \quad \langle H_b'(u), \varphi\rangle \stackrel{\text{def}}{=} \frac{d}{ds}H_b(u+s\varphi)|_{s=0}\,, \tag{1.10}$$

and $\partial_x H_b'(u)$ is the Hamilton vector field of H_b, Ξ_{H_b}, with respect to ω:

$$\omega(\varphi, \Xi_{H_b}(u)) = \langle H_b'(u), \varphi\rangle\,.$$

For $b = 0$, Ξ_{H_0} is tangent to the manifold of solitons (1.7). Also, M is symplectic with respect to ω, that is, ω is nondegenerate on T_uM, $u \in M$. Using the stability theory for 2-solitons based on the work of Maddocks-Sachs [23], and energy methods (enhanced and simplified using algebraic identities coming from complete integrability of mKdV) we will show that the solution to (1.1) with initial data on M stays close to M for $t \le \log(1/h)/h$.

A basic intuition coming from symplectic geometry then indicates that $u(t)$ stays close to an integral curve on M of the Hamilton vector field (defined using $\omega|_M$) of H_b *restricted* to M:

$$\begin{gathered}
H_{\text{eff}}(a,c) \stackrel{\text{def}}{=} H_b|_M(a,c) = H_0|_M(a,c) + \frac{1}{2}\int b(x)q_2(x,a,c)^2dx\,,\\
H_0|_M(a,c) = -\frac{1}{3}(|c_1|^3 + |c_2|^3)\,,\\
\omega|_M = da_1\wedge d|c_1| + da_2 \wedge d|c_2|\,,\\
\Xi_{H_{\text{eff}}} = \sum_{j=1}^{2}\operatorname{sgn}(c_j)(\partial_{a_j}H_{\text{eff}}\,\partial_{c_j} - \partial_{c_j}H_{\text{eff}}\,\partial_{a_j})\,.
\end{gathered} \tag{1.11}$$

The effective equations of motion (1.5) follow. This simple but crucial observation was made in [18],[19] and it did not seem to be present in earlier mathematical work on solitons in external fields [13].

The condition made in the theorem, that $|c_1(t) \pm c_2(t)|$ and $|c_j(t)|$ are bounded away from zero for $t < T_0(h)/h$ (where $T_0(h)$ could be ∞), follows from a condition involving a simpler system of decoupled h-independent ODEs – see Appendix B. Here we state a condition which gives an h-independent T_0 appearing in (1.6).

Suppose we are given $b(x,t) = b_0(hx, ht)$ in (1.1) and the initial condition is given by $q_2(x, \bar a, \bar c)$, $\bar a = (\bar a_1, \bar a_2)$, $\bar c = (\bar c_1, \bar c_2)$, $|\bar c_1 \pm \bar c_2| > \delta_0$, $|\bar c_j| > \delta_0$, We consider an h-independent system of two decoupled differential equations for

$$A(T) = (A_1(T), A_2(T))\,, \quad C(T) = (C_1(T), C_2(T))\,,$$

given by

$$\begin{cases} \partial_T A_j = C_j^2 - b_0(A_j, T) \\ \partial_T C_j = C_j \partial_x b_0(A_j, T) \end{cases}, \qquad A(0) = \bar a h\,, \quad C(0) = \bar c\,, \quad j = 1, 2\,. \tag{1.12}$$

Then, for a given $\delta_1 < \delta_0$, $T_0(h)$ in (1.6) can be replaced by

$$T_0 \stackrel{\text{def}}{=} \sup\{T \;:\; |C_1(T) \pm C_2(T)| > \delta_1\,,\ |C_j(T)| > \delta_1\,, \quad j = 1, 2\}\,. \tag{1.13}$$

1.3. **Outline of the proof.** To obtain the effective dynamics we follow a long tradition (see [13] and references given there) and define the *modulation parameters*

$$a(t) = (a_1(t), a_2(t))\,, \quad c(t) = (c_1(t), c_2(t))\,,$$

be demanding that

$$v(x,t) = u(x,t) - q(x, a(t), c(t))\,, \quad q = q_2\,,$$

satisfies symplectic orthogonality conditions:

$$\begin{aligned} &\omega(v, \partial_{a_1} q) = 0 \qquad\qquad &\omega(v, \partial_{a_2} q) = 0 \\ &\omega(v, \partial_{c_1} q) = 0 \qquad\qquad &\omega(v, \partial_{c_2} q) = 0 \end{aligned}$$

These can be arranged by the implicit function theorem thanks to the nondegeneracy of $\omega|_M$. This makes q the symplectic orthogonal projection of u onto the manifold of solitons M.

Since $u = q + v$ and u solves mKdV, we have

$$\partial_t v = \partial_x(\mathcal{L}_{c,a} v - 6qv^2 - 2v^3 + bv) - F_0\,, \tag{1.14}$$

where

$$\mathcal{L}_{c,a} = -\partial_x^2 - 6q(x,a,c)^2 v\,,$$

and F_0 results from the perturbation and ∂_t landing on the parameters:

$$F_0 = \sum_{j=1}^{2} (\dot a_j - c_j^2)\partial_{a_j} q + \sum_{j=1}^{2} \dot c_j \partial_{c_j} q - \partial_x(bq)\,.$$

We decompose $F_0 = F_\parallel + F_\perp$, where $F_\parallel$ is symplectic projection of F_0 onto $T_q M$, and $F_\perp$ is the symplectic projection onto its symplectic orthogonal $(T_q M)^\perp$. As seen in (5.4), $F_\parallel \equiv 0$ is equivalent to the equations of motion (1.5) (we assume in the proof that $c_2 > c_1 > 0$).

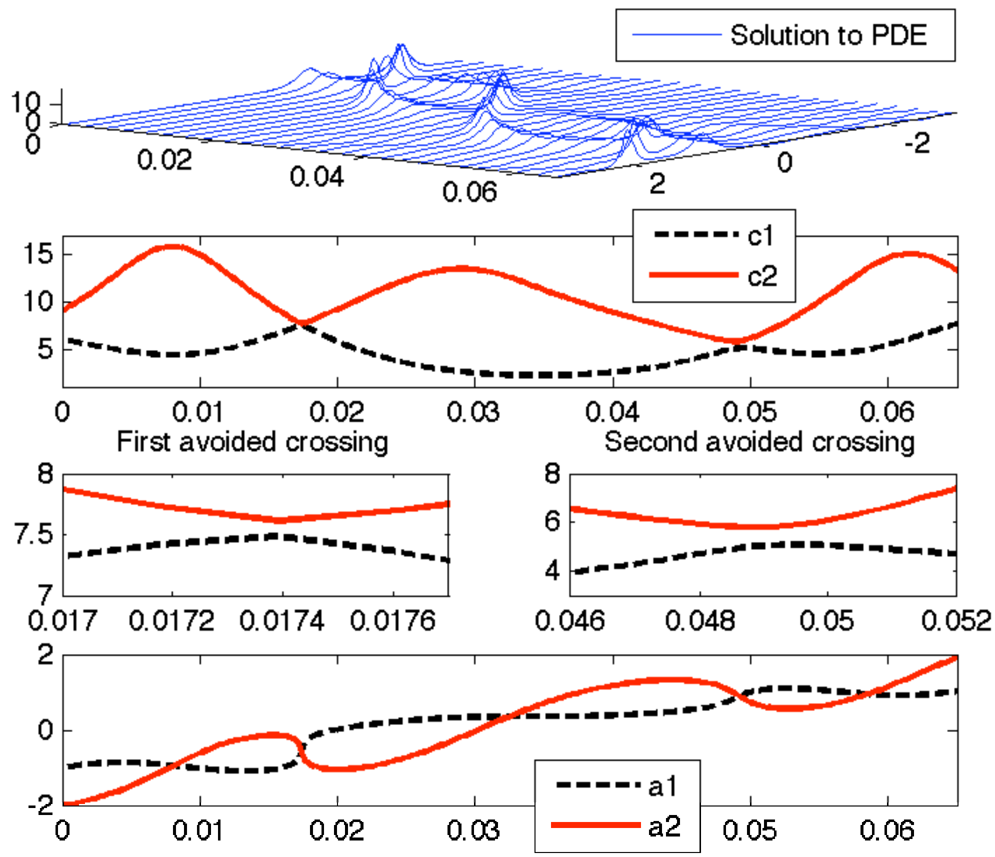

FIGURE 3. The plots of c and a for the external potential given by the last $b(x,t)$ in (1.16), and $\bar{c} = (6,10)$, $\bar{a} = (-1,-2)$. We see the avoided crossings near times at which the decoupled dynamics (1.12) would give a crossing of c_j's (see also Fig.6). The crossings are avoided with $\exp(-1/Ch)$ width and $a_1 = a_2$ at the crossings. These cases are not yet covered by our theory. Of the five crossings of a_j's in the bottom figure, three do not involve crossings of c_j's are hence the description by effective dynamics there is covered by our theorem. However, in the absence of avoided crossing of c_j's the solitons can interact only once.

Using the properties of q, we show that $F_\perp$ is $\mathcal{O}(h^2)$. In fact it is important to obtain a specific form for the $\mathcal{O}(h^2)$ term so that it is amenable to finding a certain correction term later – see §6.

The estimates for $F_{\|}$ are obtained using the symplectic orthogonality properties of v. For example, $0 = \langle v, \partial_x^{-1}\partial_{a_j} q\rangle$ implies

$$0 = \partial_t \langle v, \partial_x^{-1}\partial_{a_j} q\rangle = \langle \underbrace{\partial_t v}_{\substack{\uparrow \\ \text{substitute equation (1.14)}}}, \partial_x^{-1}\partial_{a_j} q\rangle + \langle v, \partial_t \partial_x^{-1}\partial_{a_j} q\rangle \,,$$

which can be used to show that

$$|F_{\|}| \leq C h^2 \|v\|_{H^2} + \|v\|_{H^2}^2 \,, \tag{1.15}$$

see §7.

The next step is to estimate v satisfying (1.14) with $v(0) = \mathcal{O}(h^2)$ (in the theorem $v(0) = 0$, but we need this relaxed assumption for the bootstrap argument). We want to show that on a time interval of length h^{-1}, that v at most doubles. The Lyapunov functional $\mathcal{E}(t)$ that we use to achieve this comes from the variational characterization of the double soliton (see [22, §2] and Lemma 4.1 below): if

$$H_c(u) = I_5(u) + (c_1^2 + c_2^2) I_3(u) + c_1^2 c_2^2 I_1(u) \,,$$

then

$$H_c'(q(\cdot, a, c)) = 0 \,, \qquad \forall \; a \in \mathbb{R}^2 \,,$$

and

$$H_c''(q(\cdot, a, c)) = \mathcal{K}_{c,a} \,,$$

where $\mathcal{K}_{c,a}$ is a fourth order operator given in (4.11) below. Hence

$$\mathcal{E}(t) \stackrel{\text{def}}{=} H_{c(t)}(q(\bullet, a(t), c(t)) + v(t)) - H_{c(t)}(q(\bullet, a(t), c(t))) \,,$$

satisfies

$$\mathcal{E}(t) \approx \langle \mathcal{K}_{c,a} v, v\rangle \,,$$

and, as in Maddocks-Sachs [23] for KdV, $\mathcal{K}_{c,a}$ has a two dimensional kernel and one negative eigenvalue. However, the symplectic orthogonality conditions on v imply that we project far enough away from these eigenspaces and hence we have the coercivity

$$\delta \|v\|_{H^2}^2 \leq \mathcal{E}(t) \,.$$

To get the upper bound on $\mathcal{E}(t)$, we compute

$$\frac{d}{dt}\mathcal{E}(t) = \mathcal{O}(h) \|v(t)\|_{H^2}^2 + \langle \mathcal{K}_{c,a} v, \; F_{\|}\rangle + \langle \mathcal{K}_{c,a} v, \; F_{\perp}\rangle \,,$$

see §9. Using (1.15) we can estimate the second term on the right-hand side but $|F_\perp| = \mathcal{O}(h^2)$ only. We improve this to h^3 using a correction term to v – see §8, and the comment at the end of this section.

All of this combined gives, on $[0, T]$,

$$\|v\|_{H^2}^2 \lesssim \|v(0)\|_{H^2}^2 + T(|F_{\|}| \|v\|_{H^2} + h^2 \|v\|_{H^2} + \|v\|_{H^2}^2) \,,$$
$$|F_{\|}| \leq C h^2 \|v\|_{H^2} + \|v\|_{H^2}^2 \,,$$

which implies

$$\|v\|_{H^2} \lesssim h^2 , \qquad |F_{\|}| \lesssim h^4 , \qquad \text{on} \ \ [0, h^{-1}] .$$

Iterating the argument $\delta \log(1/h)$ times gives a slightly weaker bound for longer times. The $O(h^4)$ errors in the ODEs can be removed without affecting the bound on v, proving the theorem.

In the proofs various facts due to complete integrability (such as the miraculous Lemma 2.1) simplify the arguments, in particular in the above energy estimate.

We conclude with the remark about the correction term added to v in order to improve the bound on $\|F_\perp\|$ from h^2 to h^3. A similar correction term was used in [19] for NLS 1-solitons. Together with the symplectic projection interpretation, it was the key to sharpening the results in earlier works. Implementing the same idea in the setting of 2-solitons is more subtle. The 2-soliton is treated as if it were the sum of two decoupled 1-solitons, the corrections are introduced for each piece, and the result is that $F_\perp$ is corrected so that

$$\|F_\perp\|_{H^2} \lesssim h^3 + h^2 e^{-\gamma |a_1 - a_2|}$$

That is, when $|a_1 - a_2| = O(1)$, there is no improvement. However, this happens only on an $O(1)$ time scale and hence does not spoil the long time estimate.

1.4. **Numerical experiments.** The mKdV equation is very friendly from the numerical point of view and `MATLAB` is sufficient for producing good results.

We first describe the simple codes on which our experiments are based. Instead of considering (1.1) on the line, we consider it on the circle identified with $[-\pi, \pi)$. To solve it numerically we adapt the code given in [33, Chapter 10] which is based on the Fast Fourier Transform in x, the method of integrating factor for the $-u_{xxx} \mapsto -ik^3 \hat{u}(k)$ term, and the fourth-order Runge-Kutta formula for the resulting ODE in time. Unless the amplitude of the solution gets large (which results in large terms in the equation due to the u^3 term) it suffices to take 2^N, $N = 8$, discretization points in x.

For $X \in [-\pi, \pi)$ we consider $B(X, T)$ periodic in X, and compute $U(X, T)$ satisfying

$$\partial_T U = -\partial_X (\partial_X^2 U + 2U^3 - B(X,T) U) , \ \ U(\pi, T) = U(-\pi, T) .$$

A simple rescaling,

$$u(x, t) = \alpha U(\alpha x, \alpha^3 t) , \ \ b(x, t) = \alpha^2 B(\alpha x, \alpha^3 t) ,$$

gives a solution of (1.1) on $[-\pi/\alpha, \pi/\alpha]$ with periodic boundary conditions. When α is small this is a good approximation of the equation on the line. If we use $U(X, T)$ in our numerical calculations with the initial data $q_2(X, A, C)$, $A \in \mathbb{R}^2$, $C \in \mathbb{R}^2 \setminus \mathcal{C}$, the initial condition on for $u(x, t)$ is given by

$$u(x, 0) = q_2(x, A/\alpha, \alpha C) .$$

If we want $\bar{c} = \alpha C$ to satisfy the assumptions (1.3), the effective small constant h becomes $h = \alpha$ and b_0 in (1.1) becomes

$$b_0(x,t) = h^2 B(x, h^2 t) \,.$$

In principle we have three scales: size of B, size of $\partial_x B$, and size of $\partial_t B$, which should correspond to three small parameters h. For simplicity we just use one scale h in the Theorem.

Figure 1 shows four examples of evolution and comparison with effective dynamics computed using the `MATLAB` codes available at [17]. The external potentials used are given by

$$\begin{aligned} B(x,t) &= 100\cos^2(x - 10^3 t) - 50\sin(2x + 10^3 t)\,, \\ B(x,t) &= 100\cos^2(x - 10^3 t) + 50\sin(2x + 10^3 t)\,, \\ B(x,t) &= 60\cos^2(x + 1 - 10^2 t) + 40\sin(2x + 2 + 10^2 t)\,, \\ B(x,t) &= 40\cos(2x + 3 - 10^2 t) + 30\sin(x + 1 + 10^2 t)\,. \end{aligned} \tag{1.16}$$

The rescaling the fixed size potential used in the theorem, $b_0(x,t) = h^2 B(x, h^2 t)$, means that our h satisfies $h \simeq 1/5$ in the last two examples. In the first two examples the scales in x are different than the ones in t: the potential is not slowly varying in t if $h \simeq 1/10$. The agreement with the main theorem is very good in all cases. However, the theorem in the current version does not apply to the two bottom figures since the condition in (1.13) is not satisfied for the full time of the experiment. See also Fig. 3 and Appendix B.

We have not exploited numerical experiments in a fully systematic way but the following conclusions can be deduced:

- For the case covered by our theorem the agreement with the numerical solution is remarkably close; the same thing is true for times longer than T_0/h, with T_0 defined by (1.13) despite the crossings of C_j's (resulting in the avoided crossing of c_j's) The agreement is weaker but the experiments involve only relatively large value of h.
- The soliton profile persists for long times but we see a deviation from the effective dynamics. This suggest the optimality of the bound $\log(1/h)/h$ in (1.4).
- The slow variation in t required in the theorem can probably be relaxed. For instance, in the top plots in Fig.1 $\max|\partial_t b_0| / \max|\partial_x b_0| \sim 10$, while the agreement with the effective dynamics is excellent. For longer times it does break down as can be seen using the `Bmovie.m` code presented in [17, §3]. An indication that slow variation in time might be removable also comes from [2].
- When the decoupled equations (1.12) predict crossing of C_j's, we observe an avoided crossing of c_j's – see Fig.3 and Fig.6 – with exponentially small width,

$\exp(-1/Ch)$. At such times we also see the crossing of a_j's, though it really corresponds to solitons changing their scale constants – see Fig.7. To have multiple interactions of a pair of solitons, this type of crossing has to occur, and it needs to be investigated further.

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2. Hamiltonian structure and conserved quantities

The symplectic form, at first defined on $\mathcal{S}(\mathbb{R};\mathbb{R})$ is given by

$$\omega(u,v) \stackrel{\text{def}}{=} \langle u, \partial_x^{-1} v\rangle\,, \quad \langle f, g\rangle = \int f g\,, \tag{2.1}$$

where

$$\partial^{-1} f(x) \stackrel{\text{def}}{=} \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{+\infty}\right) f(y)\,dy$$

Then the mKdV (equation (1.1) with $b \equiv 0$) is the Hamiltonian flow $\partial_t u = \partial_x H_0'(u)$ and (1.1) is the Hamiltonian flow $\partial_t u = \partial_x H_b'(u)$, where

$$H_0 = \frac{1}{2}\int (u_x^2 - u^4) \qquad H_b = \frac{1}{2}\int (u_x^2 - u^4 + bu^2)$$

Solutions to mKdV have infinitely many conserved integrals and the first four are given by

$$\begin{aligned} I_0(u) &= \int u\,dx\,, \\ I_1(u) &= \int u^2\,dx\,, \\ I_3(u) &= \int (u_x^2 - u^4)\,dx\,, \\ I_5(u) &= \int (u_{xx}^2 - 10u_x^2u^2 + 2u^6)\,dx\,, \end{aligned}$$

which are the mass, momentum, energy, and second energy, respectively. In this paper we will only use these particular conserved quantities.

We write $I_j(u) = \int A_j(u)$, which means that $A_j(u)$ denotes the j-th Hamiltonian *density*.

For future reference, we record the expressions appearing in the Taylor expansions of these densities,

$$A_j(q+v) = A_j(q) + A_j'(q)(v) + \frac{1}{2}A''(q)(v,v) + \mathcal{O}(v^3)\,, \tag{2.2}$$

$$\begin{aligned}A_1'(q)(v) &= 2qv\,,\\ A_3'(q)(v) &= 2q_xv_x - 4q^3v\,,\\ A_5'(q)(v) &= 2q_{xx}v_{xx} - 20q_xq^2v_x - 20q_x^2qv + 12q^5v\,,\end{aligned}$$

and

$$\begin{aligned}A_1''(q)(v,v) &= 2v^2\,,\\ A_3''(q)(v,v) &= 2v_x^2 - 12q^2v^2\,,\\ A_5''(q)(v,v) &= 2v_{xx}^2 - 20q^2v_x^2 - 20q_x^2v^2 - 80qq_xvv_x + 60q^4v^2\,.\end{aligned}$$

The differentials, $I_j'(q)$, are identified with functions by writing:

$$\langle I_j'(q), v\rangle = \int A_j'(q)(v)\,.$$

It is useful to record a formal expression for $I_j'(q)$'s valid when $A_j(q)$'s are polynomials in $\partial_x^\ell q$:

$$I_j'(q) = \sum_{\ell\geq 0}(-\partial_x)^\ell \frac{\partial A_j(q)}{\partial q_x^{(\ell)}}\,, \quad q_x^{(\ell)} = \partial_x^\ell q\,. \tag{2.3}$$

The Hessians, $I_j''(q)$, are the (self-adjoint) operators given by

$$\langle I_j''(q)v, v\rangle = \int A_j''(q)(v,v)\,.$$

One way to generate the mKdV energies is as follows (see Olver [28]). Let us put

$$\Lambda(u) = -\partial_x^2 - 4u^2 - 4u_x\partial_x^{-1}u\,,$$

and recall that $\Lambda(u)\partial_x$ is skew-adjoint:

$$\begin{aligned}\Lambda(u)\partial_x &= -\partial_x^3 - 4u^2\partial_x - 4u_x\partial_x^{-1}u\partial_x\\ &= -\partial_x^3 - 4u^2\partial_x - 4u_xu + 4u_x\partial_x^{-1}u_x\,,\end{aligned}$$

where we used the formal integration by parts $\partial_x^{-1}(uf_x) = -\partial_x^{-1}(u_xf) + uf$.

With this notation we have the fundamental recursive identity:

$$\partial_x I_{2k+1}'(u) = \Lambda(u)\partial_x I_{2k-1}'(u)\,, \tag{2.4}$$

which together with skew-adjointness of $\Lambda(u)\partial_x$ shows that

$$\langle I_j'(u), \partial_x I_k'(u)\rangle = \langle I_{j-2}'(u), \partial_x I_{k+2}'(u)\rangle\,,$$

for j and k odd (if we use (2.4) with m even the choice $I_{2m}(u) = 0$, for $m > 0$ is consistent). By iteration this shows that

$$\langle I_j'(u), \partial_x I_k'(u)\rangle = 0\,, \quad \forall\, j\,,k\,. \tag{2.5}$$

In fact, since j and k are odd we can iterate all the way down to $j = 1$ and apply (2.3):

$$\begin{aligned}\langle I_1'(u), \partial_x I_{k+j-1}'(u)\rangle &= -\langle \partial_x u_x^{(\ell)}, \sum_{\ell\geq 0} \partial A_{j+k-1}(u)/\partial u_x^{(\ell)}\rangle \\ &= -\int \partial_x(A_{j+k-1}(u))dx = 0 \ .\end{aligned}$$

If u solves mKdV, then $\partial_t u = \frac{1}{2}\partial_x I_3'(u)$ and hence by (2.5) we obtain

$$\partial_t I_j(u) = \langle I_j'(u), \partial_t u\rangle = \frac{1}{2}\langle I_j'(u), \partial_x I_3'(u)\rangle = 0\,.$$

The following identities related to the conservation laws will be needed in §9. Recalling the definition (2.2) of A_j, we have:

Lemma 2.1. *For any function $u \in \mathcal{S}$, and for $b \in C^\infty \cap \mathcal{S}'$, we have*

$$\begin{aligned}&\langle I_1'(u), (bu)_x\rangle = \langle b_x, A_1(u)\rangle \\ &\langle I_3'(u), (bu)_x\rangle = 3\langle b_x, A_3(u)\rangle - \langle b_{xxx}, A_1(u)\rangle \\ &\langle I_5'(u), (bu)_x\rangle = 5\langle b_x, A_5(u)\rangle - 5\langle b_{xxx}, A_3(u)\rangle + \langle b_{xxxxx}, A_1(u)\rangle\end{aligned}$$

Proof. By taking arbitrary $b \in \mathcal{S}$, we see that the claimed formulae are equivalent to

$$\begin{aligned}&u\partial_x I_1'(u) = \partial_x A_1(u)\,, \\ &u\partial_x I_3'(u) = 3\partial_x A_3(u) - \partial_x^3 A_1(u)\,, \\ &u\partial_x I_5'(u) = 5\partial_x A_5(u) - 5\partial_x^3 A_3(u) + \partial_x^5 A_1(u)\,,\end{aligned}$$

and these can be checked by direct computation. □

Lemma 2.2. *For any function $u, q \in \mathcal{S}$, and for $b \in C^\infty \cap \mathcal{S}'$, we have*

$$\begin{aligned}&\langle I_1''(q)v, (bq)_x\rangle - \langle \partial_x I_1'(q), bv\rangle = \langle b_x, A_1'(q)(v)\rangle \\ &\langle I_3''(q)v, (bq)_x\rangle - \langle \partial_x I_3'(q), bv\rangle = 3\langle b_x, A_3'(q)(v)\rangle - \langle b_{xxx}, A_1'(q)(v)\rangle \\ &\langle I_5''(q)v, (bq)_x\rangle - \langle \partial_x I_5'(q), bv\rangle = 5\langle b_x, A_5'(q)(v)\rangle - 5\langle b_{xxx}, A_3'(q)(v)\rangle \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \langle b_{xxxxx}, A_1'(q)(v)\rangle\end{aligned}$$

Proof. Differentiate the formulæ in Lemma 2.1 with respect to u at q in the direction of v. □

3. Double soliton profile and properties

Here we record some properties of mKdV and its double soliton solutions. The parametrization of the family of double solitons follows the presentation for NLS in Faddeev–Takhtajan [12].

The double-soliton is defined in terms of the profile $q(x,a,c)$, where

$$\begin{gathered} a = (a_1, a_2) \in \mathbb{R}^2 , \quad c = (c_1, c_2) \in \mathbb{R}^2 \setminus \mathcal{C} , \\ \mathcal{C} \stackrel{\text{def}}{=} \{(c_1, c_2) \ : \ c_1 = \pm c_2\} \cup \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R} . \end{gathered} \tag{3.1}$$

The profile $q = q_2$ (from now on we drop the subscript 2) is defined by

$$q(x,a,c) = \frac{\det M_1}{\det M} \tag{3.2}$$

where

$$M = [M_{ij}]_{1\le i,j\le 2} , \qquad M_{ij} = \frac{1+\gamma_i\gamma_j}{c_i+c_j} , \qquad M_1 = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{M}} & \gamma_1 \\ & & \gamma_2 \\ \hline 1 & 1 & 0 \end{array} \right]$$

and

$$\gamma_j = (-1)^{j-1} \exp(-c_j(x-a_j)), \quad j = 1, 2 .$$

For conveninece we will consider the

$$0 < c_1 < c_2$$

connected component of $\mathbb{R}^2 \setminus \mathcal{C}$ throughout the paper. Since

$$\begin{aligned} q(x, a_1, a_2, c_1, c_2) &= -q(x, a_2, a_1, c_2, c_1) , \\ q(x, a_1, a_2, -c_1, -c_2) &= -q(-x, -a_1, -a_2, c_1, c_2) , \end{aligned}$$

the only other component to consider would be, say, $0 < -c_1 < c_2$ (see Fig.2), and the analysis is similar.

We should however mention that in numerical experiments it is more useful to introduce a phase parameter $\epsilon = (\epsilon_1, \epsilon_2)$, $\epsilon_j = \pm 1$, and define $\tilde q(x,a,c,\epsilon)$ by (3.2) but with γ_j's replaced by

$$\tilde\gamma_j = (-1)^{j-1} \epsilon_j \exp(-c_j(x-a_j)), \quad j = 1, 2 .$$

We can then check that

$$\tilde q(x,a,c,\epsilon) = q(x,a,(\epsilon_1 c_1, \epsilon_2 c_2)) ,$$

but $\tilde q$ seems more stable in numerical calculations.

The corresponding *double-soliton*

$$u(x,t) = q(x, a_1 + c_1^2 t, a_2 + c_2^2 t, c_1, c_2) \tag{3.3}$$

is an exact solution to mKdV. For the double soliton this can be checked by an explicit calculation but it is a consequence of the inverse scattering method. This is the only place in this paper where we appeal directly to the inverse scattering method. Fig. 4 illustrates some aspects of this evolution.

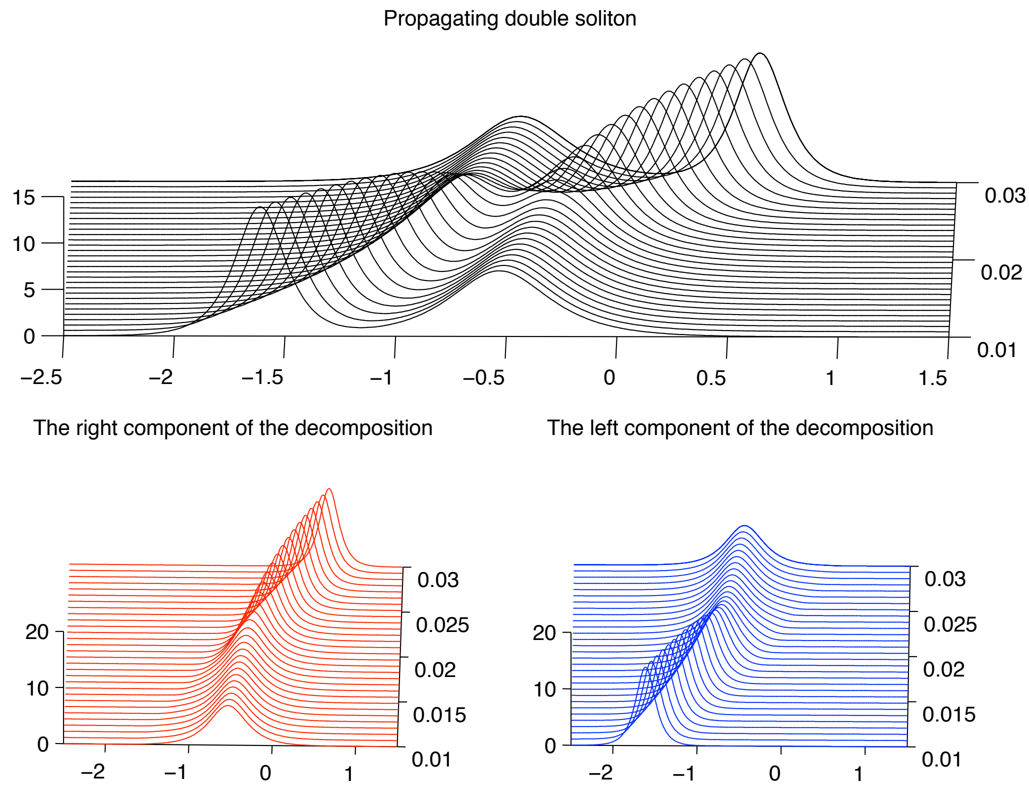

FIGURE 4. A depiction of the double soliton solution given by (3.3). The top figure shows the evolution of a double soliton. The bottom two figures show the evolution of its two components defined using (3.11). One possible "particle-like" interpretation of the two soliton interaction [4] is that the slower soliton, shown in the left bottom plot is hit by the fast soliton shown in the right bottom plot. Just like billiard balls, the slower one picks up speed, and the fast one slows down. But unlike billiard balls, the solitons simply switch velocities.

The scaling properties of mKdV imply that

$$
\begin{aligned}
q(x+t, a+(t,t), c) &= q(x,a,c)\,, \\
q(tx, ta, c/t) &= q(x,a,c)/t\,.
\end{aligned}
\tag{3.4}
$$

Both properties also follow from the formula for q, with the second one being slightly less obvious:

$$
\begin{aligned}
q(tx, ta, c/t) &= \frac{1}{\det tM} \det \left[\begin{array}{c|c} tM & \begin{matrix} \gamma_1 \\ \gamma_2 \end{matrix} \\ \hline 1 \quad 1 & 0 \end{array} \right] \\
&= \frac{1}{\det tM} \det \left(\left[\begin{array}{cc|c} t & 0 & 0 \\ 0 & t & 0 \\ \hline 0 & 0 & 1 \end{array} \right] M_1 \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1/t \end{array} \right] \right) \\
&= q(x, a, c)/t \,.
\end{aligned}
$$

Now we discuss in more detail the properties of the profile q. Recalling that we suppose that $c_2 > c_1 > 0$, let

$$
\alpha_1 \stackrel{\text{def}}{=} \frac{1}{c_1} \log \left(\frac{c_1 + c_2}{c_2 - c_1} \right) , \qquad \alpha_2 \stackrel{\text{def}}{=} \frac{1}{c_2} \log \left(\frac{c_2 - c_1}{c_1 + c_2} \right) , \tag{3.5}
$$

noting that for $c_2 > c_1 > 0$, $\alpha_1 > 0$ and $\alpha_2 < 0$. Fix a smooth function, $\theta \in C^\infty(\mathbb{R}, [0,1])$, such that

$$
\theta(s) = \begin{cases} \ \ 1 & \text{for} \quad s \leq -1 \,, \\ -1 & \text{for} \quad s \geq 1 \,. \end{cases} \tag{3.6}
$$

Define the shifted positions as

$$
\hat{a}_j \stackrel{\text{def}}{=} a_j + \alpha_j \theta(a_2 - a_1) \tag{3.7}
$$

that is,

$$
\hat{a}_j = \begin{cases} a_j + \alpha_j \,, & a_2 \ll a_1 \,, \\ a_j - \alpha_j \,, & a_2 \gg a_1 \,. \end{cases}
$$

see Fig. 5. We note that $\hat{a}_j = \hat{a}_j(a_j, c_1, c_2)$.

Let $\mathcal{S}$ denote the Schwartz space. We will next introduce function classes $\mathcal{S}_{\text{sol}}$ and $\mathcal{S}_{\text{err}}$, and then show that $q \in \mathcal{S}_{\text{sol}}$ and give an approximate expression for q with error in $\mathcal{S}_{\text{err}}$.

Definition 3.1. *Let $\mathcal{S}_{\text{err}}$ denote the class of functions, $\varphi = \varphi(x, a, c)$, $x \in \mathbb{R}$, $a \in \mathbb{R}^2$, $0 < \delta < c_1 < c_2 - \delta < 1/\delta$ (for any fixed δ) satisfying*

$$
\left| \partial_x^\ell \partial_c^k \partial_a^p \varphi \right| \leq C_2 \exp(-(|x - a_1| + |x - a_2|)/C_1) \,,
$$

where C_j depend on δ, ℓ, k, and p only.

Let $\mathcal{S}_{\text{sol}}$ denote the class of functions of (x, a, c) of the form

$$
p_1(c_1, c_2)\varphi_1(c_1(x - \hat{a}_1)) + p_2(c_1, c_2)\varphi_2(c_2(x - \hat{a}_2)) + \varphi(x, a, c)
$$

where

(1) $|\partial_k^\ell \varphi_j(k)| \leq C_\ell \exp(-|k|/C)$, *for some* C,
(2) $p_j \in C^\infty(\mathbb{R}^2 \setminus \mathcal{C})$.

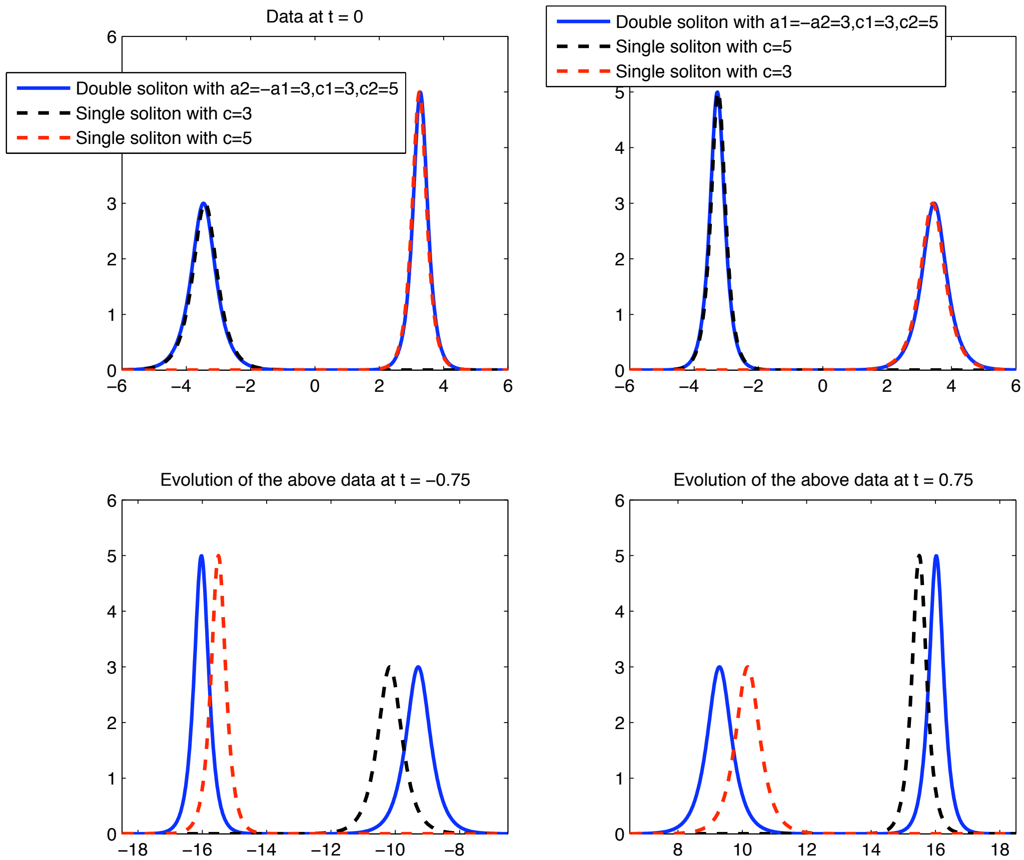

FIGURE 5. The top plots show show $q(x, 3, 5, \mp 3, \pm 3)$, the corresponding $\eta(x, \hat{a}_j, c_j)$ given by Lemma 3.2. The bottom plots show the post-interaction pictures at times ∓ 0.75. Since the sign of $a_2 - a_1$ changes after the interaction we see the shift compared to the evotion of $\eta(x, \hat{a}_j, c_j)$'s.

(3) $\varphi \in \mathcal{S}_{\rm err}$.

Some elementary properties of $\mathcal{S}_{\rm sol}$ and $\mathcal{S}_{\rm err}$ are given in the following.

Lemma 3.1 (properties of $\mathcal{S}_{\rm err}$).

(1) $\partial_x \mathcal{S}_{\rm err} \subset \mathcal{S}_{\rm err}$, $\partial_{a_j} \mathcal{S}_{\rm err} \subset \mathcal{S}_{\rm err}$, $\partial_{c_j} \mathcal{S}_{\rm err} \subset \mathcal{S}_{\rm err}$.
(2) $(x - a_j)\mathcal{S}_{\rm err} \subset \mathcal{S}_{\rm err}$ *and* $(x - \hat{a}_j)\mathcal{S}_{\rm err} \subset \mathcal{S}_{\rm err}$.
(3) *If* $f \in \mathcal{S}_{\rm err}$ *and* $\int_{-\infty}^{+\infty} f = 0$, *then* $\partial_x^{-1} f \in \mathcal{S}_{\rm err}$.

The class $\mathcal{S}_{\rm err}$ allows to formulate the following

Lemma 3.2 (asymptotics for q)**.** *Suppose that* $0 < c_1 < c_2 < c_1/\epsilon < 1/\epsilon^2$, *for* $\epsilon > 0$. *Then for* $|a_2 - a_1| \geq C_0/(c_1 + c_2)$,

$$\left| \partial_x^\ell \partial_c^k \partial_a^p \left(q(x, a, c) - \sum_{j=1}^{2} \eta(x, \hat{a}_j, c_j) \right) \right| \leq C_2 \exp(-(|x - a_1| + |x - a_2|)/C_1) \,, \tag{3.8}$$

where C_2 *depends on* k, ℓ, p *and* ϵ, *and* C_0, C_1 *on* ϵ *only. In other words,*

$$q(x, a, c) - \sum_{j=1}^{2} \eta(x, \hat{a}_j, c_j) \in \mathcal{S}_{\text{err}} \,.$$

Corollary 3.3. $\partial_x^{-1} \partial_{a_j} q$, $\partial_x^{-1} \partial_{c_j} q \in \mathcal{S}_{\text{sol}}$.

Proof. By Lemma 3.2, we have

$$\partial_{c_j} q = \partial_{c_j} \sum_{j=1}^{2} \eta(\cdot, \hat{a}_j, c_j) + f$$

where $f \in \mathcal{S}_{\text{err}}$. By direct computation with the η terms, we find that

$$\int_{-\infty}^{+\infty} \partial_{c_j} \sum_{j=1}^{2} \eta(\cdot, \hat{a}_j, c_j) = 0 \,.$$

By the remark in Lemma 3.5, we have $\int_{-\infty}^{+\infty} \partial_{c_j} q = 0$. Hence $\int_{-\infty}^{+\infty} f = 0$. By Lemma 3.1(3), we have $\partial_x^{-1} f \in \mathcal{S}_{\text{err}}$. Hence

$$\partial_x^{-1} \partial_{c_j} q = \partial_x^{-1} \partial_{c_j} \sum_{j=1}^{2} \eta(\cdot, \hat{a}_j, c_j) + \mathcal{S}_{\text{err}}$$

and the right side is clearly in $\mathcal{S}_{\text{sol}}$. ☐

Proof of Lemma 3.2. We define

$$Q(x, \alpha, \delta) \stackrel{\text{def}}{=} q(x, -\alpha, \alpha, 1 - \delta, 1 + \delta) \,, \tag{3.9}$$

so that, using (3.4),

$$\begin{gathered} q(x, a_1, a_2, c_1, c_2) = \frac{c_1 + c_2}{2} Q \left(\left(\frac{c_1 + c_2}{2} \right) \left(x - \frac{a_1 + a_2}{2} \right), \alpha, \delta \right) \,, \\ \alpha = \left(\frac{c_1 + c_2}{2} \right) \left(\frac{a_2 - a_1}{2} \right) \,, \quad \delta = \frac{c_2 - c_1}{c_2 + c_1} \,. \end{gathered} \tag{3.10}$$

Hence it is enough to study the more symmetric expression (3.9). We decompose it in the same spirit as the decomposition of double solitons for KdV was performed in [4]:

$$Q(x, \alpha, \delta) = \tau(x, \alpha, \delta) + \tau(-x, -\alpha, \delta) \,, \tag{3.11}$$

where

$$\tau(x,\alpha,\delta) = \frac{1}{2}\frac{(1+\delta)\exp((1-\delta)(x+\alpha)) + (1-\delta)\exp((1+\delta)(x-\alpha))}{\delta\operatorname{sech}^2(x-\delta\alpha) + \delta^{-1}\cosh^2(\delta x - \alpha)}\,. \tag{3.12}$$

This follows from a straightforward but tedious calculation which we omit.

Thus, to show (3.8) we have to show that

$$\begin{aligned} &|\partial_x^\ell \partial_\alpha^p \partial_\delta^k (\tau(x,\alpha,\delta) - \eta(x - |\alpha| - \log(1/\delta)/(1\pm\delta), 1\pm\delta))| \\ &\quad \le C_2 \exp(-(|x|+|\alpha|)/C_1)\,, \quad \pm\alpha \gg 1\,, \end{aligned} \tag{3.13}$$

uniformly for $0 < \delta \le 1-\epsilon$.

To see this put $\gamma = (1-\delta)/(1+\delta)$, and multiply the numerator and denominator of (3.12) by $e^{-(1+\delta)(x-\alpha)}$:

$$\tau(x,\alpha,\delta) = \frac{2(1-\delta)\left(1+\gamma^{-1}e^{2\alpha-2\delta x}\right)}{\delta e^{(1-\delta)(x+\alpha)}(1-e^{-2x+2\delta\alpha})^2 + \delta^{-1}e^{-(1-\delta)(x+\alpha)}(1+e^{-2\delta x+2\alpha})^2}\,. \tag{3.14}$$

Similarly, the multiplication by $e^{-(1+\delta)(x-\alpha)}$ gives

$$\begin{aligned} \tau(x,\alpha,\delta) &= \frac{2(1+\delta)\left(1+\gamma e^{-2\alpha+2\delta x}\right)}{\delta e^{(1+\delta)(x-\alpha)}(1-e^{-2x+2\delta\alpha})^2 + \delta^{-1}e^{-(1+\delta)(x-\alpha)}(1+e^{-2\delta x-2\alpha})^2} \\ &= \frac{2(1+\delta)\left(1+\gamma e^{-2\alpha+2\delta x}\right)(1+e^{-2\delta x-2\alpha})^{-2}}{\delta e^{(1+\delta)(x-\alpha)}\left((1-e^{-2x+2\delta\alpha})/(1+e^{-2\delta x-2\alpha})\right)^2 + \delta^{-1}e^{-(1+\delta)(x-\alpha)}}\,. \end{aligned} \tag{3.15}$$

This shows that for negative values of x, τ is negligible: multiplying the numerator and denominator by δ and using (3.14) for $\alpha \le 0$ and (3.15) for $\alpha \ge 0$, gives

$$\tau(x,\alpha,\delta) \le \begin{cases} \delta(1+\delta)(1+e^{-2(|\alpha|+\delta|x|)})e^{-(1+\delta)(|x|+|\alpha|)}\,, & \alpha \ge 0\,, \\ \delta(1+\delta)(1+e^{2\delta|x|-2|\alpha|})^{-1}e^{-(1-\delta)(|x|+|\alpha|)}\,, & \alpha \le 0\,, \end{cases} \tag{3.16}$$

and in fact this is valid uniformly for $0 \le \delta \le 1$. Similar estimates hold also for derivatives.

For $x \ge 0$, $0 \le \delta \le 1-\epsilon$, and for $\alpha \ll -1$, we use (3.14) to obtain,

$$\tau(x,\alpha,\delta) = (1-\delta)\operatorname{sech}\left((1-\delta)\left(x - |\alpha| - \frac{1}{1-\delta}\log\frac{1}{\delta}\right)\right) + \epsilon_-(x,\alpha,\delta)\,,$$

and for $\alpha \gg 1$, (3.15):

$$\tau(x,\alpha,\delta) = (1+\delta)\operatorname{sech}\left((1+\delta)\left(x - |\alpha| - \frac{1}{1+\delta}\log\frac{1}{\delta}\right)\right) + \epsilon_+(x,\alpha,\delta)\,,$$

where

$$|\partial_x^k \epsilon_\pm| \le C_k \exp(-(|x|+|\alpha|)/c)\,, \quad c > 0\,,$$

uniformly in δ, $0 < \delta < 1-\epsilon$. Inserting the resulting decomposition into (3.10) completes the proof. □

Lemma 3.4 (fundamental identities for q). *With $q = q(\cdot, a, c)$, we have*

$$\partial_x I_3'(q) = 2\partial_x(-\partial_x^2 q - 2q^3) = 2\sum_{j=1}^{2} c_j^2 \partial_{a_j} q\,, \tag{3.17}$$

$$\partial_x I_1'(q) = 2\partial_x q = -2\sum_{j=1}^{2} \partial_{a_j} q\,, \tag{3.18}$$

$$q = \sum_{j=1}^{2}(x - a_j)\partial_{a_j} q + \sum_{j=1}^{2} c_j \partial_{c_j} q\,. \tag{3.19}$$

These three identities are analogues of the following three identities for the single-soliton $\eta = \eta(\cdot, a, c)$, which are fairly easily verified by direct inspection.

$$\partial_x I_1'(\eta) = \partial_x \eta = -\partial_a \eta$$

$$\partial_x I_3'(\eta) = \partial_x(-\partial_x^2 \eta - 2\eta^3) = c^2 \partial_a \eta$$

$$\eta = (x - a)\partial_a \eta + c\partial_c \eta$$

Proof. The first identity is just the statement that (3.3) solves mKdV and we take it on faith from the inverse scattering method (or verify it by a computation). To see (3.18) and (3.19) we differentiate (3.4) with respect to t. $\square$

The value of $I_j(q)$ for all j is recorded in the next lemma.

Lemma 3.5 (values of $I_j(q)$).

$$I_0(q) = 2\pi \tag{3.20}$$

For $j = 1, 3, 5$, we have

$$I_j(q) = 2(-1)^{\frac{j-1}{2}} \frac{c_1^j + c_2^j}{j}\,. \tag{3.21}$$

Also,

$$\int x q(x, a, c)^2 \, dx = 2a_1 c_1 + 2a_2 c_2\,. \tag{3.22}$$

Note that by (3.20),

$$\int_{-\infty}^{+\infty} \partial_{a_j} q = 0, \quad \int_{-\infty}^{+\infty} \partial_{c_j} q = 0, \quad j = 1, 2\,.$$

from which it follows that $\partial_x^{-1}(\partial_{a_j} q)$ and $\partial_x^{-1}(\partial_{c_j} q)$ are Schwartz class functions.

Proof. We prove (3.21), (3.20) by reduction to the 1-soliton case. Let $u(t) = q(\cdot, a_1 + tc_1^2, a_2 + tc_2^2, c_1, c_2)$. Then by the asymptotics in Lemma 3.2,

$$I_j(q) = I_j(u(0)) = I_j(u(t)) = \sum_{k=1}^{2} I_j(\eta(\cdot, (a_k + c_k^2 t)\hat{}, c_k)) + \omega(t)$$

where

$$|\omega(t)| \lesssim \langle c_2((a_1 + tc_1^2) - (a_2 + tc_2^2))\rangle^{-2}$$

But note that by scaling,

$$I_j(\eta(\cdot, (a_k + c_k^2 t)\hat{}, c_k)) = c_k^j I_j(\eta)$$

By sending $t \to +\infty$, we find that

$$I_j(q) = (c_1^j + c_2^j) I_j(\eta)$$

To compute $I_j(\eta)$, we let $\eta_c(x) = c\eta(cx)$. By scaling $I_j(\eta_c) = c^j I_j(\eta)$. Hence

$$\begin{aligned} jI_j(\eta) &= \partial_c\big|_{c=1} I_j(\eta_c) = \langle I_j'(\eta), \partial_c\big|_{c=1}\eta_c\rangle \\ &= \langle I_j'(\eta), (x\eta)_x\rangle = 2(-1)^{\frac{j-1}{2}}\langle \eta, (x\eta)_x\rangle = 2(-1)^{\frac{j-1}{2}}\,, \end{aligned}$$

where we have used the identity

$$I_j'(\eta) = 2(-1)^{\frac{j-1}{2}}\eta\,, \tag{3.23}$$

which follows from the energy hierarchy. In fact, $I_1'(\eta) = 2\eta$ is just the definition of I_1'. Assuming that $I_j'(\eta) = 2(-1)^{\frac{j-1}{2}}\eta$, we compute

$$\begin{aligned} \partial_x I_{j+2}'(\eta) &= \Lambda(\eta)\partial_x I_j'(\eta) \\ &= 2(-1)(-1)^{\frac{j-1}{2}}(\partial_x^2 + 4\eta^2 + 4\eta_x\partial_x^{-1}\eta)\eta_x \\ &= 2(-1)^{\frac{j+1}{2}}\partial_x(\eta_{xx} + 2\eta^3) \\ &= 2(-1)^{\frac{j+1}{2}}\partial_x\eta \end{aligned}$$

We now prove (3.22). By direct computation, if $u(t)$ solves mKdV, then $\partial_t \int xu^2 = -3I_3(u)$. Again let $u(t) = q(\cdot, a_1 + tc_1^2, a_2 + tc_2^2, c_1, c_2)$. By (3.21) with $j = 3$, we have

$$\int xq(x,a,c)^2\,dx = \int xu(0,x)^2\,dx = \int xu(t,x)^2\,dx - 2(c_1^3 + c_2^3)t$$

By the asymptotics in Lemma 3.2,

$$\int xu(t,x)^2 = \sum_{j=1}^{2} \int x\eta(x, (a_j + tc_j^2)\hat{}, c_j)^2 + \omega(t)$$

where

$$|\omega(t)| \leq (a_1 + tc_1^2)\langle c_2((a_1 + c_1^2 t) - (a_2 + tc_2^2))\rangle^{-2}$$

But

$$\int x\eta(x,\hat{a}_j,c_j)^2 = 2c_j\hat{a}_j$$

Combining, and using that $c_1\hat{a}_1 + c_2\hat{a}_2 = c_1a_1 + c_2a_2$, we obtain

$$\int xq(x,a,c)^2\,dx = 2(c_1a_1 + c_2a_2) + \omega(t)$$

Send $t \to +\infty$ to obtain the result. □

We define the four-dimensional manifold of 2-solitons M as

$$M = \{\, q(\cdot,a,c) \mid a = (a_1,a_2) \in \mathbb{R}^2\,, c = (c_1,c_2) \in (\mathbb{R})^2 \setminus \mathcal{C}\,\}$$

Lemma 3.6. *The symplectic form* (2.1) *restricted to the manifold of* 2*-solitons is given by*

$$\omega|_M = \sum_{j=1}^{2} da_j \wedge dc_j\,.$$

In particular, it is nondegenerate and M is a symplectic manifold.

Proof. By (3.21) with $j=1$ and (3.18),

$$\begin{aligned} 0 &= \frac{1}{2}\partial_{a_1} I_1(q) = \frac{1}{2}\langle I_1'(q), \partial_{a_1} q\rangle = \langle \partial_{a_1} q, \partial_x^{-1}\partial_{a_1} q\rangle + \langle \partial_{a_2} q, \partial_x^{-1}\partial_{a_1} q\rangle \\ &= \langle \partial_{a_2} q, \partial_x^{-1}\partial_{a_1} q\rangle \end{aligned}$$

Again by (3.21) with $j=1$ and (3.18),

$$1 = \frac{1}{2}\partial_{c_1} I_1(q) = \frac{1}{2}\langle I_1'(q), \partial_{c_1} q\rangle = \langle \partial_{a_1} q, \partial_x^{-1}\partial_{c_1} q\rangle + \langle \partial_{a_2} q, \partial_x^{-1}\partial_{c_1} q\rangle \tag{3.24}$$

By (3.21) with $j=3$ and (3.17),

$$-c_1^2 = \frac{1}{2}\partial_{c_1} I_3(q) = \frac{1}{2}\langle I_3'(q), \partial_{c_1} q\rangle = -c_1^2\langle \partial_{a_1} q, \partial_x^{-1}\partial_{c_1} q\rangle - c_2^2\langle \partial_{a_2} q, \partial_x^{-1}\partial_{c_1} q\rangle \tag{3.25}$$

Solving (3.24) and (3.25), we obtain that $\langle \partial_{a_1} q, \partial_x^{-1}\partial_{c_1} q\rangle = 1$ and $\langle \partial_{a_2} q, \partial_x^{-1}\partial_{c_1} q\rangle = 0$. We similarly obtain that $\langle \partial_{a_2} q, \partial_x^{-1}\partial_{c_2} q\rangle = 1$ and $\langle \partial_{a_1} q, \partial_x^{-1}\partial_{c_2} q\rangle = 0$. It remains to show that $\langle \partial_{c_1} q, \partial_x^{-1} \partial_{c_2} q \rangle = 0$:

$$\begin{aligned}
\langle \partial_{c_1} q, \partial_x^{-1} \partial_{c_2} q \rangle &= \frac{1}{c_1} \langle \sum_{j=1}^{2} c_j \partial_{c_j} q, \partial_x^{-1} \partial_{c_2} q \rangle \\
&= \frac{1}{c_1} \langle q - \sum_{j=1}^{2} (x - a_j) \partial_{a_j} q, \partial_x^{-1} \partial_{c_2} q \rangle && \text{by (3.19)} \\
&= \frac{1}{c_1} \langle q + x q_x, \partial_x^{-1} \partial_{c_2} q \rangle + \frac{1}{c_1} \sum_{j=1}^{2} a_j \langle \partial_{a_j} q, \partial_x^{-1} \partial_{c_2} q \rangle && \text{by (3.18)} \\
&= -\frac{1}{2c_1} \partial_{c_2} \int x q^2 + \frac{a_2}{c_1} \\
&= 0 && \text{by (3.22)}
\end{aligned}$$

□

Remark. If $|a_1 - a_2| \gg 2$, and $c_1 < c_2$ then, in the notation of (3.7),

$$\sum_{j=1,2} da_j \wedge dc_j = \sum_{j=1,2} d\hat{a}_j \wedge dc_j \,,$$

that is the map $(a, c) \mapsto (\hat{a}, c)$ is symplectic.

The nondegeneracy of the symplectic form (2.1) restricted to the manifold of 2-olitons, M shows that H^2 functions close to M can be uniquely decomposed into an element q, of M and a function symplectically orthogonal $T_q M$. We recall this standard fact in the following

Lemma 3.7 (Symplectic orthogonal decomposition)**.** *Given $\tilde{c}$, there exist constants $\delta > 0$, $C > 0$ such that the following holds. If $u = q(\cdot, \tilde{a}, \tilde{c}) + \tilde{v}$ with $\|\tilde{v}\|_{H^2} \leq \delta$, then there exist unique a, c such that*

$$|a - \tilde{a}| \leq C \|\tilde{v}\|_{H^2} \,, \qquad |c - \tilde{c}| \leq C \|\tilde{v}\|_{H^2}$$

and $v \stackrel{\text{def}}{=} u - q(\cdot, a, c)$ satisfies

$$\langle v, \partial_x^{-1} \partial_{a_j} q \rangle = 0 \text{ and } \langle v, \partial_x^{-1} \partial_{c_j} q \rangle = 0 \,,\ j = 1, 2 \,. \tag{3.26}$$

Proof. Let $\varphi : H^2 \times \mathbb{R}^2 \times (\mathbb{R}_+)^2 \to \mathbb{R}^4$ be defined by

$$\varphi(u, a, c) = \begin{bmatrix} \langle u - q(\cdot, a, c), \partial_x^{-1} \partial_{a_1} q \rangle \\ \langle u - q(\cdot, a, c), \partial_x^{-1} \partial_{a_2} q \rangle \\ \langle u - q(\cdot, a, c), \partial_x^{-1} \partial_{c_1} q \rangle \\ \langle u - q(\cdot, a, c), \partial_x^{-1} \partial_{c_2} q \rangle \end{bmatrix}$$

Using that $\omega\big|_M = da_1 \wedge dc_1 + da_2 \wedge dc_2$, we compute the Jacobian matrix of φ with respect to (a, c) at $(q(\cdot, \tilde{a}, \tilde{c}), \tilde{a}, \tilde{c})$ to be

$$D_{a,c}\varphi(q(\cdot, \tilde{a}, \tilde{c}), \tilde{a}, \tilde{c}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

By the implicit function theorem, the equation $\varphi(u, a, c) = 0$ can be solved for (a, c) in terms of u in a neighbourhood of $q(\cdot, \tilde{a}, \tilde{c})$. □

We also record the following lemma which will be useful in the next section:

Lemma 3.8. *Suppose v solves a linearized equation*

$$\partial_t v = \frac{1}{2}\partial_x I_3''(q(t))v = \partial_x(-\partial_x^2 - 6q(t)^2)v\,, \quad q(x,t) = q(x, a_j + tc_j^2, c_j)\,.$$

Then

$$\partial_t \langle v(t), \partial_x^{-1}(\partial_{c_j} q)(t)\rangle = \partial_t \langle v(t), \partial_x^{-1}(\partial_{a_j} q)(t)\rangle = 0\,,$$

where $(\partial_{c_j} q)(t) = (\partial_{c_j} q)(x, a_j + tc_j^2, c_j)$ (and not $\partial_{c_j}(q(x, a_j + tc_j^2, c_j))$). In addition, for $v(0) = \partial_{a_j} q$, $v(t) = (\partial_{a_j} q)(t)$, and for $v(0) = \partial_{c_j} q$,

$$v(t) = (\partial_{c_j} q)(t) + 2c_j t(\partial_{a_j} q)(t)\,.$$

4. Lyapunov functional and coercivity

In this section we introduce the function H_c adapted from the KdV theory of Maddocks-Sachs [23]. We will build our Lyapunov functional $\mathcal{E}$ from H_c.

Thus let

$$H_c(u) \stackrel{\text{def}}{=} I_5(u) + (c_1^2 + c_2^2)I_3(u) + c_1^2 c_2^2 I_1(u)\,.$$

We give a direct proof that $q(\cdot, a, c)$ is a critical point of H_c:

Lemma 4.1 (q is a critical point of H)**.** *We have*

$$H_c'(q(\cdot, a, c)) = 0\,, \tag{4.1}$$

that is

$$I_5'(q) + (c_1^2 + c_2^2)I_3'(q) + c_1^2 c_2^2 I_1'(q) = 0\,.$$

Proof. We follow Lax [22, §2]: we want to find $A = A(q)$ and $B = B(q)$ such that

$$H'(q) \stackrel{\text{def}}{=} I_5'(q) + AI_3'(q) + BI_1'(q) = 0\,,$$

for all $q = q(x, a, c) \in M$. If we consider the mKdV evolution of q given by (3.3), then Lemma 3.2 shows that as $t \to \pm\infty$ we can express $H'(q)$ asymptotically using $H'(\eta_{c_1})$ and $H'(\eta_{c_2})$. From (3.23) we see that

$$H'(\eta_c) = I_5'(\eta_c) + AI_3'(\eta_c) + BI_1'(\eta_c) = 2(c^4 - Ac^2 + B)\eta_c\,.$$

Two parameters c_1 and c_2 are roots of this equation if $A = c_1^2 + c_2^2$ and $B = c_1^2 c_2^2$ and this choice gives

$$
\begin{aligned}
H'(q(t)) = r(t)\,, \ \ \|r(t)\|_{L^2} \leq C \exp(-|t|/C)\,, \\
q(t) \stackrel{\text{def}}{=} q(x, a_1 + c_1^2 t, a_2 + c_2^2 t, c_1, c_2)\,,
\end{aligned}
\tag{4.2}
$$

where the exponential decay of $r(t)$ comes from Lemma 3.2 and the fact that $c_1 \neq c_2$.

To prove (4.1) we need to show that $r(0) \equiv 0$. For the reader's convenience we provide a direct proof of this widely accepted fact. Since it suffices to prove that $\langle r(0), w\rangle = 0$, for all $w \in \mathcal{S}$, we consider the mKdV linearized equation at $q(t)$,

$$
v_t = \frac{1}{2}\partial_x I_3''(q(t))v\,, \quad v(0) = w \in \mathcal{S}\,, \tag{4.3}
$$

and will show that

$$
\partial_t \langle r(t), v(t)\rangle = \partial_t \langle H'(q(t)), v(t)\rangle = 0\,. \tag{4.4}
$$

The conclusion $\langle r(0), w\rangle = 0$ will then follow from showing that

$$
\langle r(t), v(t)\rangle \to 0\,, \quad t \to \infty\,. \tag{4.5}
$$

We first claim that

$$
\partial_t \langle I_k'(q), v\rangle = 0\,, \quad \forall\, k\,.
$$

In fact, from (2.5) we have $\langle I_k'(\varphi), \partial_x I_3'(\varphi)\rangle = 0$ for all $\varphi \in \mathcal{S}$. Differentiating with respect to φ in the direction of v, we obtain

$$
\langle I_k''(\varphi)v, \partial_x I_3'(\varphi)\rangle = -\langle I_k'(\varphi), \partial_x I_3''(\varphi)v\rangle\,.
$$

Applying this with $v = v(t)$ and $\varphi = q(t)$ we conclde that

$$
\begin{aligned}
\partial_t \langle I_k'(q), v\rangle &= \langle I_k''(q)\partial_t q, v\rangle + \frac{1}{2}\langle I_k'(q), \partial_x I_3''(q)v\rangle \\
&= \frac{1}{2}\langle I_k''(q)\partial_x I_3'(q), v\rangle + \frac{1}{2}\langle I_k'(q), \partial_x I_3''(q)v\rangle\,, \\
&= 0\,.
\end{aligned}
$$

Since H is a linear combination of I_k's, $k = 1, 3, 5$, this gives (4.4).

We now want to use the exponential decay of $\|r(t)\|_{L^2}$ in (4.2), and (4.4) to show (4.5). Clearly, all we need is a subexponential estimate on $v(t)$, that is

$$
\forall\, \epsilon > 0\ \exists\, t_0\,, \quad \|v(t)\|_{L^2} \leq e^{\epsilon t}\,, \quad t > t_0\,. \tag{4.6}
$$

Let ψ be a smooth function such that $\psi(x) = 1$ for all $|x| \leq 1$ and $\psi(x) \sim e^{-2|x|}$ for $|x| \geq 1$. With the notation of Lemma 3.2 define

$$
\psi_j(x, t) = \psi(\delta(x - (a_j + c_j^2 t)\widehat{\ }))\,.
$$

for $0 < \delta \ll 1$ to be selected below and $j = 1, 2$. We now establish that

$$\left|\partial_t \left(\|v\|_{L^2}^2 + \|v_x\|_{L^2}^2 + 6 \int q^2 v^2 \right)\right| \lesssim \sum_{j=1}^{2} \|\psi_j v\|_{L^2}^2 . \tag{4.7}$$

To prove (4.7), apply ∂_x^{-1} to (4.3) and pair with v_t to obtain

$$0 = \langle \partial_x^{-1} v_t, v_t \rangle + \langle v_{xx}, v_t \rangle + \langle 6q^2 v, v_t \rangle$$

which implies

$$\partial_t \left(\frac{1}{2} \|v_x\|_{L^2}^2 + 3 \int q^2 v^2 \right) = 6 \int q q_t v^2 \tag{4.8}$$

Next, pair (4.3) with v to obtain

$$0 = \langle v_t, v \rangle + \langle v_{xxx}, v \rangle + 6 \langle \partial_x (q^2 v), v \rangle$$

which implies

$$\partial_t \|v\|_{L^2}^2 = -12 \int q q_x v^2 \tag{4.9}$$

Summing (4.8) and (4.9) gives (4.7).

The inequality (4.7) shows that we need to control is $\|\psi_j v(t)\|$, $j = 1, 2$. For t large ψ_j provides a localization to the region where q decomposes into an approximate sum of decoupled solitons (see Lemma 3.2). Hence we define

$$\mathcal{L}_j = c_j^2 - \partial_x^2 - 6\eta^2(x, (a_j + tc_j^2)\widehat{\ }, c_j)$$

(see also §8 below for a use of similar operators). A calculation shows that

$$t \geq T(\delta) \implies \partial_t \langle \mathcal{L}_j \psi_j v, \psi_j v \rangle = \mathcal{O}(\delta) \|v\|_{H^1}^2 , \tag{4.10}$$

where $T(\delta)$ is large enough to ensure that the supports of ψ_j's are separated. It suffices to assume that $v(0) = w$ satisfies $\langle w, \partial_x^{-1} \partial_{a_j} q \rangle = 0$ and $\langle w, \partial_x^{-1} \partial_{c_j} q \rangle = 0$, since Lemma 3.8 already showed that the evolutions of $\partial_{a_j} q$ and $\partial_{c_j} q$ are linearly bounded in t. Under this assumption, we have by Lemma 3.8 that $\langle v(t), \partial_x^{-1} \partial_{a_j} q(t) \rangle = 0$ and $\langle v(t), \partial_x^{-1} \partial_{c_j} q(t) \rangle = 0$.

We now want to invoke the well known coercivity estimates for operators $\mathcal{L}_j$ – see for instance [18, §4] for a self contained presentation. For that we need to check that

$$|\langle \psi_j v, \partial_x^{-1} \partial_a \eta(\hat{a}_j + tc_j^2, c_j) \rangle| \ll 1 , \quad |\langle \psi_j v, \partial_x^{-1} (\partial_c \eta(\hat{a}_j + tc_j^2, c_j)|\rangle| \ll 1 .$$

This follows from the fact that v is symplectically orthogonal to $(\partial_{c_j} q)(t)$ and $\partial_{a_j} q(t)$ (Lemma 3.8 again), the fact that q decouples into two solitons for t large, and from the remark after the proof of Lemma 3.6.

Hence,

$$\langle \mathcal{L}_j \psi_j v, \psi_j v \rangle \gtrsim \|\psi_j v\|_{H^1}^2 .$$

We now sum (4.7) and (4.10) multiplied by $\delta^{-\frac{1}{2}}$ to obtain, for t suffieciently large (depending on δ),

$$F'(t) \leq C\delta^{\frac{1}{2}} F(t) \,,$$
$$F(t) \stackrel{\text{def}}{=} \|v(t)\|_{H^1}^2 + 6 \int q^2(t)v(t)^2 + \delta^{-\frac{1}{2}} \langle \mathcal{L}_j(t)\psi_j(t)v(t), \psi_j(t)v(t)\rangle$$

(where we added the additional $\int q^2v^2$ term to the right hand side at no cost). Consequently, $F(t) \leq \exp(C'\delta^{\frac{1}{2}}t)$, for $t > T_1(\delta)$.

We recall that this implies (4.6) and going back to (4.4) show that $r(0) = 0$, and hence $H'(q) = 0$. □

We denote the Hessian of H_c at $q(\bullet, a, c)$ by $\mathcal{K}_{c,a}$:

$$\mathcal{K}_{c,a} = I_5''(q) + (c_1^2 + c_2^2)I_3''(q) + c_1^2c_2^2 I_1''(q)$$

It is a fourth order self-adjoint operator on $L^2(\mathbb{R})$ and a calculation shows that

$$\begin{aligned} \frac{1}{2}\mathcal{K}_{c,a} = \ & (-\partial_x^2 + c_1^2)(-\partial_x^2 + c_2^2) \\ & + 10\partial_x\, q^2\partial_x + 10(-q_x^2 + (q^2)_{xx} + 3q^4) - 6(c_1^2 + c_2^2)q^2 \end{aligned} \tag{4.11}$$

Lemma 4.2 (mapping properties of $\mathcal{K}$)**.** *The kernel of $\mathcal{K}_{c,a}$ in $L^2(\mathbb{R})$ is spanned by $\partial_{a_j}q$:*

$$\mathcal{K}_{c,a}\partial_{a_j} q = 0, \tag{4.12}$$

and

$$\mathcal{K}_{c,a}\partial_{c_j} q = 4(-1)^j c_j(c_1^2 - c_2^2)\partial_x^{-1}\partial_{a_j} q \tag{4.13}$$

Proof. Equations (4.12) follow from differentiation of (4.1) with respect to a_j. As $x \to \infty$, the leading part of $\mathcal{K}_{c,a}$ is given by $(-\partial_x^2+c_1^2)(-\partial_x^2+c_2^2)$ and hence the kernel in L^2 is at most two dimensional.

To see (4.13) recall that

$$I_1'(q) = 2q = -2\partial_x^{-1}(\partial_{a_1}q + \partial_{a_2}q)$$

$$I_3'(q) = -2q'' - 4q^3 = 2\partial_x^{-1}(c_1^2\partial_{a_1}q + c_2^2\partial_{a_2}q)\,,$$

where we used Lemma 3.4. By differentiating $H'(q) = I_5'(q) + (c_1^2 + c_2^2)I_3'(q) + c_1^2c_2^2I_1'(q) = 0$ with respect to c_j, we obtain

$$\mathcal{K}(\partial_{c_1}q) = -2c_1(I_3'(q) + c_2^2 I_1'(q))\,, \quad \mathcal{K}(\partial_{c_2}q) = -2c_2(I_3'(q) + c_1^2 I_1'(q))\,. \tag{4.14}$$

Inserting the above formulæ for $I_1'(q)$ and $I_2'(q)$ gives (4.13). □

The main result of this section is the following coercivity result:

Proposition 4.3 (coercivity of $\mathcal{K}$)**.** *There exists $\delta = \delta(c) > 0$ such that for all $v \in H^2$ satisfying the symplectic orthogonality conditions*

$$\langle v, \partial_x^{-1} \partial_{a_j} q \rangle = 0 \ \textit{and} \ \langle v, \partial_x^{-1} \partial_{c_j} q \rangle = 0 \,, \ j = 1, 2 \,,$$

we have

$$\delta \|v\|_{H^2}^2 \leq \langle \mathcal{K}_{c,a} v, v \rangle \,. \tag{4.15}$$

The proposition is proved in a few steps. In Lemma 4.2 we already described the kernel $\mathcal{K}_{c,a}$ and now we investigate the negative eigenvalues:

Proposition 4.4 (Spectrum of $\mathcal{K}$)**.** *The operator $\mathcal{K}_{c,a}$ has a single negative eigenvalue, $h \in L^2(\mathbb{R})$:*

$$\mathcal{K}_{c,a} h = -\mu h \,, \quad \mu > 0 \,. \tag{4.16}$$

In addition, for

$$0 < \delta < c_1 < c_2 - \delta < 1/\delta \,,$$

there exists a constant, ρ, depending only on δ, such that

$$\min\{\lambda > 0 \ : \ \lambda \in \sigma(\mathcal{K}_{c,a})\} > \rho \,, \quad a \in \mathbb{R}^2 \,, \tag{4.17}$$

Proof. As always we assume $0 < c_1 < c_2$. We know the continuous spectrum of $\mathcal{K}_{c,a}$,

$$\sigma_{\rm ac}(\mathcal{K}_{c,a}) = [2c_1^2 c_2^2, +\infty)$$

and that for all a, c, there is a two-dimensional kernel given by $\operatorname{span}\{\partial_{a_1} q, \partial_{a_2} q\}$. The eigenvalues depend continuously on a, c, and hence the constant dimension of the kernel shows that the number of negative eigenvalues is constant (since the creation or annihilation of a negative eigenvalue would increase the dimension of $\ker \mathcal{K}_{c,a}$.)

Hence it suffices to determine the number of negative eigenvalues of $\mathcal{K}$ for any convenient values of a, c. To do that we use the following fact:

Lemma 4.5 (Maddocks-Sachs [23, Lemma 2.2])**.** *Suppose that $\mathcal{K}$ is a self-adjoint, 4th order operator of the form*

$$\mathcal{K} = 2(-\partial_x^2 + c_1^2)(-\partial_x^2 + c_2^2) + p_0(x) - \partial_x p_1(x) \partial_x \,,$$

where the coefficients $p_j(x)$ are smooth, real, and rapidly decaying as $x \to \pm\infty$. Let $r_1(x)$, $r_2(x)$ be two linearly independent solutions of $\mathcal{K} r_j = 0$ such that $r_j \to 0$ as $x \to -\infty$.

Then the number of negative eigenvalues of $\mathcal{K}$ is equal to

$$\sum_{x \in \mathbb{R}} \dim \ker \begin{bmatrix} r_1(x) & r_1'(x) \\ r_2(x) & r_2'(x) \end{bmatrix} \,. \tag{4.18}$$

We apply this lemma with $\mathcal{K} = \mathcal{K}_{c,a}$, in which case

$$p_1 = 20q^2 \,, \quad p_0 = 40q_{xx}q + 20q_x^2 + 60q^4 - 12(c_1^2 + c_2^2)q^2 \,, \quad q = q(\bullet, a, c) \,.$$

Convenient values of a and c are provided by $a_1 = a_2 = 0$ and $c_1 = 0.5$, $c_2 = 1.5$. In the notation of (3.9) we then have $q(x, a, c) = Q(x, 0, 0.5)$, and since

$$\partial_x Q = -\partial_{a_1} q - \partial_{a_2} q \,, \quad \partial_\alpha Q = -\partial_{a_1} q + \partial_{a_2} q \,,$$

we can take $r_1 = \partial_x Q$ and $r_2 = \partial_\alpha Q$. A computation based on (3.11) and (3.12) shows that

$$Q(x, 0.5, 0) = \operatorname{sech}(x/2) \,, \quad \partial_x Q(x, 0.5, 0) = -\frac{\sinh(x/2)}{2\cosh^2(x/2)} \,,$$

$$\begin{aligned} \partial_\alpha Q(x, 0.5, 0) &= \frac{\sinh(x/2)}{4\cosh^4(x/2)}(9 - 2\cosh^2(x/2)) \\ &= \frac{9\sinh(x/2)}{4\cosh^4(x/2)} + \partial_x Q(x, 0.5, 0) \,. \end{aligned} \tag{4.19}$$

Since $x \mapsto y = \sinh(x/2)$ is invertible, we only need to check the dimension of the kernel the Wronskian matrix of

$$\tilde{r}_1(y) = \frac{y}{1 + y^2} \,, \quad \tilde{r}_2(y) = \frac{y}{(1 + y^2)^2} \,,$$

and that is equal to 1 at $y = 0$ and 0 on $\mathbb{R} \setminus \{0\}$. In view of (4.18) this completes the proof of (4.16)

To prove (4.17) we first note that by rescaling (3.10) we only need to prove the estimate for

$$K(c, \alpha) \stackrel{\text{def}}{=} \mathcal{K}_{((c,1),(-\alpha,\alpha))} \,, \quad c \in [\delta, 1 - \delta] \,, \quad 0 < \delta < 1/2 \,.$$

For that we introduce another operator

$$P(c) \stackrel{\text{def}}{=} (-\partial_x^2 + 1)(-\partial_x^2 + c^2) + 10\partial_x \eta^2 \partial_x + 10(3\eta^2 - 2\eta^4) - 6(1 + c^2)\eta^2 \,, \tag{4.20}$$

where

$$\eta = \operatorname{sech} x \,, \quad c \in \mathbb{R}_+ \setminus \{1\} \,.$$

The operator $P(c)$ is the Hessian of $H_{(c,1)}$ at η, which is also a critical point for $H_{(c,1)}$. In particular,

$$P(c)\partial_x \eta = 0 \,.$$

Putting,

$$U_\alpha f(x) \stackrel{\text{def}}{=} f(x + \alpha + \log((1 + c)/(1 - c)))) \,,$$

and

$$P_+(c, \alpha) \stackrel{\text{def}}{=} U_\alpha^* P(c) U_\alpha \,,$$

we see that

$$K(c, \alpha) = 2P_+(c, \alpha) + \mathcal{O}(e^{-(\alpha + |x|)/C})\partial_x^2 + \mathcal{O}(e^{-(\alpha + |x|)/C}) \,, \quad x \geq 0 \,.$$

Similarly, if

$$T_c f(x) \stackrel{\text{def}}{=} \sqrt{c} f(cx) ,$$

and

$$P_-(c,\alpha) \stackrel{\text{def}}{=} c^2 U_\alpha T_c P(1/c) T_c^* U_\alpha^* ,$$

then

$$K(c,\alpha) = 2P_-(c,\alpha) + \mathcal{O}(e^{-(\alpha+|x|)/C})\partial_x^2 + \mathcal{O}(e^{-(\alpha+|x|)/C}) , \quad x \leq 0 .$$

We reduce the estimate (4.17) to a spectral fact about the operators $P(c)$ and $P(1/c)$:

Lemma 4.6. *Suppose that there exists*

$$\alpha \longmapsto \lambda(c,\alpha) \in \mathbb{R} \setminus \{0\}$$

such that

$$\lambda(c,\alpha) \in \sigma(K(c,\alpha)) , \quad \lambda(c,\alpha) \longrightarrow 0 , \; \alpha \longrightarrow \infty .$$

Then we have

$$\dim \ker_{L^2} P(c) + \dim \ker_{L^2} P(1/c) > 2 , \tag{4.21}$$

where $\ker_{L^2}$ *means the kernel in* L^2*.*

Proof. The assumption that $0 \neq \lambda(c,\alpha) \to 0$ as $\alpha \to \infty$ implies that there exists a family of quasimodes f_α, $\|f_\alpha\|_{L^2} = 1$,

$$\|K(c,\alpha) f_\alpha\|_{L^2} = o(1) , \quad \alpha \longrightarrow \infty , \qquad f_\alpha \perp \ker_{L^2} K(c,\alpha) . \tag{4.22}$$

Since we know that the kernel of $K(c,\alpha)$ is spanned by $U_\alpha^* \partial_x \eta + \mathcal{O}(e^{-(|x|+\alpha)/C})$ and $U_\alpha T_c \partial_x \eta + \mathcal{O}(e^{-(|x|+\alpha)/C})$, we can modify f_α and replace the orthogonality condition by

$$f_\alpha \perp \operatorname{span}(U_\alpha^* \partial_x \eta, U_\alpha T_c \partial_x \eta) .$$

The estimate in (4.22), and $\|f_\alpha\|_{L^2} = \mathcal{O}(1)$, imply that

$$\|f_\alpha\|_{H^2} = \mathcal{O}(1) , \quad \alpha \longrightarrow \infty . \tag{4.23}$$

We first claim that

$$\int_{-1}^{1} |f_\alpha(x)|^2 dx = o(1) , \quad \alpha \longrightarrow \infty . \tag{4.24}$$

In fact, on $[-\alpha/2, \alpha/2]$,

$$K(c,\alpha) = (-\partial_x^2 + c^2)(-\partial_x^2 + 1) + \mathcal{O}(e^{-\alpha/C})\partial_x^2 + \mathcal{O}(e^{-\alpha/c}) ,$$

and hence, using (4.23),

$$(-\partial_x^2 + c^2)(-\partial_x^2 + 1) f_\alpha = r_\alpha , \quad \|r_\alpha\|_{L^2([-\alpha/2,\alpha/2])} = o(1) .$$

Putting

$$e_\alpha \stackrel{\text{def}}{=} [(-\partial_x^2 + c^2)(-\partial_x^2 + 1)]^{-1} \left(r_\alpha 1\!\!1_{[-\alpha/2,\alpha/2]} \right) , \quad \|e_\alpha\|_{H^2} = o(1) ,$$

we see that $f_\alpha = g_\alpha + e_\alpha$ where

$$(-\partial_x^2 + c^2)(-\partial_x^2 + 1) g_\alpha(x) = 0 \,, \quad |x| < \alpha/2 \,. \tag{4.25}$$

Suppose now that (4.24) were not valid. Then the same would be true for g_α, and there would exist a constant $c_0 > 0$, and a sequence $\alpha_j \to \infty$, for which $\|g_{\alpha_j}\|_{L^2([-1,1])} > c_0$. In view of (4.25) this implies that

$$g_{\alpha_j}(x) = \sum_{\pm} \left(a_j^\pm e^{\pm x} + b_j^\pm e^{\pm c x} \right) \,, \quad |x| < \alpha/2 \,, \quad |a_j^\pm|, |b_j^\pm| = \mathcal{O}(1) \,,$$

and for at least one choice of sign,

$$|a_j^\pm|^2 + |b_j^\pm|^2 > c_1 > 0 \,.$$

We can choose a subsequence so that this is true for a fixed sign, say, $+$, for all j. In that case, a simple calculation shows that for $M_j \to \infty$, $M_j \le \alpha_j/2$,

$$\begin{aligned}
\int_0^{M_j} |g_{\alpha_j}(x)|^2 dx &\ge \frac{1}{2} |a_j^+|^2 e^{2M_j} + \frac{1}{2c} |b_j^+|^2 e^{2cM_j} - \frac{2}{c+1} |a_j^+||b_j^+| e^{(c+1)M_j} \\
&\quad - \frac{2}{1-c} |a_j^+||b_j^-| e^{(1-c)M_j} - \mathcal{O}(1) \\
&\ge \frac{1}{2} \left(\frac{1-c}{1+c} \right)^2 \left(|a_j^+|^2 e^{2M_j} + \frac{1}{c} |b_j^+|^2 e^{2M_j c} \right) \\
&\quad - \frac{4}{(1-c)^2} |a_j^+|^2 e^{2(1-c)M_j} - \mathcal{O}(1) \,,
\end{aligned}$$

where we used the fact that $0 < \delta < c < 1 - \delta$. Hence

$$\begin{aligned}
\|f_{\alpha_j}\|_{L^2} &\ge \int_0^{M_j} |f_{\alpha_j}(x)|^2 dx \ge \int_0^{M_j} |g_{\alpha_j}(x)|^2 dx - o(1) \\
&\ge \frac{1}{2} \left(\frac{1-c}{1+c} \right)^2 \gamma e^{2M_j c} - \mathcal{O}(1) \longrightarrow \infty \,, \quad j \to \infty \,.
\end{aligned}$$

Since $\|f_\alpha\|_{L^2} = 1$ we obtain a contradiction proving (4.24).

Now let $\chi_\pm C^\infty(\mathbb{R})$ be supported in $\pm[-1, \infty)$, and satisfy $\chi_+^2 + \chi_-^2 = 1$. Then (4.24) (and the corresponding estimates for derivatives obtained from (4.22)) shows that

$$\|P_\pm(c, \alpha)(\chi_\pm f_\alpha)\|_{L^2} = o(1) \,, \quad \alpha \longrightarrow \infty \,.$$

For at least one of the signs we must have $\|\chi_\pm f_\alpha\|_{L^2} > 1/3$ (if α is large enough), and hence we obtain a quasimode for $P_\pm(c, \alpha)$, orthogonal to the known element of the kernel of $P_\pm(c, \alpha)$. This means that $P_\pm(c, \alpha)$, for at least one of the signs has an additional eigenvalue approaching 0 as $\alpha \to \infty$. Since the spectrum of $P_\pm(c, \alpha)$ is independent of α it follows that for at least one sign the kernel is two dimensional. This proves (4.21). □

The next lemma shows that (4.21) is impossible:

Lemma 4.7. *For* $c \in \mathbb{R}_+ \setminus \{1\}$

$$\ker_{L^2} P(c) = \mathbb{C} \cdot \partial_x \eta \,. \tag{4.26}$$

Proof. Let $\mathcal{L} \stackrel{\text{def}}{=} (I_3''(\eta) + I_1''(\eta))/2$:

$$\mathcal{L} v = -v_{xx} - 6\eta^2 v + v \,, \quad \eta(x) = \operatorname{sech}(x) \,.$$

We recall (see the comment after (4.20)) that

$$P(c) = \frac{1}{2} H''_{(c,1)}(\eta) = \frac{1}{2} \left(I_5''(\eta) + (1+c^2) I_3''(\eta) + c^2 I_1''(\eta) \right) \,.$$

We already noted that

$$\mathcal{L}(\partial_x \eta) = P(c) \partial_x \eta = 0 \,,$$

and proceeding as in (4.14) we also have

$$\mathcal{L}(\partial_x(x\eta)) = -2\eta \,, \quad P(c)(\partial_x(x\eta)) = 2(1-c^2)\eta \,. \tag{4.27}$$

We claim that

$$P(c) \partial_x \mathcal{L} = \mathcal{L} \partial_x P(c) \tag{4.28}$$

Since $I_j'(\eta + tv) = t I_j''(\eta) v + \mathcal{O}(t^2)$, $v \in \mathcal{S}$, the equation (2.5) implies that

$$\langle I_j''(\eta) v, \partial_x I_k''(\eta) v \rangle = 0 \,, \quad \forall\, j, k \,, \quad v \in \mathcal{S} \,.$$

From this we see that

$$\langle P(c) v, \partial_x \mathcal{L} v \rangle = 0 \,, \quad \forall v \in \mathcal{S} \,,$$

and hence by polarization,

$$\langle P(c) v, \partial_x \mathcal{L} w \rangle = -\langle P(c) w, \partial_x \mathcal{L} v \rangle = \langle \partial_x P(c) w, \mathcal{L} v \rangle \,.$$

which implies (4.28).

Suppose now that $\dim \ker_{L^2} P(c) = 2$ for some $c \neq 1$, and let η_x and ψ be the basis of this kernel. Since $P(c)$ is symmetric with respect to the reflection $x \mapsto -x$, ψ can be chosen to be either even or odd. Applying (4.28) to ψ we get $P(c)\partial_x \mathcal{L} \psi = 0$ and hence

$$\partial_x \mathcal{L} \psi = \alpha \eta_x + \beta \psi \,,$$

for some $\alpha, \beta \in \mathbb{R}$.

If ψ is odd then $\partial_x \mathcal{L}\psi$ is even, and therefore $\alpha = \beta = 0$. But then $\psi \in \ker_{L^2} \mathcal{L} = \mathbb{C} \cdot \eta_x$, giving a contradiction.

If ψ is even then $\partial_x \mathcal{L} \psi$ is odd, $\beta = 0$ and $\mathcal{L}\psi = \alpha\eta$. We have $\alpha \neq 0$ since ψ is orthogonal to the kernel of $\mathcal{L}$, spanned by $\partial_x \eta$. From (4.27) we obtain

$$\psi = -\frac{\alpha}{2} \partial_x (x\eta) \,.$$

Applying the second equation in (4.27) we then obtain

$$P(c)\psi = -\alpha(1-c^2)\eta \,,$$

contradicting $\psi \in \ker_{L^2} P(c)$. □

With this lemma we complete the proof of Proposition 4.4. □

To obtain the coercivity statement in Proposition 4.3 we first obtain coercivity under a different orthogonality condition:

Lemma 4.8. *There exists a constant $\rho > 0$ depending only on c_1, c_2, such that the following holds: If $\langle u, \partial_x^{-1}\partial_{a_1} q\rangle = 0$, $\langle u, \partial_x^{-1}\partial_{a_2} q\rangle = 0$, $\langle u, \partial_{a_1} q\rangle = 0$, $\langle u, \partial_{a_2} q\rangle = 0$, then $\langle \mathcal{K}_{c,a} u, u\rangle \geq \rho \|u\|_{L^2}^2$.*

Proof. To simplify notation we put $\mathcal{K} = \mathcal{K}_{c,a}$ in the proof. Using (4.13) and the expression for the symplectic form, $\omega\big|_M = da_1 \wedge dc_1 + da_2 \wedge dc_2$, we have

$$\langle \mathcal{K}\partial_{c_1} q, \partial_{c_1} q\rangle = -4c_1(c_1^2 - c_2^2)\langle \partial_x^{-1}\partial_{a_1} q, \partial_{c_1} q\rangle = 4c_1(c_1^2 - c_2^2)$$

and similarly

$$\langle \mathcal{K}\partial_{c_2} q, \partial_{c_2} q\rangle = -4c_2(c_1^2 - c_2^2)\,. \tag{4.29}$$

Since we assumed that $c_1 < c_2$, $\langle \mathcal{K}\partial_{c_1} q, \partial_{c_1} q\rangle < 0$.

Let $\widetilde{\partial_{c_1} q}$ be the orthogonal projection of $\partial_{c_1} q$ on $(\ker \mathcal{K})^\perp$. We first claim that there exists a constant α such that $u = \tilde{u} + \alpha\widetilde{\partial_{c_1} q}$ with $\langle \tilde{u}, h\rangle = 0$, where μ and h are defined in Proposition 4.4.

To prove this, decompose $\partial_{c_1} q$ as $\partial_{c_1} q = \xi + \beta h$ with $\langle \xi, h\rangle = 0$. Then by (4.29)

$$\begin{aligned} 0 &> \langle \mathcal{K}\partial_{c_1} q, \partial_{c_1} q\rangle \\ &= \langle \mathcal{K}\xi, \xi\rangle + 2\beta\langle \mathcal{K}h, \xi\rangle + \beta^2\langle \mathcal{K}h, h\rangle \\ &= \langle \mathcal{K}\xi, \xi\rangle - \mu\beta^2 \end{aligned}$$

Since $\langle \mathcal{K}\xi, \xi\rangle \geq 0$, we must have that $\beta \neq 0$. Hence there exists u' and α such that $u = u' + \alpha\partial_{c_1} q$ with $\langle u', h\rangle = 0$. Now take $\tilde{u}$ to be the projection of u' away from the kernel of $\mathcal{K}$. This completes the proof of the claim.

We have that

$$\langle u, \mathcal{K}\partial_{c_1} q\rangle = -4c_1(c_2^2 - c_1^2)\langle u, \partial_x^{-1}\partial_{a_1} q\rangle = 0$$

by (4.13) and hypothesis. Substituting $u = \tilde{u} + \alpha\widetilde{\partial_{c_1} q}$, we obtain

$$\langle \tilde{u}, \mathcal{K}\partial_{c_1} q\rangle = -\alpha\langle \widetilde{\partial_{c_1} q}, \mathcal{K}\partial_{c_1} q\rangle = -\alpha\langle \partial_{c_1} q, \mathcal{K}\partial_{c_1} q\rangle \tag{4.30}$$

Now let $\tilde{\rho}$ denote the bottom of the positive spectrum of $\mathcal{K}$. We have

$$\begin{aligned} \langle \mathcal{K}u, u\rangle &= \langle \mathcal{K}(\tilde{u} + \alpha\widetilde{\partial_{c_1} q}), (\tilde{u} + \alpha\widetilde{\partial_{c_1} q})\rangle \\ &= \langle \mathcal{K}\tilde{u}, \tilde{u}\rangle + 2\alpha\langle \mathcal{K}\tilde{u}, \partial_{c_1} q\rangle + \alpha^2\langle \mathcal{K}\partial_{c_1} q, \partial_{c_1} q\rangle \\ &= \langle \mathcal{K}\tilde{u}, \tilde{u}\rangle - \alpha^2\langle \mathcal{K}\partial_{c_1} q, \partial_{c_1} q\rangle && \text{by (4.30)} \\ &\geq \tilde{\rho}\|\tilde{u}\|_{L^2}^2 + 4c_1(c_2^2 - c_1^2)\alpha^2 \\ &\geq \tilde{C}(\|\tilde{u}\|_{L^2}^2 + \alpha^2) \end{aligned}$$

where $\tilde{C}$ depends on c_1, c_2 and $\tilde{\rho}$. However, since $u = \tilde{u} + \alpha\widetilde{\partial_{c_1} q}$, we have

$$\|u\|_{L^2}^2 \leq C(\|\tilde{u}\|_{L^2}^2 + \alpha^2)$$

where C depends on c_1, c_2 which completes the proof. $\square$

We now put

$$\begin{aligned} E = E_{a,c} &= \ker \mathcal{K} = \operatorname{span}\{\partial_{a_1} q, \partial_{a_2} q\}\,, \\ F = F_{a,c} &= \operatorname{span}\{\partial_x^{-1}\partial_{c_1} q, \partial_x^{-1}\partial_{c_2} q\}\,, \\ G = G_{a,c} &= \operatorname{span}\{\partial_x^{-1}\partial_{a_1} q, \partial_x^{-1}\partial_{a_2} q\}\,. \end{aligned} \tag{4.31}$$

In this notation Lemma 4.8 states that

$$u \perp (E+G) \implies \langle \mathcal{K}u, u\rangle \geq \theta \|u\|_{L^2}^2\,,$$

while to establish Proposition 4.3 we need

$$u \perp (F+G) \Longrightarrow \langle \mathcal{K}u, u\rangle \geq \tilde{\theta}\|u\|_{L^2}^2\,.$$

That is, we would like to replace orthogonality with the kernel E by orthogonality with a "nearby" subspace F. For this, we apply the following analysis with $D = F^\perp$.

Definition 4.1. *Suppose that D and E are two closed subspaces in a Hilbert space. Then $\alpha(D,E)$, the* angle between D and E, *is*

$$\alpha(D,E) \stackrel{\text{def}}{=} \cos^{-1} \sup_{\substack{\|d\|=1,\, d\in D \\ \|e\|=1,\, e\in E}} \langle d, e\rangle$$

It is clear that $0 \leq \alpha(D,E) \leq \pi/2$, $\alpha(D,E) = \alpha(E,D)$, and that $\alpha(E,D) = \pi/2$ if and only if $E \perp D$. We will need slightly more subtle properties stated in the following

Lemma 4.9. *Suppose that D and E are two closed subspaces in a Hilbert space. Then*

$$\alpha(D,E) = \cos^{-1} \sup_{\|d\|=1, d\in D} \|P_E d\|\,, \quad \alpha(D,E) = \sin^{-1} \inf_{\|d\|=1, d\in D} \|P_{E^\perp} d\|\,. \tag{4.32}$$

In addition if E is finite dimensional then

$$\alpha(D,E) = 0 \iff D \cap E \neq \{0\}\,. \tag{4.33}$$

Proof. To see (4.32) let $d \in D$, with $\|d\| = 1$. By the definition of the projection operator,

$$\begin{aligned} 1 - \|P_E d\|^2 = \|d - P_E d\|^2 = \inf_{e\in E} \|d-e\|^2 &= \inf_{\substack{e\in E \\ \|e\|=1}} \inf_{\alpha\in\mathbb{R}} \|d - \alpha e\|^2 \\ &= \inf_{\substack{e\in E \\ \|e\|=1}} \inf_{\alpha\in\mathbb{R}} (1 - 2\alpha\langle d, e\rangle + \alpha^2) = \inf_{\substack{e\in E \\ \|e\|=1}} (1 - \langle d, e\rangle^2) \\ &= 1 - \sup_{\substack{e\in E \\ \|e\|=1}} \langle d, e\rangle^2 \end{aligned}$$

and consequently,

$$\|P_E d\| = \sup_{\substack{e\in E\\ \|e\|=1}} \langle d, e\rangle \,,$$

from which the first formula in (4.32) follows. The second one is a consequence of the first one as $1 = \|P_E d\|^2 + \|P_{E^\perp} d\|^2$.

The $\Leftarrow$ implication in (4.33) is clear. To see the other implication, we observe that if $D \cap E = \{0\}$ and E is finite dimensional then

$$\inf_{\substack{y\in E\\ \|y\|=1}} d(y, D) > 0 \,,$$

where $d(y,D) = \inf_{z\in D} \|y - z\|$ is the distance from y to D. This implies that

$$\begin{aligned} 0 < \inf_{\substack{y\in E\\ \|y\|=1}} \inf_{z\in D} \|y-z\|^2 &= \inf_{\substack{y\in E\\ \|y\|=1}} \inf_{z\in D} (1 - 2\langle y, z\rangle + \|z\|^2) \\ \leq \inf_{\substack{y\in E\\ \|y\|=1}} \inf_{\substack{z\in D\\ \|z\|=1}} (2 - 2\langle y, z\rangle) &= 2(1 - \sup_{\substack{y\in E\\ \|y\|=1}} \sup_{\substack{z\in D\\ \|z\|=1}} \langle y, z\rangle) \\ = 2(1 - \cos\alpha(D,E)) \,. \end{aligned}$$

Thus if $D \cap E = \{0\}$ then $\alpha(D,E) > 0$. But that is the $\Rightarrow$ implication in (4.33). $\square$

In the notation of (4.31), the translation symmetry gives

$$\alpha(E_{a,c}, F_{a,c}^\perp) = F(c_1, c_2, a_1 - a_2) \,,$$

where F is a continuous fuction in $\mathcal{C} \times \mathbb{R}$. We claim that

$$F(c_1, c_2, \alpha) \geq \kappa_\delta > 0 \quad \text{for} \quad \delta \leq c_1 \leq c_1 + \delta \leq c_2 \leq \delta^{-1} \,. \tag{4.34}$$

Consider now the case $|a_1 - a_2| \leq A$ (where A is chosen large below), and hence c_1, c_2, and $a_1 - a_2$ vary within a compact set. Thus it suffices to check that $\alpha(E_{a,c}, F_{a,c}^\perp)$ is nowhere zero and this amounts to checking $E \cap F^\perp = \{0\}$.

Suppose the contrary, that is that there exists

$$u = z_1 \partial_{a_1} q + z_2 \partial_{a_2} q \in F^\perp \,.$$

Since $\omega\big|_M = da_1 \wedge dc_1 + da_2 \wedge dc_2$,

$$z_j = \langle u, \partial_x^{-1} \partial_{c_j} q\rangle = 0 \,.$$

This proves (4.34). To complete the argument in the case $|a_1 - a_2| \leq A$, we need:

Lemma 4.10. *Let $E = \ker \mathcal{K}$, and suppose that G is a subspace such that $E \perp G$ and the following holds:*

$$u \perp (E + G) \implies \langle \mathcal{K} u, u\rangle \geq \theta \|u\|_{L^2}^2 \,.$$

Then, for any other subspace F we have

$$u \perp (F + G) \implies \langle K u, u\rangle \geq \theta \sin^2 \alpha(E, F^\perp) \, \|u\|_{L^2}^2 \,.$$

Proof. Suppose $u \perp (F+G)$ and consider its orthogonal decomposition, $u = P_E u + \tilde{u}$. Since $E \perp G$ and $u \perp G$, we have $\tilde{u} \perp (E+G)$. Hence, by the hypothesis we have

$$\langle \mathcal{K}u, u\rangle = \langle \mathcal{K}\tilde{u}, \tilde{u}\rangle \geq \theta \|\tilde{u}\|_{L^2}^2 = \theta \|P_{E^\perp} u\|_{L^2}^2 \,.$$

An application of (4.32),

$$\sin\alpha(E, F^\perp) = \inf_{\substack{\|d\|=1 \\ d\in F^\perp}} \|P_{E^\perp} d\|_{L^2} \leq \frac{\|P_{E^\perp} u\|_{L^2}}{\|u\|_{L^2}} \,,$$

concludes the proof. □

5. Set-up of the proof

Recall the definition of T_0 (for given $\delta_0 > 0$ and $\bar{a}$, $\bar{c}$) stated in the introduction. Recall

$$B(a,c,t) \stackrel{\text{def}}{=} \int b(x,t) q^2(x,a,c)\, dx \,.$$

In the next several sections, we establish the key estimates required for the proof of the main theorem. Let us assume that on some time interval $[0,T]$, there are C^1 parameters $a(t) \in \mathbb{R}^2$, $c(t) \in \mathbb{R}^2$ such that, if we set

$$v(\cdot,t) \stackrel{\text{def}}{=} u(\cdot,t) - q(\cdot, a(t), c(t)) \tag{5.1}$$

then the symplectic orthogonality conditions (3.26) hold. Since u solves (1.1), $v(t)$ satisfies

$$\partial_t v = \partial_x(-\partial_x^2 v - 6q^2 v - 6qv^2 - 2v^3 + bv) - F_0 \tag{5.2}$$

where F_0 results from the perturbation and ∂_t landing on the parameters:

$$F_0 \stackrel{\text{def}}{=} \sum_{j=1}^{2} (\dot{a}_j - c_j^2)\partial_{a_j} q + \sum_{j=1}^{2} \dot{c}_j \partial_{c_j} q - \partial_x(bq) \tag{5.3}$$

Now decompose

$$F_0 = F_\parallel + F_\perp$$

where $F_\parallel$ is symplectically parallel to M and $F_\perp$ is symplectically orthogonal to M. Explicitly,

$$F_\parallel = \sum_{j=1}^{2} (\dot{a}_j - c_j^2 + \tfrac{1}{2}\partial_{c_j} B)\partial_{a_j} q + \sum_{j=1}^{2} (\dot{c}_j - \tfrac{1}{2}\partial_{a_j} B)\partial_{c_j} q \tag{5.4}$$

$$F_\perp = -\partial_x(bq) + \frac{1}{2}\sum_{j=1}^{2} [-(\partial_{c_j} B)\partial_{a_j} q + (\partial_{a_j} B)\partial_{c_j} q] \tag{5.5}$$

All implicit constants will depend upon $\delta_0 > 0$ and L^∞ norms of $b_0(x,t)$ and its derivatives. We further assume that

$$\delta_0 \le c_1(t) \le c_2(t) - \delta_0 \le \delta_0^{-1} \tag{5.6}$$

holds on all of $[0,T]$.

In §6 we will estimate $F_\perp$ using the properties of q recalled in §3. We note that $F_\parallel \equiv 0$ would mean that the parameters solve the effective equations of motion (1.5). Hence the estimates on $F_\parallel$ are related to the quality of our effective dynamics and they are provided in §7. In §8 we then construct a correction term which removes the leading non-homogeneous terms from the equation for v. Finally energy estimates in §9 based on the coercivity of $\mathcal{K}$ lead to the final bootstrap argument in §10.

6. Estimates on $F_\perp$

Using the identities in Lemma 3.4, we will prove that $F_\perp$ is $\mathcal{O}(h^2)$; in fact, we obtain more precise information. For notational convenience, we will drop the t dependence in $b(x,t)$, and will write b', b'', b''', to represent x-derivatives.

We will use the following consequences of Lemma 3.2:

$$\partial_{a_j} q = -\partial_x \eta(\cdot, \hat{a}_j, c_j) + \mathcal{S}_{\rm err} \tag{6.1}$$

and

$$\begin{aligned} c_j \partial_{c_j} q = \; & \partial_x[(x-a_j)\eta(x,\hat{a}_j,c_j)] + \frac{2c_{3-j}\theta(a_2-a_1)}{(c_1+c_2)(c_1-c_2)} \partial_x \eta(x,\hat{a}_j,c_j) \\ & - \frac{2c_j\theta(a_2-a_1)}{(c_1+c_2)(c_1-c_2)} \partial_x \eta(x, \hat{a}_{3-j}, c_{3-j}) + \mathcal{S}_{\rm err} \,, \end{aligned} \tag{6.2}$$

where θ is given by (3.6).

Importantly, as the last formula shows, $\partial_{c_j} q$ is not localized around $\hat{a}_j$ due to the c_j-dependence of $\hat{a}_{3-j}$. Also note that it is $(x-a_j)$ and not $(x-\hat{a}_j)$ in the first term inside the brackets.

Definition 6.1. *Let $\mathcal{A}$ denote the class of functions of a, c that are of the form*

$$h^2 \varphi(a_1 - a_2, a, c) + \nu(a,c) h^3 \,,$$

$a = (a_1, a_2) \in \mathbb{R}^2$, $0 < \delta < c_1 < c_2 - \delta < 1/\delta$, *where*

$$\left|\partial_\alpha^\ell \partial_c^k \partial_a^p \varphi(\alpha, a, c)\right| \le C \langle \alpha \rangle^{-N} \,, \quad \left|\partial_c^k \partial_a^p \nu(a,c)\right| \le C \,,$$

where C depends on δ, N, ℓ, k, and p only.

We note that if $f \in \mathcal{S}_{\rm err}$, then $\int f(x)dx$ has the form $\varphi(a_1 - a_2, a, c)$, $\varphi \in \mathcal{A}$. The most important feature of the class $\mathcal{A}$ is that for $f \in \mathcal{A}$,

$$|\partial_{a_j}^k \partial_{c_j}^\ell f| \lesssim h^2 \langle a_1 - a_2 \rangle^{-N} + h^3$$

with implicit constant depending on c_1, c_2.

Lemma 6.1. *We have*

$$\partial_{a_j} B(a,c,\cdot) = 2c_j b'(\hat{a}_j) + \mathcal{A} \tag{6.3}$$

$$\begin{aligned}\partial_{c_j} B(a,c,\cdot) = &\; 2b(\hat{a}_j) + 2b'(\hat{a}_j)(a_j - \hat{a}_j) - \frac{\pi^2}{12} b''(\hat{a}_j) c_j^{-2} \\ &- \frac{2(-1)^j c_{3-j}(b'(\hat{a}_2) - b'(\hat{a}_1))\theta}{(c_1+c_2)(c_1-c_2)} + \mathcal{A}\end{aligned} \tag{6.4}$$

Proof. First we compute $\partial_{a_j} B(a,c,t)$. We have that $\partial_{a_j} q$ is exponentially localized around $\hat{a}_j$. Substituting the Taylor expansion of b around $\hat{a}_j$, we obtain

$$\begin{aligned}\partial_{a_j} B(a,c,t) = &\; b(\hat{a}_j) \int \partial_{a_j} q^2 + b'(\hat{a}_j) \int (x-\hat{a}_j) \partial_{a_j} q^2 \\ &+ \frac{1}{2} b''(\hat{a}_j) \int (x-\hat{a}_j)^2 \partial_{a_j} q^2 + \mathcal{O}(h^3) \\ = &\; \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathcal{O}(h^3)\end{aligned}$$

Terms I and II are straightforward. Using (3.21) and (3.22),

$$\begin{aligned}&\mathrm{I} = b(\hat{a}_j) \partial_{a_j} \int q^2 = 0 \\ &\mathrm{II} = b'(\hat{a}_j) \left(\partial_{a_j} \int x q^2 - \hat{a}_j \partial_{a_j} \int q^2 \right) = 2c_j b'(\hat{a}_j)\end{aligned}$$

For III, we will substitute (6.1) and hence pick up $\mathcal{O}(h^2)\langle a_1 - a_2 \rangle^{-N}$ errors.

$$\mathrm{III} = -\frac{1}{2} b''(\hat{a}_j) \int (x - \hat{a}_j)^2 \partial_x \eta^2(x, \hat{a}_j, c_j)\, dx + \mathcal{A} = \mathcal{A}$$

Thus, we obtain (6.3). Next, we compute $\partial_{c_j} B(a,c,t)$. Note that $\partial_{c_j} q$ is *not* localized around $\hat{a}_j$. Begin by rewriting $\partial_{c_j} B$ as

$$\partial_{c_j} B = \int b(\hat{a}_j) \partial_{c_j} q^2 + \int b'(\hat{a}_j)(x - \hat{a}_j) \partial_{c_j} q^2 + \int \tilde{b}_j \, \partial_{c_j} q^2$$

where

$$\tilde{b}_j(x) \stackrel{\text{def}}{=} b(x) - b(\hat{a}_j) - b'(\hat{a}_j)(x - \hat{a}_j) \,.$$

Now substitute (6.2) into the last term and note that the $\mathcal{S}_{\mathrm{err}}$ term in (6.2) produces an $\mathcal{A}$ term here.

$$\begin{aligned}
\partial_{c_j} B = & \int b(\hat{a}_j)\partial_{c_j} q^2 + \int b'(\hat{a}_j)(x-\hat{a}_j)\partial_{c_j} q^2 \\
& + \frac{2}{c_j}\int \tilde{b}_j(x)\, \partial_x[(x-a_j)\eta(x,\hat{a}_j,c_j)]\eta(x,\hat{a}_j,c_j) \\
& + \frac{c_{3-j}\theta}{c_j(c_1+c_2)(c_1-c_2)}\int \tilde{b}_j(x)\partial_x \eta^2(x,\hat{a}_j,c_j) \\
& - \frac{\theta}{(c_1+c_2)(c_1-c_2)}\int \tilde{b}_j(x)\partial_x\eta^2(x,\hat{a}_{3-j},c_{3-j}) + \mathcal{A} \\
= & \, \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathcal{A}
\end{aligned}$$

where terms I-V are studied separately below.

$$\begin{aligned}
\mathrm{I} &= b(\hat{a}_j)\partial_{c_j}\int q^2 = 2b(\hat{a}_j) \\
\mathrm{II} &= b'(\hat{a}_j)\left(\partial_{c_j}\int xq^2 - \hat{a}_j\partial_{c_j}\int q^2\right) \\
&= 2b'(\hat{a}_j)(a_j-\hat{a}_j)
\end{aligned}$$

Term III is localized around $\hat{a}_j$, and thus we integrate by parts in x and Taylor expand $\tilde{b}_j$ around $\hat{a}_j$ to obtain

$$\begin{aligned}
\mathrm{III} &= \frac{1}{c_j}\int \left(-\tilde{b}_j'(x)(x-a_j) + \tilde{b}_j(x)\right)\eta^2(x,\hat{a}_j,c_j) \\
&= -\frac{1}{2}\frac{b''(\hat{a}_j)}{c_j}\int (x-\hat{a}_j)^2\eta^2(x,\hat{a}_j,c_j) \\
&\quad - b''(\hat{a}_j)(\hat{a}_j-a_j)\int (x-\hat{a}_j)\eta^2(x,\hat{a}_j,c_j) + \mathcal{O}(h^3) \\
&= -\frac{\pi^2}{12} b''(\hat{a}_j)c_j^{-2} + \mathcal{O}(h^3)
\end{aligned}$$

Term IV is localized around $\hat{a}_j$, and thus we integrate by parts in x and Taylor expand $\tilde{b}_j$ around $\hat{a}_j$ to obtain

$$\begin{aligned}
\int \tilde{b}_j(x)\,\partial_x\eta^2(x,\hat{a}_j,c_j) &= -\int (b'(x)-b(\hat{a}_j))\eta^2(x,\hat{a}_j,c_j) \\
&= -\frac{1}{2}b''(\hat{a}_j)\int (x-\hat{a}_j)\eta^2(x,\hat{a}_j,c_j) + \mathcal{O}(h^3) \\
&= \mathcal{O}(h^3)
\end{aligned}$$

Term V is localized around $\hat{a}_{3-j}$, and thus we integrate by parts in x and Taylor expand $\tilde{b}_j$ around $\hat{a}_{3-j}$.

$$\begin{aligned}\int \tilde{b}_j(x)\partial_x \eta^2(x,\hat{a}_{3-j},c_{3-j}) &= -\int (b'(x)-b(\hat{a}_j))\eta^2(x,\hat{a}_{3-j},c_{3-j}) \\ &= -(b'(\hat{a}_{3-j})-b'(\hat{a}_j))\int \eta^2(x,\hat{a}_{3-j},c_{3-j}) \\ &\quad - b''(\hat{a}_{3-j})\int (x-\hat{a}_{3-j})\eta^2(x,\hat{a}_{3-j},c_{3-j}) + \mathcal{O}(h^3) \\ &= -2c_{3-j}(b'(\hat{a}_{3-j})-b'(\hat{a}_j)) + \mathcal{O}(h^3)\end{aligned}$$

□

Lemma 6.2 (estimates on $F_\perp$)**.**

$$\partial_x^{-1}\partial_{a_j}F_\perp = \mathcal{O}(h^2)\cdot\mathcal{S}_{\rm sol}\,, \qquad \partial_x^{-1}\partial_{c_j}F_\perp = \mathcal{O}(h^2)\cdot\mathcal{S}_{\rm sol}\,, \quad j=1,2 \tag{6.5}$$

$$F_\perp = -\frac{1}{2}\sum_{j=1}^{2}\frac{b''(\hat{a}_j)}{c_j^2}\partial_x\tau(\cdot,\hat{a}_j,c_j) + \mathcal{A}\cdot\mathcal{S}_{\rm sol} \tag{6.6}$$

where

$$\tau \stackrel{\rm def}{=} \left(\frac{\pi^2}{12}+x^2\right)\eta(x)\,, \qquad \tau(x,\hat{a}_j,c_j) \stackrel{\rm def}{=} c_j\tau(c_j(x-\hat{a}_j))\,. \tag{6.7}$$

In light of the above lemma, we introduce the notation $F_\perp = (F_\perp)_0 + \tilde{F}_\perp$, where

$$(F_\perp)_0 = -\frac{1}{2}\sum_{j=1}^{2}\frac{b''(\hat{a}_j)}{c_j^2}\partial_x\tau(\cdot,\hat{a}_j,c_j) \tag{6.8}$$

and $\tilde{F}_\perp \in \mathcal{A}\cdot\mathcal{S}_{\rm sol}$. We make use of (6.5) in §7 and (6.6) in §8–9.

Proof. We begin by proving (6.6). By (3.18), (3.19),

$$\begin{aligned}\partial_x(bq) &= (\partial_x b)q + b(\partial_x q) \\ &= (\partial_x b)\sum_{j=1}^{2}((x-a_j)\partial_{a_j}q + c_j\partial_{c_j}q) - b\sum_{j=1}^{2}\partial_{a_j}q \\ &= \sum_{j=1}^{2}(-b+(\partial_x b)(x-a_j))\partial_{a_j}q + \sum_{j=1}^{2}(\partial_x b)c_j\partial_{c_j}q + \mathcal{O}(h^3)\cdot\mathcal{S}_{\rm sol}\end{aligned} \tag{6.9}$$

The $\partial_{a_j}q$ term is well localized around $\hat{a}_j$, and thus we can Taylor expand the coefficients around $\hat{a}_j$. The $\partial_{c_j}q$ term we leave alone for the moment.

We have $\partial_x(bq) =$

$$\sum_{j=1}^{2}\Big(-b(\hat{a}_j)+b'(\hat{a}_j)(\hat{a}_j-a_j)+b''(\hat{a}_j)(\hat{a}_j-a_j)(x-\hat{a}_j)+\frac{1}{2}b''(\hat{a}_j)(x-\hat{a}_j)^2\Big)\partial_{a_j}q + \sum_{j=1}^{2} b'(x)c_j\partial_{c_j}q + \mathcal{A}\cdot\mathcal{S}_{\mathrm{sol}}$$

Substituting the above together with (6.3) and (6.4) into (5.5), we obtain

$$F_\perp = \frac{1}{2}\sum_{j=1}^{2} b''(\hat{a}_j)\Big(\frac{\pi^2}{12}c_j^{-2} - 2(\hat{a}_j-a_j)(x-\hat{a}_j)-(x-\hat{a}_j)^2\Big)\partial_{a_j}q + \frac{(b'(\hat{a}_2)-b'(\hat{a}_1))\theta}{(c_1+c_2)(c_1-c_2)}\sum_{j=1}^{2}(-1)^j c_{3-j}\partial_{a_j}q - \sum_{j=1}^{2}(b'(x)-b'(\hat{a}_j))c_j\partial_{c_j}q + \mathcal{A}\cdot\mathcal{S}_{\mathrm{sol}}$$

We now substitute (6.1) and (6.2) recognizing that this will only generate errors of type $\mathcal{A}$ times a Schwartz class function. We also Taylor expand around $\hat{a}_j$ or $\hat{a}_{3-j}$ depending upon the localization.

$$\begin{aligned}
F_\perp = {} & \frac{1}{2}\sum_{j=1}^{2} b''(\hat{a}_j)\Big(-\frac{\pi^2}{12}c_j^{-2} + 2(\hat{a}_j-a_j)(x-\hat{a}_j)+(x-\hat{a}_j)^2\Big)\partial_x\eta(x,\hat{a}_j,c_j) && \leftarrow \mathrm{I}\\
& - \frac{(b'(\hat{a}_2)-b'(\hat{a}_1))\theta}{(c_1+c_2)(c_1-c_2)}\sum_{j=1}^{2}(-1)^j c_{3-j}\partial_x\eta(x,\hat{a}_j,c_j) && \leftarrow \mathrm{II}\\
& - \sum_{j=1}^{2} b''(\hat{a}_j)(x-\hat{a}_j)\partial_x[(x-a_j)\eta(x,\hat{a}_j,c_j)] && \leftarrow \mathrm{III}\\
& - \sum_{j=1}^{2}\frac{c_{3-j}\theta}{(c_1+c_2)(c_1-c_2)} b''(\hat{a}_j)(x-\hat{a}_j)\partial_x\eta(x,\hat{a}_j,c_j) && \leftarrow \mathrm{IV}\\
& + \sum_{j=1}^{2}\frac{c_j\theta}{(c_1+c_2)(c_1-c_2)} b''(\hat{a}_{3-j})(x-\hat{a}_{3-j})\partial_x\eta(x,\hat{a}_{3-j},c_{3-j}) && \leftarrow \mathrm{V}\\
& + \sum_{j=1}^{2}\frac{c_j\theta}{(c_1+c_2)(c_1-c_2)}(b'(\hat{a}_{3-j})-b'(a_j))\partial_x\eta(x,\hat{a}_{3-j},c_{3-j}) && \leftarrow \mathrm{VI}\\
& + \mathcal{A}\cdot\mathcal{S}_{\mathrm{sol}}
\end{aligned}$$

We have that $\mathrm{IV}+\mathrm{V}=0$ and $\mathrm{II}+\mathrm{VI}=0$. Hence

$$\begin{aligned} F_\perp &= \mathrm{I}+\mathrm{III}+\mathcal{A}\cdot\mathcal{S}_{\rm sol} \\ &= -\frac{1}{2}\sum_{j=1}^{2} b''(\hat a_j)\partial_x\left(\left(\frac{\pi^2}{12}c_j^{-2}+(x-\hat a_j)^2\right)\eta(x,\hat a_j,c_j)\right) \end{aligned}$$

This completes the proof of (6.6). To obtain (6.5), we note that a consequence of (6.6) is $F_\perp=\mathcal{O}(h^2)f$, where $f\in\mathcal{S}_{\rm sol}$. By the definition (5.5) of $F_\perp$ and Corollary 3.3, we have $\partial_x^{-1}F_\perp\in\mathcal{S}_{\rm sol}$, and hence $f\in\mathcal{S}_{\rm sol}$. □

7. Estimates on the parameters

The equations of motion are recovered (in approximate form) using the symplectic orthogonality properties (3.26) of v and the equation (5.2) for v. For a function G of the form

$$G=g_1\partial_{a_1}q+g_2\partial_{a_2}q+g_3\partial_{c_1}q+g_4\partial_{c_2}q$$

with $g_j=g_j(a,c)$, define

$$\operatorname{coef}(G)=(g_1,g_2,g_3,g_4)\,.$$

Lemma 7.1. *Suppose we are given $\delta_0>0$ and $b_0(x,t)$, and parameters $a(t)$, $c(t)$ such that v defined by (5.1) satisfies the symplectic orthogonality conditions (3.26). Suppose, moreover, that the amplitude separation condition (5.6) holds. Then (with implicit constants depending upon $\delta_0>0$ and L^∞ norms of b_0 and its derivatives), if $\|v\|_{H^2}\lesssim 1$, then we have*

$$|\operatorname{coef}(F_\parallel)|\lesssim h^2\|v\|_{H^1}+\|v\|_{H^1}^2\,. \tag{7.1}$$

Proof. Since $\langle v,\partial_x^{-1}\partial_{a_j}q\rangle=0$, we have upon substituting (5.2)

$$\begin{aligned} 0 &= \partial_t\langle v,\partial_x^{-1}\partial_{a_j}q\rangle \\ &= \langle\partial_t v,\partial_x^{-1}\partial_{a_j}q\rangle+\langle v,\partial_t\partial_x^{-1}\partial_{a_j}q\rangle \\ &= \langle(\partial_x^2 v+6q^2)v,\partial_{a_j}q\rangle+\langle(6qv^2+2v^3),\partial_{a_j}q\rangle && \leftarrow \mathrm{I}+\mathrm{II} \\ &\quad -\langle bv,\partial_{a_j}q\rangle-\langle F_\parallel,\partial_x^{-1}\partial_{a_j}q\rangle-\langle F_\perp,\partial_x^{-1}\partial_{a_j}q\rangle && \leftarrow \mathrm{III}+\mathrm{IV}+\mathrm{V} \\ &\quad +\langle v,\partial_x^{-1}\partial_{a_j}\left(\sum_{k=1}^{2}\partial_{a_k}q\,\dot a_k+\sum_{k=1}^{2}\partial_{c_k}q\,\dot c_k\right)\rangle && \leftarrow \mathrm{VI} \end{aligned}$$

We have, by (3.17),

$$\begin{aligned} \mathrm{I} &= \langle v,\partial_{a_j}(\partial_x^2 q+2q^3)\rangle \\ &= -\frac{1}{2}\langle v,\partial_x^{-1}\partial_{a_j}\partial_x I_3'(q)\rangle \\ &= -\langle v,\partial_x^{-1}\partial_{a_j}\sum_{k=1}^{2}c_k^2\partial_{a_k}q\rangle \end{aligned}$$

Also, by (5.5)

$$\begin{aligned}\text{III} &= -\langle bv, \partial_{a_j} q\rangle \\ &= -\langle v, \partial_{a_j}(bq)\rangle \\ &= -\langle v, \partial_x^{-1}\partial_{a_j}\partial_x(bq)\rangle \\ &= -\langle v, \partial_x^{-1}\partial_{a_j}\big(-F_\perp - \tfrac{1}{2}\sum_{k=1}^{2}(\partial_{c_k}B)\partial_{a_k}q + \tfrac{1}{2}\sum_{k=1}^{2}(\partial_{a_k}B)\partial_{c_k}q\big)\rangle\end{aligned}$$

Thus

$$\begin{aligned}|\text{I}+\text{III}+\text{VI}| &= |\langle v, \partial_x^{-1}\partial_{a_j}F_\perp\rangle + \langle v, \partial_x^{-1}\partial_{a_j}F_\parallel\rangle| \\ &\leq \|v\|_{L^2}(\|\partial_x^{-1}\partial_{a_j}F_\perp\|_{L^2} + \|\partial_x^{-1}\partial_{a_j}F_\parallel\|) \\ &\leq \|v\|_{L^2}(h^2 + |\operatorname{coef}(F_\parallel)|)\end{aligned}$$

Next, we note that by Cauchy-Schwarz,

$$|\text{II}| \lesssim \|v\|_{H^1}^2 .$$

Next, observe from (5.4) and Lemma 3.6 that

$$\text{IV} = \langle F_\parallel, \partial_x^{-1}\partial_{a_j}q\rangle = -(\dot c_j - \frac{1}{2}\partial_{a_j}B) .$$

Of course, we have $\text{V} = \langle F_\perp, \partial_x^{-1}\partial_{a_j}q\rangle = 0$. Combining, we obtain

$$\left|\dot c_j - \frac{1}{2}\partial_{a_j}B\right| \lesssim \|v\|_{H^1}(h^2 + |\operatorname{coef}(F_\parallel)|) + \|v\|_{H^1}^2 . \tag{7.2}$$

A similar calculation, applying ∂_t to the identity $0 = \langle v, \partial_x^{-1}\partial_{c_j}q\rangle$, yields

$$\left|\dot a_j - c_j^2 + \frac{1}{2}\partial_{c_j}B\right| \lesssim \|v\|_{H^1}(h^2 + |\operatorname{coef}(F_\parallel)|) + \|v\|_{H^1}^2 . \tag{7.3}$$

Combining (7.2) and (7.3) gives (7.1). □

8. Correction term

Recall the definition (6.7) of τ. Let ρ be the unique function solving

$$(1 - \partial_x^2 - 6\eta^2)\rho = \tau ,$$

see [19, Proposition 4.2] for the properties of this equation. The function ρ is smooth, exponentially decaying at ∞, and satisfies the symplectic orthogonality conditions

$$\langle \rho, \eta\rangle = 0 , \qquad \langle \rho, x\eta\rangle = 0 \tag{8.1}$$

Set

$$\rho(x, \hat a_j, c_j) \stackrel{\text{def}}{=} c_j^{-1}\rho(c_j(x - \hat a_j))$$

and note that

$$(c_j^2 - \partial_x^2 - 6\eta^2(\cdot, \hat a_j, c_j))\rho(\cdot, \hat a_j, c_j) = \tau(\cdot, \hat a_j, c_j)$$

Define the symplectic projection operator

$$Pf \stackrel{\text{def}}{=} \sum_{j=1}^{2} \langle f, \partial_x^{-1} \partial_{c_j} q \rangle \partial_{a_j} q + \sum_{j=1}^{2} \langle f, \partial_x^{-1} \partial_{a_j} q \rangle \partial_{c_j} q \,.$$

Define

$$w \stackrel{\text{def}}{=} -\frac{1}{2}(I-P) \sum_{j=1}^{2} \frac{b''(\hat{a}_j)}{c_j^2} \rho(\cdot, \hat{a}_j, c_j) \tag{8.2}$$

Note that $w = \mathcal{O}(h^2)$ and clearly now w satisfies

$$\langle w, \partial_x^{-1} \partial_{a_j} q \rangle = 0 \,, \qquad \langle w, \partial_x^{-1} \partial_{c_j} q \rangle = 0 \,. \tag{8.3}$$

Recall the definition (6.8) of $(F_\perp)_0$.

Lemma 8.1. *If* $\dot{c}_j = \mathcal{O}(h)$, *and* $\dot{a}_j = c_j - b(\hat{a}_j) + \mathcal{O}(h)$, *then*

$$\partial_t w + \partial_x(\partial_x^2 w + 6q^2 w - bw) = -(F_\perp)_0 - G + \mathcal{A} \cdot \mathcal{S}_{\text{sol}} \,. \tag{8.4}$$

where G *is an* $\mathcal{O}(h^2)$ *term that is symplectically parallel to* M, *i.e.*

$$G \in \operatorname{span}\{\partial_x^{-1}\partial_{a_1} q, \partial_x^{-1}\partial_{a_2} q, \partial_x^{-1}\partial_{c_1} q, \partial_x^{-1}\partial_{c_2} q\} \,.$$

Proof. Let

$$w_j = \frac{b''(\hat{a}_j)}{c_j^2} \rho(\cdot, \hat{a}_j, c_j)$$

Then

$$\begin{aligned} \partial_t w_j = \; & b'''(\hat{a}_j)\dot{\hat{a}}_j c_j^{-2} \rho(\cdot, \hat{a}_j, c_j) - 2b''(\hat{a}_j) c_j^{-3} \dot{c}_j \rho(\cdot, \hat{a}_j, c_j) \\ & + b''(\hat{a}_j) c_j^{-2} \dot{\hat{a}}_j \partial_{a_j} \rho(\cdot, \hat{a}_j, c_j) + b''(\hat{a}_j) c_j^{-2} \dot{\hat{c}}_j \partial_{c_j} \rho(\cdot, \hat{a}_j, c_j) + \partial_t b''(\hat{a}_j) c_j^{-2} \rho(\cdot, \hat{a}_j, c_j) \\ = & -\dot{a}_j \partial_x w_j + \mathcal{A} \cdot \mathcal{S}_{\text{sol}} \end{aligned}$$

Also, we have

$$\begin{aligned} (\partial_x^2 + 6q^2) w_j &= (\partial_x^2 + 6\eta^2(\cdot, \hat{a}_j, c_j)) w_j + \mathcal{A} \cdot \mathcal{S}_{\text{sol}} \\ &= c_j^2 w_j - b''(\hat{a}_j) c_j^{-2} \tau(\cdot, \hat{a}_j, c_j) + \mathcal{A} \cdot \mathcal{S}_{\text{sol}} \end{aligned}$$

Also,

$$bw_j = b(\hat{a}_j) w_j + \mathcal{A} \cdot \mathcal{S}_{\text{sol}}$$

Combining, we obtain

$$\begin{aligned} & \partial_t w_j + \partial_x(\partial_x^2 w_j + 6q^2 w_j - bw_j) \\ & \quad = -b''(\hat{a}_j) c_j^{-2} \partial_x \tau(\cdot, \hat{a}_j, c_j) + (-\dot{a}_j + c_j^2 - b(\hat{a}_j)) \partial_x w_j + \mathcal{A} \cdot \mathcal{S}_{\text{sol}} \\ & \quad = -b''(\hat{a}_j) c_j^{-2} \partial_x \tau(\cdot, \hat{a}_j, c_j) + \mathcal{A} \cdot \mathcal{S}_{\text{sol}} \end{aligned}$$

Now we discuss $\partial_t P w_j$.

$$\begin{aligned}\partial_t P w_j = &\ \langle \partial_t w_j, \partial_x^{-1}\partial_{a_1} q\rangle \partial_{c_1} q + \langle w_j, \partial_t \partial_x^{-1}\partial_{a_1} q\rangle \partial_{c_j} q + \text{similar} \\ &+ \langle w_j, \partial_x^{-1}\partial_{a_1} q\rangle \partial_t \partial_{c_1} q + \text{similar}\end{aligned}$$

The first line of terms is symplectically parallel to M. For the second line, note that by (8.1), we have $\langle w_j, \partial_x^{-1}\partial_{a_1} q\rangle = \mathcal{A}$. Consequently,

$$\partial_t P w_j = T_q M + \mathcal{A}\cdot \mathcal{S}_{\text{sol}}$$

□

Define $\tilde{u}$ and $\tilde{v}$ by

$$u = \tilde{u} + w\,, \qquad v = \tilde{v} + w\,. \tag{8.5}$$

Of course, it follows that $\tilde{u} = q + \tilde{v}$. Note that by (3.26) and (8.3), we have

$$\langle \tilde{v}, \partial_x^{-1}\partial_{a_j} q\rangle = 0 \text{ and } \langle \tilde{v}, \partial_x^{-1}\partial_{c_j} q\rangle = 0\,,\ j = 1, 2\,. \tag{8.6}$$

Note that $\tilde{u}$ solves

$$\partial_t \tilde{u} = -\partial_x(\partial_x^2 \tilde{u} + 2\tilde{u}^3 - b\tilde{u}) - \partial_t w - \partial_x(\partial_x^2 w + 6\tilde{u}^2 w - bw) + \mathcal{O}(h^4) \tag{8.7}$$

where the $\mathcal{O}(h^4)$ terms arise from w^2 and w^3. Moreover, if we make the mild assumption that $\tilde{v} = \mathcal{O}(h)$, then $\tilde{u}^2 w = q^2 w + \mathcal{O}(h^3)$. By (8.7) and (8.4), we have

$$\partial_t \tilde{u} = -\partial_x(\partial_x^2 \tilde{u} + 2\tilde{u}^3 - b\tilde{u}) + (F_\perp)_0 + G + \mathcal{A}\cdot\mathcal{S}_{\text{sol}} \tag{8.8}$$

Since $\tilde{u} = q + \tilde{v}$, we have (in analogy with (5.2))

$$\partial_t \tilde{v} = \partial_x(-\partial_x^2 \tilde{v} - 6q^2 \tilde{v} + b\tilde{v}) - F_\parallel - \tilde{F}_\perp + G + \mathcal{A}\cdot\mathcal{S}_{\text{sol}} + O(h^3)H^1 \tag{8.9}$$

where we have made the assumption that $\tilde{v} = \mathcal{O}(h^{3/2})$ in order to discard the $\tilde{v}^2$ and $\tilde{v}^3$ terms. We thus see that, in comparison to v, the equation for $\tilde{v}$ has a lower-order inhomogeneity, but still satisfies the symplectic orthogonality conditions (8.6) and $v = \tilde{v} + \mathcal{O}(h^2)$.

9. Energy estimate

Since $w = \mathcal{O}(h^2)$, to obtain the desired bound on v it will suffice to obtain a bound for $\tilde{v}$. This will be achieved by the "energy method."

Lemma 9.1. *Suppose we are given $\delta_0 > 0$ and $b_0(x,t)$, and parameters $a(t)$, $c(t)$ such that v defined by* (5.1) *satisfies the symplectic orthogonality conditions* (3.26) *on $[0,T]$. Suppose, moreover, that the amplitude separation condition* (5.6) *holds on $[0,T]$. Then (with implicit constants depending upon $\delta_0 > 0$ and L^∞ norms of b_0 and its derivatives), if $\|v\|_{H^2} \lesssim 1$ and $T \ll h^{-1}$, then*

$$\|v\|^2_{L^\infty_{[0,T]}H^2} \lesssim \|v(0)\|^2_{H^2} + h^4\left(1 + \int_0^T \langle a_1 - a_2\rangle^{-N}\, dt\right)^2 .$$

Proof. Recall that we have defined

$$H_c(u) = I_5(u) + (c_1^2 + c_2^2) I_3(u) + c_1^2 c_2^2 I_1(u) .$$

With w given by (8.2) and $\tilde{u}$ given by (8.5), let

$$\mathcal{E}(t) = H_c(\tilde{u}) - H_c(q) .$$

Then

$$\begin{aligned} \partial_t \mathcal{E} = \; & \langle H_c'(\tilde{u}), \partial_t \tilde{u} \rangle - \langle H_c'(q), \partial_t q \rangle + 2(c_1 \dot{c}_1 + c_2 \dot{c}_2)(I_3(\tilde{u}) - I_3(q)) \\ & + 2 c_1 c_2 (c_1 \dot{c}_2 + \dot{c}_1 c_2)(I_1(\tilde{u}) - I_1(q)) \\ = \; & \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} \end{aligned}$$

Note that $\mathrm{II} = 0$ since Lemma 4.1 showed that $H_c'(q) = 0$. For III, we have by (3.17) and the orthogonality conditions (8.6),

$$\begin{aligned} \mathrm{III} &= 2(c_1 \dot{c}_1 + c_2 \dot{c}_2)(\langle I_3'(q), \tilde{v} \rangle + \mathcal{O}(\|\tilde{v}\|_{H^1}^2)) \\ &= 4(c_1 \dot{c}_1 + c_2 \dot{c}_2) \langle \sum_{j=1}^{2} c_j^2 \partial_x^{-1} \partial_{a_j} q, \tilde{v} \rangle + \mathcal{O}((|\dot{c}_1| + |\dot{c}_2|) \|\tilde{v}\|_{H^1}^2) \\ &= \mathcal{O}((|\dot{c}_1| + |\dot{c}_2|) \|\tilde{v}\|_{H^1}^2) \end{aligned}$$

Term IV is bounded similarly. It remains to study Term I. Writing (8.8) as $\partial_t \tilde{u} = \frac{1}{2} \partial_x I_3'(\tilde{u}) + \partial_x (b \tilde{u}) + (F_\perp)_0 + G + \mathcal{A} \cdot \mathcal{S}_{\mathrm{sol}}$ and appealing to (2.5), we have by Lemma 2.1 (with u replaced by $\tilde{u}$ in that lemma) that

$$\begin{aligned} \mathrm{I} &= \langle H_c'(\tilde{u}), \partial_x (b \tilde{u}) \rangle + \langle H_c'(\tilde{u}), (F_\perp)_0 + \mathcal{A} \cdot \mathcal{S}_{\mathrm{sol}} \rangle \\ &= 5 \langle b_x, A_5(\tilde{u}) \rangle - 5 \langle b_{xxx}, A_3(\tilde{u}) \rangle + \langle b_{xxxxx}, A_1(\tilde{u}) \rangle \\ &\quad + (c_1^2 + c_2^2)(3 \langle b_x, A_3(\tilde{u}) \rangle - \langle b_{xxx}, A_1(\tilde{u}) \rangle) + c_1^2 c_2^2 \langle b_x, A_1(\tilde{u}) \rangle \\ &\quad + \langle H_c'(\tilde{u}), (F_\perp)_0 + \mathcal{A} \cdot \mathcal{S}_{\mathrm{sol}} \rangle \end{aligned}$$

Expand $A_j(\tilde{u}) = A_j(q + \tilde{v}) = A_j(q) + A_j'(q)(\tilde{v}) + \mathcal{O}(\tilde{v}^2)$ and $H_c'(\tilde{u}) = H_c'(q) + \mathcal{K}_{c,a} \tilde{v} + \mathcal{O}(\tilde{v}^2) = \mathcal{K}_{c,a} \tilde{v} + \mathcal{O}(\tilde{v}^2)$ to obtain $\mathrm{I} = \mathrm{IA} + \mathrm{IB} + \mathrm{IC}$, where

$$\begin{aligned} \mathrm{IA} = \; & 5 \langle b_x, A_5(q) \rangle - 5 \langle b_{xxx}, A_3(q) \rangle + \langle b_{xxxxx}, A_1(q) \rangle \\ & + (c_1^2 + c_2^2)(3 \langle b_x, A_3(q) \rangle - \langle b_{xxx}, A_1(q) \rangle) + c_1^2 c_2^2 \langle b_x, A_1(q) \rangle \\ \mathrm{IB} = \; & 5 \langle b_x, A_5'(q)(\tilde{v}) \rangle - 5 \langle b_{xxx}, A_3'(q)(\tilde{v}) \rangle + \langle b_{xxxxx}, A_1'(q)(\tilde{v}) \rangle \\ & + (c_1^2 + c_2^2)(3 \langle b_x, A_3'(q)(\tilde{v}) \rangle - \langle b_{xxx}, A_1'(q)(\tilde{v}) \rangle) + c_1^2 c_2^2 \langle b_x, A_1'(q)(\tilde{v}) \rangle \\ \mathrm{IC} = \; & \langle \mathcal{K}_{c,a} \tilde{v}, (F_\perp)_0 \rangle + \mathcal{O}(h \|\tilde{v}\|_{H^2}^2) + \mathcal{O}(\mathcal{A} \cdot \|\tilde{v}\|_{H^2}) \end{aligned}$$

Then reapply Lemma 2.1 (with u replaced by q in that lemma) to obtain that IA $= -\langle H_c'(q), \partial_x(bq)\rangle = 0$. Applying Lemma 2.2,

$$\begin{aligned} \text{IB} &= \langle \mathcal{K}_{c,a}\tilde{v}, (bq)_x\rangle - \langle \partial_x H_c'(q), b\tilde{v}\rangle \\ &= \langle \mathcal{K}_{c,a}\tilde{v}, (bq)_x\rangle \end{aligned}$$

In summary thus far, we have obtained that

$$\partial_t \mathcal{E} = \langle \mathcal{K}_{c,a}\tilde{v}, (bq)_x + (F_\perp)_0\rangle + \mathcal{O}(h\|\tilde{v}\|_{H^2}^2) + \mathcal{O}(\mathcal{A}\|\tilde{v}\|_{H^2})$$

By (4.12), (4.13), and (8.6) (recalling the definition (5.3) of F_0), we obtain

$$\langle \mathcal{K}_{c,a}\tilde{v}, \partial_x(bq)\rangle = -\langle \mathcal{K}_{c,a}\tilde{v}, F_0\rangle = -\langle \mathcal{K}_{c,a}\tilde{v}, F_\parallel + F_\perp\rangle$$

Hence

$$\partial_t \mathcal{E} = -\langle \mathcal{K}_{c,a}\tilde{v}, F_\parallel + \tilde{F}_\perp\rangle + \mathcal{O}(h\|\tilde{v}\|_{H^2}^2) + \mathcal{O}(\mathcal{A}\|\tilde{v}\|_{H^2})$$

It follows from Lemma 7.1 and $\tilde{F}_\perp \in \mathcal{A}\cdot\mathcal{S}_{\text{sol}}$ (see (6.6), (6.8)) that

$$|\partial_t \mathcal{E}| \lesssim (h^2\langle a_1 - a_2\rangle^{-N} + h^3)\|\tilde{v}\|_{H^2} + h\|\tilde{v}\|_{H^2}^2$$

If $T = \delta h^{-1}$,

$$\mathcal{E}(T) = \mathcal{E}(0) + h^2\left(1 + \int_0^T \langle a_1 - a_2\rangle^{-N}\right)\|\tilde{v}\|_{L^\infty_{[0,T]}H^2_x} + h\|\tilde{v}\|^2_{L^\infty_{[0,T]}H^2}.$$

By Lemma 4.1, the definition of $\mathcal{E}$ and $\mathcal{K}_{c,a}$, and the fact that $\tilde{u} = q + \tilde{v}$, we have

$$|\mathcal{E} - \langle \mathcal{K}_{c,a}\tilde{v}, \tilde{v}\rangle| \lesssim \|\tilde{v}\|_{H^2}^3.$$

Applying this at time 0 and T, together with the coercivity of $\mathcal{K}$ (Proposition 4.3),

$$\|\tilde{v}(T)\|_{H^2}^2 \lesssim \|\tilde{v}(0)\|_{H^2}^2 + h^2\left(1 + \int_0^T \langle a_1 - a_2\rangle^{-N}\right)\|\tilde{v}\|_{L^\infty_{[0,T]}H^2_x} + h\|\tilde{v}\|^2_{L^\infty_{[0,T]}H^2}.$$

Replacing T by T' such that $0 \le T' \le T$, and taking the supremum in T' over $0 \le T' \le T$, we obtain

$$\|\tilde{v}\|^2_{L^\infty_{[0,T]}H^2} \lesssim \|\tilde{v}(0)\|_{H^2}^2 + h^2\left(1 + \int_0^T \langle a_1 - a_2\rangle^{-N}\right)\|\tilde{v}\|_{L^\infty_{[0,T]}H^2_x} + h\|\tilde{v}\|^2_{L^\infty_{[0,T]}H^2}.$$

By selecting δ small enough, we obtain

$$\|\tilde{v}\|^2_{L^\infty_{[0,T]}H^2} \lesssim \|\tilde{v}(0)\|_{H^2}^2 + h^4\left(1 + \int_0^T \langle a_1 - a_2\rangle^{-N}\,dt\right)^2$$

Finally, using that $\|w\|_{H^2} \sim h^2$, and $v = \tilde{v} + w$, we obtained the claimed estimate. □

10. Proof of the main theorem

We start with the proposition which links the ODE analysis with the estimates on the error term v:

Proposition 10.1. *Suppose we are given $b_0 \in C_b^\infty(\mathbb{R}^2)$ and $\delta_0 > 0$. (Implicit constants below depend only on b_0 and δ_0). Suppose that we are further given $\bar{a} \in \mathbb{R}^2$, $\bar{c} \in \mathbb{R}^2 \backslash \mathcal{C}$, $\kappa \geq 1$, $h > 0$, and v_0 satisfying (3.26), such that*

$$0 < h \lesssim \kappa^{-1}, \qquad \|v_0\|_{H_x^2} \leq \kappa h^2 .$$

Let $u(t)$ be the solution to (1.1) with $b(x,t) = b_0(hx, ht)$ and initial data $\eta(\cdot, \bar{a}, \bar{c}) + v_0$. Then there exist a time $T' > 0$ and trajectories $a(t)$ and $c(t)$ defined on $[0, T']$ such that $a(0) = \bar{a}$, $c(0) = \bar{c}$ and the following holds, with $v \stackrel{\text{def}}{=} u - \eta(\cdot, a, c)$:

1. *On $[0, T']$, the orthogonality conditions (3.26) hold.*
2. *Either $c_1(T') = \delta_0$, $c_1(T') = c_2(T') - \delta_0$, $c_2(T') = \delta_0^{-1}$, or $T' = \omega h^{-1}$, where $\omega \ll 1$.*
3. *$|\dot{a}_j - c_j^2 + b(a_j, t)| \lesssim h$.*
4. *$|\dot{c}_j - c_j b'(a_j)| \lesssim h^2$.*
5. *$\|v\|_{L^\infty_{[0,T']} H_x^2} \leq \alpha \kappa h^2$, where $\alpha \gg 1$.*

Here α and ω are constants depending only on b_0 and δ_0 (independent of κ, etc)

Proof. Recall our convention that implicit constants depend only on b_0 and δ. By Lemma 3.7 and the continuity of the flow $u(t)$ in H^2, there exists some $T'' > 0$ on which $a(t)$, $c(t)$ can be defined so that (3.26) hold. Now take T'' to be the maximal time on which $a(t)$, $c(t)$ can be defined so that (3.26) holds. Let T' be first time $0 \leq T' \leq T''$ such that $c_1(T') = \delta_0$, $c_1(T') = c_2(T') - \delta_0$, $c_2(T') = \delta_0^{-1}$, $T' = T''$, or ωh^{-1} (whichever comes first). Here, $0 < \omega \ll 1$ is a constant that we will chosen suitably small at the end of the proof (depending only upon implicit constants in the estimates, and hence only on b_0 and δ).

Remark 10.2. We will show that on $[0, T']$, we have $\|v(t)\|_{H_x^2} \lesssim \kappa h^2$, and hence by Lemma 3.7 and the continuity of the $u(t)$ flow, it must be the case that either $c_1(T') = \delta_0$, $c_1(T') = c_2(T') - \delta_0$, $c_2(T') = \delta_0^{-1}$, or ωh^{-1} (i.e. the case $T' = T''$ does not arise).

Let T, $0 < T \leq T'$, be the maximal time such that

$$\|v\|_{L^\infty_{[0,T]} H_x^2} \leq \alpha \kappa h^2 , \tag{10.1}$$

where α is suitably large constant related to the implicit constants in the estimates (and thus dependent only upon b_0 and $\delta_0 > 0$).

Remark 10.3. We will show, assuming that (10.1) holds, that $\|v\|_{L^\infty_{[0,T]} H_x^1} \leq \frac{1}{2} \alpha \kappa h^{1/2}$ and thus by continuity we must have $T = T'$.

In the remainder of the proof, we work on the time interval $[0, T]$, and we are able to assume that the orthogonality conditions (3.26) hold, $\delta_0 \leq c_1(t) \leq c_2(t) - \delta_0 \leq \delta_0^{-1}$, and that (10.1) holds. By Lemma 7.1 and Taylor expansion, we have (since $\kappa^2 h^4 \lesssim h^2$)

$$\begin{cases} \dot{a}_j = c_j^2 - b(a_j, t) + \mathcal{O}(h) \\ \dot{c}_j = c_j \partial_x b(a_j, t) + \mathcal{O}(h^2)\,, \end{cases} \tag{10.2}$$

with initial data $a_j(0) = \bar{a}_j$, $c_j(0) = \bar{c}_j$. Let

$$\xi(t) \stackrel{\text{def}}{=} \frac{b(a_1(t), t) - b(a_2(t), t)}{a_1(t) - a_2(t)}$$

and let $\Xi(t)$ denote an antiderivative. By the mean-value theorem $|\xi| \lesssim h$, and since $T \leq \omega h^{-1}$, we have $e^{\Xi} \sim 1$. We then have

$$\frac{d}{dt}\left(e^{\Xi}(a_2 - a_1)\right) = e^{\Xi}(c_2^2 - c_1^2) + \mathcal{O}(h)\,.$$

Since $\delta_0^2 \leq c_2^2 - c_1^2$, we see that $e^{\Xi}(a_2 - a_1)$ is strictly increasing. Let $0 \leq t_1 \leq T$ denote the unique time at which $e^{\Xi}(a_2 - a_1) = 0$ (if the quantity is always positive, take $t_1 = 0$, and if the quantity is always negative, take $t_1 = T$, and make straightforward modifications to the argument below). If $t < t_1$, integrating from t to t_1 we obtain

$$\delta_0^2(t_1 - t) \lesssim -e^{\Xi(t)}(a_2(t) - a_1(t)) = e^{\Xi(t)}|a_2(t) - a_1(t)|$$

If $t > t_1$, integrating from t_1 to t we obtain

$$\delta_0^2(t - t_1) \lesssim e^{\Xi(t)}(a_2(t) - a_1(t))\,.$$

Hence,

$$\int_0^T \langle a_2(t) - a_1(t)\rangle^{-2} \lesssim 1\,.$$

By Lemma 9.1, we conclude that

$$\|v\|_{L_T^\infty H_x^2} \leq \frac{\alpha}{4}(\|v(0)\|_{H^2} + h^2) \leq \frac{\alpha}{4}(\kappa h^2 + h^2) \leq \frac{1}{2}\alpha\kappa h^2\,.$$

□

We can now complete

Proof of the main Theorem. Suppose that $\|v_0\|_{H^2} \leq h^2$. Iterate Prop. 10.1, as long as the condition

$$\delta_0 \leq c_1 \leq c_2 - \delta_0 \leq \delta_0^{-1} \tag{10.3}$$

remains true, as follows: for the k-th iterate, put $\kappa = \alpha^k$ in Prop. 10.1 and advance from time $t_k = k\omega h^{-1}$ to time $t_{k+1} = (k+1)\omega h^{-1}$. At time t_k, we have $\|v(t_k)\|_{H^2} \leq \alpha^k h^2$, and we find from Prop. 10.1 that $\|v\|_{L^\infty_{[t_k, t_{k+1}]} H_x^2} \leq \alpha^{k+1} h^2$. Provided (10.3) holds on all of $[0, t_K]$, we can continue until $\kappa^{-1} \sim h$, i.e. $K \sim \log h^{-1}$.

Recall (1.6), and $A_j(T)$, $C_j(T)$ defined by (1.12). Let $\hat{a}_j(t) = h^{-1}A_j(ht)$, $\hat{c}_j(t) = C_j(ht)$. Then $\hat{a}_j$, $\hat{c}_j$ solve

$$\begin{cases} \dot{\hat{a}}_j = \hat{c}_j^2 - b(\hat{a}_j, t) \\ \dot{\hat{c}}_j = \hat{c}_j \partial_x b(\hat{a}_j, t) \end{cases}$$

with initial data $\hat{a}_j(0) = \bar{a}_j$, $\hat{c}_j(0) = \bar{c}_j$. We know that (10.3) holds for $\hat{c}_j$ on $[0, h^{-1}T_0]$. Let $\tilde{a}_j = a_j - \hat{a}_j$, $\tilde{c}_j = c_j - \hat{c}_j$ denote the differences. Let

$$\gamma(t) \stackrel{\text{def}}{=} \frac{b(a_j,t) - b(\hat{a}_j,t)}{a_j - \hat{a}_j}$$

$$\sigma(t) \stackrel{\text{def}}{=} \frac{\partial_x b(a_j,t) - \partial_x b(\hat{a}_j,t)}{a_j - \hat{a}_j}\,.$$

By the mean-value theorem, $|\gamma(t)| \lesssim h$ and $|\sigma(t)| \lesssim h^2$. We have

$$\begin{cases} \dot{\tilde{a}}_j = \tilde{c}_j^2 + 2\hat{c}_j\tilde{c}_j - \gamma\tilde{a}_j + \mathcal{O}(h) \\ \dot{\tilde{c}}_j = \tilde{c}_j(\partial_x b)(a_j,t) + \hat{c}_j\sigma\tilde{a}_j + \mathcal{O}(h^2)\,. \end{cases} \tag{10.4}$$

We conclude that $|\tilde{a}_j| \lesssim e^{Cht}$ and $|\tilde{c}_j| \lesssim he^{Cht}$. This is proved by Gronwall's method and a bootstrap argument. Since (10.3) holds for $\hat{c}_j$ on $[0, h^{-1}T_0]$, it holds for c_j on the same time scale if $T_0 < \infty$, and up to the maximum time allowable by the above iteration argument, $\epsilon h^{-1}\log h^{-1}$, if $T_0 = +\infty$. □

Appendix A. Local and global well-posedness

In this appendix, we will prove that (1.1) is globally well-posed in H^k, $k \geq 1$ provided

$$M(T) \stackrel{\text{def}}{=} \sum_{j=0}^{k+1} \|\partial_x^j b(x,t)\|_{L^\infty_{[0,T]}L^\infty_x} < \infty\,. \tag{A.1}$$

for all $T > 0$. This is proved for $k = 1$ under the additional assumption that $\|b\|_{L^2_x L^\infty_T} < \infty$ in the appendix of Dejak-Sigal [11]. ‡ The removal of the assumption $\|b\|_{L^2_x L^\infty_T} < \infty$ is convenient since it allows for us to consider potentials that asymptotically in x converge to a nonzero number, rather than decay. Moreover, our argument is self-contained.

Well-posedness for KdV (nonlinearity $\partial_x u^2$) with $b \equiv 0$ was obtained by Bona-Smith [5] via the energy method, using the vanishing viscosity technique for construction and a regularization argument for uniqueness. Although their argument adapts to include $b \neq 0$ and to mKdV (1.1), it applies only for $k > \frac{3}{2}$ due to the derivative in the nonlinearity. Kenig-Ponce-Vega [21, 20] reduced the regularity requirements (for $b \equiv 0$) below $k = 1$ by introducing new local smoothing and maximal function

‡It is further assumed in [11] that $\|b\|_{L^\infty_T L^\infty_x}$ is small, although this appears to be unnecessary in their argument.

estimates and applying the contraction method. These estimates were obtained by Fourier analysis (Plancherel's theorem, van der Corput lemma). At the H^1 level of regularity (and above) for mKdV, the full strength of the maximal function estimate in [21, 20] is not needed. Here, we prove a local smoothing estimate and a (weak) maximal function estimate (see (A.2) and (A.3) in Lemma A.1 below) instead by the integrating factor method, which easily accomodates the inclusion of a potential term since integration by parts can be applied. The estimates proved by Kenig-Ponce-Vega were directly applied by Dejak-Sigal, treating the potential term as a perturbation, which required introducing the norm $\|b\|_{L^2_x L^\infty_T}$. Our argument does not apply directly to KdV since we are lacking the (strong) maximal function estimate used by [21, 20].

Let $Q_n = [n-\frac{1}{2}, n+\frac{1}{2}]$ so that $\mathbb{R} = \cup Q_n$. Let $\tilde{Q}_n = [n-1, n+1]$. An example of our notation is:

$$\|u\|_{\ell^\infty_n L^2_T L^2_{Q_n}} = \sup_n \|u\|_{L^2_{(0,T)} L^2_{Q_n}} .$$

We will use variants like $\ell^2_n L^\infty_T L^2_{Q_n}$ etc. Note that due to the finite incidence of overlap, we have

$$\|u\|_{\ell^\infty_n L^2_T L^2_{Q_n}} \sim \|u\|_{\ell^\infty_n L^2_T L^2_{\tilde{Q}_n}}$$

Theorem A.1 (local well-posedness)**.** *Take $k \in \mathbb{Z}$, $k \geq 1$. Suppose that*

$$M \stackrel{\text{def}}{=} \sum_{j=0}^{k+1} \|\partial_x^j b(x,t)\|_{L^\infty_{[0,1]} L^\infty_x} < \infty .$$

For any $R \geq 1$, take

$$T \lesssim \min(M^{-1}, R^{-4}) .$$

1. *If $\|u_0\|_{H^k} \leq R$, there exists a solution $u(t) \in C([0,T]; H^k_x)$ to (1.1) on $[0,T]$ with initial data $u_0(x)$ satisfying*

$$\|u\|_{L^\infty_T H^k_x} + \|\partial_x^{k+1} u\|_{\ell^\infty_n L^2_T L^2_{Q_n}} \lesssim R .$$

2. *This solution $u(t)$ is unique among all solutions in $C([0,T]; H^1_x)$.*
3. *The data-to-solution map $u_0 \mapsto u(t)$ is continuous as a mapping $H^k \to C([0,T]; H^k_x)$.*

The main tool in the proof of Theorem A.1 is the local smoothing estimate (A.2) below.

Lemma A.1. *Suppose that*

$$v_t + v_{xxx} - (bv)_x = f .$$

We have, for

$$T \lesssim (1 + \|b_x\|_{L^\infty_T L^\infty_x} + \|b\|_{L^\infty_T L^\infty_x})^{-1} ,$$

the energy and local smoothing estimates

$$\|v\|_{L^\infty_T L^2_x} + \|v_x\|_{\ell^\infty_n L^2_T L^2_{Q_n}} \lesssim \|v_0\|_{L^2_x} + \begin{cases} \|\partial_x^{-1} f\|_{\ell^1_n L^2_T L^2_{Q_n}} \\ \|f\|_{L^1_T L^2_x} \end{cases} \tag{A.2}$$

and the maximal function estimate

$$\|v\|_{\ell^2_n L^\infty_T L^2_{Q_n}} \lesssim \|v_0\|_{L^2_x} + T^{1/2}\|v\|_{L^2_T H^1_x} + T^{1/2}\|f\|_{L^2_T L^2_x}\,. \tag{A.3}$$

The implicit constants are independent of b.

Proof. Let $\varphi(x) = -\tan^{-1}(x-n)$, and set $w(x,t) = e^{\varphi(x)}v(x,t)$. Note that $0 < e^{-\frac{\pi}{2}} \leq e^{\varphi(x)} \leq e^{\frac{\pi}{2}} < \infty$, so the inclusion of this factor is harmless in the estimates, although has the benefit of generating the "local smoothing" term in (A.2). We have

$$\partial_t w + w_{xxx} - 3\varphi' w_{xx} + 3(-\varphi'' + (\varphi')^2)w_x + (-\varphi''' + 3\varphi''\varphi' - (\varphi')^3)w - (bw)_x + \varphi' bw = e^\varphi f\,.$$

This equation and manipulations based on integration by parts show that

$$\begin{aligned}\partial_t\|w\|_{L^2_x} = {} & 6\langle\varphi', w_x^2\rangle - 3\langle(-\varphi'' + (\varphi')^2)', w^2\rangle + 2\langle-\varphi''' + 3\varphi''\varphi' - (\varphi')^3, w^2\rangle \\ & - \langle b_x, w^2\rangle + 2\langle b\varphi', w^2\rangle + 2\langle w, e^\varphi f\rangle\,.\end{aligned}$$

We integrate the above identity over $[0,T]$, move the smoothing term $6\int_0^T \langle\varphi', w_x^2\rangle_x\, dt$ over to the left side, and estimate the remaining terms to obtain:

$$\begin{aligned}&\|w(T)\|^2_{L^2_x} + 6\|\langle x-n\rangle^{-1} w_x\|^2_{L^2_T L^2_x} \\ &\qquad \leq \|w_0\|^2_{L^2_x} + CT(1 + \|b_x\|_{L^\infty_T L^\infty_x} + \|b\|_{L^\infty_T L^\infty_X})\|w\|^2_{L^\infty_T L^2_x} + C\int_0^T \left|\int e^\varphi f w\, dx\right| dt\,.\end{aligned}$$

Replacing T by T', and taking the supremum over $T' \in [0,T]$, we obtain, for $T \lesssim (1 + \|b_x\|_{L^\infty_T L^\infty_x} + \|b\|_{L^\infty_{[0,T]} L^\infty_x})^{-1}$, the estimate

$$\|w\|^2_{L^\infty_T L^2_x} + \|\langle x-n\rangle^{-1} w_x\|^2_{L^2_T L^2_x} \lesssim \|w_0\|^2_{L^2_x} + \int_0^T \left|\int e^\varphi f w\, dx\right| dt$$

Using that $0 < e^{-\pi/2} \leq e^\varphi \leq e^{\pi/2} < \infty$, this estimate can be converted back to an estimate for v:

$$\|v\|^2_{L^\infty_T L^2_x} + \|v_x\|^2_{L^2_T L^2_{Q_n}} \lesssim \|v_0\|^2_{L^2_x} + \int_0^T \left|\int e^{2\varphi} f v\, dx\right| dt\,.$$

Estimating as

$$\int_0^T \left|\int e^{2\varphi} f v\, dx\right| dt \lesssim \|f\|_{L^1_T L^2_x}\|v\|_{L^\infty_T L^2_x}\,,$$

and then taking the supremum in n yields the second bound in (A.2). Estimating instead as:

$$\begin{aligned}\int_0^T \left|\int e^{2\varphi} f v\,dx\right| dt &= \int_0^T \left|\int e^{2\varphi}(\partial_x\partial_x^{-1}f)v\,dx\right| dt \\ &\le \int_0^T \left|\int (\partial_x^{-1}f)\,\partial_x(e^{2\varphi}v)\,dx\right| dt \\ &\le \sum_m \|\partial_x^{-1}f\|_{L_T^2L_{Q_m}^2}\|\langle\partial_x\rangle v\|_{L_T^2L_{Q_m}^2} \\ &\le \|\partial_x^{-1}f\|_{\ell_m^1L_T^2L_{Q_m}^2}\|\langle\partial_x\rangle v\|_{\ell_m^\infty L_T^2L_{Q_m}^2}\end{aligned}$$

and taking the supremum in n yields the second bound in (A.2).

For the estimate (A.3), we take $\psi(x) = 1$ on $[n-\frac{1}{2}, n+\frac{1}{2}]$ and 0 outside $[n-1, n+1]$, set $w = \psi v$, and compute, similarly to the above,

$$\|v\|^2_{L_T^\infty L_{Q_n}^2} \lesssim \|v_0\|^2_{L^2_{\tilde{Q}_n}} + T\|v_x\|^2_{L_T^2L^2_{\tilde{Q}_n}} + T\|f\|^2_{L_T^2L^2_{\tilde{Q}_n}}$$

The proof is completed by summing in n. □

Proof of Theorem A.1. We prove the existence by contraction in the space X, where

$$X = \{\, u \mid \|u\|_{C([0,T];H_x^k)} + \|\partial_x^{k+1}u\|_{\ell_n^\infty L_T^2L_{Q_n}^2} + \sup_{\alpha\le k-1}\|\partial_x^\alpha u\|_{\ell_n^2L_T^\infty L_{Q_n}^2} \le CR \,\}\,.$$

Here C is just chosen large enough to exceed the implicit constant in (A.2). Given $u \in X$, let $\varphi(u)$ denote the solution to

$$\partial_t\varphi(u) + \partial_x^3\varphi(u) - \partial_x(b\varphi(u)) = -2\partial_x(u^3)\,. \tag{A.4}$$

with initial condition $\varphi(u)(0) = u_0$. A fixed point $\varphi(u) = u$ in X will solve (1.1). We separately treat the case $k = 1$ for clarity of exposition.

Case $k = 1$. Applying ∂_x to (A.4) gives, with $v = \varphi(u)_x$,

$$v_t + v_{xxx} - (bv)_x = -2(u^3)_{xx} + (b_x\varphi(u))_x\,.$$

Now, (A.2) gives

$$\begin{aligned}&\|\varphi(u)_x\|_{L_T^\infty L_x^2} + \|\varphi(u)_{xx}\|_{\ell_n^\infty L_T^2L_{Q_n}^2} \lesssim \\ &\qquad \|u_0\|_{H_x^1} + \|(u^3)_x\|_{\ell_n^1L_T^2L_{Q_n}^2} + \|(b_x\varphi(u))_x\|_{L_T^1L_x^2}\,.\end{aligned} \tag{A.5}$$

Using that $\|u\|^2_{L_Q^\infty} \lesssim (\|u\|_{L^2_{\tilde{Q}}} + \|u_x\|_{L^2_{\tilde{Q}}})\|u\|_{L^2_{\tilde{Q}}}$, we also have

$$\|(u^3)_x\|_{L_Q^2} \lesssim \|u_x\|_{L_Q^2}\|u\|^2_{L_Q^\infty} \lesssim \|u_x\|_{L_Q^2}\|u\|_{L^2_{\tilde{Q}}}(\|u\|_{L^2_{\tilde{Q}}} + \|u_x\|_{L^2_{\tilde{Q}}})\,.$$

Taking the L_T^2 norm and applying the Hölder inequality, we obtain

$$\|(u^3)_x\|_{L_T^2L_Q^2} \lesssim \|u_x\|_{L_T^\infty L_Q^2}\|u\|_{L_T^\infty L^2_{\tilde{Q}}}(\|u\|_{L_T^2L^2_{\tilde{Q}}} + \|u_x\|_{L_T^2L^2_{\tilde{Q}}})\,.$$

Taking the ℓ^1_n norm and applying the Hölder inequality again yields

$$\|(u^3)_x\|_{\ell^1 L^2_T L^2_{Q_n}} \lesssim \|u_x\|_{\ell^\infty_n L^\infty_T L^2_{Q_n}} \|u\|_{\ell^2_n L^\infty_T L^2_{\tilde{Q}_n}} (\|u\|_{\ell^2_n L^2_T L^2_{\tilde{Q}_n}} + \|u_x\|_{\ell^2_n L^2_T L^2_{\tilde{Q}_n}}) \,.$$

Using the straightforward bounds $\|u_x\|_{\ell^\infty_n L^\infty_T L^2_{Q_n}} \lesssim \|u_x\|_{L^\infty_T L^2_x}$,

$$\|u\|_{\ell^2_n L^2_T L^2_{\tilde{Q}_n}} \lesssim \|u\|_{L^2_T L^2_x} \lesssim T^{1/2} \|u\|_{L^\infty_T L^2_x}$$

and

$$\|u_x\|_{\ell^2_n L^2_T L^2_{\tilde{Q}_n}} \lesssim \|u_x\|_{L^2_T L^2_x} \lesssim T^{1/2} \|u_x\|_{L^\infty_T L^2_x} \,,$$

we obtain

$$\|(u^3)_x\|_{\ell^1_n L^2_T L^2_{Q_n}} \lesssim T^{1/2} \|u\|^2_{L^\infty_T H^1_x} \|u\|_{\ell^2_n L^\infty_T L^2_{Q_n}} \,.$$

Inserting these bounds into (A.5),

$$\begin{aligned} \|\varphi(u)_x\|_{L^\infty_T L^2_x} + \|\varphi(u)_{xx}\|_{\ell^\infty_n L^2_T L^2_{Q_n}} \lesssim\ & \|u_0\|_{H^1_x} + T^{1/2} \|u\|^2_{L^\infty_T H^1_x} \|u\|_{\ell^2_n L^\infty_T L^2_{Q_n}} \\ & + T(\|b_x\|_{L^\infty_x} + \|b_{xx}\|_{L^\infty_x}) \|\varphi(u)\|_{H^1_x} \,. \end{aligned} \tag{A.6}$$

The local smoothing estimate (A.2) applied to $v = \varphi(u)$ (not $v = \varphi(u)_x$ as above), and the estimate

$$\|(u^3)_x\|_{L^1_T L^2_x} \lesssim T \|u\|^3_{L^\infty_T H^1_x} \,,$$

provides the estimate

$$\|\varphi(u)\|_{L^\infty_T L^2_x} \lesssim T \|u\|^3_{L^\infty_T H^1_x} \tag{A.7}$$

The maximal function estimate (A.3) applied to $v = \varphi(u)$ and the estimate

$$\|(u^3)_x\|_{L^2_T L^2_x} \lesssim T^{1/2} \|u\|^3_{L^\infty_T H^1_x} \,,$$

give the estimate

$$\|\varphi(u)\|_{\ell^2_n L^\infty_T L^2_{Q_n}} \lesssim \|u_0\|_{L^2_x} + T \|\varphi(u)\|_{L^\infty_T H^1_x} + T \|u\|^3_{L^\infty_T H^1_x} \,. \tag{A.8}$$

Summing (A.6), (A.7), (A.8), we obtain that $\|\varphi(u)\|_X \leq CR$ if $\|u\|_X \leq CR$ provided T is as stated above. Thus $\varphi : X \to X$. A similar argument establishes that φ is a contraction on X.

Case $k \geq 2$. Differentiating (A.4) k times with respect to x we obtain, with $v = \partial^k_x \varphi(u)$,

$$\partial_t v + \partial^3_x v - \partial_x(bv) = -2\partial^{k+1}_x (u^3) - 2\partial_x \sum_{\substack{\alpha+\beta \leq k+1 \\ \beta \leq k-1}} \partial^\alpha_x b \, \partial^\beta_x \varphi(u) \,.$$

Using (A.2) gives

$$\begin{aligned} & \|\partial^k_x \varphi(u)\|_{L^\infty_T L^2_x} + \|\partial^{k+1}_x \varphi(u)\|_{\ell^\infty_n L^2_T L^2_{Q_n}} \lesssim \\ & \|\partial^k_x u^3\|_{\ell^1_n L^2_T L^2_{Q_n}} + \sup_{\substack{\alpha+\beta \leq k+1 \\ \beta \leq k-1}} \|\partial_x (\partial^\alpha_x b \, \partial^\beta_x \varphi(u))\|_{L^1_T L^2_x} \,. \end{aligned}$$

Expanding, and applying Leibniz rule gives

$$\partial_x^k u = \sum_{\substack{\alpha+\beta+\gamma=k \\ \alpha\leq\beta\leq\gamma}} c_{\alpha\beta\gamma}\partial_x^\alpha u\, \partial_x^\beta u\, \partial_x^\gamma u\,,$$

which is then estimated as follows

$$\|\partial_x^k u\|_{\ell_n^1 L_T^2 L_{Q_n}^2} \lesssim \sum_{\substack{\alpha+\beta+\gamma=k \\ \alpha\leq\beta\leq\gamma}} \|\partial_x^\alpha u\|_{\ell_n^2 L_T^\infty L_{Q_n}^\infty} \|\partial_x^\beta u\|_{\ell_n^2 L_T^2 L_{Q_n}^\infty} \|\partial_x^\gamma u\|_{\ell_n^\infty L_T^\infty L_{Q_n}^2}\,.$$

By the Sobolev embedding theorem (as in the $k=1$ case) we obtain

$$\|\partial_x^k u^3\|_{\ell_n^1 L_T^2 L_{Q_n}^2} \lesssim \sum_{\substack{\alpha+\beta+\gamma=k \\ \alpha\leq\beta\leq\gamma}} \left(\sup_{\sigma\leq\alpha+1} \|\partial_x^\sigma u\|_{\ell_n^2 L_T^\infty L_{Q_n}^2}\right)\left(\sup_{\sigma\leq\beta+1} \|\partial_x^\sigma u\|_{\ell_n^2 L_T^2 L_{Q_n}^2}\right)\|\partial_x^\gamma u\|_{L_T^\infty L_x^2}$$

When $k\geq 2$, we have $\alpha\leq[[\frac{1}{3}k]]\leq k-2$ and $\beta\leq[[\frac{1}{2}k]]\leq k-1$, and therefore

$$\|\partial_x^k u^3\|_{\ell_n^1 L_T^2 L_{Q_n}^2} \lesssim T^{1/2}\left(\sup_{\alpha\leq k-1}\|\partial_x^\alpha u\|_{\ell_n^2 L_T^\infty L_{Q_n}^2}\right)\|u\|_{L_T^\infty H_x^k}^2\,.$$

Also,

$$\|\partial_x(\partial_x^\alpha b\, \partial_x^\beta \varphi(u))\|_{L_T^1 L_x^2} \leq T\left(\sup_{\alpha\leq k+1}\|\partial_x^\alpha b\|_{L_T^\infty L_x^\infty}\right)\|\varphi(u)\|_{L_T^\infty H_x^k}$$

Combining these estimates, we obtain

(A.9)
$$\begin{aligned}\|\partial_x^k\varphi(u)\|_{L_T^\infty L_x^2} + \|\partial_x^{k+1}\varphi(u)\|_{\ell_n^\infty L_T^2 L_{Q_n}^2} \lesssim \|u_0\|_{H_x^k} \\ + T^{1/2}\left(\sup_{\alpha\leq k-1}\|\partial_x^\alpha u\|_{\ell_n^2 L_T^\infty L_{Q_n}^2}\right)\|u\|_{L_T^\infty H_x^k}^2 + T\left(\sup_{\alpha\leq k+1}\|\partial_x^\alpha b\|_{L_T^\infty L_x^\infty}\right)\|\varphi(u)\|_{L_T^\infty H_x^k}\end{aligned}$$

The local smoothing $\|(u^3)_x\|_{L_T^1 L_x^2}\lesssim T\|u\|_{L_T^\infty H_x^1}^3$ to obtain

$$\|\varphi(u)\|_{L_T^\infty L_x^2} \lesssim T\|u\|_{L_T^\infty H_x^1}^3 \tag{A.10}$$

We apply the maximal function estimate (A.3) to $v=\partial_x^\alpha\varphi(u)$ for $\alpha\leq k-1$ and use that $\|\partial_x^{\alpha+1}u^3\|_{L_T^1 L_x^2}\leq T\|u\|_{L_T^\infty H_x^k}^3$ and

$$\|\partial_x^{\alpha+1}(b\varphi(u))\|_{L_T^1 L_x^2} \leq T\left(\sup_{\beta\leq k}\|\partial_x^\beta b\|_{L_T^\infty L_x^\infty}\right)\|\varphi(u)\|_{L_T^\infty H_x^k}$$

to obtain

$$\begin{aligned}\|\partial_x^\alpha\varphi(u)\|_{\ell_n^2 L_T^\infty L_{Q_n}^2} \lesssim\ & \|u_0\|_{H_x^{k-1}} + T\|\varphi(u)\|_{L_T^\infty H_x^k} + T\|u\|_{L_T^\infty H_x^k}^3 \\ & + T\left(\sup_{\beta\leq k}\|\partial_x^\beta b\|_{L_T^\infty L_x^\infty}\right)\|\varphi(u)\|_{L_T^\infty H_x^k}\end{aligned} \tag{A.11}$$

Summing (A.9), (A.10), (A.11), we obtain that $\varphi: X\to X$, and a similar argument shows that φ is a contraction. This concludes the case $k\geq 2$.

To establish uniqueness within the broader class of solutions belonging merely to $C([0,T];H^1_x)$, we argue as follows. Suppose $u,v\in C([0,T];H^1_x)$ solve (1.1). By (A.3),

$$\|v\|_{\ell^2_n L^\infty_T L^2_{Q_n}} \lesssim \|v_0\|_{L^2} + T\|v\|_{L^\infty_T H^1_x} + T\|v\|^3_{L^\infty_T H^1_x}\,.$$

By taking T small enough in terms of $\|v\|_{L^\infty_T H^1_x}$, we have that

$$\|v\|_{\ell^2_n L^\infty_T L^2_{Q_n}} \lesssim \|v\|_{L^\infty_T H^1_x}\,. \tag{A.12}$$

Similarly,

$$\|u\|_{\ell^2_n L^\infty_T L^2_{Q_n}} \lesssim \|u\|_{L^\infty_T H^1_x}\,. \tag{A.13}$$

Set $w=u-v$. Then, with $g=(u^3-v^3)/(u-v)=u^2+uv+v^2$, we have

$$w_t + w_{xxx} - (bw)_x \pm (gw)_x = 0\,.$$

Apply (A.2) to $v=w_x$ to obtain

$$\|w_x\|_{L^\infty_T L^2_x} + \|w_{xx}\|_{\ell^\infty_n L^2_T L^2_{Q_n}} \lesssim \|(gw)_x\|_{\ell^1_n L^2_T L^2_{Q_n}} + \|(b_x w)_x\|_{L^1_T L^2_x} \tag{A.14}$$

The terms of $\|(gw)_x\|_{\ell^1_n L^2_T L^2_{Q_n}}$ are bounded following the method used above:

$$\begin{aligned}\|u_x v w\|_{\ell^1_n L^2_T L^2_{Q_n}} &\lesssim \|u_x\|_{\ell^\infty_n L^\infty_T L^2_{Q_n}} \|vw\|_{\ell^1_n L^2_T L^\infty_{Q_n}} \\ &\lesssim \|u_x\|_{\ell^\infty_n L^\infty_T L^2_{Q_n}} (\|vw\|_{\ell^1_n L^2_T L^1_{Q_n}} + \|(vw)_x\|_{\ell^1_n L^2_T L^1_{Q_n}})\end{aligned}$$

The term in parentheses is bounded by

$$\|v\|_{\ell^2_n L^2_T L^2_{Q_n}} \|w\|_{\ell^2_n L^\infty_T L^2_{Q_n}} + \|v_x\|_{\ell^2_n L^2_T L^2_{Q_n}} \|w\|_{\ell^2_n L^\infty_T L^2_{Q_n}} + \|v\|_{\ell^2_n L^\infty_T L^2_{Q_n}} \|w_x\|_{\ell^2_n L^2_T L^2_{Q_n}}$$

which leads to the bound

$$\|u_x v w\|_{\ell^1_n L^2_T L^2_{Q_n}} \lesssim T^{1/2} \|u\|_{L^\infty_T H^1_x} (\|v\|_{L^\infty_T H^1_x} \|w\|_{\ell^2_n L^\infty_T L^2_{Q_n}} + \|v\|_{\ell^2_n L^\infty_T L^2_{Q_n}} \|w\|_{L^\infty_T H^1_x}) \tag{A.15}$$

We now allow implicit constants to depend upon $\|u\|_{L^\infty_T H^1_x}$ and $\|v\|_{L^\infty_T H^1_x}$. Appealing to (A.14), (A.15) (and analogous estimates for other terms in gw), (A.12), (A.13) to obtain

$$\|w\|_{L^\infty_T H^1_x} \lesssim T^{1/2} (\|w\|_{\ell^2_n L^\infty_T L^2_{Q_n}} + \|w\|_{L^\infty_T H^1_x})$$

Combining this estimate with the maximal function estimate (A.3) applied to w yields

$$\|w\|_{\ell^2_n L^\infty_T L^2_{Q_n}} \lesssim T^{1/2} \|w\|_{L^\infty_T H^1_x} + T\|g\|_{L^\infty_T H^1_x} \|w\|_{L^\infty_T H^1_x}\,.$$

This gives $w\equiv 0$ for T sufficiently small. The continuity of the data-to-solution map is proved using similar arguments. □

Next, we prove global well-posedness in H^k by proving *a priori* bounds. Theorem A.1 shows that doing it suffices for global well-posedness

Theorem A.2 (global well-posedness). *Fix $k\geq 1$ and suppose $M(T)<\infty$ for all $T\geq 0$, where $M(T)$ is defined in* (A.1). *For $u_0\in H^k$, there is a unique global solution $u\in C_{loc}([0,+\infty);H^k_x)$ to* (1.1) *with $\|u\|_{L^\infty_T H^k_x}$ controlled by $\|u_0\|_{H^k}$, T, and $M(T)$.*

Proof. Before beginning, we note that by the Gagliaro-Nirenberg inequality, $\|u\|_{L^4}^4 \lesssim \|u\|_{L^2}^3 \|u_x\|_{L^2}$, we have (in the focusing case)

$$\|u_x\|_{L^2}^2 - \|u_x\| \|u\|_{L^2}^3 \le I_3(u) \le \|u_x\|_{L^2}^2 \,.$$

With $\alpha = \|u_x\|_{L^2}^2 / \|u\|_{L^2}^6$ and $\beta = I_3(u)/\|u\|_{L^2}^6$, this is $\alpha - \alpha^{1/2} \le \beta \le \alpha$, which implies that $\langle \alpha \rangle \sim \langle \beta \rangle$, i.e.

$$\|u_x\|_{L^2}^2 + \|u\|_{L^2}^6 \sim I_3(u) + \|u\|_{L^2}^6$$

The same statement holds in the defocusing case.

Another fact we need is based on the

$$\begin{aligned}\frac{d}{dt} I_j(u) &= \langle I_j'(u), \partial_t u \rangle \\ &= \langle I_j'(u), -u_{xxx} - 2(u^3)_x + (bu)_x \rangle \\ &= \langle I_j'(u), \partial_x I_3'(u) \rangle + \langle I_j'(u), (bu)_x \rangle \\ &= \langle I_j'(u), (bu)_x \rangle\end{aligned}$$

For $u(t) \in L^2$, we compute near conservation of momentum and energy from Lemma 2.1:

$$\frac{d}{dt} I_1(u) = \langle b_x, A_1(u) \rangle$$

Estimate $|\langle b_x, A_1(u) \rangle| \le \|b_x\|_{L^\infty} I_1(u)$, and apply Gronwall to obtain a bound on $\|u\|_{L_T^\infty L_x^2}$ in terms of $\|b_x\|_{L_T^\infty L^\infty}$ and $\|u_0\|_{L^2}$. For $u(t) \in H^1$, we compute near conservation of energy from Lemma 2.1:

$$\frac{d}{dt} I_3(u) = 3\langle b_x, A_3(u) \rangle - \langle b_{xxx}, A_1(u) \rangle \,.$$

We have

$$\begin{aligned}|\langle b_x, A_3(u) \rangle| &\lesssim \|b_x\|_{L^\infty} (\|u_x\|_{L^2}^2 + \|u\|_{L^4}^4) \\ &\lesssim \|b_x\|_{L^\infty} (\|u_x\|_{L^2}^2 + \|u_x\|_{L^2} \|u\|_{L^2}^3) \\ &\lesssim \|b_x\|_{L^\infty} (\|u_x\|_{L^2}^2 + \|u\|_{L^2}^6) \\ &\lesssim \|b_x\|_{L^\infty} (I_3(u) + \|u\|_{L^2}^6)\end{aligned}$$

and

$$|\langle b_{xxx}, A_1(u) \rangle| \lesssim \|b_{xxx}\|_{L^\infty} \|u\|_{L^2}^2 \,.$$

Combining these gives

$$\left| \frac{d}{dt} I_3(u) \right| \lesssim \|b_x\|_{L^\infty} I_3(u) + \|b_x\|_{L^\infty} \|u\|_{L^2}^6 + \|b_{xxx}\|_{L^\infty} \|u\|_{L^2}^2$$

Gronwall's inequality, combined with the previous bound on $\|u\|_{L^2}$, gives the bound on $I_3(u)$ and hence $\|u\|_{H^1}$.

For $u(t) \in H^2$, we apply Lemma 2.1 to obtain

$$\begin{aligned}\frac{d}{dt} I_5(u) &= \langle I_5'(u), (bu)_x \rangle \\ &= 5\langle b_x, A_5(u)\rangle - 5\langle b_{xxx}, A_3(u)\rangle + \langle b_{xxxxx}, A_1(u)\rangle\end{aligned}$$

We have

$$\begin{aligned}|\langle b_x, A_5(u)\rangle| &\lesssim \|b_x\|_{L^\infty}(\|u_{xx}\|_{L^2}^2 + \|u\|_{H^1}^4 + \|u\|_{H^1}^6) \\ &\lesssim \|b_x\|_{L^\infty} I_5(u) + \|b_x\|_{L^\infty}(\|u\|_{H^1}^4 + \|u\|_{H^1}^6)\end{aligned}$$

Also,

$$|\langle b_{xxx}, A_3(u)\rangle| \lesssim \|b_{xxx}\|_{L^\infty}(\|u\|_{H^1}^2 + \|u\|_{H^1}^4)$$

and

$$|\langle b_{xxxxx}, A_1(u)\rangle| \lesssim \|b_{xxx}\|_{L^\infty}\|(u^2)_{xx}\|_{L^2} \lesssim \|b_{xxx}\|_{L^\infty}\|u\|_{H^2}\|u\|_{L^2}$$

Combining, applying Gronwall's inequality, and appealing to the bound on $\|u\|_{H^1}$ obtained previously, we obtain the claimed *a priori* bound in the case $k = 2$.

Bounds on H^k for $k \geq 3$ can be obtained by the above method appealing to higher-order analogues of the identities in Lemma 2.1. However, starting with $k = 3$, we do not need such refined information. By direct computation from (1.1),

$$\frac{d}{dt}\|\partial_x^k u\|_{L^2}^2 = -\int \partial_x^{k+1}(bu)\,\partial_x^k u + 2\int \partial_x^{k+1} u^3 \, \partial_x^k u$$

In the Leibniz expansion of $\partial_x^{k+1} u^3$, we isolate two cases:

$$\partial_x^{k+1} u^3 = 3u^2 \partial_x^{k+1} u + \sum_{\substack{\alpha+\beta+\gamma=k+1 \\ \alpha\leq\beta\leq\gamma\leq k}} c_{\alpha\beta\gamma}\partial_x^\alpha u \, \partial_x^\beta u \, \partial_x^\gamma u$$

For the first term,

$$\left|\int u^2 \, \partial_x^{k+1} u \, \partial_x^k u\right| = \left|\int (u^2)_x (\partial_x^k u)^2\right| \lesssim \|u\|_{H^2}^2 \|u\|_{H^k}^2$$

By the Hölder's inequality and interpolation, if $\alpha + \beta + \gamma = k + 1$ and $\gamma \leq k$,

$$\|\partial_x^\alpha u \, \partial_x^\beta u \, \partial_x^\gamma u\|_{L^2} \lesssim \|u\|_{H^2}^2 \|u\|_{H^k}$$

Thus we have

$$\left|\int \partial_x^{k+1} u^3 \, \partial_x^k u\right| \lesssim \|u\|_{H^2}^2 \|u\|_{H^k}^2$$

Similarly, we can bound

$$\left|\int \partial_x^{k+1}(bu)\,\partial_x^k u\right| \lesssim M(t)\|u\|_{H^k}^2$$

by separately considering the term $b\,\partial_x^{k+1} u \, \partial_x^k u$ and integrating by parts. We obtain

$$\left|\frac{d}{dt}\|\partial_x^k u\|_{L^2}^2\right| \lesssim (M + \|u\|_{H^2}^2)\|u\|_{H^k}^2$$

and can apply the Gronwall inequality to obtain the desired *a priori* bound. □

Appendix B. Comments about the effective ODEs

Here we make some comments about the differential equations for the parameters a and c.

B.1. **Conditions on T_0.** First we give a reason for replacing $T_0(h)$ in the definition of $T(h)$ (1.6) by T_0 defined by (1.13). In (10.2) we have seen that the a and c solving the system (1.5) give the following equations for $\widetilde{A} = ha$, $\widetilde{C} = c$, $T = ht$:

$$\begin{cases} \partial_T \widetilde{A}_j = \widetilde{C}_j^2 - b_0(\widetilde{A}_j, T) + \mathcal{O}(h) \\ \partial_T \widetilde{C}_j = \widetilde{C}_j \partial_x b_0(\widetilde{A}_j, T) + \mathcal{O}(h) \end{cases}, \qquad \widetilde{A}(0) = \bar{a}h\,, \quad \widetilde{C}(0) = \bar{c}\,, \quad j = 1, 2\,.$$

This can also be seen by analysing (B.6) using Lemma 3.2.

As in (10.4) we can write the equations for $\widetilde{A}_j - A_j$ and $\widetilde{C}_j - C_j$:

$$\begin{cases} \partial_T(\widetilde{A}_j - A_j) = (\widetilde{C}_j - C_j)^2 + 2C_j(\widetilde{C}_j - C_j) + \gamma_0(\widetilde{A}_j - A_j) + \mathcal{O}(h) \\ \partial_T(\widetilde{C}_j - C_j) = (\widetilde{C}_j - C_j)(\partial_x b_0)(A_j, t) + C_j\sigma_0(\widetilde{A}_j - A_j) + \mathcal{O}(h)\,, \\ \widetilde{A}_j(0) - A_j(0) = 0\,, \quad \widetilde{C}_j(0) - C_j(0) = 0\,, \end{cases}$$

where $\gamma_0, \sigma_0 = \mathcal{O}(1)$. This implies that

$$\begin{cases} \widetilde{A}_j(T) - A_j(T) = \mathcal{O}(h)e^{CT}\,, \\ \widetilde{C}_j(T) - C_j(T) = \mathcal{O}(h)e^{CT}\,. \end{cases}$$

This means that for $T < \delta \log(1/h)$, we have $C_j(T) = \widetilde{C}_j(T) + \mathcal{O}(h^{1-\delta C})$. Hence, if δ is small enough, then for small h we have that $T_0(h)$ defined in (1.6) and T_0 in (1.13) can be interchanged.

B.2. **Examples with C_j going to 0.** In the decoupled equations (1.12) we can have

$$C_j(T) \to 0\,, \quad T \to \infty\,,$$

which implies that $T_0 < \infty$ in the definition (1.13). That prevents $\log(1/h)/h$ lifespan of the approximation (1.4).

Let us put

$$a = A_j\,, \quad c = C_j\,,$$

so that the system (1.12) becomes

$$a'_T = c^2(T) - b_0(a, T)\,, \quad c'_T = c\,\partial_a b_0(a, T)\,. \tag{B.1}$$

For simplicity we consider the case of $b_0(a, T) = b_0(a)$. In that case the Hamiltonian

$$E(a, c) = -\frac{1}{3}c^3 + cb_0(a)$$

is conserved in the evolution and we have

$$\exp(T \min \partial_a b) \leq |c(T)| \leq \exp(T \max \partial_a b) \,. \tag{B.2}$$

In particular this means that $c > \delta > 0$ if $T < T_1(\delta)$.

We cannot improve on (B.2), and in general we may have

$$|c(T)| \leq e^{-\gamma T} \,, \quad T \to \infty \,,$$

but this behaviour is rare. First we note that the conservation of E shows that if $c(T_j) \to 0$ for some sequence $T_j \to \infty$, then $E = 0$. We can then solve for c, and the equation reduces to $da/dT = 2b_0(a)$, $c^2 = 3b_0(a)$, that is to

$$\frac{1}{2} \int_{a_0}^{a} \frac{d\tilde{a}}{b_0(\tilde{a})} = T \,, \quad b(a(0)) > 0 \,. \tag{B.3}$$

If $b_0(a) > 0$ in this set of values a then

$$a(T) \to \infty \,, \; T \to \infty \,, \tag{B.4}$$

and $c(T) = (3b_0(a(T)))^{\frac{1}{2}}$.

If $b_0(a) = 0$ for some $a > a(0)$ ($a'_T = 2b_0 > 0$), then we denote a_1, the smallest such a and assume that the order of vanishing of b_0 there is ℓ_1. The analysis of (B.3) shows that

$$a(T) = a_1 + \mathcal{O}(1) \begin{cases} Ke^{-\gamma T} & \ell_1 = 1 \,, \\ \\ KT^{-1/(\ell_1 - 1)} & \ell_1 > 1 \,, \end{cases}$$

which gives the rate of decay of $c(T)$.

Hence we have shown the following statement which is almost as long to state as to prove:

Lemma B.1. *Suppose that in* (B.1) $b_0 = b_0(a)$. *Then*

$$E \neq 0 \,, \quad |c(0)| > \delta_0 > 0 \implies \exists \, \delta > 0 \; \forall \, T > 0 \,, \quad |c(T)| > \delta \,.$$

If $E = 0$, let

$$a_1 = \min\{ a \;:\; a > a(0) \,, \; b_0(a) = 0 \} \,,$$

with a_1 not defined if the set is empty (note that $c(0) \neq 0$ and $E = 0$ imply that $b_0(a(0)) > 0$). Now suppose that a_1 exists, and that

$$\partial^\ell b_0(a_1) = 0 \,, \; \ell < \ell_1 \,, \quad \partial^{\ell_1} b_0(a_1) \neq 0 \,.$$

Then as $T \to \infty$,

$$|c(T)| \leq \begin{cases} Ke^{-\gamma T} & \ell_1 = 1 \,, \\ \\ KT^{-\ell_1/(\ell_1 - 1)} & \ell_1 > 1 \,, \end{cases}$$

for some constants γ and K, and $a(T) \to a_1$.

If a_1 does not exist then $c(T) = (3b_0(a(T)))^{\frac{1}{2}}$, $a(T) \to \infty$, $T \to \infty$.

We excluded the case of infinite order of vanishing since it is very special from our point of view.

The lemma suggests that $c \to 0$ is highly nongeneric but it can occur for our system. Since for the original time t in (1.1) we would like to go up to time $\delta \log(1/h)/h$ we cannot do it in some cases as then

$$c(t)|_{t=\delta\log(1/h)/h} \sim \begin{cases} h^{\gamma\delta/2} & \ell_1 = 1\,, \\ \log^{-\frac{1}{2}\ell_1/(\ell_1-1)}(1/h) & \ell_1 > 1\,. \end{cases}$$

B.3. **Avoided crossing for the effective equations of motion.** Here we make some comments about the puzzling avoided crossing which needs further investigation.

For the decoupled equations it is easy to find examples in which

$$c_1(T_0) = c_2(T_0)\,. \tag{B.5}$$

One is shown in Fig.6. We take b_0 independent of T and equal to $\cos^2 x$. If we choose the initial conditions so that $c_j^2 = 3\cos^2 A_j$, $A_j = ha_j$ as in (1.12), and $-\pi/2 < A_1 < -A_2 < 0$, then when $A_1(T_0) = -A_2(T_0)$ we have (B.5) (this also provides an example of $c_2(T) \to 0$ as $T \to \infty$).

The decoupled equations (1.12) should be compared the rescaled version of (1.5):

$$\begin{gathered} \partial_T c_j = \partial_{x_j} B_0(c, A, h)\,, \quad \partial_T A_j = c_j^2 - \partial_{c_j} B_0(c, A, h)\,, \\ B_0(c, A, h) \stackrel{\text{def}}{=} \frac{1}{2}\int q_2(x/h, c, A/h) b_0(x) dx\,. \end{gathered} \tag{B.6}$$

For the example above the comparison between the solutions of the decoupled h-independent equations and solutions to the equation (B.6) are shown in Fig.6 (the solutions (1.12) are shown as a single curve which both solutions with these initial data follow).

The dramatic avoided crossings shown in Fig.6 (and also, for a different, time dependent b_0 in Fig.3) are not seen in the behaviour of $q_2(x, c, A/h)$ which is the approximation of the solution to (1.1) – see Fig.7. The masses of the right and left solitons are switched and that corresponds to the switch of positions of A_1 and A_2. It is possible that a different parametrization of double solitons would resolve this problem. Another possibility is to study the decomposition (3.11) in the proof of Lemma 3.2 uniformly $\alpha \to 0$ (corresponding to $a_2 - a_1 \to 0$).

We conclude with two heuristic observations. If the decoupled equations lead to (B.5) and $|A_1 - A_2| > \epsilon > 0$ (which is the case when we approach the crossing in Fig.6) then equations (B.6) differ from (1.12) by terms of size

$$h \log\left(\frac{c_2 - c_1}{c_1 + c_2}\right)\,,$$

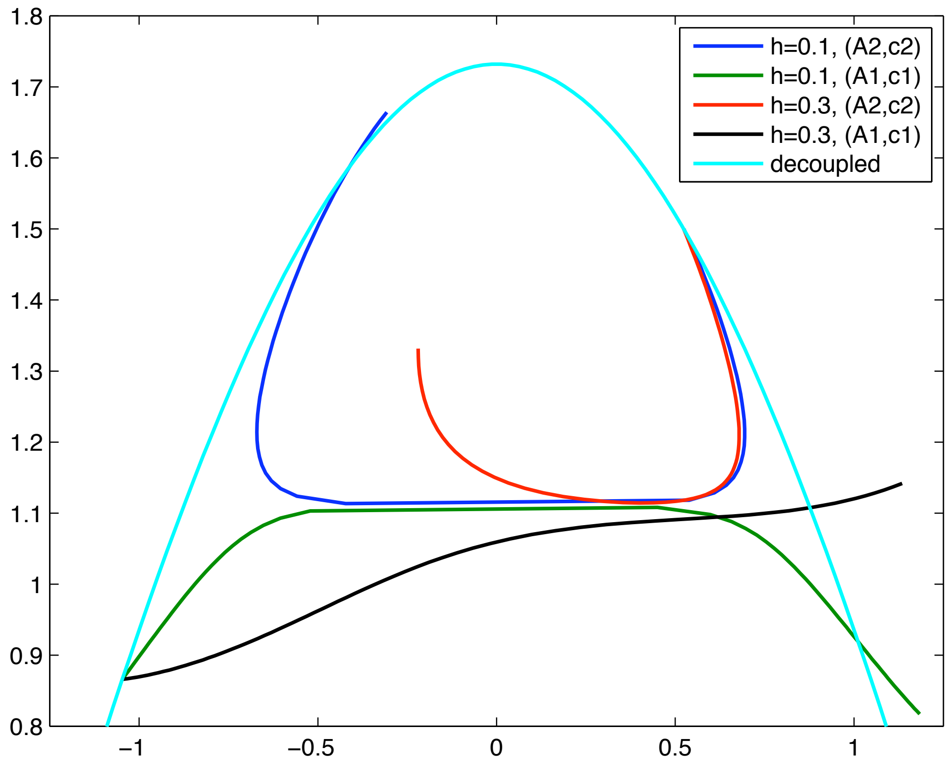

FIGURE 6. The plots of (A_j, c_j), $j = 1, 2$, solving (B.6) for for $b_0(x,t) = \cos^2 x$ and initial data $A_1(0) = -\pi/3$, $A_2(0) = \pi/6$, and $c_1(0) = \sqrt{3}\cos(\pi/3)$, $c_2(0) = \sqrt{3}\cos(\pi/6)$. The "decoupled" curve corresponds to solving (1.12). Because of the choice of initial conditions, (A_j, c_j), $j = 1, 2$ line on the same curve.

see Lemma 3.2. For this to affect the motion of trajectories on finite time scales in T we need

$$c_2 - c_1 \simeq \exp\left(-\frac{\gamma}{h}\right) . \tag{B.7}$$

This means that c_j's have to get exponentially close to each other (but does not explain avoided crossing).

On the other hand if $|a_1 - a_2| > \epsilon > 0$, where a_j's are the original variables in (1.5), $A_j(0) = ha_j(0)$, then we can use the decomposition in Lemma 3.2 and variables $\hat{a}_j$ defined by (3.7). The remark after the proof of Lemma 3.6 shows that the equations of motion take essentially the same form written in terms of $\widehat{a}_j$'s and c_j's and hence $\hat{a}_j$ has to stay bounded. And that means that $c_2 - c_1$ is bounded away from 0. Hence, when $c_2 - c_1 \to 0$ we must also have $a_2 - a_1 \to 0$ as seen in Fig.3 and Fig.6.

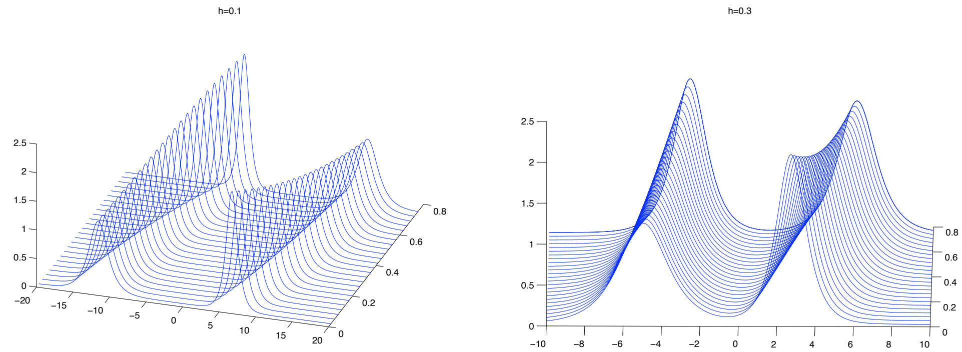

FIGURE 7. The plots of $q_2(x, c, A/h)$ for (A_j, c_j), $j = 1, 2$, solving (B.6) for for $b_0(x, t) = \cos^2 x$ and initial data $A_1(0) = -\pi/3$, $A_2(0) = \pi/6$, and $c_1(0) = \sqrt{3}\cos(\pi/3)$, $c_2(0) = \sqrt{3}\cos(\pi/6)$. On the left $h = 0.1$ and on the right $h = 0.3$.

APPENDIX C. ALTERNATIVE PROOF OF LEMMA 4.7 (WITH BERND STURMFELS)

We note that the standard substition reduces the equation $P(c)u = 0$, where $P(c)$ is defined in (4.20), to an equation with rational coefficients:

$$z = \tanh x\,, \quad \partial_x = (1 - z^2)\partial_z\,, \quad \eta^2 = 1 - z^2\,.$$

This means that $P(c)u = 0$ is equivalent to $Q(c)v = 0$, $u(x) = v(\tanh x)$, where

$$Q(c) = (L^2 + 1)(L^2 + c^2) - 10LR(z)L + 10(3R(z) - 2R(z)^2) - 6(1 + c^2)R(z)\,,$$

and

$$L = \frac{1}{i}(1 - z^2)\partial_z\,, \quad R(z) = 1 - z^2\,, \quad -1 < z < 1\,.$$

Lemma 4.7 will follow from finding a basis of solutions of $Q(c)v = 0$ and from seeing that the only bounded solution is the one corresponding to $\partial_x\eta$, that is, to

$$v(z) = z(1 - z^2)^{\frac{1}{2}}\,.$$

Remarkably, and no doubt because of some deeper underlying structure due to complete integrability, this can be achieved using `MAPLE` package `DEtools`.

First, the operator $Q(c)$ is brought to a convenient form

$$\begin{aligned}
Q = {} & (z-1)^4(z+1)^4\frac{d^4}{dz^4}f(z) + 12z(z-1)^3(z+1)^3\frac{d^3}{dz^3}f(z) \\
& + (z-1)^2(z+1)^2(26z^2 - c^2 + 1)\frac{d^2}{dz^2}f(z) \\
& - 2z(z-1)(z+1)(8z^2 - 11 + c^2)\frac{d}{dz}f(z) \\
& + (4 - 20z^2 + 6c^2z^2 - 5c^2 + 16z^2)f(z)
\end{aligned}$$

Applying the `MAPLE` command `DFactorsols(Q,f(z))` gives the following explicit basis of solutions to $Q(c)v = 0$, $c \neq 1$:

$$\begin{aligned}
v_1(z) &= (1-z^2)^{\frac{1}{2}} z\,, \\
v_2(z) &= (1+z)^{-\frac{c}{2}}(1-z)^{\frac{c}{2}}((c+z)^2+z^2-1)\,, \\
v_3(z) &= v_2(-z) = (1+z)^{\frac{c}{2}}(z-1)^{-\frac{c}{2}}((c-z)^2+z^2-1)\,, \\
v_4(z) &= (1-z^2)^{-\frac{1}{2}}\left(-3zc^2+3z^3c^2-7z^3+7z\right)\log\frac{z+1}{z-1} \\
&\qquad + (1-z^2)^{-\frac{1}{2}}\left(4c^2-6c^2z^2+14z^2-12\right)\,.
\end{aligned}$$

For $c \neq 1$ these solutions are linearly independent and only v_1 vanishes at $z = \pm 1$ (or is bounded). Hence $\ker_{L^2} P(c)$ is one dimensional proving Lemma 4.7.

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