DIVERGENCE OF INFINITE-VARIANCE NONRADIAL SOLUTIONS TO THE 3D NLS EQUATION

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Abstract. We consider solutions $u(t)$ to the 3d focusing NLS equation $i\partial_t u+\Delta u+|u|^2u=0$ such that $\|xu(t)\|_{L^2}=\infty$ and $u(t)$ is nonradial. Denoting by $M[u]$ and $E[u]$, the mass and energy, respectively, of a solution u, and by $Q(x)$ the ground state solution to $-Q+\Delta Q+|Q|^2Q=0$, we prove the following: if $M[u]E[u]<M[Q]E[Q]$ and $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}>\|Q\|_{L^2}\|\nabla Q\|_{L^2}$, then either $u(t)$ blows-up in finite positive time or $u(t)$ exists globally for all positive time and there exists a sequence of times $t_n\to+\infty$ such that $\|\nabla u(t_n)\|_{L^2}\to\infty$. Similar statements hold for negative time.

1. Introduction

The 3d focusing nonlinear Schrödinger equation (NLS) is

$$i\partial_t u+\Delta u+|u|^2u=0\,, \tag{1.1}$$

where $u=u(x,t)\in\mathbb{C}$ and $(x,t)\in\mathbb{R}^{3+1}$. We shall denote the initial data $u_0(x)=u(x,0)$. The standard local theory in H^1 is based upon the Strichartz estimates (see Cazenave [1], Tao [20]), and asserts the existence of a maximal forward time $T^*\gtrsim\|\nabla u_0\|_{L^2}^{-4}$ such that $u(t)\in C([0,T^*);H^1_x)$. If $T^*<\infty$, then it follows from the local theory that $\|\nabla u(t)\|_{L^2}\to+\infty$ as $t\nearrow T^*$, and we say that $u(t)$ *blows-up* in finite forward time. If, on the other hand, $T^*=+\infty$, then we say that $u(t)$ exists globally in (forward) time. In this case, the local theory gives us no information about the behavior of $\|\nabla u(t)\|_{L^2}$ as $t\to+\infty$. Analogous statements hold backwards in time. In fact, if $u(t)$ solves NLS, then $\bar u(-t)$ solves NLS, and thus, it suffices to study the forward-in-time case[1].

Solutions to (1.1) in H^1 satisfy mass, energy, and momentum conservation, given respectively by

$$M[u]=\|u\|_{L^2}^2,\quad E[u]=\frac12\|\nabla u\|_{L^2}^2-\frac14\|u\|_{L^4}^4,\quad P[u]=\operatorname{Im}\int\bar u\,\nabla u\,.$$

There exists a ground state (minimal L^2 norm) solution $Q=Q(x)$ to the (stationary) nonlinear elliptic equation

$$-Q+\Delta Q+|Q|^2Q=0\,,$$

[1]This is not to say that a given solution $u(t)$ must have the same forward-in-time and backward-in-time behavior; however, if u_0 is real-valued, then $u(t)=\bar u(-t)$.

which is unique modulo translation and gauge symmetry. This Q is radial, smooth, positive, and behaves as $Q(x) \sim e^{-|x|}$ for $|x| \to +\infty$. It gives rise to a solution $u(x,t) = e^{it}Q(x)$ to (1.1) called the ground state soliton.

In Holmer-Roudenko [9], we proved that if $u_0 \in H^1$, $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} > \|Q\|_{L^2}\|\nabla Q\|_{L^2}$ and $M[u]E[u] < M[Q]E[Q]$, then $u(t)$ blows-up in finite forward (and finite backward) time, provided that either (1) $\|xu_0\|_{L^2} < \infty$, that is, the initial data (and hence, the whole flow $u(t)$) has finite variance, or (2) u_0 (and hence, the whole flow $u(t)$) is radial. Moreover, it is sharp in the sense that $u(t) = e^{it}Q(x)$ solves NLS and does not blow-up in finite time. Via the Galilean transform and momentum conservation, if $P[u] \neq 0$, this can be refined to the following: if $M[u]E[u] - \frac{1}{2}P[u]^2 < M[Q]E[Q]$ and $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} > \|Q\|_{L^2}\|\nabla Q\|_{L^2}$, then the above conclusions hold (see Appendix B for clarification). These results are essentially classical. The finite variance case follows from the virial identity [21], [6]:

$$\partial_t^2 \|xu(t)\|_{L^2}^2 = 24E[u] - 4\|\nabla u\|_{L^2}^2$$

and the sharp Gagliardo-Nirenberg inequality [22]. The radial case follows from a localized virial identity and a radial Gagliardo-Nirenberg inequality [19]. The radial case is an extension of a result of Ogawa-Tsutsumi [17], who proved the case $E[u] < 0$. Martel in [13] showed that in the case of $E < 0$ either finite variance or radiality assumptions can be relaxed to nonisotropic ones, namely, if (1) $\| \, |y| \, u_0\|_{L^2_x} < \infty$ where $y = (x_1, x_2)$, or (2) $u_0(x_1, x_2, x_3) = u_0(|y|, x_3)$.

In this paper, we drop the additional hypothesis of finite variance and radiality and obtain the following conclusion:

Theorem 1.1. *Suppose $u_0 \in H^1$, $M[u]E[u] < M[Q]E[Q]$ and $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} > \|Q\|_{L^2}\|\nabla Q\|_{L^2}$. Then either $u(t)$ blows-up in finite forward time or $u(t)$ is forward global and there exists a sequence $t_n \to +\infty$ such that $\|\nabla u(t_n)\|_{L^2} \to +\infty$. A similar statement holds for negative time.*

It is still possible, as far as we know, that a given solution satisfying the hypothesis might, say, blow-up in finite negative time but be global in forward time with the existence of a sequence $t_n \to +\infty$ such that $\|\nabla u_n(t)\|_{L^2} \to +\infty$. In other words, a given solution might have different behavior in forward and backward times.

The above remarks regarding the refinement for $P[u] \neq 0$, by applying a Galilean transformation to convert to a solution with $P[u] = 0$, apply in the context of Theorem 1.1 as well. In fact, we will always assume $P[u] = 0$ in this paper (see Appendix B for the standard details).

A result similar to Theorem 1.1 was obtained by Glangetas-Merle [5] for the case of $E[u] < 0$ (see also Nawa [16]). However, our proof is different in structure and uses a different form of concentration compactness machinery. Our proof is more akin to the proof of the scattering result we have in [9], [2], appealing to (suitable adaptations of) the profile decomposition results of Keraani [11], nonlinear perturbation theory based upon the Strichartz estimates, and rigidity theorems based upon the localized virial identity. Our scattering result was in turn modeled on a similar result by Kenig-Merle [10] for the energy-critical NLS equation. In his various lectures, Kenig refers to this scheme as the "concentration compactness–rigidity method" and discusses a "road map" for applying it to various problems. We believe that this method applied to prove Theorem 1.1 has more potential for generalization. In particular, it could perhaps provide an affirmative answer to:

Weak conjecture. Under the hypothesis of Theorem 1.1, either $u(t)$ blows-up in finite forward time or $\|\nabla u(t)\|_{L^2} \to \infty$ as $t \to +\infty$.

Strong conjecture. Under the hypothesis of Theorem 1.1, $u(t)$ blows-up in finite forward time.

Why are we interested in removing the finite-variance hypothesis from our earlier result? The assumption $\|xu_0\|_{L^2} < \infty$ might be considered unnatural on the grounds that blow-up is a local-in-space phenomenon and should not be dictated, in such a strong sense, by the size of the initial data at spatial infinity. In the case $\|xu_0\|_{L^2} < \infty$ addressed in [9], the proof given via the virial identity actually provides, once the solution is scaled so that $M[u] = M[Q]$, an upper bound T_b on the blow-up time T^*, where T_b is given as:

$$T_b = r'(0) + \sqrt{r'(0)^2 + 2r(0)}\,,$$

and

$$r(0) = c_1 \|xu_0\|_{L^2}^2\,, \quad r'(0) = 4c_1 \operatorname{Im} \int (x\cdot \nabla u_0)\, \bar{u}_0\,.$$

Here, c_1 is a constant depending on $E[u]$ that diverges as $E[u] \nearrow E[Q]$. We carry out this classical argument in Prop. 3.1. This upper bound is actually an estimate for the time at which $\|xu(t)\|_{L^2} = 0$ if $u(t)$ were to continue to exist up to that time. However, numerics show that even if blow-up occurs at the origin, the variance $\|xu(t)\|_{L^2}$ actually does not go to zero at the blow-up time due to radiated mass ejected from the blow-up core, and thus, blow-up occurs before the time predicted by this method. This suggests that the full variance $\|xu(t)\|_{L^2}$ is not the correct quantity on which to base a blow-up theory. An analysis of the radial case using the radial Gagliardo-Nirenberg inequality (carried out in Prop. 3.3) reveals that there is an upper bound expressible entirely in terms of a spatially truncated version of $r'(0)$ as well as the proximity of $E[u]$ to $E[Q]$. Thus, the size of the initial variance does not appear at all, and $r'(0)$ can be thought of as measuring the degree and sign of quadratic phase.[2] Theorem 1.1 might be considered the first step in assessing the

[2] The relevance of quadratic phase seems very important from our numerics, see forthcoming paper [7]. We remark that in the 2d case it is exactly quantifiable via the pseudoconformal transformation.

relevance of the variance in blow-up theory of nonradial solutions, even though it is, unfortunately, nonquantitative.[3]

Another motivation is that there exist equations with less structure that NLS, such as the Zakharov system, for which the assumption of finite variance is not known to be of assistance in proving that negative energy solutions blow-up. Merle [14] proved using a localized virial-type identity that *radial* negative energy solutions of the 3d Zakharov system behave according to the conclusion of Theorem 1.1. No result is known for nonradial solutions (finite-variance or not) and it is conceivable that the concentration compactness methods of this paper might be of assistance in addressing this case. Even for the 3d NLS equation (1.1) itself, there are studies in the behavior of finite-time blow-up solutions, such as the divergence of the critical L^3 norm proved for radial solutions in Merle-Raphael [15], for which concentration compactness methods might enable one to remove the radiality assumption.

The paper is structured as follows. §2–6 are devoted to preparatory material; §7–9 are devoted the proof of Theorem 1.1. In §2, we review the dichotomy and scattering result we obtained in [9], [2]. In §3 we deduce some blow-up theorems for the virial identity and its localized versions – in the nonradial case, we are forced to assume an *a priori* uniform-in-time localization on the solution under consideration. In §4, we rewrite the variational characterization of the ground state Q from Lions [12] in a form that is more compatible with the scale-invariant perspective of this paper; this material is needed for §5. In §5, we carry out the base-case of the inductive argument that follows in §7–9. Under the assumption that Theorem 1.1 is false, we are able to construct a special "critical" solution that remains uniformly-in-time concentrated in H^1. Such a solution would contradict the results of §3, and hence, cannot exist.

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2. Ground state and dichotomy

We begin by recalling a few basic facts about the ground state Q, the minimal mass $H^1(\mathbb{R}^3)$ solution of $-Q + \Delta Q + |Q|^2 Q = 0$.

Weinstein [22] proved that the sharp constant c_{GN} of Gagliardo-Nirenberg inequality

$$\|u\|_{L^4(\mathbb{R}^3)}^4 \leq c_{GN} \|u\|_{L^2(\mathbb{R}^3)} \|\nabla u\|_{L^2(\mathbb{R}^3)}^3 \tag{2.1}$$

[3]Another problem we face in the nonradial case is that of predicting the *location* of the blow-up. Nothing says that blow-up should occur at the origin, even if $P[u] = 0$.

is achieved by taking $u = Q$. Using the Pohozhaev identities

$$\|\nabla Q\|_2^2 = 3\|Q\|_2^2, \quad \|Q\|_4^4 = 4\|Q\|_2^2,$$

we can express c_{GN} as

$$c_{GN} = \frac{4}{3\|Q\|_2\|\nabla Q\|_2}\,. \tag{2.2}$$

The Pohozhaev identities also give:

$$E[Q] = \frac{1}{6}\|\nabla Q\|_{L^2}^2\,. \tag{2.3}$$

Let

$$\eta(t) = \frac{\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}}{\|Q\|_{L^2}\|\nabla Q\|_{L^2}}\,. \tag{2.4}$$

By (2.1), (2.2), and (2.3), we have

$$3\eta(t)^2 \geq \frac{M[u]E[u]}{M[Q]E[Q]} \geq 3\eta(t)^2 - 2\eta(t)^3, \tag{2.5}$$

see Figure 1.

Suppose that $M[u]E[u]/M[Q]E[Q] < 1$. Then we have 2 cases:

- If $0 \leq M[u]E[u]/M[Q]E[Q] < 1$, then there exist two solutions (see Figure 2) $0 \leq \lambda_- < 1 < \lambda$ to

$$\frac{M[u]E[u]}{M[Q]E[Q]} = 3\lambda^2 - 2\lambda^3\,. \tag{2.6}$$

- If $E[u] < 0$, then there exists exactly one solution $\lambda > 1$ to (2.6).

By the H^1 local theory, there exist $-\infty \leq T_* < 0 < T^* \leq \infty$ such that $T_* < t < T^*$ is the maximal time interval of existence for $u(t)$ solving (1.1). Moreover,

$$T^* < +\infty \implies \|\nabla u(t)\|_{L^2} \geq \frac{c}{(T^* - t)^{1/4}} \quad \text{as } t \nearrow T^*$$

with a similar statement holding if $T_* \neq -\infty$.

The following is a consequence of the continuity of the flow $u(t)$ (see Figures 1–2). The proof is carried out in [9, Theorem 4.2].

Proposition 2.1 (dichotomy)**.** *Let $M[u]E[u] < M[Q]E[Q]$ and $0 \leq \lambda_- < 1 < \lambda$ be defined as above. Then exactly one of the following holds:*

(1) *The solution $u(t)$ is global (i.e., $T_* = -\infty$ and $T^* = +\infty$) and*

$$\forall\, t \in (-\infty, +\infty)\,, \quad \frac{1}{3} \cdot \frac{M[u]E[u]}{M[Q]E[Q]} \leq \eta(t)^2 \leq \lambda_-^2\,.$$

(2) $\forall\, t \in (T_*, T^*)\,, \quad \lambda \leq \eta(t)\,.$

The first case is only possible for $0 \leq M[u]E[u]/M[Q]E[Q] \leq 1$.

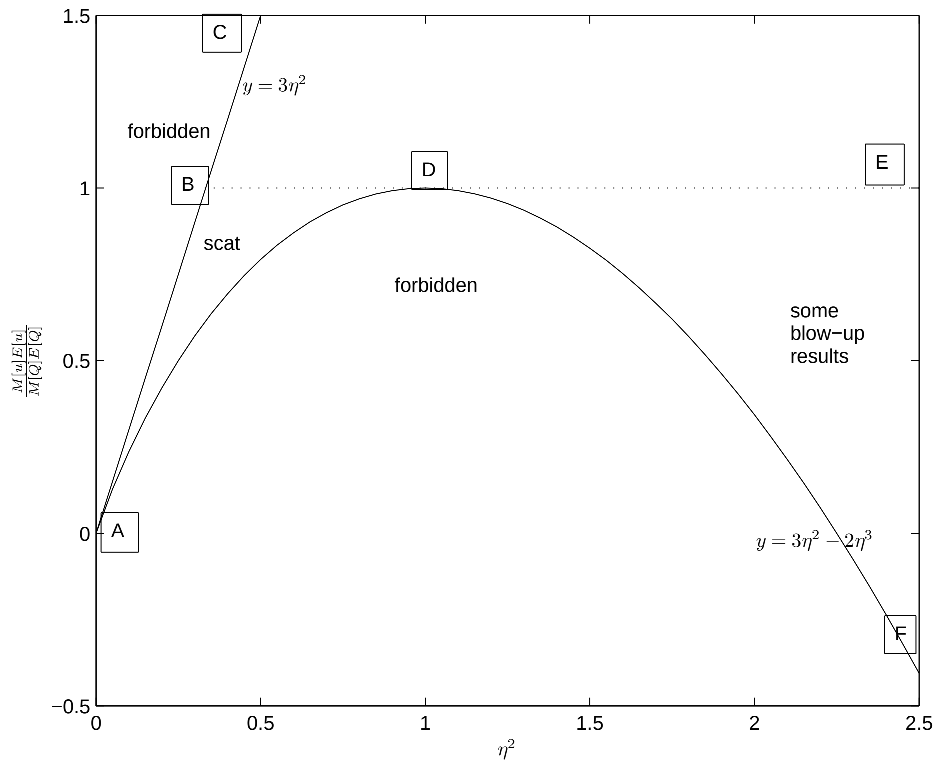

FIGURE 1. A plot of $M[u]E[u]/M[Q]E[Q]$ versus η^2, where η is defined by (2.4). The area to the left of line ABC and inside region ADF are excluded by (2.5). The region inside ABD corresponds to case (1) of Prop. 2.1 and Theorem 2.2 (solutions scatter). The region EDF corresponds to case (2) of Prop. 2.1 and Theorem 1.1 (solutions either blow-up in finite time or diverge in H^1 along a sequence $t_n \to \infty$), Prop. 3.1 (finite-variance solutions blow-up in finite time), and Prop. 3.3 (radial solutions blow-up in finite-time). Behavior of solutions on the dotted line (mass-energy threshold line) is given by [3, Theorem 3-4].

Naturally, one can check the initial data (the value of $\eta(0)$) to determine whether the solution is of the first or second type in Prop. 2.1. Note that the second case does not assert finite-time blow-up (this is the subject of this paper). In the first case, we proved in [9], [2] that more holds.

Theorem 2.2 (scattering)**.** *If $0 < M[u]E[u]/M[Q]E[Q] < 1$ and the first case of Prop. 2.1 holds, then $u(t)$ scatters as $t \to +\infty$ or $t \to -\infty$. This means that there*

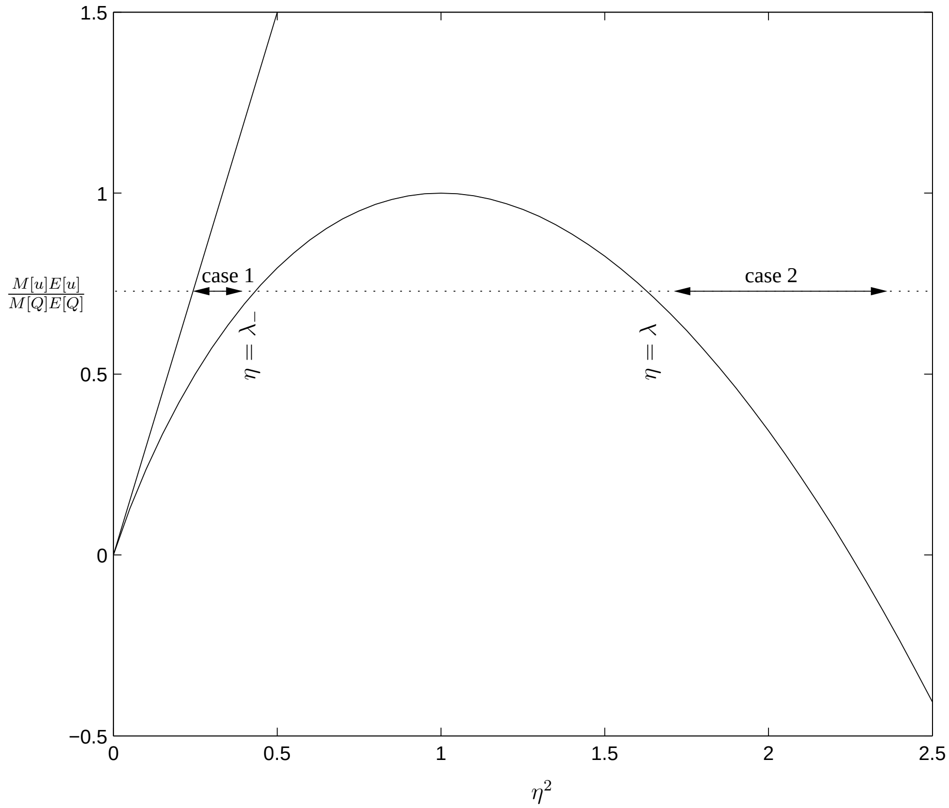

FIGURE 2. On the plot of $M[u]E[u]/M[Q]E[Q]$ versus η^2, indicates how a choice of $M[u]E[u]/M[Q]E[Q]$ determines via (2.6) (at most) two special values of η, namely $\eta = \lambda_-$ and $\eta = \lambda$. In the NLS flow in Case 1 and Case 2 of Prop. 2.1, $\eta(t)$ moves along the indicated horizontal lines. Note that Theorem 2.2 states that in Case 1, $\eta(t)$ approaches the left endpoint as $t \to \pm\infty$. Theorem 1.1 states that in Case 2, there exists a sequence of times $t_n \to +\infty$ along which $\eta(t_n) \to +\infty$.

exist $\psi_\pm \in H^1$ *such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{-it\Delta}\psi_\pm\|_{H^1} = 0\,. \tag{2.7}$$

Consequently, we have that

$$\lim_{t\to\pm\infty} \|u(t)\|_{L^4} = 0 \tag{2.8}$$

and

$$\lim_{t\to\pm\infty} \eta(t)^2 = \frac{1}{3}\cdot\frac{M[u]E[u]}{M[Q]E[Q]}\,. \tag{2.9}$$

Let us justify (2.8)–(2.9) since they are not mentioned in [9], [2]. By (2.7), the Gagliardo-Nirenberg inequality, and mass conservation for the linear and nonlinear flows, we have

$$\lim_{t\to\pm\infty} \|u(t) - e^{-it\Delta}\psi_\pm\|_{L^4} = 0\,.$$

The statement in (2.8) then follows by the linear decay estimate $\|e^{-it\Delta}\psi\|_{L^4} \leq t^{-3/4}\|\psi\|_{L^{4/3}}$ and an approximation argument (to deal with the fact that $\psi \notin L^{4/3}$).[4] By (2.8), we have

$$\lim_{t\to\pm\infty} \|\nabla u(t)\|_{L^2}^2 = 2E[u] + \frac{1}{2}\lim_{t\to\pm\infty} \|u(t)\|_{L^4}^4 = 2E[u]\,.$$

Multiply by $M[u]/M[Q]E[Q]$ and use the Pohozhaev identities to obtain (2.9).

3. Virial identity and blow-up conditions

Now we turn our attention to the second case of Prop. 2.1. We begin by giving the classical derivation, using the virial identity, of the upper bound on the (finite) blow-up time under the finite variance hypothesis.

Proposition 3.1 (Finite-variance blow-up time)**.** *Let $M[u] = M[Q]$ and $E[u]/E[Q] < 1$ and suppose that the second case of Prop. 2.1 holds (take $\lambda > 1$ to be as defined in (2.6)). Define $r(t)$ to be the scaled variance:*

$$r(t) = \frac{\|xu\|_{L^2}^2}{48E[Q]\lambda^2(\lambda-1)}\,.$$

Then blow-up occurs in forward time before t_b (i.e., $T^* \leq t_b$), where

$$t_b = r'(0) + \sqrt{r'(0)^2 + 2r(0)}.$$

Note that

$$r(0) = \frac{1}{48E[Q]\lambda^2(\lambda-1)}\|xu_0\|_{L^2}^2$$

and

$$r'(0) = \frac{1}{12E[Q]\lambda^2(\lambda-1)}\,\mathrm{Im}\int (x\cdot\nabla u_0)\,\bar{u}_0.$$

As we remarked in the introduction, we feel that the dependence of t_b on $r'(0)$ (or ideally a spatially truncated version of it) is quite natural, but the dependence on $r(0)$ seems unsubstantiated, placing a very strong weight on the size of the solution at spatial infinity.

[4] As a result of this approximation argument, we lose the quantitative estimate of $t^{-3/4}$ on the rate of decay.

Proof. The virial identity gives

$$r''(t) = \frac{1}{48E[Q]\lambda^2(\lambda-1)}(24E[u] - 4\|\nabla u\|_{L^2}^2).$$

By the Pohozhaev identities,

$$r''(t) = \frac{1}{2\lambda^2(\lambda-1)}\left(\frac{E[u]}{E[Q]} - \frac{\|\nabla u\|_{L^2}^2}{\|\nabla Q\|_{L^2}^2}\right).$$

By definition of λ and η,

$$r''(t) = \frac{1}{2\lambda^2(\lambda-1)}(3\lambda^2 - 2\lambda^3 - \eta(t)^2)\,.$$

Since $\eta(t) \geq \lambda$ (and $\lambda > 1$), we have

$$r''(t) \leq -1\,.$$

Integrating in time twice gives

$$r(t) \leq -\frac{1}{2}t^2 + r'(0)t + r(0)\,.$$

The positive root of the polynomial on the right-hand side is t_b as given in the proposition statement. □

We next review the local virial identity. Let $\varphi \in C_c^\infty(\mathbb{R}^3)$ be radial such that

$$\varphi(x) = \begin{cases} |x|^2 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 \end{cases}.$$

For $R > 0$ define

$$z_R(t) = \int R^2 \varphi\left(\frac{x}{R}\right)|u(x,t)|^2\,dx. \tag{3.1}$$

Then direct calculation gives the local virial identity:

$$z_R''(t) = 4\int \partial_j\partial_k\psi\Big(\frac{x}{R}\Big)\,\partial_j u\,\partial_k\bar{u} - \int \Delta\psi\Big(\frac{x}{R}\Big)\,|u|^4 - \frac{1}{R^2}\int \Delta^2\psi\Big(\frac{x}{R}\Big)\,|u|^2\,. \tag{3.2}$$

Note that

$$z_R''(t) = (24E[u] - 4\|\nabla u(t)\|_{L^2}^2) + A_R(u(t)),$$

where, for suitable φ [5]

$$A_R(u(t)) \lesssim \frac{1}{R^2}\|u\|_{L^2(|x|\geq R)}^2 + \|u\|_{L^4(|x|\geq R)}^4. \tag{3.3}$$

[5]Note that in the upper bound we do not need the term $\|\nabla u\|_{L^2(|x|\geq R)}^2$. This term was needed in the *lower* bound that was applied in the proof of the scattering theorem [2].

Using the local virial identity, we can prove a version of Prop. 3.1, valid without the assumption of finite variance but assuming that the solution is suitably localized in H^1 for all times. Define

$$\eta_{\geq R}(t) = \frac{\|u\|_{L^2(|x|\geq R)}\|\nabla u\|_{L^2(|x|\geq R)}}{\|Q\|_{L^2}\|\nabla Q\|_{L^2}}\,.$$

Proposition 3.2 (Blow-up time for *a priori* localized solutions)**.** *Let $M[u]=M[Q]$ and $E[u]<E[Q]$ and suppose that the second case of Prop. 2.1 holds (take $\lambda>1$ to be as defined in* (2.6)*). Select γ such that $0<\gamma<\min(\lambda-1,\gamma_0)$, where γ_0 is an absolute constant. Suppose that there is a radius $R\gtrsim\gamma^{-1/2}$ such that for all t, there holds $\eta_{\geq R}(t)\lesssim\gamma$. Define $\tilde{r}(t)$ to be the scaled local variance:*

$$\tilde{r}(t) = \frac{z_R(t)}{48E[Q]\lambda^2(\lambda-1-\gamma)}\,.$$

Then blow-up occurs in forward time before t_b (i.e., $T^*\leq t_b$), where

$$t_b = \tilde{r}'(0)+\sqrt{\tilde{r}'(0)^2+2\tilde{r}(0)}\,.$$

One could, in fact, define $\eta_{\geq R}(t)=\|u(t)\|_{L^3(|x|\geq R)}$ and obtain the same statement with a similar proof but a different Gagliardo-Nirenberg inequality.

Proof. By the local virial identity and the same steps used in the proof of Prop. 3.1,

$$\tilde{r}''(t) = \frac{1}{2\lambda^2(\lambda-1-\gamma)}\left(3\lambda^2-2\lambda^3-\eta(t)^2\right)+\frac{A_R(u(t))}{48E[Q]\lambda^2(\lambda-1-\gamma)}.$$

By the estimates (the first one is the exterior version of Gagliardo-Nirenberg)

$$\|u\|_{L^4(|x|\geq R)}^4 \lesssim \|u\|_{L^2(|x|\geq R)}\|\nabla u\|_{L^2(|x|\geq R)}^3 \lesssim \eta_{\geq R}(t)\,\eta(t)^2 \lesssim \gamma\,\eta(t)^2 \tag{3.4}$$

and

$$\frac{1}{R^2}\|u\|_{L^2(R\leq|x|\leq 2R)}^2 \leq \frac{1}{R^2}M[Q] \lesssim \gamma \lesssim \gamma\,\eta(t)^2\,, \tag{3.5}$$

applied to control the A_R term, and using that $\eta(t)\geq\lambda$, we obtain

$$\tilde{r}''(t)\leq -1\,.$$

The remainder of the argument is the same as in the proof of Prop. 3.1. □

For comparison purposes, we review the quantified proof of finite-time blow-up for *radial* solutions presented in [9].

Proposition 3.3 (Radial blow-up time)**.** *Let $M[u]=M[Q]$ and $E[u]<E[Q]$ and suppose that the second case of Prop. 2.1 holds (take $\lambda>1$ to be as defined in* (2.6)*). Suppose that u is radial. Let*

$$R = c_2\max\left(1,\frac{1}{\lambda^{1/2}(\lambda-1)^{1/2}}\right)\,,$$

where c_2 is an appropriately large, but absolute, constant. Define $\tilde{r}(t)$ to be the scaled local variance:

$$\tilde{r}(t) = \frac{z_R(t)}{48E[Q]\lambda^2(\lambda-1)}\,.$$

Then blow-up occurs in forward time before t_b (i.e., $T^* \leq t_b$), where

$$t_b = \tilde{r}'(0) + \sqrt{\tilde{r}'(0)^2 + 2\,\tilde{r}(0)}\,.$$

We have that

$$t_b \lesssim c_\lambda \left(1 + r'(0)^2\right)^{1/2}\,.$$

where $c_\lambda \nearrow \infty$ as $\lambda \searrow 1$ (i.e., as $E[u] \nearrow E[Q]$).

Proof. We modify the proof of Prop. 3.2 only in (3.4) and (3.5) by using the radial Gagliardo-Nirenberg inequality [19] instead of (3.4)

$$\|u\|^4_{L^4(|x|\geq R)} \lesssim \frac{1}{R^2}\|u\|^3_{L^2(|x|\geq R)}\|\nabla u\|_{L^2(|x|\geq R)} \lesssim \frac{\eta(t)}{R^2}$$

and also

$$\frac{1}{R^2}\|u\|^2_{L^2(R\leq|x|\leq 2R)} \lesssim \frac{1}{R^2} \lesssim \frac{\eta(t)}{R^2}.$$

Then we have, for some absolute constant c_1,

$$\tilde{r}''(t) \leq \frac{1}{2\lambda^2(\lambda-1)}\left(3\lambda^2 - 2\lambda^3 - \eta(t)\left(\eta(t) - \frac{c_1}{R^2}\right)\right)\,.$$

We require that R is large enough so that $c_1/R^2 \leq 1$. Since $\eta(\eta - c_1/R^2)$ increases as $\eta \geq 1$ increases, and $\eta \geq \lambda$, we have

$$\eta(t)\left(\eta(t) - \frac{c_1}{R^2}\right) \geq \lambda\left(\lambda - \frac{c_1}{R^2}\right)\,.$$

This gives

$$\tilde{r}''(t) \leq -\frac{1}{(\lambda-1)}\left(\lambda - 1 - c_1\frac{1}{2\lambda\,R^2}\right) = -1 + \frac{c_1}{2\lambda(\lambda-1)R^2}.$$

The restriction on R in the proposition statement is such that

$$\frac{c_1}{2\lambda(\lambda-1)R^2} \leq \frac{1}{2},$$

from which it follows that

$$r''(t) \leq -\frac{1}{2}.$$

The remainder of the argument is the same as in the proof of Prop. 3.1. □

4. Variational characterization of the ground state

For now, write $u = u(x)$ (time dependence plays no role) in what follows in this section. The goal of this section is a variational characterization of the ground state Q stated below as Prop. 4.1. For the proof we will just show how it follows from scaling, the bounds depicted in Figure 1, and an existing characterization of Q appearing in Lions [12, Theorem I.2]. Prop. 4.1 will be one of the main ingredients in our treatment of the "near boundary case" in §5.

Proposition 4.1 (Variational characterization of the ground state)**.** *There exists a function $\epsilon(\rho)$ with $\epsilon(\rho) \to 0$ as $\rho \to 0$ such that the following holds: Suppose there is $\lambda > 0$ such that*

$$\left| \frac{M[u]E[u]}{M[Q]E[Q]} - (3\lambda^2 - 2\lambda^3) \right| \leq \rho\lambda^3 \,, \tag{4.1}$$

and

$$\left| \frac{\|u\|_{L^2}\|\nabla u\|_{L^2}}{\|Q\|_{L^2}\|\nabla Q\|_{L^2}} - \lambda \right| \leq \rho \begin{cases} \lambda^2 & \text{if } \lambda \leq 1 \\ \lambda & \text{if } \lambda \geq 1 \end{cases} . \tag{4.2}$$

Then there exists $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$ such that

$$\|u(x) - e^{i\theta}\lambda^{3/2}\beta^{-1}Q(\lambda(\beta^{-1}x - x_0))\|_{L^2_x} \leq \beta^{1/2}\epsilon(\rho) \tag{4.3}$$

and

$$\left\|\nabla\left[u(x) - e^{i\theta}\lambda^{3/2}\beta^{-1}Q(\lambda(\beta^{-1}x - x_0))\right]\right\|_{L^2_x} \leq \lambda\beta^{-1/2}\epsilon(\rho), \tag{4.4}$$

where $\beta = M[u]/M[Q]$.

Remark 4.2. Note that the right-hand side bounds in (4.1) and (4.2) do not depend on the mass. Moreover, the conclusion (4.3) and (4.4) could be replaced with the weaker statement

$$\|u(x) - e^{i\theta}\lambda^{3/2}\beta^{-1}Q(\lambda(\beta^{-1}x - x_0))\|_{L^2_x} \left\|\nabla\left[u(x) - e^{i\theta}\lambda^{3/2}\beta^{-1}Q(\lambda(\beta^{-1}x - x_0))\right]\right\|_{L^2_x} \leq \epsilon(\rho),$$

which also has a right-hand side independent of the mass.

Remark 4.3. Define $v(x) = \beta u(\beta x)$ and note that $M[v] = \beta^{-1}M[u] = M[Q]$. Now we can restate Proposition 4.1 as follows:

Suppose $\|v\|_{L^2} = \|Q\|_{L^2}$ and there is $\lambda > 0$ such that

$$\left| \frac{E[v]}{E[Q]} - (3\lambda^2 - 2\lambda^3) \right| \leq \lambda^3\rho \,, \tag{4.5}$$

and

$$\left| \frac{\|\nabla v\|_{L^2}}{\|\nabla Q\|_{L^2}} - \lambda \right| \leq \rho \begin{cases} \lambda^2 & \text{if } \lambda \leq 1 \\ \lambda & \text{if } \lambda \geq 1 \end{cases} . \tag{4.6}$$

Then there exists $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$ such that

$$\|v - e^{i\theta}\lambda^{3/2}Q(\lambda(\bullet - x_0))\|_{L^2} \le \epsilon(\rho) \tag{4.7}$$

and

$$\left\|\nabla\left[v - e^{i\theta}\lambda^{3/2}Q(\lambda(\bullet - x_0))\right]\right\|_{L^2} \le \lambda\,\epsilon(\rho). \tag{4.8}$$

In fact, Prop. 4.1 is equivalent to the above scaled statement.

We first restate the result from Lions [12, Theorem I.2] below as Prop. 4.4 and then show how the proof of Prop. 4.1 follows from Prop. 4.4.

Proposition 4.4. [12, Theorem I.2] *There exists a function $\epsilon(\rho)$, defined for small $\rho > 0$ and such that $\lim_{\rho\to 0}\epsilon(\rho) = 0$, such that for all $u \in H^1$ with*

$$\Big|\|u\|_4 - \|Q\|_4\Big| + \Big|\|u\|_2 - \|Q\|_2\Big| + \Big|\|\nabla u\|_2 - \|\nabla Q\|_2\Big| \le \rho, \tag{4.9}$$

there exist $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$ such that

$$\left\|u - e^{i\theta_0}Q(\bullet - x_0)\right\|_{H^1} \le \epsilon(\rho). \tag{4.10}$$

Proof of Prop. 4.1. We prove Remark 4.3 which is equivalent to Prop. 4.1 by rescaling off the mass. Set $\tilde{u}(x) = \lambda^{-3/2}v(\lambda^{-1}x)$. Then (4.6) implies

$$\left|\frac{\|\nabla\tilde{u}\|_{L^2}}{\|\nabla Q\|_{L^2}} - 1\right| \le \rho. \tag{4.11}$$

Next, by (4.5) and (4.6) we have

$$\begin{aligned}
2\left|\frac{\|v\|_{L^4}^4}{\|Q\|_{L^4}^4} - \lambda^3\right| &\le \left|\frac{E[v]}{E[Q]} - (2\lambda^3 - 3\lambda^2)\right| + 3\left|\frac{\|\nabla v\|_{L^2}^2}{\|\nabla Q\|_{L^2}^2} - \lambda^2\right| \\
&\le \rho\left(\lambda^3 + 3\begin{cases}\lambda^3 & \text{if } \lambda \le 1\\ \lambda^2 & \text{if } \lambda \ge 1\end{cases}\right) \\
&\le 4\lambda^3\rho.
\end{aligned}$$

Thus, in terms of $\tilde{u}$, we obtain

$$\left|\frac{\|\tilde{u}\|_{L^4}^4}{\|Q\|_{L^4}^4} - 1\right| \le 2\rho. \tag{4.12}$$

Hence, (4.11) and (4.12) imply the condition (4.9) for $\tilde{u}$ (the factors in front of ρ in both inequalities can be inconsequentially incorporated into ρ), and by Proposition 4.4, there exist $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$ such that (4.10) holds for $\tilde{u}$. Rescaling back to v, we obtain exactly (4.7) and (4.8). □

5. Near-boundary case

We know by Prop. 2.1 that if $M[u]=M[Q]$ and $E[u]/E[Q]=3\lambda^2-2\lambda^3$ for some $\lambda>1$ and $\|\nabla u_0\|_{L^2}/\|\nabla Q\|_{L^2}\geq 1$, then $\|\nabla u(t)\|_{L^2}/\|\nabla Q\|_{L^2}\geq\lambda$ for all t. The next result says that $\|\nabla u(t)\|_{L^2}/\|\nabla Q\|_{L^2}$ cannot, globally in time, remain near λ.

Proposition 5.1 (Near boundary case)**.** *Let $\lambda_0>1$. There exists $\rho_0=\rho_0(\lambda_0)>0$ (with the property that $\rho_0\to 0$ as $\lambda_0\searrow 1$) such that for any $\lambda\geq\lambda_0$, the following holds: There does* NOT *exist a solution $u(t)$ of NLS with $P[u]=0$ satisfying $\|u\|_{L^2}=\|Q\|_{L^2}$,*

$$\frac{E[u]}{E[Q]}=3\lambda^2-2\lambda^3\,, \tag{5.1}$$

and

$$\lambda\leq\frac{\|\nabla u(t)\|_{L^2}}{\|\nabla Q\|_{L^2}}\leq\lambda(1+\rho_0)\quad\text{for all } t\geq 0\,. \tag{5.2}$$

Of course, the assertion is equivalent to: For every solution $u(t)$ of NLS with $P[u]=0$ satisfying $\|u\|_{L^2}=\|Q\|_{L^2}$,

$$\frac{E[u]}{E[Q]}=3\lambda^2-2\lambda^3\,,$$

and

$$\lambda\leq\frac{\|\nabla u(t)\|_{L^2}}{\|\nabla Q\|_{L^2}}\quad\text{for all } t\geq 0,$$

there exists a time $t_0\geq 0$ such that

$$\frac{\|\nabla u(t_0)\|_{L^2}}{\|\nabla Q\|_{L^2}}\geq\lambda(1+\rho_0),$$

equivalently, there exists a sequence $t_n\to+\infty$ such that

$$\frac{\|\nabla u(t_n)\|_{L^2}}{\|\nabla Q\|_{L^2}}\geq\lambda(1+\rho_0)$$

for all n. This seemingly stronger statement is seen to be equivalent by "resetting" the initial time $\tilde{u}(t)=u(t-t_0-1)$ for $t\geq 0$.

We shall need a version of Lemma 5.1 from [2].

Lemma 5.2. *Suppose that $u(t)$ with $P[u]=0$ solving* (1.1) *satisfies, for all t,*

$$\|u(t)-e^{i\theta(t)}Q(\bullet-x(t))\|_{H^1}\leq\epsilon \tag{5.3}$$

for some continuous functions $\theta(t)$ and $x(t)$. Then

$$\frac{|x(t)|}{t}\lesssim\epsilon^2\quad\textit{as}\quad t\to+\infty.$$

The proof of this lemma is very similar to that of Lemma 5.1 in [2]. For the reader's convenience, we carry it out in Appendix A.

Proof of Prop. 5.1. The constants c_j we introduce below are absolute constants. To the contrary, suppose that $u(t)$ is a solution of the type described in the proposition statement, i.e., $\|u\|_{L^2} = \|Q\|_{L^2}$, $E[u]/E[Q] = 3\lambda^2 - 2\lambda^3$ and

$$\lambda \leq \frac{\|\nabla u(t)\|_{L^2}}{\|\nabla Q\|_{L^2}} \leq \lambda(1+\rho_0) \quad \text{for all } t \geq 0\,. \tag{5.4}$$

Since $\|\nabla u(t)\|_{L^2}^2 \geq \lambda^2 \|\nabla Q\|_{L^2}^2 = 6\lambda^2 E[Q]$, we have

$$24E[u] - 4\|\nabla u(t)\|_{L^2}^2 \leq -48E[Q]\lambda^2(\lambda-1).$$

By Prop. 4.1, there exist functions $x(t)$ and $\theta(t)$ such that

$$\|u(t) - e^{i\theta(t)}\lambda^{3/2}Q(\lambda(\bullet + x(t)))\|_{L^2} \leq \epsilon(\rho), \tag{5.5}$$

$$\|u(t) - e^{i\theta(t)}\lambda^{3/2}Q(\lambda(\bullet + x(t)))\|_{\dot H^1} \leq \lambda\epsilon(\rho). \tag{5.6}$$

By continuity of the $u(t)$ flow, we may assume that $\theta(t)$ and $x(t)$ are continuous. Let

$$R(T) = \max\left(\max_{0\leq t\leq T} |x(t)|,\ \log \epsilon(\rho)^{-1}\right)\,.$$

Fix $T > 0$. Take $R = 2R(T)$ in the local virial identity (3.2). By (5.5)-(5.6), there exists $c_2 > 0$ such that

$$|A_R(u(t))| \leq \tfrac{1}{2}c_2\lambda^2\left(\epsilon(\rho) + e^{-R(T)}\right)^2 \leq c_2\lambda^2\epsilon(\rho)^2.$$

Consequently, by taking ρ_0 small enough, we can make $\epsilon(\rho)$ small enough so that for all $0 \leq t \leq T$,

$$z_R''(t) \leq -24E[Q]\lambda^2(\lambda-1).$$

(Note that here, the closer $\lambda > 1$ is to 1, the smaller ρ_0 needs to be taken.) By integrating in time over $[0,T]$ twice, we obtain that

$$\frac{z_R(T)}{T^2} \leq \frac{z_R(0)}{T^2} + \frac{z_R'(0)}{T} - 12E[Q]\lambda^2(1-\lambda)\,.$$

We have

$$|z_R(0)| \leq c_3R^2\|u_0\|_{L^2}^2 = c_3\|Q\|_{L^2}^2 R^2,$$

and

$$|z_R'(0)| \leq c_3R\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} \leq c_3\|Q\|_{L^2}\|\nabla Q\|_{L^2}(1+\rho_0)R,$$

and as a result

$$z_{2R(T)}(T) \leq c_4\left(\frac{R(T)^2}{T^2} + \frac{R(T)}{T}\right) - 12E[Q]\lambda^2(\lambda-1).$$

By taking T sufficiently large and applying Lemma 5.2, we obtain

$$0 \leq z_{2R(T)}(T) \leq c_4\epsilon(\rho)^2 - 12E[Q]\lambda^2(\lambda-1) < 0$$

provided ρ_0 is selected small enough so that $c_4\epsilon(\rho)^2 \leq 6E[Q]\lambda^2(\lambda-1)$. Note this selection of ρ_0 is independent of T. This is a contradiction. $\square$

6. Profile decomposition

Let us recall the Keraani-type profile decomposition lemma and some associated results from [9], [2]. We first need to review the Strichartz norm notation from [9].

We say that (q,r) is $\dot H^s$ Strichartz admissible (in 3d) if

$$\frac{2}{q}+\frac{3}{r}=\frac{3}{2}-s.$$

Let

$$\|u\|_{S(L^2)}=\sup_{\substack{(q,r)\ L^2\text{ admissible}\\ 2\le r\le 6,\ 2\le q\le\infty}}\|u\|_{L^q_tL^r_x}.$$

Define

$$\|u\|_{S(\dot H^{1/2})}=\sup_{\substack{(q,r)\ \dot H^{1/2}\text{ admissible}\\ 3\le r\le 6^-,\ 4^+\le q\le\infty}}\|u\|_{L^q_tL^r_x},$$

where 6^- is an arbitrarily preselected and fixed number <6; similarly for 4^+. Now we consider dual Strichartz norms. Let

$$\|u\|_{S'(L^2)}=\inf_{\substack{(q,r)\ L^2\text{ admissible}\\ 2\le q\le\infty,\ 2\le r\le 6}}\|u\|_{L^{q'}_tL^{r'}_x},$$

where (q',r') is the Hölder dual to (q,r). Also define

$$\|u\|_{S'(\dot H^{-1/2})}=\inf_{\substack{(q,r)\ \dot H^{-1/2}\text{ admissible}\\ \frac{4}{3}^+\le q\le 2^-,\ 3^+\le r\le 6^-}}\|u\|_{L^{q'}_tL^{r'}_x}.$$

We extend our notation $S(\dot H^s)$, $S'(\dot H^s)$ as follows: If a time interval is not specified (that is, if we just write $S(\dot H^s)$, $S'(\dot H^s)$), then the t-norm is evaluated over $(-\infty,+\infty)$. To indicate a restriction to a time subinterval $I\subset(-\infty,+\infty)$, we will write $S(\dot H^s;I)$ or $S'(\dot H^s;I)$. We shall also use the notation NLS(t) to indicate the nonlinear flow map associated to (1.1).

The following proposition incorporates results from our earlier papers. The basic form of the (linear) profile decomposition is proved in [9, Lemma 5.2], [2, Lemma 2.1] (and the proof given there was modeled on a similar result of Keraani [11]). The proof of (6.2) is given in [2, Lemma 2.3] and the method of replacing linear flows by nonlinear flows appears as part of [9, Prop. 5.4, 5.5].

Proposition 6.1. *Suppose that $\phi_n=\phi_n(x)$ is a bounded sequence in H^1. There exist a subsequence of ϕ_n (still denoted ϕ_n), profiles ψ^j in H^1, and parameters x_n^j, t_n^j so that for each M,*

$$\phi_n=\sum_{j=1}^M \mathrm{NLS}(-t_n^j)\,\psi^j(\bullet-x_n^j)+W_n^M,$$

where (as $n\to\infty$):

- *For each j, either $t_n^j = 0$* [6]*, $t_n^j \to +\infty$, or $t_n^j \to -\infty$.*
- *If $t_n^j \to +\infty$, then $\| \mathrm{NLS}(-t)\, \psi^j \|_{S(\dot H^{1/2};[0,+\infty))} < \infty$ and if $t_n^j \to -\infty$, then $\| \mathrm{NLS}(-t)\, \psi^j \|_{S(\dot H^{1/2};(-\infty,0])} < \infty$.*
- *For $j \neq k$,*
$$|t_n^j - t_n^k| + |x_n^j - x_n^k| \to +\infty.$$
- *$\mathrm{NLS}(t) W_n^M$ is global for M large enough with $\left(\lim_n \| \mathrm{NLS}(t)\, W_n^M \|_{S(\dot H^{1/2})}\right) \to 0$ as $M \to \infty$. (Note: we do not claim that $\lim_n \|W_n^M\|_{H^1} \to 0$.)*

We also have the $\dot H^s$ Pythagorean decomposition: For fixed M and $0 \le s \le 1$, we have

$$\|\phi_n\|_{\dot H^s}^2 = \sum_{j=1}^{M} \| \mathrm{NLS}(-t_n^j)\, \psi^j \|_{\dot H^s}^2 + \|W_n^M\|_{\dot H^s}^2 + o_n(1). \tag{6.1}$$

We also have the energy Pythagorean decomposition [7]*:*

$$E[\phi_n] = \sum_{j=1}^{M} E[\psi^j] + E[W_n^M] + o_n(1). \tag{6.2}$$

A similar statement to (6.2) was proved in [2, Lemma 2.3] for the linear flows $e^{-it_n^j\Delta}\psi^j$ by establishing the L^4 orthogonal decomposition, and implicitly (by the existence of wave operators and the long-term perturbation argument) for the nonlinear flow:

$$\|\phi_n\|_{L^4}^4 = \sum_{j=1}^{M} \| \mathrm{NLS}(-t_n^j)\, \psi^j \|_{L^4}^4 + \|W_n^M\|_{L^4}^4 + o_n(1), \tag{6.3}$$

and thus, the energy Pythagorean decomposition (6.2) follows.

The next lemma is taken from [9, Prop. 2.3] (the statement is slightly different, but the proof given there actually establishes the statement given below):

Lemma 6.2 (perturbation theory)**.** *For each $A \gg 1$, there exists $\epsilon_0 = \epsilon_0(A) \ll 1$ and $c = c(A)$ such that the following holds. Fix $T > 0$. Let $u = u(x,t) \in L^\infty_{[0,T]} H^1_x$ solve*

$$i\partial_t u + \Delta u + |u|^2 u = 0$$

on $[0,T]$. Let $\tilde u(x,t) \in L^\infty_{[0,T]} H^1_x$ and define

$$e = i\partial_t \tilde u + \Delta \tilde u + |\tilde u|^2 \tilde u.$$

For each $\epsilon \le \epsilon_0$, if

$$\|\tilde u\|_{S(\dot H^{1/2};[0,T])} \le A, \quad \|e\|_{S'(\dot H^{1/2};[0,T])} \le \epsilon, \quad \text{and} \quad \|e^{it\Delta}(u(0) - \tilde u(0))\|_{S(\dot H^{1/2};[0,T])} \le \epsilon\,,$$

[6] This is done by passing to another subsequence in n and adjusting the profiles ψ^j; see also comment in Step 1 of the proof [2, Lemma 2.3].

[7] By energy conservation $E[\psi^j] = E[\mathrm{NLS}(-t_n^j)\, \psi^j]$.

then

$$\|u-\tilde{u}\|_{S(\dot{H}^{1/2};[0,T])} \le c(A)\epsilon\,.$$

We remark that T does not actually enter into the parameter dependence in any way: ϵ_0 depends only on A, not on T. In fact, in [9, Prop. 2.3], $T=+\infty$. Now, in our application below, it will turn out that $A=A(T)$, so ultimately there will be dependence upon T, but it is only through A.

The equation (6.1) gives $\dot{H}^1$ asymptotic orthogonality at $t=0$, but we will need to extend this to the NLS flow for $0\le t\le T$. This is the subject of the next lemma, which does not appear in our previous papers.

Lemma 6.3 ($\dot{H}^1$ Pythagorean decomposition along the NLS flow)**.** *Suppose (as in Prop. 6.1) ϕ_n is a bounded sequence in H^1. Fix any time $0<T<\infty$. Suppose that $u_n(t)\equiv \mathrm{NLS}(t)\phi_n$ exists up to time T for all n and*

$$\lim_{n\to\infty} \|\nabla u_n(t)\|_{L^\infty_{[0,T]}L^2_x} < \infty.$$

Let $W_n^M(t)\equiv \mathrm{NLS}(t)W_n^M$ (which we know is global and, in fact, scattering). Then, for all j, $v^j(t)\equiv \mathrm{NLS}(t)\,\psi^j$ exist up to time T and for all $t\in[0,T]$,

$$\|\nabla u_n(t)\|_{L^2}^2 = \sum_{j=1}^M \|\nabla v^j(t-t_n^j)\|_{L^2}^2 + \|\nabla W_n^M(t)\|_{L^2_x}^2 + o_n(1). \tag{6.4}$$

Here, $o_n(1)\to 0$ uniformly on $0\le t\le T$.

Proof. Let M_0 be such that for $M\ge M_0$, we have $\|\mathrm{NLS}(t)\,W_n^M\|_{S(\dot{H}^{1/2})}\le \delta_{\mathrm{sd}}$ (δ_{sd} is the small data scattering threshold defined in [9]). Reorder the first M_0 profiles and introduce an index M_2, $0\le M_2\le M$, so that

(1) For each $1\le j\le M_2$, we have $t_n^j=0$. If $M_2=0$, that means there are no j in this category.
(2) For each $M_2+1\le j\le M_0$, we have $|t_n^j|\to+\infty$. If $M_2=M_0$, that means there are no j in this category.

We then know from the profile construction that the $v^j(t)$ for $j>M_0$ are scattering (in both time directions). It follows from Prop. 6.1 that for fixed T and $M_2+1\le j\le M_0$, we have $\|v^j(t-t_n^j)\|_{S(\dot{H}^{1/2};[0,T])}\to 0$ as $n\to+\infty$. Indeed, consider the case $t_n^j\to+\infty$ and $\|v^j(-t)\|_{S(\dot{H}^{1/2};[0,+\infty))}<\infty$. Then for $q<\infty$, it is immediate from dominated convergence that $\|v^j(-t)\|_{L^q_{[0,+\infty)}L^r_x}<\infty$ implies $\|v^j(t-t_n^j)\|_{L^q_{[0,T]}L^r_x}\to 0$. Since v^j is constructed in Prop. 6.1 via the existence of wave operators [9, Prop. 4.6] to converge in H^1 to a linear flow at $-\infty$, it follows from the L^3_x decay of the linear flow that $\|v^j(t-t_n^j)\|_{L^\infty_{[0,T]}L^3_x}\to 0$.

Let $B=\max(1,\lim_n\|\nabla u_n(t)\|_{L^\infty_{[0,T]}L^2_x})<\infty$. For each $1\le j\le M_2$, define T^j to be the maximal forward time $\le T$ on which $\|\nabla v^j\|_{L^\infty_{[0,T^j]}L^2_x}\le 2B$. Let $\tilde{T}=\min_{1\le j\le M_2}T^j$

(if $M_2 = 0$, then just take $\tilde{T} = T$.) We will begin by proving that (6.4) holds for $T = \tilde{T}$. It will then follow from (6.4) that for each $1 \le j \le M_2$, we have $T^j = T$, and hence, $\tilde{T} = T$. Thus, for the remainder of the proof, we work on $[0, \tilde{T}]$. For each $1 \le j \le M_2$, we have

$$\begin{aligned}\|v^j(t)\|_{S(\dot{H}^{1/2};[0,\tilde{T}])} &\lesssim \|v^j\|_{L^\infty_{[0,\tilde{T}]} L^3_x} + \|v^j\|_{L^4_{[0,\tilde{T}]} L^6_x} \\ &\lesssim \|v^j\|^{1/2}_{L^\infty_{[0,\tilde{T}]} L^2_x} \|\nabla v^j\|^{1/2}_{L^\infty_{[0,\tilde{T}]} L^2_x} + \tilde{T}^{1/4} \|\nabla v^j\|_{L^\infty_{[0,\tilde{T}]} L^2_x} \\ &\lesssim \langle \tilde{T}^{1/4} \rangle B,\end{aligned}$$

where we have used that $\|v^j\|_{L^\infty_{[0,\tilde{T}]} L^2_x} = \|\psi^j\|_{L^2_x} \le \lim_n \|\phi_n\|_{L^2}$ by (6.1) with $s = 0$.

Let

$$\tilde{u}_n(x,t) = \sum_{j=1}^{M} v^j(x - x_n^j, t - t_n^j).$$

Of course, $\tilde{u}_n$ also depends upon M but we suppress this dependence from the notation. Also, let

$$e_n = i\partial_t \tilde{u}_n + \Delta \tilde{u}_n + |\tilde{u}_n|^2 \tilde{u}_n.$$

We now outline a series of claims, which we do not prove here since the proofs closely follow the proof of [9, Prop. 5.4].

Claim 1. There exists $A = A(\tilde{T})$ (independent of M but dependent on $\tilde{T}$) such that for all $M > M_0$, there exists $n_0 = n_0(M)$ such that for all $n > n_0$,

$$\|\tilde{u}_n\|_{S(\dot{H}^{1/2};[0,\tilde{T}])} \le A\,.$$

Claim 2. For each $M > M_0$ and $\epsilon > 0$, there exists $n_1 = n_1(M, \epsilon)$ such that for $n > n_1$,

$$\|e_n\|_{L^{10/3}_{[0,\tilde{T}]} L^{5/4}_x} \le \epsilon\,.$$

Remark 3. Note that since $u_n(0) - \tilde{u}_n(0) = W_n^M$, there exists $M' = M'(\epsilon)$ sufficiently large so that for each $M > M'$ there exists $n_2 = n_2(M)$ such that $n > n_2$ implies

$$\|e^{it\Delta}(u_n(0) - \tilde{u}_n(0))\|_{S(\dot{H}^{1/2};[0,\tilde{T}])} \le \epsilon\,.$$

Recall we are given $\tilde{T}$, and thus, by Claim 1, there is a large number $A(\tilde{T})$. Then the statement of Lemma 6.2 gives us $\epsilon_0 = \epsilon_0(A)$. Now select an arbitrary $\epsilon \le \epsilon_0$, and obtain from Remark 3 an index $M' = M'(\epsilon)$. Now select an arbitrary $M > M'$. Set $n' = \max(n_0, n_1, n_2)$. Then we conclude from Claims 1-2, Remark 3, and Lemma 6.2, that for $n > n'(M, \epsilon)$,

$$\|u_n - \tilde{u}_n\|_{S(\dot{H}^{1/2};[0,\tilde{T}])} \le c(\tilde{T})\,\epsilon\,, \tag{6.5}$$

where $c = c(A) = c(\tilde{T})$.

Now we prove (6.4) on $[0,\tilde{T}]$. We know that for each $1 \le j \le M_2$, we have $\|\nabla v^j(t)\|_{L^\infty_{[0,\tilde{T}]}L^2_x} \le 2B$. Let us discuss $j \ge M_2+1$. As we've noted, $\|v^j(t-t_n^j)\|_{S(\dot{H}^{1/2};[0,\tilde{T}])} \to 0$ as $n\to+\infty$. By the Strichartz estimates, $\|\nabla v^j(t-t_n^j)\|_{L^\infty_{[0,\tilde{T}]}L^2_x} \lesssim \|\nabla v^j(-t_n^j)\|_{L^2_x}$. By the pairwise divergence of parameters,

$$\begin{aligned}
\|\nabla \tilde{u}_n(t)\|^2_{L^\infty_{[0,\tilde{T}]}L^2_x} &= \sum_{j=1}^{M_2} \|\nabla v^j(t)\|^2_{L^\infty_{[0,\tilde{T}]}L^2_x} + \sum_{j=M_2+1}^{M} \|\nabla v^j(t-t_n^j)\|^2_{L^\infty_{[0,\tilde{T}]}L^2_x} + o_n(1)\\
&\lesssim M_2B^2 + \sum_{j=M_2+1}^{M} \|\nabla \operatorname{NLS}(-t_n^j)\,\psi^j\|^2_{L^2_x} + o_n(1)\\
&\le M_2B^2 + \|\nabla\phi_n\|^2_{L^2_x} + o_n(1)\\
&\le M_2B^2 + B^2 + o_n(1).
\end{aligned}$$

From (6.5), we conclude that

$$\begin{aligned}
\|u_n-\tilde{u}_n\|_{L^\infty_{[0,\tilde{T}]}L^4_x} &\lesssim \|u_n-\tilde{u}_n\|^{1/2}_{L^\infty_{[0,\tilde{T}]}L^3_x}\|\nabla(u_n-\tilde{u}_n)\|^{1/2}_{L^\infty_{[0,\tilde{T}]}L^2_x}\\
&\le c(\tilde{T})^{1/2}(M_2B^2+2B^2+o_n(1))^{1/4}\epsilon^{1/2}.
\end{aligned}$$

An argument similar to the proof of (6.3) now establishes that, for each $t\in[0,\tilde{T}]$,

$$\|u_n(t)\|^4_{L^4} = \sum_{j=1}^{M}\|v^j(t-t_n^j)\|^4_{L^4} + \|W_n^M(t)\|^4_{L^4} + o_n(1). \tag{6.6}$$

By (6.2) and energy conservation ($E[\psi^j]=E[v^j(t-t_n^j)]$, etc.), we have

$$E[u_n(t)] = \sum_{j=1}^{M} E[v^j(t-t_n^j)] + E[W_n^M(t)] + o_n(1). \tag{6.7}$$

Combining (6.6) and (6.7) gives (6.4). □

Lemma 6.4 (profile reordering)**.** *Suppose that $\phi_n=\phi_n(x)$ is an H^1 bounded sequence to which we apply the Prop. 6.1 out to a given M. Let $\lambda_0>1$. Suppose that $M[\phi_n]=M[Q]$, $E[\phi_n]/E[Q]=3\lambda_n^2-2\lambda_n^3$ with $\lambda_n\ge\lambda_0>1$ and $\|\nabla\phi_n\|_{L^2}/\|\nabla Q\|_{L^2}\ge\lambda_n$ for each n. Then, the profiles can be reordered so that there exists $1\le M_1\le M_2\le M$ and*

1. *For each $1\le j\le M_1$, we have $t_n^j=0$ and $v^j(t)\equiv \operatorname{NLS}(t)\psi^j$ does not scatter as $t\to+\infty$. (In particular, we are asserting the existence of at least one j that falls into this category.)*
2. *For each $M_1+1\le j\le M_2$, we have $t_n^j=0$ and $v^j(t)$ scatters as $t\to+\infty$. (If $M_1=M_2$, there are no j with this property.)*
3. *For each $M_2+1\le j\le M$, we have that $|t_n^j|\to+\infty$. (If $M_2=M$, there are no j with this property.)*

Proof. We first prove that there exists at least one j such that t_n^j converges as $n \to +\infty$. Indeed, it follows that

$$\begin{aligned}\frac{\|\phi_n\|_{L^4}^4}{\|Q\|_{L^4}^4} &= -\frac{E[\phi_n]}{2E[Q]} + \frac{3\|\phi_n\|_{L^2}^2}{2\|\nabla Q\|_{L^2}^2} \\ &\geq -\frac{1}{2}(3\lambda_n^2 - 2\lambda_n^3) + \frac{3}{2}\lambda_n^2 \\ &= \lambda_n^3 \geq \lambda_0^3 > 1.\end{aligned}$$

Now if j is such that $|t_n^j| \to \infty$, then $\|\operatorname{NLS}(-t_n^j)\psi^j\|_{L^4} \to 0$. The claim now follows from (6.3). Note that if j is such that t_n^j converges as $n \to +\infty$, then we might as well WLOG assume that $t_n^j = 0$ (see also footnote 7).

Reorder the profiles ψ^j so that for $1 \leq j \leq M_2$, we have $t_n^j = 0$, and for $M_2+1 \leq j \leq M$, we have $|t_n^j| \to \infty$. It only remains to show that there exists one j, $1 \leq j \leq M_2$ such that $v^j(t)$ is nonscattering. If not, then for all $1 \leq j \leq M_2$, we have that all v^j are scattering, and thus, $\lim_{t\to+\infty} \|v^j(t)\|_{L^4} = 0$. Let t_0 be large enough so that, for all $1 \leq j \leq M_2$, we have $\|v^j(t_0)\|_{L^4}^4 \leq \epsilon/M_2$. By the L^4 orthogonality (6.6) along the NLS flow, we have

$$\begin{aligned}\lambda_0^3\|Q\|_{L^4}^4 &\leq \|u_n(t_0)\|_{L^4}^4 \\ &= \sum_{j=1}^{M_2} \|v^j(t_0)\|_{L^4}^4 + \sum_{j=M_2+1}^{M} \|v^j(t_0 - t_n^j)\|_{L_x^4}^4 + \|W_n^M(t_0)\|_{L_x^4}^4 + o_n(1).\end{aligned}$$

As $n \to +\infty$, we have $\sum_{j=M_2+1}^{M} \|v^j(t_0 - t_n^j)\|_{L_x^4}^4 \to 0$, and thus, the last line

$$\leq \epsilon + \|W_n^M(t_0)\|_{L_x^4}^4 + o_n(1).$$

This gives a contradiction. □

7. Outline of the inductive argument

Having developed several preliminaries in §2–6, we now begin the proof of Theorem 1.1.

Consider the following statement:

Definition 7.1. *Let $\lambda > 1$. We say that* $\exists\mathrm{GB}(\lambda, \sigma)$ holds *if there exists a solution $u(t)$ to NLS such that*

$$M[u] = M[Q], \quad \frac{E[u]}{E[Q]} = 3\lambda^2 - 2\lambda^3\,,$$

and

$$\lambda \leq \frac{\|\nabla u(t)\|_{L^2}}{\|\nabla Q\|_{L^2}} \leq \sigma \quad \textit{for all } t \geq 0.$$

$\exists\mathrm{GB}(\lambda,\sigma)$ can be read "there exist solutions at energy $3\lambda^2-2\lambda^3$ globally bounded by σ."

By Prop. 5.1, $\exists\mathrm{GB}(\lambda,\lambda(1+\rho_0(\lambda_0)))$ is *false* for all $\lambda\geq\lambda_0>1$.

Note that the statement "$\exists\mathrm{GB}(\lambda,\sigma)$ is false" is equivalent to the statement: For every solution $u(t)$ to NLS such that $M[u]=M[Q]$ and $E[u]/E[Q]=3\lambda^2-2\lambda^3$ such that $\lambda\leq\|\nabla u(t)\|_{L^2}/\|\nabla Q\|_{L^2}$ for all t, there exists a time $t_0\geq 0$ such that $\|\nabla u(t_0)\|_{L^2}/\|\nabla Q\|_{L^2}\geq\sigma$. (In fact, there exists a sequence $t_n\to+\infty$ such that $\|\nabla u(t_n)\|_{L^2}/\|\nabla Q\|_{L^2}\geq\sigma$ for all n. This follows by resetting the initial time.)

We will induct on the statement "$\exists\mathrm{GB}(\lambda,\sigma)$ is false." Note that if $\lambda\leq\sigma_1\leq\sigma_2$, then "$\exists\mathrm{GB}(\lambda,\sigma_2)$ is false" implies "$\exists\mathrm{GB}(\lambda,\sigma_1)$ is false", as is easily understood by writing down the contrapositive. We now define a threshold – see the illustration in Figure 3.

Definition 7.2 (The critical threshold)**.** *Fix $\lambda_0>1$. Let $\sigma_c=\sigma_c(\lambda_0)$ be the supremum of all $\sigma>\lambda_0$ such that* $\exists\mathrm{GB}(\lambda,\sigma)$ is false *for all λ such that $\lambda_0\leq\lambda\leq\sigma$. The notation σ_c stands for "σ-critical."*

By Prop. 5.1, we know that $\sigma_c(\lambda_0)>\lambda_0$.

Suppose $\lambda_0>1$ and $\sigma_c(\lambda_0)=\infty$. Let $u(t)$ be any solution with $E[u]/E[Q]\leq 3\lambda_0^2-2\lambda_0^3$, $M[u]=M[Q]$, and $\|\nabla u_0\|_{L^2}/\|\nabla Q\|_{L^2}>1$. We claim there exists a sequence of times t_n such that $\|\nabla u(t_n)\|_{L^2}\to\infty$. Indeed, suppose not, and let $\lambda\geq\lambda_0$ be such that $E[u]/E[Q]=3\lambda^2-2\lambda^3$. Since there is no sequence t_n along which $\|\nabla u(t_n)\|_{L^2}\to+\infty$, there exists $\sigma<\infty$ such that $\lambda\leq\|\nabla u(t)\|_{L^2}/\|\nabla Q\|_{L^2}\leq\sigma$ for all $t\geq 0$. But this means that $\exists\mathrm{GB}(\lambda,\sigma)$ holds true, and thus, $\sigma_c(\lambda_0)\leq\sigma<\infty$. Thus, in order to prove our theorem, we need to show that for every $\lambda_0>1$, we have $\sigma_c(\lambda_0)=\infty$.

Hence, we shall now fix $\lambda_0>1$ and assume that $\sigma_c(\lambda_0)<\infty$, and work toward a contradiction. Clearly, it suffices to do this for λ_0 close to 1, and thus, we shall make the assumption that $\lambda_0<\frac{3}{2}$. As we'll see, this will be convenient later.

8. Existence of a critical solution

Lemma 8.1 (Existence of a critical solution)**.** *There exist initial data $u_{c,0}$ and $\lambda_c\in[\lambda_0,\sigma_c(\lambda_0)]$ such that $u_c(t)\equiv\mathrm{NLS}(t)\,u_{c,0}$ is global, $M[u_c]=M[Q]$, $E[u_c]=3\lambda_c^2-2\lambda_c^3$ and*

$$\lambda_c\leq\frac{\|\nabla u_c(t)\|_{L^2}}{\|\nabla Q\|_{L^2}}\leq\sigma_c\quad \textit{for all } t\geq 0\,.$$

We have that for all $\sigma<\sigma_c$ and all $\lambda_0\leq\lambda\leq\sigma$, $\exists\mathrm{GB}(\lambda,\sigma)$ is false, i.e., there are no solutions $u(t)$ for which $M[u]=M[Q]$, $E[u]/E[Q]=3\lambda^2-2\lambda^3$ and

$$\lambda\leq\frac{\|\nabla u(t)\|_{L^2}}{\|\nabla Q\|_{L^2}}\leq\sigma\quad\text{for all } t\geq 0.$$

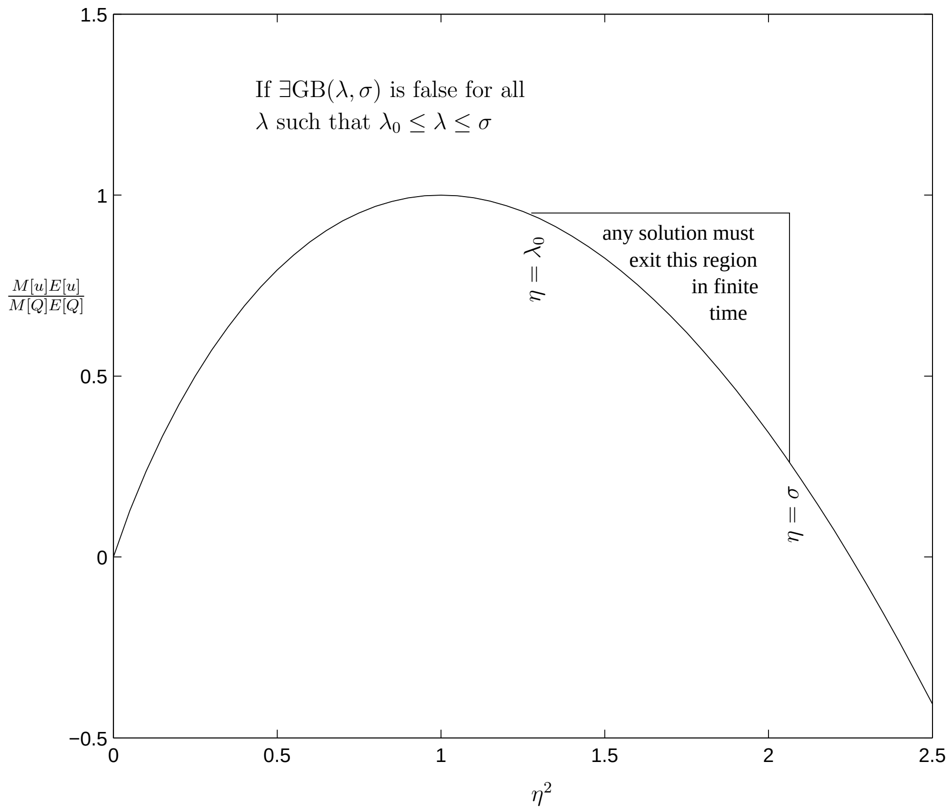

FIGURE 3. A depiction of the meaning of the statement "$\exists\mathrm{GB}(\lambda,\sigma)$ is false for all λ such that $\lambda_0 \le \lambda \le \sigma$." It means that for any solution $u(t)$ with $\eta(t) > \lambda$ (when λ is defined by (2.6)) if the path $(\eta(t), 3\lambda^2 - 2\lambda^3))$ is plotted here, it must escape (along the horizontal line) the indicated triangular region at some finite time. The value σ_c is the largest σ for which this statement holds.

But on the other hand, we have found a solution $u_c(t)$ such that $M[u_c] = M[Q]$, $E[u_c] = 3\lambda_c^2 - 2\lambda_c^3$ and

$$\lambda_c \le \frac{\|\nabla u_c(t)\|_{L^2}}{\|\nabla Q\|_{L^2}} \le \sigma_c \quad \text{for all } t \ge 0\,.$$

Thus, we call this the "critical solution" or "threshold solution". In §9, we shall show that these properties induce a uniform-in-time concentration property of $u_c(t)$, and we then observe that all of the alleged properties of $u_c(t)$ are inconsistent with the local virial identity (in particular, Prop. 3.2).

Proof. By definition of σ_c, there exist sequences λ_n and σ_n such that $\lambda_0 \le \lambda_n \le \sigma_n$ and $\sigma_n \searrow \sigma_c$ for which $\exists\mathrm{GB}(\lambda_n, \sigma_n)$ holds. This means that there exists $u_{n,0}$ with

$u_n(t) = \mathrm{NLS}(t)\, u_{n,0}$ such that $u_n(t)$ is global, $M[u_n] = M[Q]$, $E[u_n]/E[Q] = 3\lambda_n^2 - 2\lambda_n^3$, and

$$\lambda_n \leq \frac{\|\nabla u_n(t)\|_{L^2}}{\|\nabla Q\|_{L^2}} \leq \sigma_n.$$

Since λ_n is bounded, we can pass to a subsequence such that λ_n converges. Let $\lambda' = \lim_n \lambda_n$. We know, of course, that $\lambda_0 \leq \lambda' \leq \sigma_c$.

In Lemma 6.4, take $\phi_n = u_{n,0}$, and henceforth adopt the notation from that lemma. For $M_1 + 1 \leq j \leq M_2$, the $v^j(t)$ scatter as $t \to +\infty$ and for $M_2 + 1 \leq j \leq M$, the v^j also scatter in one or the other time direction – see Prop. 6.1. Thus, for all $M_1 + 1 \leq j \leq M$, we have $E[\psi^j] = E[v^j] \geq 0$. By (6.2),

$$\sum_{j=1}^{M_1} E[\psi^j] \leq E[\phi_n] + o_n(1).$$

For at least one $1 \leq j \leq M_1$, we have

$$E[\psi^j] \leq \max\left(\lim_n E[\phi_n], 0\right).$$

We might as well take, WLOG, $j = 1$. Since we also have $M[\psi^1] \leq \lim_n M[\phi_n] = M[Q]$, we have

$$\frac{M[\psi^1]E[\psi^1]}{M[Q]E[Q]} \leq \max\left(\lim_n \frac{E[\phi_n]}{E[Q]}, 0\right).$$

Thus,

$$\frac{M[\psi^1]E[\psi^1]}{M[Q]E[Q]} = 3\lambda_1^2 - 2\lambda_1^3.$$

for some $\lambda_1 \geq \lambda_0$. [8] (In the case $\lim_n E[\phi_n] \geq 0$, we will have $\lambda_1 \geq \lambda' \geq \lambda_0$. In the case $\lim_n E[\phi_n] < 0$, we will have $\lambda_1 > \frac{3}{2} > \lambda_0$ but might not have $\lambda_1 \geq \lambda'$). Since v^1 is a nonscattering solution, we cannot have $\|\psi^1\|_{L^2}\|\nabla\psi^1\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$, since it would contradict Theorem 2.2. We therefore must have $\|\psi^1\|_{L^2}\|\nabla\psi^1\|_{L^2} > \lambda_1\|Q\|_{L^2}\|\nabla Q\|_{L^2}$.

Two cases emerge:

Case 1. $\lambda_1 \leq \sigma_c$. Since $\exists\mathrm{GB}(\lambda_1, \sigma_c - \delta)$ is false for each $\delta > 0$ (the inductive hypothesis), there exists a nondecreasing sequence t_k of times such that

$$\lim_k \frac{\|v^1(t_k)\|_{L^2}\|\nabla v^1(t_k)\|_{L^2}}{\|Q\|_{L^2}\|\nabla Q\|_{L^2}} \geq \sigma_c.$$

[8]This λ_1 is of course different from the λ_1 in the sequence λ_n used above.

Hence,

$$
\begin{aligned}
\sigma_c^2 - o_k(1) &\leq \frac{\|v^1(t_k)\|_{L^2}^2 \|\nabla v^1(t_k)\|_{L^2}^2}{\|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}^2} \\
&\leq \frac{\|\nabla v^1(t_k)\|_{L^2}^2}{\|\nabla Q\|_{L^2}^2} \\
&\leq \frac{\sum_{j=1}^M \|\nabla v^j(t_k - t_n^j)\|_{L^2}^2 + \|\nabla W_n^M(t_k)\|_{L^2}^2}{\|\nabla Q\|_{L^2}^2} \quad \text{(recall that } t_n^1 = 0) \\
&\leq \frac{\|\nabla u_n(t)\|_{L^2}^2}{\|\nabla Q\|_{L^2}^2} + o_n(1) \quad \text{(by Lemma 6.3, taking } n = n(k) \text{ large)} \\
&\leq \sigma_c^2 + o_n(1).
\end{aligned} \tag{8.1}
$$

Send $k \to +\infty$ (and hence, $n(k) \to +\infty$). We conclude that all inequalities must be equalities. In particular, we conclude that $W_n^M(t_k) \to 0$ in H^1 norm [9], that $v^j \equiv 0$ for all $j \geq 2$, and that $M[v_1] = M[Q]$. Moreover, by Lemma 6.3, we have that for all t,

$$\frac{\|\nabla v^1(t)\|_{L_x^2}}{\|\nabla Q\|_{L^2}} \leq \lim_n \frac{\|u_n(t)\|_{L_x^2}}{\|\nabla Q\|_{L^2}} \leq \sigma_c.$$

Hence, we take $u_{c,0} = v^1(0)$ $(= \psi^1)$, $\lambda_c = \lambda_1$.

Case 2. $\lambda_1 > \sigma_c$. Then we do not have access to the inductive hypothesis, but we do know that for all t,

$$\lambda_1^2 \leq \frac{\|v^1(t)\|_{L^2}^2 \|\nabla v^1(t)\|_{L^2}^2}{\|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}^2}.$$

Replace the first line of (8.1) by the above inequality; the rest of the inequalities in (8.1) still hold (we might as well now take $t_k = 0$). Send $n \to +\infty$ to get $\lambda_1 \leq \sigma_c$, a contradiction. Thus, this case does not arise.

□

9. Concentration of critical solutions

In this section, we take $u(t) = u_c(t)$ to be a critical solution, as provided by Lemma 8.1.

Lemma 9.1. *There exists a path $x(t)$ in $\mathbb{R}^3$ such that*

$$K \equiv \{\, u(t, \bullet - x(t)) \mid t \geq 0 \,\} \subset H^1$$

has compact closure in H^1.

[9] This implies that $W_n^M(0) = W_n^M \to 0$ in H^1, since we know that $W_n^M(t)$ is a scattering solution and have the bounds depicted in Fig. 1. We do not need this observation for the current proof, but do for the proof of Lemma 9.1.

Proof. As we showed in [2, Appendix A] it suffices to show that for each sequence of times $t_n \to \infty$, there exists (passing to a subsequence) a sequence x_n such that $u(t_n, \bullet - x_n)$ converges in H^1.

Take $\phi_n = u(t_n)$ in Lemma 6.4. Arguing similarly to the proof of Lemma 8.1, we obtain that $\psi^j = 0$ for $j \geq 2$ and $W_n^M \to 0$ in H^1 as $n \to \infty$. Hence, $u(t_n, \bullet - x_n) \to \psi^1$ in H^1.

□

As a result of Lemma 9.1, we have a uniform-in-time H^1 concentration of $u_c(t)$.

Corollary 9.2. *For each $\epsilon > 0$, there exists $R > 0$ such that for all t, $\|u(t, \bullet - x(t))\|_{H^1(|x|\geq R)} \leq \epsilon$*

The proof is elementary, but is given in [2, Cor. 3.3].

We next observe that the localization property of $u_c(t)$ given by Corollary 9.2 implies that $u_c(t)$ blows-up in finite time by Prop. 3.2. But this contradicts the boundedness of $u_c(t)$ in H^1, and hence, $u_c(t)$ cannot exist.

This contradiction completes the proof of Theorem 1.1.

Appendix A. Proof of Lemma 5.2

Here we will carry out the proof of Lemma 5.2, which closely follows the proof of Lemma 5.1 in [2]. We will adopt the notation from that paper.

Without loss of generality we may assume that $x(0) = 0$. Let $R(T) = \max_{0\leq t\leq T} |x(t)|$. It suffices to prove that there is an absolute constant $c > 0$ such that for each T with $R(T) = |x(T)| \gg 1$, we have

$$|x(T)| \leq c\,T \left(e^{-|x(T)|} + \epsilon\right)^2 . \tag{A.1}$$

Consider such a $T > 0$. We know that for $0 \leq t \leq T$, we have $|x(t)| \leq R(T)$. By (5.4) in [2] (and adopting the definition of $z_R(T)$ in that paper), there is an absolute constant c_1 such that

$$|z'_{2R(T)}(t)| \leq c_1 \int_{|x|\geq 2R(T)} \left(|\nabla u(t)|^2 + |u(t)|^2\right)\, dx.$$

By (5.3), for all $0 \leq t \leq T$ there holds

$$|z'_{2R(T)}(t)| \leq c_2 \left(\epsilon + \|Q\|_{H^1(|x|\geq R(T))}\right)^2 .$$

Owing to the exponential localization of $Q(x)$, we have upon integrating the above inequality over $[0, T]$ the bound

$$|z_{2R(T)}(t) - z_{2R(T)}(0)| \leq c_3 T \left(\epsilon + e^{-R(T)}\right)^2 . \tag{A.2}$$

Due to the fact that $|x(T)| = R(T)$, we have that there exists an absolute constant c_4 such that

$$|z_{2R(T)}(T)| \geq c_4 R(T)\,. \tag{A.3}$$

Moreover, since $x(0) = 0$, we have the simple bound

$$|z_{2R(T)}(0)| \leq c_5 \left(1 + R(T)\epsilon^2\right). \tag{A.4}$$

By combining (A.2), (A.3), and (A.4), we obtain (A.1).

Appendix B. Nonzero momentum

Suppose that we have a solution $u(x,t)$ with $M[u] = M[Q]$ and $P[u] \neq 0$. We apply the Galilean transformation to the solution $u(t)$ as in Section 4 of [2] to obtain a new solution $\tilde{u}(x,t)$:

$$\tilde{u}(x,t) = e^{ix\xi_0} e^{-it|\xi_0|^2} u(x - 2\xi_0 t, t) \quad \text{with} \quad \xi_0 = -\frac{P[u]}{M[u]}.$$

Then

$$P[\tilde{u}] = 0\,, \quad M[\tilde{u}] = M[u] = M[Q]\,,$$
$$E[\tilde{u}] = E[u] - \frac{1}{2}\frac{P[u]^2}{M[u]}\,,$$

and

$$\|\nabla \tilde{u}\|_2^2 = \|\nabla u\|_2^2 - \frac{P[u]^2}{M[u]}\,.$$

This choice of ξ_0 furnishes the lowest value of $E[\tilde{u}]$ under any choice of ξ_0. It is easier to have $E[\tilde{u}] < E[Q]$ than $E[u] < E[Q]$, suggesting that we should always implement this transformation to maximize the applicability of Prop. 2.1. However, one should show for consistency that if the dichotomy of Prop. 2.1 was already valid for u before the Galilean transformation was applied (i.e. $E[u] < E[Q]$), then the selection of case (1) versus (2) in Prop. 2.1 is preserved.

Suppose $M[u] = M[Q]$, $E[u] < E[Q]$ and $P[u] \neq 0$. Define $\tilde{u}$ as above. Let λ_-, λ be defined in terms of $E[u]$ by (2.6), and let $\eta(t)$ be defined in terms of $u(t)$ by (2.4). Let $\tilde{\lambda}_-$, $\tilde{\lambda}$ and $\tilde{\eta}(t)$ be the same quantities associated to $\tilde{u}$.

Suppose that case (1) of Prop. 2.1 holds for u. This implies, in particular, that $\eta(t) \leq 1$ for all t. But clearly $\tilde{\eta}(t) \leq \eta(t) \leq 1$, and thus, case (1) of Prop. 2.1 holds for $\tilde{u}$ also.

Now suppose that case (1) of Prop. 2.1 holds for $\tilde{u}$. Then $\tilde{\eta}(t) \leq \tilde{\lambda}_-$ for all t. We must show that

$$\eta^2 = \tilde{\eta}^2 + \frac{P[u]^2}{\|Q\|_{L^2}^2 \|\nabla Q\|_{L^2}^2} \leq \tilde{\lambda}_-^2 + \frac{P[u]^2}{6M[Q]E[Q]} \leq \lambda_-^2.$$

This reduces to an algebraic problem. For convenience, let $\alpha = E[u]/E[Q]$ and $\beta = P[u]^2/M[Q]E[Q]$. Then $\tilde{\lambda}_-$ is the smaller of the two roots of $3\tilde{\lambda}_-^2 - 2\tilde{\lambda}_-^3 = \alpha - \frac{1}{2}\beta$

while λ_- is the smaller of the two roots of $3\lambda_-^2 - 2\lambda_-^3 = \alpha$. In moving ρ forward from $\tilde{\lambda}_-^2$ to $\tilde{\lambda}_-^2 + \frac{1}{6}\beta$, we increment the function $3\rho - 2\rho^{3/2}$ by an amount at most $\frac{1}{6}\beta(3 - 3\rho^{1/2}) \leq \frac{1}{2}\beta$. This completes the argument.

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