THE UNIVERSITY OF CHICAGO

UNIFORM ESTIMATES FOR THE ZAKHAROV SYSTEM AND THE INITIAL-BOUNDARY VALUE PROBLEM FOR THE KORTEWEG-DE VRIES AND NONLINEAR SCHRÖDINGER EQUATIONS

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BY
JUSTIN HOLMER

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CHAPTER 1
INTRODUCTION

In Chapter 1, we derive estimates for the 1D Zakharov system, a PDE system consisting of a nonlinear Schrödinger (NLS) equation and a nonlinear wave equation, that are *uniform* as the wave speed approaches ∞. Let $u_0 : \mathbb{R} \to \mathbb{C}$, and $n_0,\ n_1 : \mathbb{R} \to \mathbb{R}$ be given initial data. The 1D Zakharov system (1D ZS_ϵ) is:

$$
\text{1D ZS}_\epsilon = \begin{cases} \partial_t u_\epsilon = i\partial_x^2 u_\epsilon \mp i n_\epsilon u_\epsilon & (1.1) \\ \epsilon^2 \partial_t^2 n_\epsilon - \partial_x^2 n_\epsilon = \partial_x^2 |u_\epsilon|^2 & (1.2) \\ u_\epsilon\big|_{t=0} = u_0 \\ n_\epsilon\big|_{t=0} = n_0 \\ \partial_t n_\epsilon\big|_{t=0} = n_1 \end{cases}
$$

where $u_\epsilon : \mathbb{R} \times [0,T] \to \mathbb{C}$, and $n_\epsilon : \mathbb{R} \times [0,T] \to \mathbb{R}$.

The modern method of applying the contraction principle in suitably defined Bourgain spaces has been used to prove local wellposedness of 1D ZS_ϵ by [BC96], [GTV97] on a time interval $[0, T_\epsilon]$ whose length depends on ϵ.

As a formal exercise, if we send $\epsilon \to 0$ in 1D ZS_ϵ, and assume that $u_\epsilon \to v$ for some v, then by setting $\epsilon = 0$ in (1.2), we expect that v solves the cubic nonlinear Schrödinger equation

$$
\text{1D NLS} = \begin{cases} \partial_t v = i\partial_x^2 v \pm i v |v|^2 \\ v\big|_{t=0} = u_0 \end{cases} \tag{1.3}
$$

In order to prove rigorous results concerning the convergence $u_\epsilon \to v$ as $\epsilon \to 0$, uniform in ϵ bounds on u_ϵ on some fixed time interval $[0,T]$ are needed. Some such uniform estimates were obtained by [AA88] from energy identities, and were applied by [AA88] and [OT92] to obtain results on the aforementioned convergence. Our objective is to obtain improved uniform bounds in order to enhance the convergence results of [OT92].

The method examined exploits local smoothing properties for the Schrödinger group $e^{it\partial_x^2}$ and uniform estimates for the inverse reduced wave operators $P_\pm$, defined so that $(\epsilon\partial_t \pm \partial_x)P_\pm z(x,t) = z(x,t)$, $P_\pm z(x,0) = 0$. By applying these estimates directly and employing a contraction argument in a suitable Banach space, [KPV95] obtained uniform estimates under a smallness assumption $\|\langle x\rangle u_0\|_{L_x^2} \leq \frac{1}{10}$. The main result of this chapter removes the smallness assumption of [KPV95] by employing a different technique, previously developed by [Chi96], [KPV98] to treat NLS equations having an order 1 nonlinearity. We introduce a pseudodifferential operator B with symbol $b(x,\xi) \in S^0$ depending on a constant M and satisfying

$$e^{-M} \leq b(x,\xi) \leq e^M$$

and apply it to the k-th derivative of (1.1) in the form

$$\partial_t u = i\partial_x^2 u \pm \tfrac{1}{2} iuP_\pm \partial_x(u\bar{u}) - ifu \tag{1.4}$$

where

$$f(x,t) = \tfrac{1}{2}n_0(x+\tfrac{t}{\epsilon}) + \tfrac{1}{2}n_0(x-\tfrac{t}{\epsilon}) + \tfrac{1}{2}\epsilon\int_{x-\frac{t}{\epsilon}}^{x+\frac{t}{\epsilon}} n_1(y)\,dy$$

The commutator $[B, i\partial_x^2]$ generates a first order term that is negative and whose size can be controlled by the constant M. In fact, by selecting $M = c\|\langle x\rangle u_0\|_{L_x^2}$, this commutator is sufficiently negative to absorb the first order terms $B\partial_x^k(\pm\frac{1}{2}iuP_\pm\partial_x(u\bar{u}))$. The key obstacle in showing this is that $[B, P_\pm]$ is not of lower order in x (nor can it be made small by any other device). It would instead suffice if the composition $BP_\pm B^{-1}$ were bounded independently of M; however, this turns out to be false as well. This problem is resolved by observing that $BP_\pm B^{-1}$ is in fact bounded independently of M if we restrict to certain spatial frequency ranges, and that $BP_\pm^* B^{-1}$ is bounded independently of M on the complementary spatial frequency ranges. The "error terms" obtained by replacing $P_\pm$ by $-P_\pm^*$ are handled using positivity properties of the operators $U_\pm = P_\pm + P_\pm^*$, and using once again that u solves (1.4).

Theorem 1. *Let $k \geq 4$, $(u_0, n_0, n_1) \in H^k \cap H^1(\langle x\rangle^2 dx) \times H^{k-\frac{1}{2}} \times H^{k-\frac{3}{2}}$, and set $M \sim \|\langle x\rangle u_0\|_{H_x^1}$. Then $\exists\, T > 0$ with*

$$T \sim \left(e^{2M} + \|u_0\|_{H_x^k} + \|n_0\|_{H_x^{k-\frac{1}{2}}} + \|n_1\|_{H_x^{k-\frac{3}{2}}}\right)^{-N}$$

(independent of ϵ) and a solution (u, n) to 1D ZS$_\epsilon$ *on* $[0, T]$ *such that* $\forall\, \epsilon,\ 0 < \epsilon \leq 1$,

$$\|u\|_{L_T^\infty H_x^k} + \|\langle x\rangle^{-1} D_x^{1/2} \partial_x^k u\|_{L_x^2 L_T^2} \leq c e^{2M} \|u_0\|_{H_x^k} \tag{1.5}$$

with c independent of ϵ.

We remark that the proof of this result can probably be adapted to yield a bound for any given time $T > 0$, provided $0 < \epsilon \leq \epsilon_0$, where

$$\epsilon_0 = \epsilon_0(T, \|u_0\|_{H_x^k}, \|\langle x\rangle u_0\|_{H_x^1}, \|n\|_{H_x^{k-\frac{1}{2}}}, \|n_1\|_{H_x^{k-\frac{3}{2}}})$$

By enhancing the argument of [OT92] in places, and using in the uniform bounds furnished by Theorem 1 in place of the energy estimates of [AA88], we obtain

Theorem 2. *For given initial data (u_0, n_0, n_1), let u_ϵ be the solution to* 1D ZS$_\epsilon$ *on a time interval* $[0, T]$*, and let v be the solution to* 1D NLS *with initial data u_0. In the* noncompatible case $(n_0 + u_0\bar{u}_0) \neq 0$, *if*

$$(u_0, n_0, n_1) \in (H^{k+1} \cap H^1(\langle x\rangle^2 dx)) \times (H^{k+\frac{1}{2}} \cap L^1) \times (H^{k-\frac{1}{2}} \cap L^1) \tag{1.6}$$

then

$$\|u_\epsilon - v\|_{L_T^\infty H_x^k} \leq c\epsilon$$

where c depends on the norms of the spaces in (1.6)*. In the* compatible case $(n_0 + u_0\bar{u}_0) = 0$, *if*

$$(u_0, n_0, n_1) \in (H^{k+2} \cap H^1(\langle x\rangle^2 dx)) \times (H^{k+\frac{3}{2}} \cap L^1) \times (H^{k+\frac{1}{2}} \cap \dot{H}^{-1} \cap L^1) \tag{1.7}$$

then

$$\|u_\epsilon - v\|_{L^\infty_T H^k_x} \le c\epsilon^2$$

where c *depends on the norms of the spaces in* (1.7).

In Chapter 2, we consider the initial-boundary value problem (IBVP) for the Korteweg de-Vries (KdV) equation on the half-line and line-segment. Let

$$S(t)\phi = \int e^{ix\xi} e^{it\xi^3} \hat{\phi}(\xi)\, d\xi \tag{1.8}$$

so that $(\partial_t + \partial_x^3)S(t)\phi = 0$ and $S(0)\phi = \phi$, and thus $u(x,t) = S(t)\phi(x)$ solves the initial-value problem for the linear KdV equation. By a change of variable in the definition (1.8), one obtains the sharp smoothing properties

$$\|S(t)\phi(x)\|_{L^\infty_x H_t^{\frac{s+1}{3}}} \le c\|\phi\|_{H^s_x}$$
$$\|\partial_x S(t)\phi(x)\|_{L^\infty_x H_t^{s/3}} \le c\|\phi\|_{H^s_x}$$

Therefore, *time traces* of solutions can be taken in $H_t^{\frac{s+1}{3}}$ and *derivative time traces* can be taken in $H_t^{s/3}$. Thus it makes sense to consider the following formulation of IBVP for KdV on the right half-line: For $f(t) \in H^{\frac{s+1}{3}}(\mathbb{R}_t^+)$, $\phi(x) \in H^s(\mathbb{R}_x^+)$, find u solving

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & \text{for } (x,t) \in (0,+\infty) \times (0,T) \\ u(0,t) = f(t) & \text{for } t \in (0,T) \\ u(x,0) = \phi(x) & \text{for } x \in (0,+\infty) \end{cases} \tag{1.9}$$

One can similarly formulate IBVP for KdV on the left-half line: For $f(t) \in H^{\frac{s+1}{3}}(\mathbb{R}_t^+)$,

$g(t) \in H^{s/3}(\mathbb{R}_t^+)$, $\phi(x) \in H^s(\mathbb{R}_x^-)$, find u solving

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & \text{for } (x,t) \in (-\infty, 0) \times (0,T) \\ u(0,t) = f(t) & \text{for } t \in (0,T) \\ \partial_x u(0,t) = g(t) & \text{for } t \in (0,T) \\ u(x,0) = \phi(x) & \text{for } x \in (-\infty, 0) \end{cases} \tag{1.10}$$

[CK02] introduce a new versatile method for treating problems of this type. In one section of their paper, they prove existence and uniqueness of a solution u to problem (1.9) in the case $s = 0$. Their method is to introduce a Duhamel forcing operator $\mathcal{L}(h)(x,t)$, for $h(t) \in H_0^{\frac{s+1}{3}}(\mathbb{R}_t)$, with the properties

$$\begin{cases} (\partial_t + \partial_x^3)\mathcal{L}(h)(x,t) = 0 & \text{for } x \neq 0 \\ \mathcal{L}(h)(0,t) = \frac{1}{3}h(t) \\ \mathcal{L}(h)(x,0) = 0 \end{cases}$$

Thus, $u(x,t) = S(t)\phi(x) + 3\mathcal{L}(f - S(t)\phi\big|_{x=0})(x,t)$ solves the linear homogeneous problem

$$\begin{cases} \partial_t u + \partial_x^3 u = 0 & \text{for } x \neq 0 \\ u(0,t) = f(t) \\ u(x,0) = \phi(x) \end{cases} \tag{1.11}$$

[CK02] prove suitable estimates on u solving (1.11) in terms of $f(t)$ and $\phi(t)$, and a solution to (1.9) for $s = 0$ is obtained by the contraction principle.

The goal of this chapter is to adapt the techniques of [CK02] to address (1.9) for $-\frac{3}{4} < s < \frac{3}{2}$ (some estimates in [CK02] fail outside $-\frac{1}{2} < s < \frac{1}{2}$), and to address (1.10), where an additional boundary condition appears, for $-\frac{3}{4} < s < \frac{3}{2}$. To accomplish this, we introduce analytic families of operators $\mathcal{L}_-^\lambda(h)(x,t)$, $\mathcal{L}_+^\lambda(h)(x,t)$, for $-2 < \text{Re}\,\lambda < 1$, with the properties

$$\begin{cases} (\partial_t + \partial_x^3)\mathcal{L}_-^\lambda(h)(x,t) = 0 & \text{for } x < 0 \\ \mathcal{L}_-^\lambda(h)(0,t) = \frac{2}{3}\sin(\frac{\pi}{3}\lambda + \frac{\pi}{6})h(t) \\ \mathcal{L}_-^\lambda(h)(x,0) = 0 \end{cases}$$

and

$$\begin{cases} (\partial_t + \partial_x^3)\mathcal{L}_+^\lambda(h)(x,t) = 0 & \text{for } x > 0 \\ \mathcal{L}_+^\lambda(h)(0,t) = \frac{1}{3}e^{\pi i\lambda}h(t) \\ \mathcal{L}_+^\lambda(h)(x,0) = 0 \end{cases}$$

The operator used by [CK02] is $\mathcal{L} = \mathcal{L}_+^0 = \mathcal{L}_-^0$. (1.9) is solved by appropriately selecting an operator from the class $\mathcal{L}_+^\lambda$, while (1.10) is solved by appropriately selecting two operators from the family $\mathcal{L}_-^\lambda$. Constraints on the eligible values of λ come from the required estimates.

Theorem 3. *Suppose $-\frac{3}{4} < s < \frac{1}{2}$. Then we have local wellposedness of* (1.9) *for $(\phi, f) \in H^s(\mathbb{R}_x^+) \times H^{\frac{s+1}{3}}(\mathbb{R}_t^+)$ and local wellposedness of* (1.10) *for $(\phi, f, g) \in H^s(\mathbb{R}_x^-) \times H^{\frac{s+1}{3}}(\mathbb{R}_t^+) \times H^{s/3}(\mathbb{R}_t^+)$. Suppose $\frac{1}{2} < s < \frac{3}{2}$. Then we have local wellposedness of* (1.9) *for $(\phi, f) \in H^s(\mathbb{R}_x^+) \times H^{\frac{s+1}{3}}(\mathbb{R}_t^+)$, provided $\phi(0) = f(0)$ and local wellposedness of* (1.10) *for $(\phi, f, g) \in H^s(\mathbb{R}_x^-) \times H^{\frac{s+1}{3}}(\mathbb{R}_t^+) \times H^{s/3}(\mathbb{R}_t^+)$, provided $\phi(0) = f(0)$.*

Finally, we consider the finite-length interval $0 < x < 1$ problem:

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & \text{for } (x,t) \in (0,1) \times (0,T) \\ u(0,t) = g_3(t) & \text{on } (0,T) \\ u(1,t) = g_1(t) & \text{on } (0,T) \\ \partial_x u(1,t) = g_2(t) & \text{on } (0,T) \\ u(x,0) = \phi & \text{on } (0,1) \end{cases} \tag{1.12}$$

with $\phi(x) \in H^s((0,1))$, $g_1(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$, $g_2 \in H^{\frac{s}{3}}(\mathbb{R}^+)$, $g_3(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$. This is accomplished by making use of two operators of type $\mathcal{L}_-^\lambda$ positioned at the right endpoint $x = 1$ and one operator of type $\mathcal{L}_+^\lambda$ positioned at the left endpoint. The equation relating the desired boundary functions to the needed "input" functions for the forcing operators is a Fredholm equation.

Theorem 4. (1.12) *is locally wellposeded for* $-\frac{3}{4} < s < \frac{3}{2}$, $s \neq \frac{1}{2}$, *for* $(\phi, g_3, g_1, g_2) \in H^s((0,1)) \times H^{\frac{s+1}{3}}(\mathbb{R}^+) \times H^{\frac{s+1}{3}}(\mathbb{R}^+) \times H^{\frac{s}{3}}(\mathbb{R}^+)$, *with the compatibility conditions* $g_3(0) = \phi(0)$ *and* $g_1(0) = \phi(1)$ *for* $\frac{1}{2} < s < \frac{3}{2}$.

In Chapter 3, we treat the initial-boundary value problem (IBVP) for the nonlinear Schrödinger (NLS) equation on the half-line. We introduce an operator analogous to the one used by [CK02], defined in terms of the Schrödinger group $e^{it\partial_x^2}$, to obtain a solution to IBVP for NLS in the cases $s = 0$, $s = 1$. This problem, for the right-half line, takes the form: Given $f \in H^{\frac{2s+1}{4}}(\mathbb{R}_t^+)$, $\phi \in H^s(\mathbb{R}^+)$, find u solving

$$\begin{cases} i\partial_t u + \partial_x^2 u + \lambda u|u|^{\alpha-1} = 0 & \text{for } (x,t) \in (0,+\infty) \times (0,T) \\ u(0,t) = f(t) & \text{for } t \in (0,T) \\ u(x,0) = \phi(x) & \text{for } x \in (0,+\infty) \end{cases} \tag{1.13}$$

The left half-line problem is actually the same problem since $u(x,t)$ solves the left-hand problem for $\phi(x)$ and $f(t)$ iff $u(-x,t)$ solves the right-hand problem for $\phi(-x)$ and $f(t)$.

The technique is a synthesis of the techniques in [CK02] with the standard proof of local wellposedness for NLS on the line $\mathbb{R}$ using the Strichartz estimates. We take the *space traces* and *mixed norm* estimates for the group $e^{it\partial_x^2}$ and the Duhamel inhomogeneous solution operator used in the standard proof and add to them local smoothing or *time traces* estimates. We also need to introduce a Duhamel forcing operator $\mathcal{L}(h)(x,t)$, satisfying

$$\begin{cases} (i\partial_t + \partial_x^2 u)\mathcal{L}(h)(x,t) = 0 & \text{for } x \neq 0 \\ \mathcal{L}(h)(0,t) = h(t) \\ \mathcal{L}(h)(x,0) = 0 \end{cases}$$

examine its continuity and decay properties for $h \in C_0^\infty(\mathbb{R})$, and prove *space traces* estimates, *time traces* estimates, and *mixed norm* estimates for it. We then present a solution to the problem (1.13) by the contraction method for $s = 0$, $1 < \alpha < 5$ and $s = 1$, $1 < \alpha < +\infty$. The L^2-critical case $s = 0$ and $\alpha = 5$ is also treated using the method of [CW89].

Theorem 5. *There is local wellposedness of* (1.13) *for* $(\phi, f) \in L^2(\mathbb{R}_x^+) \times H^{1/4}(\mathbb{R}_t^+)$ *and* $1 < \alpha < 5$ *and for* $(\phi, f) \in H^1(\mathbb{R}_x^+) \times H^{3/4}(\mathbb{R}_t^+)$ *and* $1 < \alpha < +\infty$.

The primary new feature of the results obtained here, in comparison with earlier work on the problem [SB01], [Fok02], is the limited regularity required on the boundary data $f(t)$.

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