GLOBAL EXISTENCE AND SCATTERING FOR ROUGH SOLUTIONS TO GENERALIZED NONLINEAR SCHRÖDINGER EQUATIONS ON $\mathbb{R}$

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ABSTRACT. We consider the Cauchy problem for a family of semilinear defocusing Schrödinger equations with monomial nonlinearities in one space dimension. We establish global well-posedness and scattering. Our analysis is based on a four-particle interaction Morawetz estimate giving *a priori* $L^8_{t,x}$ spacetime control on solutions.

1. INTRODUCTION

We consider the initial value problem for the one-dimensional defocusing nonlinear Schrödinger (NLS) equation,

$$\begin{cases} iu_t + \Delta u = |u|^{2k}u \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $k \in \mathbb{N}$ with $k \geq 3$ and u is a complex-valued function on spacetime $\mathbb{R}_t \times \mathbb{R}_x$. This problem is known to be locally wellposed for initial data in $H^s(\mathbb{R})$ for $s \geq s_c := \frac{1}{2} - \frac{1}{k}$; see [4, 5]. The scaling invariant Sobolev index s_c is distinguished in the theory by the invariance of the $\dot{H}^{s_c}_x$ norm under the scaling symmetry of solutions to (1.1): If u solves (1.1) then

$$u^\lambda(t,x) := \lambda^{-\frac{1}{k}} u(\lambda^{-2}t, \lambda^{-1}x) \tag{1.2}$$

also solves (1.1).

The following quantities, if finite for the initial data, are time invariant:

$$\text{Mass} := M[u(t)] := \|u(t)\|^2_{L^2_x},$$

$$\text{Energy} := E[u(t)] := \frac{1}{2}\|\nabla u(t)\|^2_{L^2_x} + \frac{1}{2k+2}\|u(t)\|^{2k+2}_{L^{2k+2}_x}.$$

The local-in-time theory in the presence of these conserved quantities iterates to prove global-in-time well-posedness for (1.1) for initial data in H^1_x. Furthermore, in this case it is known that these global-in-time solutions are bounded in the associated scaling-invariant diagonal Strichartz space $L^{3k}_{t,x}$ and scatter; see [18]. It is conjectured that global well-posedness and scattering also hold for solutions to (1.1) with initial data in $\dot{H}^{s_c}(\mathbb{R})$.

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This work makes partial progress toward this conjecture by establishing these properties for solutions to (1.1) with initial data in $H^{\frac{8}{9}}(\mathbb{R})$. In fact, for all values k considered we establish global well-posedness and scattering for (1.1) with initial data in $H^s(\mathbb{R})$ for $s > s_k$, where $s_k := \frac{8k-16}{9k-14} < \frac{8}{9}$.

Theorem 1.1. *For each $k \in \{3, 4, \dots\}$ there is a regularity threshold $s_k = \frac{8k-16}{9k-14}$ such that the initial value problem (1.1) is globally wellposed and scatters for initial data $u_0 \in H^s(\mathbb{R})$, provided $s > s_k$. In particular, there exist $u_\pm \in H^s(\mathbb{R})$ such that*

$$\|u(t) - e^{it\Delta}u_\pm\|_{H^s(\mathbb{R})} \to 0 \quad \text{as } t \to \pm\infty.$$

Our approach to proving this result is based on the proof of a similar statement for the defocusing cubic nonlinear Schrödinger equation on $\mathbb{R}^3$ in [11]. The analysis in [11] is based on an *a priori* two-particle interaction Morawetz estimate. We derive a four-particle interaction Morawetz inequality which provides $L^8_{t,x}$ spacetime control on solutions to (1.1). Our analysis relies on this improved *a priori* control.

As a consequence of the four-particle interaction Morawetz inequality, we are in fact able to offer a new proof of scattering for a class of one-dimensional defocusing nonlinear Schrödinger equations with initial data in $H^1(\mathbb{R})$; see [18] for the original proof.

Theorem 1.2 (Scattering in $H^1(\mathbb{R})$)**.** *Let $u_0 \in H^1(\mathbb{R})$. Then, there exists a unique global solution u to the initial value problem*

$$\begin{cases} iu_t + \Delta u = |u|^{2p}u, \quad p > 0, \\ u(0, x) = u_0(x). \end{cases} \tag{1.3}$$

Moreover, if $p > 2$ there exist $u_\pm \in H^1(\mathbb{R})$ such that

$$\|u(t) - e^{it\Delta}u_\pm\|_{H^1(\mathbb{R})} \to 0 \quad \text{as } t \to \pm\infty.$$

We briefly explain our strategy for proving Theorem 1.1 and Theorem 1.2.

The interaction Morawetz inequality we derive in Section 3 provides *a priori* $L^8_{t,x}$ spacetime control on solutions to (1.3) (and hence on solutions to (1.1)), provided that $\|u(t)\|_{H^{1/2}_x}$ stays bounded. In particular, if the initial data $u_0 \in H^1_x$, we immediately obtain that the unique global H^1_x solution enjoys the global $L^8_{t,x}$ estimate. In Section 4, for $p > 2$ we upgrade this estimate to stronger Strichartz norm control from which scattering in H^1_x follows, thus establishing Theorem 1.2. A similar argument in higher dimensions, $n \geq 3$, relying on the two-particle Morawetz inequality, can be found in [20].

If we are in the H^s_x setting (rather than the H^1_x setting) with s being defined in Theorem 1.1, we know the problem is H^s_x subcritical and, as a consequence, the length of the local well-posedness time interval of the unique H^s_x solution depends only on the H^s_x norm of the initial data. Thus, in order to prove global well-posedness we only need to control the H^s_x norm of the solution. This is not immediate as the H_x^s norm is not conserved. In order to derive the desired control over the H_x^s norm of the solution, we will use the 'I-method'.

The idea behind the 'I-method' ([9, 11]) is to smooth out the initial data in order to get access to the good local and global theory available at H_x^1 regularity. To this end, one introduces the Fourier multiplier I which is the identity on low frequencies and behaves like a fractional integral operator of order $1-s$ on high frequencies. Thus, the operator I maps H_x^s to H_x^1 and the H_x^s norm of u can be controlled by the H_x^1 norm of the modified solution Iu. However, Iu is not a solution to (1.1) and hence one cannot use the conservation of energy to derive a bound on the H_x^1 norm of Iu. In fact, we expect an increment in the energy of Iu. This increment is proved to be under control provided the Morawetz norm is finite; see Section 5. But in order for the Morawetz norm to be finite we need to control the $H_x^{1/2}$ norm of the solution. This sets us up for a bootstrap argument which will be carried out in Section 6.

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2. Preliminaries

In this section, we introduce notations and some basic estimates we will invoke throughout this paper.

We will often use the notation $X \lesssim Y$ whenever there exists some constant $C > 0$ so that $X \leq CY$. Similarly, we will use $X \sim Y$ if $X \lesssim Y \lesssim X$. We will use $X \ll Y$ if $X \leq cY$ for some very small constant $c > 0$. We will sometimes denote partial derivatives with subscripts ($a_j(x) := \partial_j a(x) := \partial_{x_j} a(x)$) and use the convention that repeated indices are implicitly summed.

We use $L_x^r(\mathbb{R})$ to denote the Banach space of functions $f : \mathbb{R} \to \mathbb{C}$ whose norm

$$\|f\|_r := \Big(\int_{\mathbb{R}} |f(x)|^r dx\Big)^{1/r}$$

is finite, with the usual modifications when $r = \infty$.

We use $L_t^q L_x^r$ to denote the spacetime norm

$$\|u\|_{q,r} := \|u\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R})} := \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |u(t,x)|^r dx\Big)^{q/r} dt\Big)^{1/q},$$

with the usual modifications when either q or r are infinity, or when the domain $\mathbb{R} \times \mathbb{R}$ is replaced by some smaller spacetime region. When $q = r$ we abbreviate $L_t^q L_x^r$ by $L_{t,x}^q$.

We define the Fourier transform on $\mathbb{R}$ to be

$$\hat{f}(\xi) := \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi} f(x)dx.$$

We will make use of the fractional differentiation operators $|\nabla|^s$ defined by

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat{f}(\xi).$$

These define the homogeneous Sobolev norms

$$\|f\|_{\dot{H}^s_x} := \||\nabla|^s f\|_{L^2_x}$$

and more general Sobolev norms

$$\|f\|_{H^{s,p}_x} := \|\langle\nabla\rangle^s f\|_p,$$

where, $\langle\nabla\rangle = (1+|\nabla|^2)^{\frac{1}{2}}$.

Let $e^{it\Delta}$ be the free Schrödinger propagator. In physical space this is given by the formula

$$e^{it\Delta}f(x) = \frac{1}{(4\pi it)^{1/2}} \int_{\mathbb{R}} e^{i|x-y|^2/4t} f(y)dy$$

for $t \neq 0$ (using a suitable branch cut to define $(4\pi it)^{1/2}$), while in frequency space one can write this as

$$\widehat{e^{it\Delta}f}(\xi) = e^{-4\pi^2 it|\xi|^2}\hat{f}(\xi). \tag{2.1}$$

In particular, the propagator obeys the *dispersive inequality*

$$\|e^{it\Delta}f\|_{L^\infty_x} \lesssim |t|^{-\frac{1}{2}}\|f\|_{L^1_x} \tag{2.2}$$

for all times $t \neq 0$.

We also recall *Duhamel's formula*

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) - i\int_{t_0}^t e^{i(t-s)\Delta}(iu_t + \Delta u)(s)ds. \tag{2.3}$$

Definition 2.1. A pair of exponents (q, r) is called *Schrödinger-admissible* if

$$\frac{2}{q} + \frac{1}{r} = \frac{1}{2}, \quad 2 \leq r \leq \infty.$$

For a spacetime slab $I \times \mathbb{R}$, we define the Strichartz norm

$$\|f\|_{S^0(I)} := \sup_{(q,r) \text{ admissible}} \|f\|_{L^q_t L^r_x(I\times\mathbb{R})}.$$

Then, we have the following Strichartz estimates (for a proof see [13, 15, 19]):

Lemma 2.1. *Let I be a compact time interval, $t_0 \in I$, $s \geq 0$, and let u be a solution to the forced Schrödinger equation*

$$iu_t + \Delta u = \sum_{i=1}^m F_i$$

for some functions $F_1, \dots, F_m$. Then,

$$\||\nabla|^s u\|_{S^0(I)} \lesssim \|u(t_0)\|_{\dot H^s_x} + \sum_{i=1}^{m} \||\nabla|^s F_i\|_{L_t^{q_i'} L_x^{r_i'}(I\times\mathbb{R})} \tag{2.4}$$

for any admissible pairs (q_i, r_i), $1 \le i \le m$. Here, p' denotes the conjugate exponent to p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

We will also need some Littlewood-Paley theory. Specifically, let $\varphi(\xi)$ be a smooth bump supported in $|\xi| \le 2$ and equalling one on $|\xi| \le 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$ we define the Littlewood-Paley operators

$$\begin{aligned}
\widehat{P_{\le N} f}(\xi) &:= \varphi(\xi/N)\hat f(\xi),\\
\widehat{P_{> N} f}(\xi) &:= [1-\varphi(\xi/N)]\hat f(\xi),\\
\widehat{P_N f}(\xi) &:= [\varphi(\xi/N)-\varphi(2\xi/N)]\hat f(\xi).
\end{aligned}$$

Similarly, we can define $P_{<N}$, $P_{\ge N}$, and $P_{M<\cdot\le N} := P_{\le N} - P_{\le M}$, whenever M and N are dyadic numbers. We will frequently write $f_{\le N}$ for $P_{\le N} f$ and similarly for the other operators. We recall the following standard Bernstein and Sobolev type inequalities:

Lemma 2.2. *For any $1 \le p \le q \le \infty$ and $s > 0$, we have*

$$\begin{aligned}
\|P_{\ge N} f\|_{L^p_x} &\lesssim N^{-s} \||\nabla|^s P_{\ge N} f\|_{L^p_x}\\
\||\nabla|^s P_{\le N} f\|_{L^p_x} &\lesssim N^{s} \|P_{\le N} f\|_{L^p_x}\\
\||\nabla|^{\pm s} P_N f\|_{L^p_x} &\sim N^{\pm s} \|P_N f\|_{L^p_x}\\
\|P_{\le N} f\|_{L^q_x} &\lesssim N^{\frac{1}{p}-\frac{1}{q}} \|P_{\le N} f\|_{L^p_x}\\
\|P_N f\|_{L^q_x} &\lesssim N^{\frac{1}{p}-\frac{1}{q}} \|P_N f\|_{L^p_x}.
\end{aligned}$$

For $N > 1$, we define the Fourier multiplier $I := I_N$ (cf. [9])

$$\widehat{I_N u}(\xi) := m_N(\xi)\hat u(\xi),$$

where m_N is a smooth radial decreasing function such that

$$m_N(\xi) = \begin{cases} 1, & \text{if } |\xi| \le N \\ \left(\frac{|\xi|}{N}\right)^{s-1}, & \text{if } |\xi| \ge 2N. \end{cases}$$

Thus, I is the identity operator on frequencies $|\xi| \le N$ and behaves like a fractional integral operator of order $1-s$ on higher frequencies. In particular, I maps H^s_x to H^1_x. We collect the basic properties of I into the following

Lemma 2.3. *Let $1 < p < \infty$ and $0 \le \sigma \le s < 1$. Then,*

$$\|If\|_p \lesssim \|f\|_p \tag{2.5}$$

$$\||\nabla|^\sigma P_{>N} f\|_p \lesssim N^{\sigma-1} \|\nabla I f\|_p \tag{2.6}$$

$$\|f\|_{H^s_x} \lesssim \|If\|_{H^1_x} \lesssim N^{1-s} \|f\|_{H^s_x}. \tag{2.7}$$

Proof. The estimate (2.5) is a direct consequence of the multiplier theorem.

To prove (2.6), we write

$$\||\nabla|^\sigma P_{>N} f\|_p = \|P_{>N}|\nabla|^\sigma (\nabla I)^{-1}\nabla I f\|_p.$$

The claim follows again from the multiplier theorem.

Now we turn to (2.7). By the definition of the operator I and (2.6),

$$\begin{aligned}\|f\|_{H^s_x} &\lesssim \|P_{\leq N} f\|_{H^s_x} + \|P_{>N} f\|_2 + \||\nabla|^s P_{>N} f\|_2\\ &\lesssim \|P_{\leq N} I f\|_{H^1_x} + N^{-1}\|\nabla I f\|_2 + N^{s-1}\|\nabla I f\|_2\\ &\lesssim \|I f\|_{H^1_x}.\end{aligned}$$

On the other hand, since the operator I commutes with $\langle\nabla\rangle^s$,

$$\|If\|_{H^1_x} = \|\langle\nabla\rangle^{1-s} I \langle\nabla\rangle^s f\|_2 \lesssim N^{1-s}\|\langle\nabla\rangle^s f\|_2 \lesssim N^{1-s}\|f\|_{H^s_x},$$

which proves the last inequality in (2.7). Note that a similar argument also yields

$$\|If\|_{\dot H^1_x} \lesssim N^{1-s}\|f\|_{\dot H^s_x}. \tag{2.8}$$

□

3. An interaction Morawetz inequality

In this section we develop an *a priori* four-particle interaction Morawetz inequality for solutions to one-dimensional defocusing nonlinear Schrödinger equations. This *a priori* control will be fundamental to our analysis.

The name Morawetz inequality derives from her work on monotonicity formulae for the wave equation. The Schrödinger version is due to Lin and Strauss, [16]. The idea of a two-particle interaction Morawetz inequality was first introduced in [11]. This two-particle style of estimate has proved invaluable in the study of NLS in dimensions three and higher. Unfortunately, there is no direct analogue of this estimate in dimensions one and two; nevertheless, several alternatives have been proposed, [18, 12]. Here we derive a Morawetz inequality based on four-particle interactions. This approach was suggested to us by Terry Tao, based on a private conversation with Andrew Hassel.

Proposition 3.1 (Interaction Morawetz estimate)**.** *Let u be an $H^{1/2}$ solution to* (1.3) *on the spacetime slab $I\times\mathbb{R}$. Then,*

$$\int_I\int_{\mathbb{R}} |u(t,x)|^8\,dx\,dt \lesssim \|u\|^2_{L^\infty_t \dot H^{1/2}_x(I\times\mathbb{R})}\|u_0\|_2^6. \tag{3.1}$$

The calculations that follow are difficult to justify without additional regularity and decay assumptions on the solution. This obstacle can be dealt with in the standard manner: mollify the initial data and the nonlinearity to make the interim calculations valid and observe that the mollifications can be removed at the end. For expository reasons, we skip the details and keep all computations on a formal level.

In order to prove Proposition 3.1 we first review general facts about the one-particle Morawetz action. Let $\phi : \mathbb{R}_t \times \mathbb{R}^4_y \to \mathbb{C}$ be a solution to the Schrödinger equation

$$i\phi_t + \Delta\phi = \mathcal{N}.$$

Let $a : \mathbb{R}^4_y \to \mathbb{R}$ be a convex weight function and define the Morawetz action to be the weighted momentum

$$M_a(t) := 2\,\mathrm{Im} \int_{\mathbb{R}^4} \overline{\phi(t,y)} \nabla a(y) \cdot \nabla\phi(t,y)\, dy.$$

A direct calculation establishes that in the $(y_1, \ldots, y_4)$ coordinate system we have

$$\begin{aligned}\partial_t M_a(t) = 2 \int_{\mathbb{R}^4} (-\Delta\Delta a(y)) |\phi(t,y)|^2\, dy + 4 \int_{\mathbb{R}^4} a_{jk}(y)\, \mathrm{Re}(\overline{\phi_j}\phi_k)(t,y)\, dy \\ + 2 \int_{\mathbb{R}^4} \nabla a(y) \cdot \{\mathcal{N}, \phi\}(t,y)\, dy,\end{aligned}$$

where the momentum bracket is defined by

$$\{f, g\} := \mathrm{Re}(f\nabla\bar{g} - g\nabla\bar{f}).$$

As the weight a is convex, the matrix $\{a_{jk}\}_{1\le j,k\le 4}$ is positive semi-definite and hence

$$\int_{\mathbb{R}^4} a_{jk}(y)\, \mathrm{Re}(\overline{\phi_j}\phi_k)(t,y)\, dy \ge 0.$$

Thus,

$$\begin{aligned}\partial_t M_a(t) \ge 2 \int_{\mathbb{R}^4} (-\Delta\Delta a(y)) |\phi(t,y)|^2\, dy \\ + 2 \int_{\mathbb{R}^4} \nabla a(y) \cdot \{\mathcal{N}, \phi\}(t,y)\, dy.\end{aligned} \tag{3.2}$$

Now we are ready to prove Proposition 3.1. Let u be a solution to (1.3) and for each $1 \le j \le 4$ let $u_j(t, x_j) := u(t, x_j)$. Define

$$w(t,x) = w(t, x_1, x_2, x_3, x_4) := \prod_{j=1}^{4} u_j(t, x_j);$$

note that w satisfies the equation

$$iw_t + \Delta_x w = \Big(\sum_{j=1}^{4} |u_j|^{2p}\Big) w.$$

Next, we perform the orthonormal change of variables

$$z = Ax \text{ with } A = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}.$$

Then $\Delta_x = \Delta_z$ and hence, for $\omega(t,z) := w(t,x(z))$, we have

$$i\omega_t + \Delta_z \omega = \big(\sum_{j=1}^4 |u_j|^{2p}\big)\omega.$$

Applying (3.2) to ω in the $(z_1,\dots,z_4)$ coordinate system with the convex weight $a(z) := (z_2^2+z_3^2+z_4^2)^{1/2}$, we get

$$\begin{aligned}\partial_t M_a(t) &\geq 2\int_{\mathbb{R}^4} (-\Delta_z\Delta_z a(z))|\omega(t,z)|^2\,dz \\ &\quad + 2\int_{\mathbb{R}^4} \nabla_z a(z)\cdot \big\{\big(\sum_{j=1}^4 |u_j|^{2p}\big)\omega,\omega\big\}(t,z)\,dz, \end{aligned} \tag{3.3}$$

where

$$M_a(t) := 2\,\mathrm{Im}\int_{\mathbb{R}^4} \overline{\omega(t,z)}\nabla_z a(z)\cdot\nabla_z\omega(t,z)\,dz.$$

A quick computation shows that

$$-\Delta_z\Delta_z a(z) = 4\pi\delta(z_2,z_3,z_4)$$

and hence, by a change of variables,

$$\begin{aligned} 2\int_{\mathbb{R}^4} (-\Delta_z\Delta_z a(z))|\omega(t,z)|^2\,dz_1 &= 8\pi\int_{\mathbb{R}} |\omega(t,z_1,0,0,0)|^2\,dz_1 \\ &= 16\pi\int_{\mathbb{R}} |w(t,z_1,z_1,z_1,z_1)|^2\,dz_1 \\ &= 16\pi\int_{\mathbb{R}} |u(t,z_1)|^8\,dz_1. \end{aligned}$$

To estimate the second term on the right-hand side of (3.3), we note that orthonormal changes of variables leave inner products invariant and hence,

$$\begin{aligned} &\int_{\mathbb{R}^4} \nabla_z a(z)\cdot \big\{\big(\sum_{j=1}^4 |u_j|^{2p}\big)\omega,\omega\big\}(t,z)\,dz \\ &\qquad\qquad = \int_{\mathbb{R}^4} \nabla_x a(x)\cdot \big\{\big(\sum_{j=1}^4 |u_j|^{2p}\big)w,w\big\}(t,x)\,dx. \end{aligned}$$

A simple computation then shows that in the $(x_1,\dots,x_4)$ coordinate system we have

$$\begin{aligned} \big\{\big(\sum_{j=1}^4 |u_j|^{2p}\big)w,w\big\}^i &= \big(\sum_{j=1}^4 |u_j|^{2p}\big)w\partial_{x_i}\bar w - w\partial_{x_i}\big[\big(\sum_{j=1}^4 |u_j|^{2p}\big)\bar w\big] \\ &= -|w|^2\partial_{x_i}\big(\sum_{j=1}^4 |u_j|^{2p}\big) \\ &= -\frac{p}{p+1}\partial_{x_i}\big(|w|^2|u_i|^{2p}\big). \end{aligned}$$

Integrating by parts, we obtain

$$\int_{\mathbb{R}^4} \nabla_x a(x) \cdot \big\{ \big(\sum_{j=1}^4 |u_j|^{2p}\big) w, w \big\}(t,x)\, dx$$
$$= \frac{p}{p+1} \int_{\mathbb{R}^4} \sum_{i=1}^4 a_{ii}(x) \big(|w|^2 |u_i|^{2p}\big)(t,x)\, dx \geq 0,$$

as a is a convex function.

Putting everything together we get

$$\partial_t M_a(t) \geq 8\pi \int_{\mathbb{R}} |u(t,x)|^8\, dx$$

and hence, by the Fundamental Theorem of Calculus,

$$\int_I \int_{\mathbb{R}} |u(t,x)|^8\, dx\, dt \lesssim \sup_{t \in I} |M_a(t)|.$$

In order to estimate the right-hand side in the inequality above, we first note that

$$\Big| \int_{\mathbb{R}^n} f(x) \frac{x}{|x|} \cdot \nabla f(x)\, dx \Big| \lesssim \|f\|^2_{\dot{H}^{1/2}(\mathbb{R}^n)}, \tag{3.4}$$

for any function $f : \mathbb{R}^n \to \mathbb{C}$ with $n \geq 3$. Indeed, by Cauchy-Schwarz,

$$\Big| \int_{\mathbb{R}^n} f(x) \frac{x}{|x|} \cdot \nabla f(x)\, dx \Big| \lesssim \|f\|_{\dot{H}^{1/2}(\mathbb{R}^n)} \Big\| \frac{x}{|x|} f \Big\|_{\dot{H}^{1/2}(\mathbb{R}^n)},$$

and (3.4) follows if we establish that the operator $T(f)(x) := \frac{x}{|x|} f(x)$ is bounded on $\dot{H}^{1/2}(\mathbb{R}^n)$. Using Hardy's inequality

$$\Big\| \frac{f}{|x|} \Big\|_2 \lesssim \|\nabla f\|_2,$$

it is easy to see that T is bounded on $L^2(\mathbb{R}^n)$ and on $\dot{H}^1(\mathbb{R}^n)$. By interpolation, this yields the claim.

Applying (3.4) (in the variables (z_2, z_3, z_4)), Plancherel, and a change of variables, we estimate

$$\begin{aligned}
|M_a(t)| &\lesssim \int_{\mathbb{R}} \|\omega(t, z_1, \cdot)\|^2_{\dot H^{1/2}(\mathbb{R}^3)}\, dz_1 \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^3} |\xi_2^2 + \xi_3^2 + \xi_4^2|^{1/2} |\tilde\omega(t, z_1, \xi_2, \xi_3, \xi_4)|^2 \, d\xi_2 \, d\xi_3 \, d\xi_4 \, dz_1 \\
&= \int_{\mathbb{R}^4} |\xi_2^2 + \xi_3^2 + \xi_4^2|^{1/2} |\hat\omega(t, \xi)|^2 \, d\xi \\
&\leq \int_{\mathbb{R}^4} |\xi| |\hat\omega(t, \xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^4} |\eta| |\hat w(t, \eta)|^2 \, d\eta \\
&= \int_{\mathbb{R}^4} |\eta| |\hat u_1(t, \eta_1)|^2 |\hat u_2(t, \eta_2)|^2 |\hat u_3(t, \eta_3)|^2 |\hat u_4(t, \eta_4)|^2 \, d\eta \\
&\leq \int_{\mathbb{R}^4} \big(|\eta_1| + |\eta_2| + |\eta_3| + |\eta_4|\big) \prod_{j=1}^4 |\hat u_j(t, \eta_j)|^2 \, d\eta \\
&\leq 4 \|u(t)\|^2_{\dot H^{1/2}} \|u(t)\|_2^6.
\end{aligned}$$

In the computations above, we used $\tilde\omega$ to denote the partial Fourier transform with respect to the variables (z_2, z_3, z_4) and $\hat\omega$ to denote the full Fourier transform. The change of variables performed was $\xi := A\eta$.

Thus, by the conservation of mass,

$$\int_I \int_{\mathbb{R}} |u(t, x)|^8 \, dx \, dt \lesssim \sup_{t \in I} \|u(t)\|^2_{\dot H^{1/2}_x} \|u(t)\|_2^6 \lesssim \|u\|^2_{L_t^\infty \dot H_x^{1/2}(I \times \mathbb{R})} \|u_0\|_2^6.$$

This concludes the proof of Proposition 3.1.

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2. Global well-posedness for (1.3) is a consequence of the fact that the equation is subcritical with respect to energy. The result and the proof are by now standard and we will not revisit them here; see [5, 14].

Scattering in the case $p > 2$ was first proved by Nakanishi, [18]. In this section we present a new proof relying on the four-particle interaction Morawetz inequality we developed in the previous section.

Indeed, by Proposition 3.1 and the conservation of mass and energy, the unique global solution to (1.3) with initial data in $H^1(\mathbb{R})$ satisfies

$$\|u\|_{L^8_{t,x}(\mathbb{R} \times \mathbb{R})} \lesssim \|u_0\|_{H^1(\mathbb{R})}. \tag{4.1}$$

In order to prove scattering, we first upgrade (4.1) to Strichartz control. Let $\delta > 0$ be a small constant to be chosen momentarily and divide $\mathbb{R}$ into

$L = L(\|u_0\|_{H^1(\mathbb{R})})$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{L^8_{t,x}(I_j\times\mathbb{R})} \sim \delta. \tag{4.2}$$

As $p > 2$, there exists $\varepsilon > 0$ such that $p > 2 + \frac{1}{8}\varepsilon$. By Lemma 2.1, Hölder, (4.2), and Sobolev embedding, on each $I_j \times \mathbb{R}$ we estimate

$$\begin{aligned}
\|\langle\nabla\rangle u\|_{S^0(I_j)} &\lesssim \|\langle\nabla\rangle u(t_j)\|_2 + \|\langle\nabla\rangle\big(|u|^{2p}u\big)\|_{L_t^{4/3}L_x^1} \\
&\lesssim \|u_0\|_{H^1(\mathbb{R})} + \||u|^{2p}\|_{L_t^2L_x^1}\|\langle\nabla\rangle u\|_{L_t^4L_x^\infty} \\
&\lesssim \|u_0\|_{H^1(\mathbb{R})} + \|u\|^{\varepsilon}_{L^8_{t,x}}\|u\|^{2p-\varepsilon}_{L_t^{\frac{8(2p-\varepsilon)}{4-\varepsilon}}L_x^{\frac{8(2p-\varepsilon)}{8-\varepsilon}}}\|\langle\nabla\rangle u\|_{S^0(I_j)} \\
&\lesssim \|u_0\|_{H^1(\mathbb{R})} + \delta^{\varepsilon}\big\||\nabla|^{\frac{8p-16-\varepsilon}{8(2p-\varepsilon)}}u\big\|^{2p-\varepsilon}_{S^0(I_j)}\|\langle\nabla\rangle u\|_{S^0(I_j)} \\
&\lesssim \|u_0\|_{H^1(\mathbb{R})} + \delta^{\varepsilon}\|\langle\nabla\rangle u\|^{2p+1-\varepsilon}_{S^0(I_j)}.
\end{aligned}$$

A standard continuity argument yields

$$\|\langle\nabla\rangle u\|_{S^0(I_j)} \lesssim \|u_0\|_{H^1(\mathbb{R})},$$

provided δ is chosen sufficiently small depending on $\|u_0\|_{H^1(\mathbb{R})}$. Summing these bounds over all subintervals I_j we derive

$$\|\langle\nabla\rangle u\|_{S^0(I_j)} \leq C(\|u_0\|_{H^1(\mathbb{R})}). \tag{4.3}$$

We now use (4.3) to prove asymptotic completeness, that is, there exist unique $u_\pm$ such that

$$\|u(t) - e^{it\Delta}u_\pm\|_{H^1(\mathbb{R})} \to 0 \quad \text{as } t \to \pm\infty. \tag{4.4}$$

By time reversal symmetry, it suffices to prove the claim for positive times only. For $t > 0$, we define $v(t) := e^{-it\Delta}u(t)$. We will show that $v(t)$ converges in H^1_x as $t \to +\infty$, and define u_+ to be the limit.

Indeed, by Duhamel's formula,

$$v(t) = u_0 - i\int_0^t e^{-is\Delta}\big(|u|^{2p}u\big)(s)\,ds. \tag{4.5}$$

Therefore, for $0 < \tau < t$,

$$v(t) - v(\tau) = -i\int_\tau^t e^{-is\Delta}\big(|u|^{2p}u\big)(s)\,ds.$$

Arguing as above, by Lemma 2.1 and Sobolev embedding,

$$\begin{aligned}
\|v(t) - v(\tau)\|_{H^1(\mathbb{R})} &\lesssim \|\langle\nabla\rangle\big(|u|^{2p}u\big)\|_{L_t^{4/3}L_x^1([t,\tau]\times\mathbb{R})} \\
&\lesssim \|u\|^{\varepsilon}_{L^8_{t,x}([t,\tau]\times\mathbb{R})}\|\langle\nabla\rangle u\|^{2p+1-\varepsilon}_{S^0([t,\tau])}.
\end{aligned}$$

Thus, by (4.1) and (4.3),

$$\|v(t) - v(\tau)\|_{H^1(\mathbb{R})} \to 0 \text{ as } \tau, t \to \infty.$$

In particular, this implies u_+ is well defined and inspecting (4.5) we find

$$u_+ = u_0 - i\int_0^\infty e^{-is\Delta}(|u|^{2p}u)(s)\,ds.$$

Using the same estimates as above, it is now an easy matter to derive (4.4). This completes the proof of Theorem 1.2.

5. Almost conservation law

As mentioned in the introduction, in order to prove global well-posedness for (1.1) it suffices to obtain *a priori* control over the H_x^s norm of solutions to (1.1). However, the H_x^s norm is not a conserved quantity. Nevertheless, it can be controlled by the H_x^1 norm of the modified solution $I_N u$ (see (2.7)). While we do have conservation of energy for (1.1), $I_N u$ is not a solution to (1.1) and hence we expect an energy increment. In this section, we prove that the energy increment is small on intervals where the Morawetz norm is small, thus transfering the problem to controlling the Morawetz norm globally.

Proposition 5.1 (Energy increment)**.** *Let $s > \frac{k-2}{2k-1}$ and let u be an H_x^s solution to* (1.1) *on the spacetime slab $[t_0, T]\times\mathbb{R}$ with $E(I_N u(t_0)) \le 1$. Suppose in addition that*

$$\|u\|_{L^8_{t,x}([t_0,T]\times\mathbb{R})} \le \eta \tag{5.1}$$

for a sufficiently small $\eta > 0$ (depending on k and on $E(I_N u(t_0))$). Then, for N sufficiently large (depending on k and on $E(I_N u(t_0))$),

$$\sup_{t\in[t_0,T]} E(I_N u(t)) = E(I_N u(t_0)) + N^{-1+}. \tag{5.2}$$

Proof. Fix $t \in [t_0, T]$ and define

$$\|u\|_{Z(t)} := \|\nabla P_{\le 1} u\|_{S^0([t_0,t])} + \sup_{(q,r)\ \text{admissible}} \Big(\sum_{N>1} \|\nabla P_N u\|^2_{L_t^q L_x^r([t_0,t]\times\mathbb{R})}\Big)^{1/2}.$$

We observe the inequality

$$\Big\|\Big(\sum_{N\in 2^{\mathbb{Z}}} |f_N|^2\Big)^{1/2}\Big\|_{L_t^q L_x^r} \le \Big(\sum_{N\in 2^{\mathbb{Z}}} \|f_N\|^2_{L_t^q L_x^r}\Big)^{1/2} \tag{5.3}$$

for all $2 \le q, r \le \infty$ and arbitrary functions f_N, which one proves by interpolating between the trivial cases $(2,2)$, $(2,\infty)$, $(\infty,2)$, and (∞,∞). In particular, (5.3) holds for all admissible exponents (q,r). Combining this with the Littlewood-Paley inequality, we find[1]

$$\|u\|_{L_t^q L_x^r} \lesssim \Big\| \Big(\sum_{N\in 2^{\mathbb{Z}}} |P_N u|^2 \Big)^{1/2} \Big\|_{L_t^q L_x^r} \lesssim \Big(\sum_{N\in 2^{\mathbb{Z}}} \|P_N u\|_{L_t^q L_x^r}^2 \Big)^{1/2}.$$

In particular,

$$\|\nabla u\|_{S^0([t_0,t])} \lesssim \|u\|_{Z(t)}.$$

Moreover, using Lemma 2.1, the fact that the Littlewood-Paley operators P_N commute with $i\partial_t + \Delta$, the Littlewood-Paley inequality, together with the dual of (5.3), we get

$$\|u\|_{Z(t)} \lesssim \|u(t_0)\|_{\dot{H}_x^1} + \|\nabla(iu_t + \Delta u)\|_{L_t^{q'} L_x^{r'}([t_0,t]\times\mathbb{R})}, \tag{5.4}$$

for any admissible pair (q, r).

Now define

$$Z_I(t) := \|I_N u\|_{Z(t)}.$$

Lemma 5.1. *Under the hypotheses of Proposition 5.1,*

$$\begin{aligned} Z_I(t) &\lesssim \|\nabla I_N u(t_0)\|_2 + N^{-(k+2)} Z_I(t)^{2k+1} + \eta^{\frac{14k}{3k-1}} Z_I(t)^{1+\frac{2k(3k-8)}{3k-1}} \\ &\quad + \eta^{\frac{16}{3}} \sup_{s\in[t_0,t]} E(I_N u(s))^{\frac{3k-8}{3(k+1)}} Z_I(t). \end{aligned} \tag{5.5}$$

Proof. Throughout this proof, all spacetime norms are on $[t_0, t] \times \mathbb{R}$. By (5.4) and Hölder's inequality, combined with the fact that ∇I_N acts as a derivative (as the multiplier of ∇I_N is increasing in $|\xi|$), we estimate

$$\begin{aligned} Z_I(t) &\lesssim \|\nabla I_N u(t_0)\|_2 + \|\nabla I_N(|u|^{2k}u)\|_{\frac{6}{5},\frac{6}{5}} \\ &\lesssim \|\nabla I_N u(t_0)\|_2 + \|u\|_{3k,3k}^{2k} \|\nabla I_N u\|_{6,6} \\ &\lesssim \|\nabla I_N u(t_0)\|_2 + \|u\|_{3k,3k}^{2k} Z_I(t). \end{aligned}$$

To estimate $\|u\|_{3k,3k}$, we decompose $u := u_{\leq 1} + u_{1<\cdot\leq N} + u_{>N}$. To estimate the low frequencies, we use interpolation, (5.1), Bernstein, and the fact that the operator I_N is the identity on frequencies $|\xi| \leq 1$ to get

$$\begin{aligned} \|u_{\leq 1}\|_{3k,3k} &\lesssim \|u_{\leq 1}\|_{8,8}^{\frac{8}{3k}} \|u_{\leq 1}\|_{\infty,\infty}^{1-\frac{8}{3k}} \\ &\lesssim \eta^{\frac{8}{3k}} \|u_{\leq 1}\|_{\infty,2k+2}^{1-\frac{8}{3k}} \\ &\lesssim \eta^{\frac{8}{3k}} \sup_{s\in[t_0,t]} E(I_N u(s))^{\frac{3k-8}{3k(2k+2)}}. \end{aligned}$$

[1]Strictly speaking, as the Littlewood-Paley square function is not bounded on L_x^∞, the inequality does not hold for the Schrödinger-admissible pair $(4, \infty)$. However, this particular estimate will not be needed in the proof of Proposition 5.1 and we thus make the convention that in the proof of this proposition alone the S^0 norm is the supremum over all admissible pairs except $(4, \infty)$.

To estimate the medium frequencies, we use interpolation, (5.1), Sobolev embedding, Bernstein, and the fact that the operator I_N is the identity on frequencies $|\xi| \leq N$

$$\begin{aligned} \|u_{1<\cdot\leq N}\|_{3k,3k} &\lesssim \|u_{1<\cdot\leq N}\|_{8,8}^{\frac{7}{3k-1}} \|u_{1<\cdot\leq N}\|_{24k,24k}^{\frac{3k-8}{3k-1}} \\ &\lesssim \eta^{\frac{7}{3k-1}} \||\nabla|^{\frac{1}{2}-\frac{1}{8k}} u_{1<\cdot\leq N}\|_{24k,\frac{12k}{6k-1}}^{\frac{3k-8}{3k-1}} \\ &\lesssim \eta^{\frac{7}{3k-1}} Z_I(t)^{\frac{3k-8}{3k-1}}. \end{aligned}$$

To estimate the high frequencies, we use Sobolev embedding and Lemma 2.3

$$\begin{aligned} \|u_{>N}\|_{3k,3k} &\lesssim \||\nabla|^{\frac{1}{2}-\frac{1}{k}} u_{>N}\|_{3k,\frac{6k}{3k-4}} \\ &\lesssim N^{-\frac{1}{2}-\frac{1}{k}} \|\nabla I_N u_{>N}\|_{3k,\frac{6k}{3k-4}} \\ &\lesssim N^{-\frac{1}{2}-\frac{1}{k}} Z_I(t). \end{aligned}$$

Putting everything together, we derive (5.5). □

Next, we control the energy increment in terms of the size of the modified solution $I_N u$.

Lemma 5.2. *Under the hypotheses of Proposition 5.1,*

$$\begin{aligned} (5.6) \qquad & \Big|\sup_{s\in[t_0,t]} E(I_N u(s)) - E(I_N u(t_0))\Big| \\ &\lesssim N^{-1+}\Big(Z_I(t)^{2k+2} + \eta^{\frac{16}{3}} Z_I(t)^2 \sup_{s\in[t_0,t]} E(I_N u(s))^{\frac{3k-8}{3(k+1)}} \\ &\qquad + \sum_{J=3}^{2k+2} \eta^{\frac{4(2k+2-J)}{2k-1}} Z_I(t)^J \sup_{s\in[t_0,t]} E(I_N u(s))^{\frac{(2k-5)(2k+2-J)}{(2k-1)(2k+2)}}\Big) \\ &\quad + N^{-1+}\Big(Z_I(t)^{2k+1} + \eta^{\frac{16}{3}} Z_I(t) \sup_{s\in[t_0,t]} E(I_N u(s))^{\frac{3k-8}{3(k+1)}}\Big) \\ &\qquad \times \Big(Z_I(t)^{2k+1} + \eta^{\frac{4}{3}} \sup_{s\in[t_0,t]} E(I_N u(s))^{\frac{6k-1}{3(2k+2)}}\Big) \\ &\quad + N^{-1+} \sum_{J=3}^{2k+2} \eta^{\frac{4(2k+2-J)}{2k-1}} Z_I(t)^{J-1} \sup_{s\in[t_0,t]} E(I_N u(s))^{\frac{(2k-5)(2k+2-J)}{(2k-1)(2k+2)}} \\ &\qquad \times \Big(Z_I(t)^{2k+1} + \eta^{\frac{4}{3}} \sup_{s\in[t_0,t]} E(I_N u(s))^{\frac{6k-1}{3(2k+2)}}\Big). \end{aligned}$$

Proof. As

$$\frac{d}{dt}E(u(t)) = \operatorname{Re}\int \bar{u}_t(|u|^{2k}u - \Delta u)\, dx = \operatorname{Re}\int \bar{u}_t(|u|^{2k}u - \Delta u - iu_t)\, dx,$$

we obtain

$$\begin{aligned}\frac{d}{dt}E(Iu(t)) &= \mathrm{Re}\int I\bar{u}_t(|Iu|^{2k}Iu - \Delta Iu - iIu_t)\,dx\\ &= \mathrm{Re}\int I\bar{u}_t(|Iu|^{2k}Iu - I(|u|^{2k}u))\,dx.\end{aligned}$$

Using the Fundamental Theorem of Calculus and Plancherel, we write[2]

$$\begin{aligned}&E(Iu(t)) - E(Iu(t_0))\\ &\quad= \mathrm{Re}\int_{t_0}^{t}\int_{\sum_{i=1}^{2k+2}\xi_i=0}\Big(1-\frac{m(\xi_2+\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\Big)\\ &\qquad\qquad \widehat{\overline{I\partial_t u}}(\xi_1)\widehat{Iu}(\xi_2)\cdots\widehat{\overline{Iu}}(\xi_{2k+1})\widehat{Iu}(\xi_{2k+2})\,d\sigma(\xi)\,ds.\end{aligned}$$

As $iu_t = -\Delta u + |u|^{2k}u$, we thus need to control

$$\begin{aligned}\Big|\int_{t_0}^{t}\int_{\sum_{i=1}^{2k+2}\xi_i=0}&\Big(1-\frac{m(\xi_2+\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\Big)\\ &\Delta\widehat{\overline{Iu}}(\xi_1)\widehat{Iu}(\xi_2)\cdots\widehat{\overline{Iu}}(\xi_{2k+1})\widehat{Iu}(\xi_{2k+2})\,d\sigma(\xi)\,ds\Big|\end{aligned}\tag{5.7}$$

and

$$\begin{aligned}\Big|\int_{t_0}^{t}\int_{\sum_{i=1}^{2k+2}\xi_i=0}&\Big(1-\frac{m(\xi_2+\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\Big)\\ &\widehat{\overline{I(|u|^{2k}u)}}(\xi_1)\widehat{Iu}(\xi_2)\cdots\widehat{\overline{Iu}}(\xi_{2k+1})\widehat{Iu}(\xi_{2k+2})\,d\sigma(\xi)\,ds\Big|.\end{aligned}\tag{5.8}$$

We first estimate (5.7). To this end, we decompose

$$u := \sum_{N\geq 1} P_N u$$

with the convention that $P_1 u := P_{\leq 1}u$. Using this notation and symmetry, we estimate

$$(5.7) \lesssim \sum_{\substack{N_1,\dots,N_{2k+2}\geq 1\\ N_2\geq N_3\geq\cdots\geq N_{2k+2}}} B(N_1,\dots,N_{2k+2}),\tag{5.9}$$

where

$$\begin{aligned}&B(N_1,\dots,N_{2k+2})\\ &\quad:= \Big|\int_{t_0}^{t}\int_{\sum_{i=1}^{2k+2}\xi_i=0}\Big(1-\frac{m(\xi_2+\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\Big)\\ &\qquad\quad \Delta\widehat{\overline{Iu_{N_1}}}(\xi_1)\widehat{Iu_{N_2}}(\xi_2)\cdots\widehat{\overline{Iu_{N_{2k+1}}}}(\xi_{2k+2})\widehat{Iu_{N_{2k+2}}}(\xi_{2k+2})d\sigma(\xi)ds\Big|.\end{aligned}$$

Case I**:** $N_1 > 1$, $N_2 \geq \cdots \geq N_{2k+2} > 1$.
Case I_a**:** $N \gg N_2$.

[2]Throughout this proof we use the abbreviation $m := m_N$.

In this case,

$$m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2}) = m(\xi_2) = \cdots = m(\xi_{2k+2}) = 1.$$

Thus,

$$B(N_1, \ldots, N_{2k+2}) = 0$$

and the contribution to the right-hand side of (5.9) is zero.

Case I_b: $N_2 \gtrsim N \gg N_3$.

As $\sum_{i=1}^{2k+2} \xi_i = 0$, we must have $N_1 \sim N_2$. Thus, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\Big|1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\Big| &= \Big|1 - \frac{m(\xi_2 + \cdots + \xi_{2k+2})}{m(\xi_2)}\Big| \\ &\lesssim \Big|\frac{\nabla m(\xi_2)(\xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2)}\Big| \lesssim \frac{N_3}{N_2}.\end{aligned}$$

Applying the multilinear multiplier theorem (cf. [7, 8]), Sobolev embedding, Bernstein, and recalling that $N_j > 1$, we estimate

$$\begin{aligned}&B(N_1, \ldots, N_{2k+2}) \\ &\quad\lesssim \frac{N_3}{N_2} \|\Delta I u_{N_1}\|_{6,6} \|I u_{N_2}\|_{6,6} \|I u_{N_3}\|_{6,6} \prod_{j=4}^{2k+2} \|I u_{N_j}\|_{2(2k-1),2(2k-1)} \\ &\quad\lesssim \frac{N_1}{N_2^2} \prod_{j=1}^{3} \|\nabla I u_{N_j}\|_{6,6} \prod_{j=4}^{2k+2} \||\nabla|^{\frac{k-2}{2k-1}} I u_{N_j}\|_{2(2k-1),\frac{2(2k-1)}{2k-3}} \\ &\quad\lesssim \frac{1}{N_2} Z_I(t)^{2k+2} \lesssim N^{-1+} N_2^{0-} Z_I(t)^{2k+2}.\end{aligned}$$

The factor N_2^{0-} allows us to sum in $N_1, N_2, \ldots, N_{2k+2}$, this case contributing at most $N^{-1+} Z_I(t)^{2k+2}$ to the right-hand side of (5.9).

Case I_c: $N_2 \gg N_3 \gtrsim N$.

As $\sum_{i=1}^{2k+2} \xi_i = 0$, we must have $N_1 \sim N_2$. Thus, as m is decreasing,

$$\Big|1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\Big| \lesssim \frac{m(\xi_1)}{m(\xi_2)\cdots m(\xi_{2k+2})}.$$

Using again the multilinear multiplier theorem, Sobolev embedding, Bernstein, and the fact that $m(\xi)|\xi|^{\frac{k+1}{2k-1}}$ is increasing for $s > \frac{k-2}{2k-1}$, we estimate

$$\begin{aligned}
&B(N_1,\dots,N_{2k+2})\\
&\lesssim \frac{m(N_1)}{m(N_2)\cdots m(N_{2k+2})}\frac{N_1}{N_2N_3}\prod_{j=1}^{3}\|\nabla Iu_{N_j}\|_{6,6}\prod_{j=4}^{2k+2}\||\nabla|^{\frac{k-2}{2k-1}}Iu_{N_j}\|_{2(2k-1),\frac{2(2k-1)}{2k-3}}\\
&\lesssim \frac{1}{N_3m(N_3)\prod_{j=4}^{2k+2}m(N_j)N_j^{\frac{k+1}{2k-1}}}\prod_{j=1}^{3}\|\nabla Iu_{N_j}\|_{6,6}\prod_{j=4}^{2k+2}\|\nabla Iu_{N_j}\|_{2(2k-1),\frac{2(2k-1)}{2k-3}}\\
&\lesssim \frac{1}{N_3m(N_3)}\|\nabla Iu_{N_1}\|_{6,6}\|\nabla Iu_{N_2}\|_{6,6}Z_I(t)^{2k}\\
&\lesssim N^{-1+}N_3^{0-}\|\nabla Iu_{N_1}\|_{6,6}\|\nabla Iu_{N_2}\|_{6,6}Z_I(t)^{2k}.
\end{aligned}$$

The factor N_3^{0-} allows us to sum over $N_3,\dots,N_{2k+2}$. To sum over N_1 and N_2, we use the fact that $N_1\sim N_2$ and Cauchy-Schwarz to estimate the contribution to the right-hand side of (5.9) by

$$N^{-1+}\Big(\sum_{N_1>1}\|\nabla Iu_{N_1}\|_{6,6}^2\Big)^{\frac12}\Big(\sum_{N_2>1}\|\nabla Iu_{N_2}\|_{6,6}^2\Big)^{\frac12}Z_I(t)^{2k}\lesssim N^{-1+}Z_I(t)^{2k+2}.$$

Case I_d: $N_2\sim N_3\gtrsim N$.

As $\sum_{i=1}^{2k+2}\xi_i=0$, we obtain $N_1\lesssim N_2$, and hence $m(N_1)\gtrsim m(N_2)$ and $m(N_1)N_1\lesssim m(N_2)N_2$. Thus,

$$\Big|1-\frac{m(\xi_2+\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\Big|\lesssim\frac{m(N_1)}{m(N_2)m(N_3)\cdots m(N_{2k+2})}.$$

Arguing as for Case I_c, we estimate

$$\begin{aligned}
B(N_1,\dots,N_{2k+2})&\lesssim\frac{m(N_1)N_1}{m(N_2)N_2m(N_3)N_3\prod_{j=4}^{2k+2}m(N_j)N_j^{\frac{k+1}{2k-1}}}Z_I(t)^{2k+2}\\
&\lesssim\frac{1}{m(N_3)N_3}Z_I(t)^{2k+2}\\
&\lesssim N^{-1+}N_3^{0-}Z_I(t)^{2k+2}.
\end{aligned}$$

The factor N_3^{0-} allows us to sum over $N_1,\dots,N_{2k+2}$. This case contributes at most $N^{-1+}Z_I(t)^{2k+2}$ to the right-hand side of (5.9).

Case II: There exists $1\le j_0\le 2k+2$ such that $N_{j_0}=1$. Recall that by our convention, $P_1:=P_{\le 1}$.

Case II_a: $N_1=1$.

Let J be such that $N_2\ge\cdots\ge N_J>1=N_{J+1}=\cdots=N_{2k+2}$. Note that we may assume $J\ge 3$ since otherwise

$$B(N_1,\dots,N_{2k+2})=0.$$

Also, arguing as for Case I_a, if $N \gg N_2$ then

$$B(N_1, \ldots, N_{2k+2}) = 0.$$

Thus, we may assume $N_2 \gtrsim N$. In this case we cannot have $N_2 \gg N_3$ since it would contradict $\sum_{i=1}^{2k+2} \xi_i = 0$ and $N_1 = 1$. Hence, we must have

$$N_2 \sim N_3 \gtrsim N.$$

As

$$\Big|1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\Big| \lesssim \frac{1}{m(N_2)m(N_3)\cdots m(N_{2k+2})},$$

we use the multilinear multiplier theorem and Sobolev embedding to estimate

$$\begin{aligned}
&B(N_1, \ldots, N_{2k+2}) \\
&\lesssim \frac{N_1}{m(N_2)N_2 m(N_3)N_3 m(N_4)\cdots m(N_{2k+2})} \prod_{j=1}^{3} \|\nabla I u_{N_j}\|_{6,6} \\
&\qquad \times \prod_{j=4}^{J} \||\nabla|^{\frac{k-2}{2k-1}} I u_{N_j}\|_{2(2k-1), \frac{2(2k-1)}{2k-3}} \prod_{j=J+1}^{2k+2} \|I u_{N_j}\|_{2(2k-1),2(2k-1)} \\
&\lesssim \frac{1}{m(N_2)N_2 m(N_3)N_3 \prod_{j=4}^{J} m(N_j) N_j^{\frac{k+1}{2k-1}}} Z_I(t)^J \prod_{j=J+1}^{2k+2} \|I u_{N_j}\|_{2(2k-1),2(2k-1)} \\
&\lesssim N^{-2+} N_2^{0-} Z_I(t)^J \prod_{j=J+1}^{2k+2} \|I u_{N_j}\|_{2(2k-1),2(2k-1)}.
\end{aligned}$$

Applying interpolation, (5.1), and Bernstein, we bound

$$\begin{aligned}
\|I u_{\leq 1}\|_{2(2k-1),2(2k-1)} &\lesssim \|I u_{\leq 1}\|_{8,8}^{\frac{4}{2k-1}} \|I u_{\leq 1}\|_{\infty,\infty}^{\frac{2k-5}{2k-1}} \\
&\lesssim \eta^{\frac{4}{2k-1}} \sup_{s \in [t_0,t]} E(Iu(s))^{\frac{2k-5}{(2k-1)(2k+2)}}.
\end{aligned} \tag{5.10}$$

Thus,

$$B(N_1, \ldots, N_{2k+2}) \lesssim N^{-2+} N_2^{0-} \eta^{\frac{4(2k+2-J)}{2k-1}} Z_I(t)^J \sup_{s \in [t_0,t]} E(Iu(s))^{\frac{(2k-5)(2k+2-J)}{(2k-1)(2k+2)}}.$$

The factor N_2^{0-} allows us to sum in $N_2, \ldots, N_J$. This case contributes at most

$$N^{-2+} \sum_{J=3}^{2k+2} \eta^{\frac{4(2k+2-J)}{2k-1}} Z_I(t)^J \sup_{s \in [t_0,t]} E(Iu(s))^{\frac{(2k-5)(2k+2-J)}{(2k-1)(2k+2)}}$$

to the right-hand side of (5.9).

Case II_b: $N_1 > 1$ and $N_2 = \cdots = N_{2k+2} = 1$.

As $\sum_{i=1}^{2k+2}\xi_i = 0$, we obtain $N_1 \lesssim 1$ and thus, taking N sufficiently large depending on k, we get

$$1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})} = 0.$$

This case contributes zero to the right-hand side of (5.9).

Case II_c: $N_1 > 1$ and $N_2 > 1 = N_3 = \cdots = N_{2k+2}$.

As $\sum_{i=1}^{2k+2}\xi_i = 0$, we must have $N_1 \sim N_2$. If $N_1 \sim N_2 \ll N$, then

$$1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})} = 0$$

and the contribution is zero. Thus, we may assume $N_1 \sim N_2 \gtrsim N$.

Applying the Fundamental Theorem of Calculus,

$$\begin{aligned}\Big|1 - \frac{m(\xi_2 + \xi_3 + \cdots + \xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\Big| &= \Big|1 - \frac{m(\xi_2 + \cdots + \xi_{2k+2})}{m(\xi_2)}\Big| \\ &\lesssim \Big|\frac{\nabla m(\xi_2)}{m(\xi_2)}\Big| \lesssim \frac{1}{N_2}.\end{aligned}$$

By the multilinear multiplier theorem,

$$\begin{aligned}B(N_1, \ldots, N_{2k+2}) &\lesssim \frac{1}{N_2}\|\Delta I u_{N_1}\|_{6,6}\|Iu_{N_2}\|_{6,6}\prod_{j=3}^{2k+2}\|Iu_{N_j}\|_{3k,3k} \\ &\lesssim \frac{N_1}{N_2^2}\|\nabla Iu_{N_1}\|_{6,6}\|\nabla Iu_{N_2}\|_{6,6}\|Iu_{\leq 1}\|_{3k,3k}^{2k} \\ &\lesssim N^{-1+}N_2^{0-}Z_I(t)^2\|Iu_{\leq 1}\|_{3k,3k}^{2k}.\end{aligned}$$

The factor N_2^{0-} allows us to sum in N_1 and N_2. Using interpolation, (2.5), (5.1), and Bernstein, we estimate

$$\begin{aligned}\|Iu_{\leq 1}\|_{3k,3k} &\lesssim \|Iu_{\leq 1}\|_{8,8}^{\frac{8}{3k}}\|Iu_{\leq 1}\|_{\infty,\infty}^{1-\frac{8}{3k}} \\ &\lesssim \eta^{\frac{8}{3k}}\|Iu_{\leq 1}\|_{\infty,2k+2}^{1-\frac{8}{3k}} \\ &\lesssim \eta^{\frac{8}{3k}}\sup_{s\in[t_0,t]}E(Iu(s))^{\frac{3k-8}{3k(2k+2)}}.\end{aligned}$$

Thus, this case contributes at most

$$N^{-1+}\eta^{\frac{16}{3}}Z_I(t)^2\sup_{s\in[t_0,t]}E(Iu(s))^{\frac{3k-8}{3(k+1)}}$$

to the right-hand side of (5.9).

Case II_d: $N_1 > 1$ and there exists $J \geq 3$ such that $N_2 \geq \cdots \geq N_J > 1 = N_{J+1} = \cdots = N_{2k+2}$.

To estimate the contribution of this case, we argue as for Case I; the only new ingredient is that the low frequencies are estimated via (5.10). This case contributes at most

$$N^{-1+}\sum_{J=3}^{2k+2}\eta^{\frac{4(2k+2-J)}{2k-1}}Z_I(t)^J\sup_{s\in[t_0,t]}E(Iu(s))^{\frac{(2k-5)(2k+2-J)}{(2k-1)(2k+2)}}$$

to the right-hand side of (5.9).

Putting everything together, we get

$$\begin{aligned}(5.7)&\lesssim N^{-1+}Z_I(t)^{2k+2}+N^{-1+}\eta^{\frac{16}{3}}Z_I(t)^2\sup_{s\in[t_0,t]}E(Iu(s))^{\frac{3k-8}{3(k+1)}}\\&\quad+N^{-1+}\sum_{J=3}^{2k+2}\eta^{\frac{4(2k+2-J)}{2k-1}}Z_I(t)^J\sup_{s\in[t_0,t]}E(Iu(s))^{\frac{(2k-5)(2k+2-J)}{(2k-1)(2k+2)}}.\end{aligned}\tag{5.11}$$

We turn now to estimating (5.8). Again we decompose

$$u:=\sum_{N\geq 1}P_Nu$$

with the convention that $P_1u:=P_{\leq 1}u$. Using this notation and symmetry, we estimate

$$(5.8)\lesssim\sum_{\substack{N_1,\ldots,N_{2k+2}\geq 1\\ N_2\geq\cdots\geq N_{2k+2}}}C(N_1,\cdots,N_{2k+2}),$$

where

$$\begin{aligned}&C(N_1,\cdots,N_{2k+2})\\&\quad:=\Big|\int_{t_0}^t\int_{\sum_{i=1}^{2k+2}\xi_i=0}\Big(1-\frac{m(\xi_2+\xi_3+\cdots+\xi_{2k+2})}{m(\xi_2)m(\xi_3)\cdots m(\xi_{2k+2})}\Big)\\&\qquad\overline{\widehat{P_{N_1}I(|u|^{2k}u)}}(\xi_1)\widehat{Iu_{N_2}}(\xi_2)\cdots\overline{\widehat{Iu_{N_{2k+1}}}}(\xi_{2k+1})\widehat{Iu_{N_{2k+2}}}(\xi_{2k+2})\,d\sigma(\xi)\,ds\Big|.\end{aligned}$$

In order to estimate $C(N_1,\cdots,N_{2k+2})$ we make the observation that in estimating $B(N_1,\cdots,N_{2k+2})$, for the term involving the N_1 frequency we only used the bound

$$\|P_{N_1}I\Delta u\|_{6,6}\lesssim N_1\|\nabla Iu_{N_1}\|_{6,6}\lesssim N_1Z_I(t).\tag{5.12}$$

Thus, to estimate (5.8) it suffices to prove

$$\|P_{N_1}I(|u|^{2k}u)\|_{6,6}\lesssim Z_I(t)^{2k+1}+\eta^{\frac{4}{3}}\sup_{s\in[t_0,t]}E(Iu(s))^{\frac{6k-1}{3(2k+2)}},\tag{5.13}$$

for then, arguing as for (5.7) and substituting (5.13) for (5.12), we obtain

$$
\begin{aligned}
(5.8) \lesssim N^{-1+}\Big(&Z_I(t)^{2k+1} + \eta^{\frac{16}{3}} Z_I(t) \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{3k-8}{3(k+1)}}\Big) \\
&\times \Big(Z_I(t)^{2k+1} + \eta^{\frac{4}{3}} \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{6k-1}{3(2k+2)}}\Big) \\
&+ N^{-1+} \sum_{J=3}^{2k+2} \eta^{\frac{4(2k+2-J)}{2k-1}} Z_I(t)^{J-1} \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{(2k-5)(2k+2-J)}{(2k-1)(2k+2)}} \\
&\times \Big(Z_I(t)^{2k+1} + \eta^{\frac{4}{3}} \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{6k-1}{3(2k+2)}}\Big).
\end{aligned}
$$

Thus, we are left to proving (5.13). Using (2.5) and the boundedness of the Littlewood-Paley operators, and decomposing $u := u_{\leq 1} + u_{>1}$, we estimate

$$
\begin{aligned}
\|P_{N_1} I(|u|^{2k}u)\|_{6,6} &\lesssim \|u\|^{2k+1}_{6(2k+1),6(2k+1)} \\
&\lesssim \|u_{\leq 1}\|^{2k+1}_{6(2k+1),6(2k+1)} + \|u_{>1}\|^{2k+1}_{6(2k+1),6(2k+1)}.
\end{aligned}
$$

Applying interpolation, (5.1), and Bernstein, we estimate

$$
\begin{aligned}
\|u_{\leq 1}\|_{6(2k+1),6(2k+1)} &\lesssim \|u_{\leq 1}\|_{8,8}^{\frac{4}{3(2k+1)}} \|u_{\leq 1}\|_{\infty,\infty}^{\frac{6k-1}{3(2k+1)}} \\
&\lesssim \eta^{\frac{4}{3(2k+1)}} \sup_{s\in[t_0,t]} E(Iu(s))^{\frac{6k-1}{3(2k+2)(2k+1)}}.
\end{aligned}
$$

Finally, by Sobolev embedding and (2.6),

$$
\|u_{>1}\|_{6(2k+1),6(2k+1)} \lesssim \||\nabla|^{\frac{1}{2}-\frac{1}{2(2k+1)}} u_{>1}\|_{6(2k+1),\frac{6(2k+1)}{6k+1}} \lesssim Z_I(t).
$$

Putting things together, we derive (5.13).

This completes the proof of Lemma 5.2. □

Next, we combine Lemmas 5.1 and 5.2 to derive Proposition 5.1. Indeed, Proposition 5.1 follows immediately from Lemmas 5.1 and 5.2, if we establish

$$
Z_I(t) \lesssim 1 \quad \text{and} \quad \sup_{s\in[t_0,t]} E(I_N u(s)) \lesssim 1 \quad \text{for all } t \in [t_0, T].
$$

As by assumption $E(I_N u(t_0)) \leq 1$, it suffices to show that

$$
Z_I(t) \lesssim \|\nabla I_N u(t_0)\|_2 \quad \text{for all } t \in [t_0, T] \tag{5.14}
$$

and

$$
\sup_{s\in[t_0,t]} E(I_N u(s)) \lesssim E(I_N u(t_0)) \quad \text{for all } t \in [t_0, T]. \tag{5.15}
$$

We achieve this via a bootstrap argument. Let

$$\Omega_1 := \{t \in [t_0, T] : Z_I(t) \le C_1 \|\nabla I_N u(t_0)\|_2, \quad \sup_{s\in[t_0,t]} E(I_N u(s)) \le C_2 E(I_N u(t_0))\}$$
$$\Omega_2 := \{t \in [t_0, T] : Z_I(t) \le 2C_1 \|\nabla I_N u(t_0)\|_2, \quad \sup_{s\in[t_0,t]} E(I_N u(s)) \le 2C_2 E(I_N u(t_0))\}.$$

In order to run the bootstrap argument successfully, we need to check four things:

- $\Omega_1 \neq \emptyset$. This is satisfied as $t_0 \in \Omega_1$ if we take C_1 and C_2 sufficiently large.
- Ω_1 is a closed set. This follows from Fatou's Lemma.
- If $t \in \Omega_1$, then there exists $\varepsilon > 0$ such that $[t, t+\varepsilon] \in \Omega_2$. This follows from the Dominated Convergence Theorem combined with (5.5) and (5.6).
- $\Omega_2 \subset \Omega_1$. This follows from (5.5) and (5.6) taking C_1 and C_2 sufficiently large depending on absolute constants (like the Strichartz constant) and choosing N sufficiently large and η sufficiently small depending on C_1, C_2, k, and $E(I_N u(t_0))$.

This finally proves Proposition 5.1.

6. Proof of Theorem 1.1

Given Proposition 5.1, the proof of global well-posedness for (1.1) is reduced to showing

$$\|u\|_{L^8_{t,x}(\mathbb{R}\times\mathbb{R})} \le C(\|u_0\|_{H^s_x}). \tag{6.1}$$

This also implies scattering, as we will see later.

By Proposition 3.1,

$$\|u\|_{L^8_{t,x}(I\times\mathbb{R})} \lesssim \|u_0\|_2^{3/4} \|u\|^{1/4}_{L^\infty_t \dot{H}^{1/2}_x(I\times\mathbb{R})} \tag{6.2}$$

on any spacetime slab $I \times \mathbb{R}$ on which the solution to (1.1) exists and lies in $H^{1/2}_x$. However, the $H^{1/2}_x$ norm of the solution is not a conserved quantity either, and in order to control it we must resort to the H^s_x bound on the solution. Thus, in order to obtain a global Morawetz estimate, we need a global H^s_x bound. This sets us up for a bootstrap argument.

Let u be the solution to (1.1). As $E(I_N u_0)$ is not necessarily small, we first rescale the solution such that the energy of the rescaled initial data satisfies the conditions in Proposition 5.1. By scaling,

$$u^\lambda(x,t) := \lambda^{-\frac{1}{k}} u(\lambda^{-2}t, \lambda^{-1}x)$$

is also a solution to (1.1) with initial data

$$u_0^\lambda(x) := \lambda^{-\frac{1}{k}} u_0(\lambda^{-1}x).$$

By (2.8) and Sobolev embedding,

$$\|\nabla I_N u_0^\lambda\|_2 \lesssim N^{1-s}\|u_0^\lambda\|_{\dot H^s_x} = N^{1-s}\lambda^{\frac12-\frac1k-s}\|u_0\|_{\dot H^s_x},$$
$$\|I_N u_0^\lambda\|_{2k+2} \lesssim \|u_0^\lambda\|_{2k+2} = \lambda^{\frac{1}{2k+2}-\frac1k}\|u_0\|_{2k+2} \lesssim \lambda^{\frac{1}{2k+2}-\frac1k}\|u_0\|_{H^s_x}.$$

As $s > \frac12 - \frac1k$, choosing λ sufficiently large (depending on $\|u_0\|_{H^s_x}$ and N) such that

$$N^{1-s}\lambda^{\frac12-\frac1k-s}\|u_0\|_{H^s_x} \ll 1 \quad \text{and} \quad \lambda^{\frac{1}{2k+2}-\frac1k}\|u_0\|_{H^s_x} \ll 1, \tag{6.3}$$

we get

$$E(I_N u_0^\lambda) \ll 1.$$

We now show that there exists an absolute constant C_1 such that

$$\|u^\lambda\|_{L^8_{t,x}(\mathbb{R}\times\mathbb{R})} \le C_1\lambda^{\frac78(\frac12-\frac1k)}. \tag{6.4}$$

Undoing the scaling, this yields (6.1).

We prove (6.4) via a bootstrap argument. By time reversal symmetry, it suffices to argue for positive times only. Define

$$\Omega_1 := \{t \in [0,\infty) : \|u^\lambda\|_{L^8_{t,x}([0,t]\times\mathbb{R})} \le C_1\lambda^{\frac78(\frac12-\frac1k)}\},$$
$$\Omega_2 := \{t \in [0,\infty) : \|u^\lambda\|_{L^8_{t,x}([0,t]\times\mathbb{R})} \le 2C_1\lambda^{\frac78(\frac12-\frac1k)}\}.$$

In order to run the bootstrap argument, we need to verify four things:

1) $\Omega_1 \neq \emptyset$. This is obvious as $0 \in \Omega_1$.

2) Ω_1 is closed. This follows from Fatou's Lemma.

3) $\Omega_2 \subset \Omega_1$.

4) If $T \in \Omega_1$, then there exists $\varepsilon > 0$ such that $[T, T+\varepsilon) \subset \Omega_2$. This is a consequence of the local well-posedness theory and the proof of 3). We skip the details.

Thus, we need to prove 3). Fix $T \in \Omega_2$; we will show that in fact, $T \in \Omega_1$. By (6.2) and the conservation of mass,

$$\begin{aligned}\|u^\lambda\|_{L^8_{t,x}([0,T]\times\mathbb{R})} &\lesssim \|u_0^\lambda\|_2^{\frac34}\|u^\lambda\|^{\frac14}_{L^\infty_t\dot H^{1/2}_x([0,T]\times\mathbb{R})}\\ &\lesssim \lambda^{\frac34(\frac12-\frac1k)}C(\|u_0\|_2)\|u^\lambda\|^{\frac14}_{L^\infty_t\dot H^{1/2}_x([0,T]\times\mathbb{R})}.\end{aligned}$$

To control the factor $\|u^\lambda\|_{L^\infty_t\dot H^{1/2}_x([0,T]\times\mathbb{R})}$, we decompose

$$u^\lambda(t) := P_{\le N}u^\lambda(t) + P_{>N}u^\lambda(t).$$

To estimate the low frequencies, we interpolate between the L^2_x norm and the $\dot H^1_x$ norm and use the fact that I_N is the identity on frequencies $|\xi| \le N$

$$\begin{aligned}\|P_{\le N}u^\lambda(t)\|_{\dot H^{1/2}_x} &\lesssim \|P_{\le N}u^\lambda(t)\|_2^{\frac12}\|P_{\le N}u^\lambda(t)\|^{\frac12}_{\dot H^1_x}\\ &\lesssim \lambda^{\frac12(\frac12-\frac1k)}C(\|u_0\|_2)\|I_N u^\lambda(t)\|^{\frac12}_{\dot H^1_x}.\end{aligned}$$

To control the high frequencies, we interpolate between the L_x^2 norm and the $\dot H_x^s$ norm and use Lemma 2.3

$$\begin{aligned}\|P_{>N}u^\lambda(t)\|_{\dot H_x^{1/2}} &\lesssim \|P_{>N}u^\lambda(t)\|_{L_x^2}^{1-\frac{1}{2s}}\|P_{>N}u^\lambda(t)\|_{\dot H_x^s}^{\frac{1}{2s}}\\ &\lesssim \lambda^{(1-\frac{1}{2s})(\frac12-\frac1k)}N^{\frac{s-1}{2s}}\|I_Nu^\lambda(t)\|_{\dot H_x^1}^{\frac{1}{2s}}\\ &\lesssim \lambda^{\frac12(\frac12-\frac1k)}\|I_Nu^\lambda(t)\|_{\dot H_x^1}^{\frac{1}{2s}}.\end{aligned}$$

Collecting all these estimates, we get

$$\|u^\lambda\|_{L^8_{t,x}([0,T]\times\mathbb{R})} \lesssim \lambda^{\frac78(\frac12-\frac1k)}C(\|u_0\|_2)\sup_{t\in[0,T]}\big(\|\nabla I_Nu^\lambda(t)\|_2^{\frac18}+\|\nabla I_Nu^\lambda(t)\|_2^{\frac{1}{8s}}\big).$$

Thus, taking C_1 sufficiently large depending on $\|u_0\|_2$, we obtain $T\in\Omega_1$, provided

$$\sup_{t\in[0,T]}\|\nabla I_Nu^\lambda(t)\|_2\le 1. \tag{6.5}$$

We now prove that $T\in\Omega_2$ implies (6.5). Indeed, let $\eta>0$ be a sufficiently small constant like in Proposition 5.1 and divide $[0,T]$ into

$$L\sim\left(\frac{\lambda^{\frac78(\frac12-\frac1k)}}{\eta}\right)^8$$

subintervals $I_j=[t_j,t_{j+1}]$ such that,

$$\|u^\lambda\|_{L^8_{t,x}(I_j\times\mathbb{R})}\le\eta.$$

Applying Proposition 5.1 on each of the subintervals I_j, we get

$$\sup_{t\in[0,T]}E(I_Nu^\lambda(t))\le E(I_Nu_0^\lambda)+E(I_Nu_0^\lambda)LN^{-1+}.$$

To maintain small energy during the iteration, we need

$$LN^{-1+}\sim\lambda^{7(\frac12-\frac1k)}N^{-1+}\ll 1,$$

which combined with (6.3) leads to

$$\left(N^{\frac{1-s}{s+\frac1k-\frac12}}\right)^{7(\frac12-\frac1k)}N^{-1+}\le c(\|u_0\|_{H_x^s})\ll 1.$$

This may be ensured by taking N large enough (depending only on k and $\|u_0\|_{H^s(\mathbb{R})}$), provided that

$$s>s(k):=\frac{8k-16}{9k-14}.$$

As can be easily seen, $s(k)\to\frac89$ as $k\to\infty$.

This completes the bootstrap argument and hence (6.4), and moreover (6.1), follow. Therefore (6.5) holds for all $T \in \mathbb{R}$ and the conservation of mass and Lemma 2.3 imply

$$\begin{aligned}
\|u(T)\|_{H^s_x} &\lesssim \|u_0\|_{L^2_x} + \|u(T)\|_{\dot{H}^s_x} \\
&\lesssim \|u_0\|_{L^2_x} + \lambda^{s-(\frac{1}{2}-\frac{1}{k})}\|u^\lambda(\lambda^2 T)\|_{\dot{H}^s_x} \\
&\lesssim \|u_0\|_{L^2_x} + \lambda^{s-(\frac{1}{2}-\frac{1}{k})}\|I_N u^\lambda(\lambda^2 T)\|_{H^1_x} \\
&\lesssim \|u_0\|_{L^2_x} + \lambda^{s-(\frac{1}{2}-\frac{1}{k})}(\|u^\lambda(\lambda^2 T)\|_{L^2_x} + \|\nabla I_N u^\lambda(\lambda^2 T)\|_{L^2_x}) \\
&\lesssim \|u_0\|_{L^2_x} + \lambda^{s-(\frac{1}{2}-\frac{1}{k})}(\lambda^{\frac{1}{2}-\frac{1}{k}}\|u_0\|_{L^2_x} + 1) \\
&\lesssim C(\|u_0\|_{H^s_x})
\end{aligned}$$

for all $T \in \mathbb{R}$. Hence,

$$\|u\|_{L^\infty_t H^s_x} \le C(\|u_0\|_{H^s_x}). \tag{6.6}$$

Finally, we prove that scattering holds in H^s_x for $s > s_k$. As the construction of the wave operators is standard (see [5]), we content ourselves with proving asymptotic completeness.

The first step is to upgrade the global Morawetz estimate to global Strichartz control. Let u be a global H^s_x solution to (1.1). Then u satisfies (6.1). Let $\delta > 0$ be a small constant to be chosen momentarily and split $\mathbb{R}$ into $L = L(\|u_0\|_{H^s_x})$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$\|u\|_{L^8_{t,x}(I_j\times\mathbb{R})} \le \delta.$$

By Lemma 2.1, (6.6), and the fractional chain rule, [6], we estimate

$$\begin{aligned}
\|\langle\nabla\rangle^s u\|_{S^0(I_j)} &\lesssim \|u(t_j)\|_{H^s_x} + \|\langle\nabla\rangle^s\big(|u|^{2k}u\big)\|_{L^{6/5}_{t,x}(I_j\times\mathbb{R})} \\
&\lesssim C(\|u_0\|_{H^s_x}) + \|u\|^{2k}_{L^{3k}_{t,x}}\|\langle\nabla\rangle^s u\|_{L^6_{t,x}(I_j\times\mathbb{R})},
\end{aligned}$$

while by Hölder and Sobolev embedding,

$$\begin{aligned}
\|u\|_{L^{3k}_{t,x}(I_j\times\mathbb{R})} &\lesssim \|u\|^{\frac{7}{3k-1}}_{L^8_{t,x}(I_j\times\mathbb{R})}\|u\|^{\frac{3k-8}{3k-1}}_{L^{24k}_{t,x}(I_j\times\mathbb{R})} \\
&\lesssim \delta^{\frac{7}{3k-1}}\||\nabla|^{\frac{1}{2}-\frac{1}{8k}}u\|_{L^{24k}_t L^{\frac{12k}{6k-1}}_x(I_j\times\mathbb{R})} \\
&\lesssim \delta^{\frac{7}{3k-1}}\|\langle\nabla\rangle^s u\|^{\frac{3k-8}{3k-1}}_{S^0(I_j)}.
\end{aligned}$$

Therefore,

$$\|\langle\nabla\rangle^s u\|_{S^0(I_j)} \lesssim C(\|u_0\|_{H^s_x}) + \delta^{\frac{14k}{3k-1}}\|\langle\nabla\rangle^s u\|^{1+\frac{2k(3k-8)}{3k-1}}_{S^0(I_j)}.$$

A standard continuity argument yields

$$\|\langle\nabla\rangle^s u\|_{S^0(I_j)} \le C(\|u_0\|_{H^s_x}),$$

provided we choose δ sufficiently small depending on k and $\|u_0\|_{H^s_x}$. Summing over all subintervals I_j, we obtain

$$\|\langle\nabla\rangle^s u\|_{S^0(\mathbb{R})} \leq C(\|u_0\|_{H^s_x}). \tag{6.7}$$

We now use (6.7) to prove asymptotic completeness, that is, there exist unique $u_\pm$ such that

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}u_\pm\|_{H^s_x} = 0. \tag{6.8}$$

Arguing as in Section 4, it suffices to see that

$$\Big\| \int_t^\infty e^{-is\Delta}\big(|u|^{2k}u\big)(s)\, ds \Big\|_{H^s_x} \to 0 \quad \text{as } t\to\infty. \tag{6.9}$$

The estimates above yield

$$\Big\| \int_t^\infty e^{-is\Delta}\big(|u|^{2k}u\big)(s)\, ds \Big\|_{H^s_x} \lesssim \|u\|^{\frac{14k}{3k-1}}_{L^8_{t,x}([t,\infty]\times\mathbb{R})} \|\langle\nabla\rangle^s u\|^{1+\frac{2k(3k-8)}{3k-1}}_{S^0([t,\infty]\times\mathbb{R})}.$$

Using (6.1) and (6.7) we derive (6.9).

This concludes the proof of Theorem 1.1.

□

References

[1] J. Bourgain, *Refinements of Strichartz's inequality and applications to 2D-NLS with critical nonlinearity* Intern. Mat. Res. Notices **5** (1998), 253–283.

[2] J. Bourgain, *Scattering in the energy space and below for 3D NLS*, Jour. D'Anal. Math. **75** (1998), 267–297.

[3] J. Bourgain, *Global solutions of nonlinear Schrödinger equations,* American Math. Society, Providence, R.I., 1999.

[4] T. Cazenave, F. Weissler, *The Cauchy problem for the nonlinear Schrödinger equation in H^1,* Manuscripta Math. **61** (1988), 477–494.

[5] T. Cazenave, *An introduction to nonlinear Schrödinger equations*, Textos de Metodos Matematicos **26**, 3rd Edition, 1996.

[6] M. Christ, M. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation*, J. Funct. Anal. **100** (1991), 87-109.

[7] R. Coifman, Y. Meyer, *Commutateurs d'intégrales singuliéres et opérateurs multilinéaires*, Ann. Inst. Fourier (Grenoble) **28** (1978), 177–202.

[8] R. Coifman, Y. Meyer, *Au delá des opérateurs pseudo-différentiels*, Astérisque **57**, Société Mathématique de France, Paris, 1978.

[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Almost conservation laws and global rough solutions to a Nonlinear Schrödinger Equation*, Math. Res. Letters **9** (2002), 1–24.

[10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Polynomial upper bounds for the instability of the nonlinear Schrödinger equation below the energy norm*, Commun. Pure Appl. Anal. **2**(1) (2003), 33–50.

[11] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$*, Comm. Pure Appl. Math. **57** (2004), 987–1014.

[12] Y. Fang, M. Grillakis, *On the global existence of rough solutions of the cubic defocusing Schrödinger equation in $\mathbb{R}^{2+1}$*, preprint.

[13] J. Ginibre, G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys. **123** (1989), 535–573.

[14] T. Kato, *On nonlinear Schrödinger equations*, Ann. Inst. H. Poincare Phys. Theor. **46** (1987), 113–129.

[15] M. Keel, T. Tao, *Endpoint Strichartz Estimates*, Amer. J. Math. **120** (1998), 955–980.

[16] J. Lin, W. Strauss, *Decay and scattering of solutions of a nonlinear Schrödinger equation*, Journ. Funct. Anal. **30** (1978), 245–263.

[17] C. Morawetz, *Time decay for the nonlinear Klein-Gordon equation*, Proc. Roy. Soc. A. **306** (1968), 291–296.

[18] K. Nakanishi, *Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions* 1 *and* 2, J. Funct. Anal. **169** (1999), 201–225.

[19] R.S. Strichartz, *Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), 705–774.

[20] T. Tao, M. Visan, X. Zhang, *The nonlinear Schrödinger equation with combined power-type nonlinearites,* `math.AP/0511070`.

[21] N. Tzirakis, *The Cauchy problem for the semilinear quintic Schrödinger equation in one dimension, the defocusing case*, `math-ph/0212061`.

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