SCATTERING FOR THE NON-RADIAL 3D CUBIC NONLINEAR SCHRÖDINGER EQUATION

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Abstract. Scattering of radial H^1 solutions to the 3D focusing cubic nonlinear Schrödinger equation below a mass-energy threshold $M[u]E[u] < M[Q]E[Q]$ and satisfying an initial mass-gradient bound $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$, where Q is the ground state, was established in Holmer-Roudenko [8]. In this note, we extend the result in [8] to non-radial H^1 data. For this, we prove a non-radial profile decomposition involving a spatial translation parameter. Then, in the spirit of Kenig-Merle [10], we control via momentum conservation the rate of divergence of the spatial translation parameter and by a convexity argument based on a local virial identity deduce scattering. An application to the defocusing case is also mentioned.

1. Introduction

We consider the Cauchy problem for the cubic focusing nonlinear Schrödinger (NLS) equation on $\mathbb{R}^3$:

$$i\partial_t u + \Delta u + |u|^2 u = 0, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{1.1}$$

$$u(x,0) = u_0 \in H^1(\mathbb{R}^3). \tag{1.2}$$

It is locally well-posed (e.g., see Cazenave [3]). The equation has 3 conserved quantities; namely, the mass $M[u]$, energy $E[u]$ and momentum $P[u]$:

$$M[u] = \int |u(x,t)|^2\, dx = M[u_0],$$

$$E[u] = \frac{1}{2}\int |\nabla u(x,t)|^2\, dx - \frac{1}{4}\int |u(x,t)|^4\, dx = E[u_0],$$

$$P[u] = \operatorname{Im}\int \bar{u}(x,t)\, \nabla u(x,t)\, dx = P[u_0].$$

The scale-invariant Sobolev norm is $\dot{H}^{1/2}$ and the scale-invariant Lebesgue norm is L^3. Let $u(x,t) = e^{it}Q(x)$; then u solves (1.1) provided Q solves the nonlinear elliptic equation

$$-Q + \Delta Q + |Q|^2 Q = 0. \tag{1.3}$$

This equation has an infinite number of solutions in $H^1(\mathbb{R}^3)$. The solution of minimal mass, hereafter denoted by $Q(x)$, is positive, radial, exponentially decaying, and is called *the ground state*. For further properties of Q, we refer to Weinstein [14], Holmer-Roudenko [8], Cazenave [3].

In Holmer-Roudenko [8, Theorem 1.1] (see also Holmer-Roudenko [7]), it was proved that under the condition $M[u]E[u] < M[Q]E[Q]$, solutions to (1.1)-(1.2) globally exist if u_0 satisfies

$$\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}, \tag{1.4}$$

and radial solutions with initial data satisfying (1.4) scatter in H^1 in both time directions. This means that there exist $\phi_\pm \in H^1$ such that

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta}\phi_\pm\|_{H^1} = 0.$$

In this note we extend the scattering result to include non-radial H^1 data.

Theorem 1.1. *Let $u_0 \in H^1$ and let u be the corresponding solution to* (1.1) *in H^1. Suppose*

$$M[u]E[u] < M[Q]E[Q]. \tag{1.5}$$

If $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$, then u scatters in H^1.

The argument of [8] in the radial case followed a strategy introduced by Kenig-Merle [9] for proving global well-posedness and scattering for the focusing energy-critical NLS. The argument begins by contradiction: suppose the threshold for scattering is strictly below that claimed. A profile decomposition lemma based on concentration compactness principles (and analogous to that of Keraani [11]) was invoked to prove the existence of a global but nonscattering solution u_c standing exactly at the threshold between scattering and nonscattering. The profile decomposition lemma is again invoked to prove that the flow of u_c is a precompact subset of H^1, which then implies that u_c remains spatially localized uniformly in time. This uniform localization enabled the use of a local virial identity to establish, with the aid of the sharp Gagliardo-Nirenberg inequality, a strictly positive lower bound on the convexity (in time) of the local mass of u_c. Mass conservation is then violated at a sufficiently large time.

In this paper, we show that the above program carries over to the non-radial setting with the addition of two key ingredients. First, in §2, we introduce a profile decomposition lemma that applies to non-radial H^1 sequences. To compensate for the lack of localization at the origin induced by radiality, a spatial translation sequence is needed. We also here adapt the proof given in [8] of the energy Pythagorean expansion (Lemma 2.3) to apply to non-radial sequences; in [8], an inessential application of the compact embedding $H^1_{\text{rad}} \to L^4$ was used at one point. The profile decomposition and concentration compactness techniques are previously used in works of Keraani [11], Gerard [5], see also Bahouri and Gerard [1]-[2], and originate from P.-L. Lions [12]-[13].

The application of the non-radial profile decomposition to time slices of the flow of the critical solution u_c yields the existence of a continuous time translation parameter $x(t)$ such that the translated flow $u_c(\cdot - x(t), t)$ is precompact in H^1 (Prop. 3.2). This implies the localization of $u_c(\cdot, t)$ near $x(t)$ (as opposed to the radial case, in which localization is obtained near the origin).

Obtaining suitable control on the behavior of $x(t)$ is the main new step beyond [8]. This is done by following a method introduced by Kenig-Merle [10] (who applied it to the energy-critical nonlinear wave equation). First, we argue that by Galilean invariance, the solution u_c must have zero momentum (see §4). An appropriate selection of the phase shift is possible in our case since our solution belongs to L^2.[1] This zero-momentum solution is then shown in §5 to have a near-conservation of localized center-of-mass, which provides the desired control on the rate of divergence of $x(t)$ (specifically, $x(t)/t \to 0$ as $t \to \infty$).

In §7, we remark on the adaptation of these techniques to the defocusing cubic NLS in 3D.

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2. Non-radial profile and energy decompositions

We will make use of the Strichartz norm notation used in [8]. We say that (q, r) is $\dot{H}^s$ Strichartz admissible (in 3D) if

$$\frac{2}{q} + \frac{3}{r} = \frac{3}{2} - s.$$

Let

$$\|u\|_{S(L^2)} = \sup_{\substack{(q,r)\ L^2 \text{ admissible} \\ 2 \le r \le 6,\ 2 \le q \le \infty}} \|u\|_{L_t^q L_x^r}.$$

Define

$$\|u\|_{S(\dot{H}^{1/2})} = \sup_{\substack{(q,r)\ \dot{H}^{1/2} \text{ admissible} \\ 3 \le r \le 6^-,\ 4^+ \le q \le \infty}} \|u\|_{L_t^q L_x^r},$$

where 6^- is an arbitrarily preselected and fixed number < 6; similarly for 4^+.

[1] It could not be applied in the Kenig-Merle paper [9] on the energy critical NLS since the argument there takes place in $\dot{H}^1$.

Lemma 2.1 (Profile expansion)**.** *Let $\phi_n(x)$ be a uniformly bounded sequence in H^1. Then for each M there exists a subsequence of ϕ_n, also denoted ϕ_n, and*

(1) *for each $1 \le j \le M$, there exists a (fixed in n) profile $\psi^j(x)$ in H^1,*
(2) *for each $1 \le j \le M$, there exists a sequence (in n) of time shifts t_n^j,*
(3) *for each $1 \le j \le M$, there exists a sequence (in n) of space shifts x_n^j,*
(4) *there exists a sequence (in n) of remainders $W_n^M(x)$ in H^1,*

such that

$$\phi_n(x) = \sum_{j=1}^{M} e^{-it_n^j\Delta}\psi^j(x - x_n^j) + W_n^M(x).$$

The time and space sequences have a pairwise divergence property, i.e., for $1 \le j \ne k \le M$, we have

$$\lim_{n\to+\infty} |t_n^j - t_n^k| + |x_n^j - x_n^k| = +\infty. \tag{2.1}$$

The remainder sequence has the following asymptotic smallness property[2]*:*

$$\lim_{M\to+\infty}\left[\lim_{n\to+\infty} \|e^{it\Delta}W_n^M\|_{S(\dot H^{1/2})}\right] = 0. \tag{2.2}$$

For fixed M and any $0 \le s \le 1$, we have the asymptotic Pythagorean expansion

$$\|\phi_n\|_{\dot H^s}^2 = \sum_{j=1}^{M} \|\psi^j\|_{\dot H^s}^2 + \|W_n^M\|_{\dot H^s}^2 + o_n(1). \tag{2.3}$$

Remark 2.2. If the assumption that ϕ_n is uniformly bounded in H^1 is weakened to the assumption that ϕ_n is uniformly bounded in $\dot H^{1/2}$, then the above profile decomposition remains valid provided a scaling parameter λ is also involved, similar to the theorem in [11]. However, it is not needed for the results of this note and for simplicity of exposition the proof is omitted.

Proof. The proof is very close to the one of [8, Lemma 5.2]. We also refer to [11] for a similar result in the energy-critical case.

Step 1. Construction of ψ_n^1. Let $A_1 = \limsup_n \|e^{it\Delta}\phi_n\|_{L_t^\infty L_x^3}$. If $A_1 = 0$, we are done. Indeed, for an arbitrary $\dot H^{1/2}$-admissible couple (q,r) we have

$$\left\|e^{it\Delta}\phi_n\right\|_{L_t^qL_x^r} \le \left\|e^{it\Delta}\phi_n\right\|_{L_t^4L_x^6}^{\theta} \left\|e^{it\Delta}\phi_n\right\|_{L_t^\infty L_x^3}^{1-\theta} \quad \text{with} \quad \theta = \frac{4}{q} \in (0,1).$$

Noting that $\left\|e^{it\Delta}\phi_n\right\|_{L_t^4L_x^6} \le C\|\phi_n\|_{\dot H^{1/2}}$, we get that $\limsup_n \|e^{it\Delta}\phi_n\|_{S(\dot H^{1/2})} = 0$, and we can take $\psi^j = 0$ for all j.

[2]We can always pass to a subsequence in n with the property that $\|e^{it\Delta}W_n^M\|_{S(\dot H^{1/2})}$ converges. Therefore, we use lim and not lim sup or lim inf. Similar remarks apply for the limits that appear in the Pythagorean expansion.

If $A_1 > 0$, let

$$c_1 = \limsup_n \|\phi_n\|_{H^1}.$$

Extracting a subsequence from ϕ_n, we show that there exist sequences t_n^1, x_n^1 and a function $\psi^1 \in H^1$ such that

$$e^{it_n^1\Delta}\phi_n(\cdot + x_n^1) \rightharpoonup \psi^1 \text{ weakly in } H^1, \tag{2.4}$$

$$Kc_1^4 \left\|\psi^1\right\|_{\dot{H}^{1/2}} \geq A_1^5, \tag{2.5}$$

where $K > 0$ is a constant independent of all parameters.

Let $r = \frac{16c_1^2}{A_1^2}$ and χ_r be a radial Schwartz function such that $\hat{\chi}_r(\xi) = 1$ for $\frac{1}{r} \leq |\xi| \leq r$, and $\operatorname{supp}\chi_r \subset \left[\frac{1}{2r}, 2r\right]$. By the arguments of [8], there exists sequences t_n^1, x_n^1 such that

$$|\chi_r * e^{it_n^1\Delta}\phi_n(x_n^1)| \geq \frac{A_1^3}{32c_1^2}.$$

Pass to a subsequence so that $e^{it_n^1\Delta}\phi_n(\cdot + x_n^1) \rightharpoonup \psi^1$ weakly in H^1. In [8] the functions ϕ_n are radial, and thus, by the radial Gagliardo-Nirenberg inequality, one can show that x_n^1 is bounded in n, which is not necessarily the case here. As in [8], the estimate $\|\chi_r\|_{\dot{H}^{-1/2}} \leq r$ yields, together with Plancherel and Cauchy-Schwarz inequalities, the estimate (2.5).

Next, define $W_n^1(x) = \phi_n(x) - e^{-it_n^1\triangle}\psi^1(x - x_n^1)$. Since $e^{it_n^1\triangle}\phi_n(\cdot + x_n^1) \rightharpoonup \psi^1$ in H^1, expanding $\|W_n^1\|_{\dot{H}^s}^2$ as an inner product and using the definition of W_n^1, we obtain

$$\lim_{n\to\infty} \|W_n^1\|_{\dot{H}^s}^2 = \lim_{n\to\infty} \|e^{it_n^1\triangle}\phi_n\|_{\dot{H}^s}^2 - \|\psi^1\|_{\dot{H}^s}^2, \quad 0 \leq s \leq 1,$$

which yields (2.3) for $M = 1$.

Step 2. Construction of ψ^j for $j \geq 2$. We construct the functions ψ^j inductively, applying Step 1 to the sequences (in n) W_n^{j-1}. Let $M \geq 2$. Assuming that ψ^j, x_n^j, t_n^j and W_n^j are known for $j \in \{1, \dots M-1\}$, we consider

$$A_M = \limsup_n \left\|W_n^{M-1}\right\|_{L_t^\infty L_x^3}.$$

If $A_M = 0$, we take, as in Step 1, $\psi^j = 0$ for $j \geq M$. Assume $A_M > 0$. Applying Step 1 to the sequence W_n^{M-1}, we obtain, extracting if necessary, sequences x_n^M, t_n^M and a function $\psi^M \in H^1$ such that

$$e^{it_n^M\Delta}W_n^{M-1}(\cdot + x_n^M) \rightharpoonup \psi^M \text{ weakly in } H^1, \tag{2.6}$$

$$Kc_M^4 \left\|\psi^M\right\|_{\dot{H}^{1/2}} \geq A_M^5, \text{ where } c_M = \limsup_n \|W_n^{M-1}\|_{H^1}. \tag{2.7}$$

We then define $W_n^M(x) = W_n^{M-1}(x) - e^{-it_n^M\triangle}\psi^M(x - x_n^M)$.

We next show (2.1) and (2.3) by induction. Assume that (2.3) holds at rank $M-1$. Expanding

$$\|W_n^M\|_{\dot{H}^s}^2 = \left\|e^{it_n^M\triangle}W_n^M(\cdot + x_n^M)\right\|_{\dot{H}^s}^2 = \left\|e^{it_n^M\triangle}W_n^{M-1}(\cdot + x_n^M) - \psi^M\right\|_{\dot{H}^s}^2$$

and using the weak convergence (2.6), we obtain directly (2.3) at rank M.

Assume that the condition (2.1) holds for $j,k \in \{1,\dots,M-1\}$. Let $j \in \{1,\dots,M-1\}$. Then (here, $W_n^0 = \phi_n$),

$$-e^{it_n^j\Delta}W_n^{M-1}(x+x_n^j) + e^{it_n^j\Delta}W_n^{j-1}(x+x_n^j) - \psi^j(x) = \sum_{k=j+1}^{M-1} e^{i(t_n^j - t_n^k)\Delta}\psi^k(x + x_n^j - x_n^k).$$

By the orthogonality condition (2.1), the right hand side converges to 0 weakly in H^1 as n tends to infinity. Furthermore, by the definition of W_n^{j-1},

$$e^{it_n^j\Delta}W_n^{j-1}(x+x_n^j) - \psi^j(x) \underset{n\to+\infty}{\rightharpoonup} 0 \text{ weakly in } H^1.$$

Thus, $e^{it_n^j\Delta}W_n^{M-1}(x+x_n^j)$ must go to 0 weakly in H^1. From (2.6), we deduce, if $\psi^M \neq 0$

$$\lim_{n\to+\infty} |x_n^j - x_n^M| + |t_n^j - t_n^M| = +\infty,$$

which shows that (2.1) must also holds for $k = M$.

It remains to show (2.2). Note that by (2.3), $c_M \le c_1$ for all M. If for all M, $A_M > 0$, we have by (2.3)

$$\sum_{M\ge 1} A_M^{10} \le K^2 c_1^8 \sum_{n \ge 1} \|\psi^M\|_{\dot{H}^{1/2}}^2 \le K^2 c_1^8 \limsup \|\phi_n\|_{\dot{H}^{1/2}}^2 < \infty,$$

which shows that A_M tends to 0 as M goes to ∞, yielding (2.2) and concluding the proof of Lemma 2.1. □

Lemma 2.3 (Energy Pythagorean expansion). *In the situation of Lemma 2.1, we have*

$$E[\phi_n] = \sum_{j=1}^{M} E[e^{-it_n^j\Delta}\psi^j] + E[W_n^M] + o_n(1). \tag{2.8}$$

Proof. According to (2.3), it suffices to establish for all $M \ge 1$,

$$\|\phi_n\|_{L^4}^4 = \sum_{j=1}^{M} \left\|e^{-it_n^j\Delta}\psi^j\right\|_{L^4}^4 + \left\|W_n^M\right\|_{L^4}^4 + o_n(1). \tag{2.9}$$

Step 1. Pythagorean expansion of a sum of orthogonal profiles. We show that if $M \ge 1$ is fixed, orthogonality condition (2.1) implies

$$\left\|\sum_{j=1}^{M} e^{-it_n^j\Delta}\psi^j(\cdot - x_n^j)\right\|_{L^4}^4 = \sum_{j=1}^{M}\left\|e^{-it_n^j\Delta}\psi^j\right\|_{L^4}^4 + o_n(1). \tag{2.10}$$

By reindexing, we can arrange so that there is $M_0 \le M$ such that

- For $1 \le j \le M_0$, we have that t_n^j is bounded in n.
- For $M_0+1 \le j \le M$, we have that $|t_n^j| \to \infty$ as $n \to \infty$.

By passing to a subsequence, we may assume that for each $1 \le j \le M_0$, t_n^j converges (in n), and by adjusting the profiles ψ^j we can take $t_n^j = 0$.

Note that

$$M_0 + 1 \le k \le M \Longrightarrow \lim_{n\to+\infty} \left\| e^{-it_n^k\Delta}\psi^k \right\|_{L^4} = 0. \tag{2.11}$$

Indeed, in this case $|t_n^k| \to \infty$ as $n \to \infty$. For a function $\tilde{\psi} \in \dot{H}^{3/4} \cap L^{4/3}$, from Sobolev embedding and the L^p space-time decay estimate of the linear flow, we obtain

$$\|e^{-it_n^k\Delta}\psi^k\|_{L^4} \le c\|\psi^k - \tilde{\psi}\|_{\dot{H}^{3/4}} + \frac{c}{|t_n^k|^{3/4}}\|\tilde{\psi}\|_{L^{4/3}}.$$

By approximating ψ^k by $\tilde{\psi} \in C_c^\infty$ in $\dot{H}^{3/4}$ and sending $n \to \infty$, we obtain (2.11).

By (2.1), if $1 \le j < k \le M_0$, $\lim_n |x_n^j - x_n^k| = +\infty$, and thus, it implies

$$\left\| \sum_{j=1}^{M_0} \psi^j\left(\cdot - x_n^j\right) \right\|_{L^4}^4 = \sum_{j=1}^{M_0} \left\|\psi^j\right\|_{L^4}^4 + o_n(1),$$

which yields, together with (2.11), expansion (2.10).

Step 2. End of the Proof. We first note

$$\lim_{M_1\to+\infty} \left(\lim_{n\to+\infty} \left\|W_n^{M_1}\right\|_{L^4} \right) = 0. \tag{2.12}$$

Indeed,

$$\begin{aligned}
\|W_n^{M_1}\|_{L_x^4} &\le \|e^{it\Delta}W_n^{M_1}\|_{L_t^\infty L_x^4} \\
&\le \|e^{it\Delta}W_n^{M_1}\|_{L_t^\infty L_x^3}^{1/2} \|e^{it\Delta}W_n^{M_1}\|_{L_t^\infty \dot{H}_x^1}^{1/2} \\
&\le \|e^{it\Delta}W_n^{M_1}\|_{L_t^\infty L_x^3}^{1/2} \sup_n \|\phi_n\|_{H^1}^{1/2}.
\end{aligned}$$

By (2.2), we get (2.12).

Let $M \ge 1$ and $\varepsilon > 0$. Note that $\{\phi_n\}_n$ is uniformly bounded in L^4, since it is uniformly bounded in H^1 by the hypothesis; furthermore, by (2.12) $\{W_n^M\}_n$ is also uniformly bounded in L^4. Thus, we can choose $M_1 \ge M$ and N_1 such that for $n \ge N_1$, we have

$$\begin{aligned}
&\left| \|\phi_n - W_n^{M_1}\|_{L^4}^4 - \|\phi_n\|_{L^4}^4 \right| + \left| \|W_n^M - W_n^{M_1}\|_{L^4}^4 - \|W_n^M\|_{L^4}^4 \right| \\
&\quad \le C\left[\left(\sup_n \|\phi_n\|_{L^4}^3 + \sup_n \|W_n^M\|_{L^4}^3 \right) \|W_n^{M_1}\|_{L^4} + \|W_n^{M_1}\|_{L^4}^4 \right] \le \varepsilon.
\end{aligned} \tag{2.13}$$

By (2.10) , we get $N_2 \geq N_1$ such that for $n \geq N_2$,

$$\left| \left\| \phi_n - W_n^{M_1} \right\|_{L^4}^4 - \sum_{j=1}^{M_1} \left\| e^{-it_n^j \Delta} \psi^j \right\|_{L^4}^4 \right| \leq \varepsilon. \tag{2.14}$$

Using the definition of W_n^j, expand $W_n^M - W_n^{M_1}$ to obtain

$$W_n^M - W_n^{M_1} = \sum_{j=M+1}^{M_1} e^{-it_n^j \Delta} \psi^j (\cdot - x_j).$$

By (2.10) there exists $N_3 \geq N_2$ such that for $n \geq N_2$,

$$\left| \left\| W_n^M - W_n^{M_1} \right\|_{L^4}^4 - \sum_{j=M+1}^{M_1} \left\| e^{-it_n^j \Delta} \psi^j \right\|_{L^4}^4 \right| \leq \varepsilon. \tag{2.15}$$

By (2.13), (2.14) and (2.15), we obtain that for $n \geq N_3$,

$$\left| \|\phi_n\|_{L^4}^4 - \sum_{j=1}^{M} \left\| e^{-it_n^j \Delta} \psi^j \right\|_{L^4}^4 - \left\| W_n^M \right\|_{L^4}^4 \right| \leq 4\varepsilon,$$

which concludes the proof of (2.9). □

3. Outline of the proof of the main result

Let $u(t)$ be the corresponding H^1 solution to (1.1)-(1.2). By Theorem 1.1(1)(a) in [8] the solution is globally well-posed, so our goal is to show that

$$\|u\|_{S(\dot{H}^{1/2})} < \infty. \tag{3.1}$$

This combined with Proposition 2.2 from [8] will give H^1 scattering. We will use the strategy of [9]. We shall say that $\mathrm{SC}(u_0)$ holds if (3.1) is true for the solution $u(t)$ generated from u_0.

By the small data theory there exists $\delta > 0$ such that if $M[u]E[u] < \delta$ and $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2}$, then (3.1) holds. For each $\delta > 0$ define the set S_δ to be the collection of all such initial data in H^1:

$$S_\delta = \{u_0 \in H^1 \quad \text{with} \quad M[u]E[u] < \delta \quad \text{and} \quad \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2}\}.$$

Next define $(ME)_{\mathrm{c}} = \sup\{\delta : u_0 \in S_\delta \implies SC(u_0) \text{ holds}\}$. If $(ME)_{\mathrm{c}} = M[Q]E[Q]$, then we are done, so we assume

$$(ME)_{\mathrm{c}} < M[Q]E[Q]. \tag{3.2}$$

Then there exists a sequence of solutions u_n to (1.1) with H^1 initial data $u_{n,0}$ (rescale all of them to have $\|u_n\|_{L^2} = 1$ for all n) such that $\|\nabla u_{n,0}\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2}$ and $E[u_n] \searrow (ME)_{\mathrm{c}}$ as $n \to +\infty$, for which $\mathrm{SC}(u_{n,0})$ does not hold for any n.

The next proposition gives the existence of an H^1 solution u_{c} to (1.1) with initial data $u_{\mathrm{c},0}$ such that $\|u_{\mathrm{c},0}\|_{L^2} \|\nabla u_{\mathrm{c},0}\|_{L^2} < \|Q\|_{L^2} \|\nabla Q\|_{L^2}$ and $M[u_{\mathrm{c}}]E[u_{\mathrm{c}}] = (ME)_{\mathrm{c}}$ for which $\mathrm{SC}(u_{c,0})$ does not hold. This will imply that $K = \{\, u_c(\cdot - x(t), t) \,|\, 0 \le t < +\infty \,\}$ is precompact in H^1 (Proposition 3.2). As a consequence (see Corollary (3.3)) we obtain that for each $\epsilon > 0$, there is an $R > 0$ such that, uniformly in t, we have

$$\int_{|x+x(t)|>R} |\nabla u_c(t,x)|^2 dx \le \epsilon. \tag{3.3}$$

This together with the hypothesis of zero momentum (which can always be achieved by Galilean invariance – see §4) provides a control on the growth of $x(t)$ (Lemma 5.1). Finally, the rigidity theorem (Theorem 6.1), which appeals to this control on $x(t)$ and the uniform localization (3.3), will lead to a contradiction that such critical element exists (unless it is identically zero) which will conclude the proof.

Proposition 3.1 (Existence of a critical solution)**.** *Assume* (3.2). *Then there exists a global* ($T^* = +\infty$) *solution* u_c *in* H^1 *with initial data* $u_{c,0}$ *such that* $\|u_{c,0}\|_{L^2} = 1$,

$$E[u_c] = (ME)_c < M[Q]E[Q],$$

$$\|\nabla u_c(t)\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2} \quad \textit{for all } 0 \le t < +\infty,$$

and

$$\|u_c\|_{S(\dot H^{1/2})} = +\infty.$$

Proof. The proof closely follows the proof of [8, Prop 5.4]. □

Proposition 3.2 (Precompactness of the flow of the critical solution)**.** *With* u_c *as in Proposition 3.1, there exists a continuous path* $x(t)$ *in* $\mathbb{R}^3$ *such that*

$$K = \{\, u_c(\cdot - x(t), t) \,|\, t \in [0, +\infty) \,\} \subset H^1$$

is precompact in H^1 *(i.e.,* $\bar K$ *is compact).*

Proof. For convenience, we write $u = u_c$. We argue by contradiction. By the arguments in Appendix A, we can assume that there exists $\eta > 0$ and a sequence t_n such that for all $n \ne n'$,

$$\inf_{x_0 \in \mathbb{R}^3} \|u(\cdot - x_0, t_n) - u(\cdot, t_{n'})\|_{H^1} \ge \eta. \tag{3.4}$$

Take $\phi_n = u(t_n)$ in the profile expansion lemma (Lemma 2.1). The remainder of the argument closely follows the proof of [8, Prop 5.5]. □

Corollary 3.3 (Precompactness of the flow implies uniform localization)**.** *Let* u *be a solution to* (1.1) *such that*

$$K = \{\, u(\cdot - x(t), t) \,|\, t \in [0, +\infty) \,\}$$

is precompact in H^1. *Then for each* $\epsilon > 0$, *there exists* $R > 0$ *so that*

$$\int_{|x+x(t)|>R} |\nabla u(x,t)|^2 + |u(x,t)|^2 + |u(x,t)|^4 \, dx \le \epsilon, \quad \textit{for all } 0 \le t < +\infty.$$

Proof. If not, then there exists $\epsilon > 0$ and a sequence of times t_n such that

$$\int_{|x+x(t_n)|>n} |\nabla u(x,t_n)|^2 + |u(x,t_n)|^2 + |u(x,t_n)|^4 \, dx \geq \epsilon,$$

or, by changing variables,

$$\int_{|x|>n} |\nabla u(x-x(t_n),t_n)|^2 + |u(x-x(t_n),t_n)|^2 + |u(x-x(t_n),t_n)|^4 \, dx \geq \epsilon. \tag{3.5}$$

Since K is precompact, there exists $\phi \in H^1$ such that, passing to a subsequence of t_n, we have $u(\cdot - x(t_n), t_n) \to \phi$ in H^1. By (3.5)

$$\forall R > 0, \quad \int_{|x|>R} |\nabla \phi(x)|^2 + |\phi(x)|^2 + |\phi(x)|^4 \geq \epsilon,$$

which is a contradiction with the fact that $\phi \in H^1$. □

4. Zero momentum of the critical solution

Proposition 4.1. *Assume* (3.2) *and let* u_c *be the critical solution constructed in Section 3. Then its conserved momentum* $P[u_c] = \operatorname{Im} \int \bar{u}_c \nabla u_c \, dx$ *is zero.*

Proof. Consider for some $\xi_0 \in \mathbb{R}^3$ the transformed solution

$$w_c(x,t) = e^{ix\cdot\xi_0} e^{-it|\xi_0|^2} u_c(x - 2\xi_0 t, t).$$

We compute

$$\|\nabla w_c\|_{L^2}^2 = |\xi_0|^2 M[u_c] + 2\xi_0 \cdot P[u_c] + \|\nabla u_c\|_{L^2}^2.$$

Observe that $M[w_c] = M[u_c]$ and

$$E[w_c] = \frac{1}{2}|\xi_0|^2 M[u_c] + \xi_0 \cdot P[u_c] + E[u_c].$$

To minimize $E[w_c]$, we take $\xi_0 = -P[u_c]/M[u_c]$.

Assume $P[u_c] \neq 0$. Choose $\xi_0 = -\frac{P[u_c]}{M[u_c]}$. Then $P[w_c] = 0$ and

$$M[w_c] = M[u_c], \; E[w_c] = E[u_c] - \frac{1}{2}\frac{P[u_c]^2}{M[u_c]}, \; \|\nabla w_c\|_{L^2}^2 = \|\nabla u_c\|_{L^2}^2 - \frac{P[u_c]^2}{M[u_c]}. \tag{4.1}$$

Thus, $M[w_c]E[w_c] < M[u_c]E[u_c]$, $\|w_c\|_{L^2}\|\nabla w_c\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$. By Proposition 3.1, $\|u_c\|_{S(\dot{H}^{1/2})} = +\infty$, and hence, $\|w_c\|_{S(\dot{H}^{1/2})} = +\infty$, which contradicts the definition of u_c. □

5. Control of the spatial translation parameter

Observe that

$$\frac{\partial}{\partial t}\int x|u(x,t)|^2\,dx = 2\,\mathrm{Im}\int \bar{u}\,\nabla u\,dx = 2P[u]. \tag{5.1}$$

Since $P[u_c]=0$ (see Prop. 4.1), it follows that $\int x|u_c(x,t)|^2\,dx = \text{const}$, provided it is finite. We will replace this identity with a version localized to a suitably large radius $R>0$. Provided the localization R is taken large enough over an interval $[t_0,t_1]$ to envelope the entire path $x(t)$ over $[t_0,t_1]$, we can exploit the localization of u_c in H^1 around $x(t)$ (induced by the precompactness of the translated flow $u_c(\cdot - x(t),t)$) and the zero-momentum property to prove that the localized center of mass is *nearly* conserved. The parameter $x(t)$ is then constrained from diverging too quickly to $+\infty$ by the localization of u_c in H^1 around $x(t)$ and the near conservation of localized center of mass. We refer to [10, Lemma 5.5] for a similar proof in the case of the energy-critical non-radial wave equation.

Lemma 5.1. *Let u be a solution of* (1.1) *defined on* $[0,+\infty)$ *such that* $P[u]=0$ *and* $K=\{u(\cdot - x(t),t)\,|\,t\in[0,\infty)\}$ *is precompact in* H^1*, for some continuous function* $x(\cdot)$*. Then*

$$\frac{x(t)}{t}\to 0 \quad \text{as } t\to+\infty. \tag{5.2}$$

Proof. Assume that (5.2) does not hold. Then there exists a sequence $t_n\to+\infty$ such that $|x(t_n)|/t_n\ge\epsilon_0$ for some $\epsilon_0>0$. Without loss of generality we may assume $x(0)=0$. For $R>0$, let

$$t_0(R)=\inf\{t\ge 0 : |x(t)|\ge R\}\,,$$

i.e., $t_0(R)$ is the first time when $x(t)$ reaches the boundary of the ball of radius R. By continuity of $x(t)$, the value $t_0(R)$ is well-defined. Moreover, the following properties hold: (1) $t_0(R)>0$; (2) $|x(t)|<R$ for $0\le t<t_0(R)$; and (3) $|x(t_0(R))|=R$.

Define $R_n=|x(t_n)|$ and $\tilde t_n=t_0(R_n)$. Note that $t_n\ge\tilde t_n$, which combined with $|x(t_n)|/t_n\ge\epsilon_0$ gives $R_n/\tilde t_n\ge\epsilon_0$. Since $t_n\to+\infty$ and $|x(t_n)|/t_n\ge\epsilon_0$, we have $R_n=|x(t_n)|\to+\infty$. Thus, $\tilde t_n=t_0(R_n)\to+\infty$. At this point, we can forget about t_n; we will work on the time interval $[0,\tilde t_n]$ and the only data that we will use in the remainder of the proof is:

(1) for $0\le t<\tilde t_n$, we have $|x(t)|<R_n$;
(2) $|x(\tilde t_n)|=R_n$;
(3) $\dfrac{R_n}{\tilde t_n}\ge\epsilon_0$ and $\tilde t_n\to+\infty$.

By the precompactness of K and Corollary 3.3, it follows that for any $\epsilon > 0$ there exists $R_0(\epsilon) \geq 0$ such that for any $t \geq 0$,

$$\int\limits_{|x+x(t)|\geq R_0(\epsilon)} \left(|u|^2 + |\nabla u|^2\right)\, dx \leq \epsilon. \tag{5.3}$$

We will select $\epsilon > 0$ appropriately later.

For $x \in \mathbb{R}$, let $\theta(x) \in C_c^\infty(\mathbb{R})$ be such that $\theta(x) = x$, for $-1 \leq x \leq 1$, $\theta(x) = 0$ for $|x| \geq 2^{1/3}$, $|\theta(x)| \leq |x|$, $\|\theta'\|_\infty \leq 4$, and $\|\theta\|_\infty \leq 2$. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, let $\phi(x) = (\theta(x_1), \theta(x_2), \theta(x_3))$. Then $\phi(x) = x$ for $|x| \leq 1$ and $\|\phi\|_\infty \leq 2$. For $R > 0$, set $\phi_R(x) = R\phi(x/R)$. Let $z_R : \mathbb{R} \to \mathbb{R}^3$ be the truncated center of mass given by

$$z_R(t) = \int \phi_R(x)\, |u(x,t)|^2\, dx\,.$$

Then $z_R'(t) = ([z_R'(t)]_1, [z_R'(t)]_2, [z_R'(t)]_3)$, where

$$[z_R'(t)]_j = 2\,\mathrm{Im} \int \theta'(x_j/R)\, \partial_j u\; \bar{u}\, dx.$$

Note that $\theta'(x_j/R) = 1$ for $|x_j| \leq 1$. By the zero momentum property,

$$\mathrm{Im} \int_{|x_j|\leq R} \partial_j u\, \bar{u} = -\,\mathrm{Im} \int_{|x_j|>R} \partial_j u\, \bar{u},$$

and thus,

$$[z_R'(t)]_j = -2\,\mathrm{Im} \int_{|x_j|\geq R} \partial_j u\; \bar{u}\, dx + 2\,\mathrm{Im} \int_{|x_j|\geq R} \theta'(x_j/R) \partial_j u\; \bar{u}\, dx\,,$$

from which we obtain by Cauchy-Schwarz,

$$|z_R'(t)| \leq 5 \int_{|x|\geq R} (|\nabla u|^2 + |u|^2)\,. \tag{5.4}$$

Set $\tilde{R}_n = R_n + R_0(\epsilon)$. Note that for $0 \leq t \leq \tilde{t}_n$ and $|x| > \tilde{R}_n$, we have $|x + x(t)| \geq \tilde{R}_n - R_n = R_0(\epsilon)$, and thus, (5.4) and (5.3) give

$$|z_{\tilde{R}_n}'(t)| \leq 5\,\epsilon. \tag{5.5}$$

Now we obtain an upper bound for $z_{\tilde{R}_n}(0)$ and a lower bound for $z_{\tilde{R}_n}(t)$.

$$z_{\tilde{R}_n}(0) = \int_{|x|<R_0(\epsilon)} \phi_{\tilde{R}_n}(x)\, |u_0(x)|^2\, dx + \int_{|x+x(0)|\geq R_0(\epsilon)} \phi_{\tilde{R}_n}(x) |u_0(x)|^2\, dx\,,$$

and hence, by (5.3), we have

$$|z_{\tilde{R}_n}(0)| \leq R_0(\epsilon) M[u] + 2\tilde{R}_n\, \epsilon. \tag{5.6}$$

For $0 \leq t \leq \tilde{t}_n$, we split $z_{\tilde{R}_n}(t)$ as

$$\begin{aligned} z_{\tilde{R}_n}(t) &= \int_{|x+x(t)|\geq R_0(\epsilon)} \phi_{\tilde{R}_n}(x)\,|u(x,t)|^2\,dx + \int_{|x+x(t)|\leq R_0(\epsilon)} \phi_{\tilde{R}_n}(x)\,|u(x,t)|^2\,dx. \\ &= \mathrm{I} + \mathrm{II} \end{aligned}$$

To estimate I, we note that $|\phi_{\tilde{R}_n}(x)| \leq 2\tilde{R}_n$ and use (5.3) to obtain $|\mathrm{I}| \leq 2\tilde{R}_n\epsilon$. For II, we first note that $|x| \leq |x+x(t)| + |x(t)| \leq R_0(\epsilon) + R_n = \tilde{R}_n$, and thus $\phi_{\tilde{R}_n}(x) = x$. We now rewrite II as

$$\begin{aligned} \mathrm{II} &= \int_{|x+x(t)|\leq R_0(\epsilon)} (x+x(t))\,|u(x,t)|^2\,dx - x(t)\int_{|x+x(t)|\leq R_0(\epsilon)} |u(x,t)|^2\,dx \\ &= \int_{|x+x(t)|\leq R_0(\epsilon)} (x+x(t))\,|u(x,t)|^2\,dx - x(t)M[u] + x(t)\int_{|x+x(t)|\geq R_0(\epsilon)} |u(x,t)|^2\,dx \\ &= \mathrm{IIA} + \mathrm{IIB} + \mathrm{IIC} \end{aligned}$$

Trivially, $|\mathrm{IIA}| \leq R_0(\epsilon)M[u]$, and by (5.3), $|\mathrm{IIC}| \leq |x(t)|\epsilon \leq \tilde{R}_n\epsilon$. Thus,

$$\begin{aligned} |z_{\tilde{R}_n}(t)| &\geq |\mathrm{IIB}| - |\mathrm{I}| - |\mathrm{IIA}| - |\mathrm{IIC}| \\ &\geq |x(t)|M[u] - R_0(\epsilon)M[u] - 3\tilde{R}_n\epsilon\,. \end{aligned}$$

Taking $t = \tilde{t}_n$, we get

$$|z_{\tilde{R}_n}(\tilde{t}_n)| \geq \tilde{R}_n(M[u] - 3\epsilon) - R_0(\epsilon)M[u]\,. \tag{5.7}$$

Combining (5.5), (5.6), and (5.7), we have

$$\begin{aligned} 5\,\epsilon\,\tilde{t}_n &\geq \int_0^{\tilde{t}_n} |z'_{\tilde{R}_n}(t)|\,dt \geq \left|\int_0^{\tilde{t}_n} z'_{\tilde{R}_n}(t)\,dt\right| \geq |z_{\tilde{R}_n}(\tilde{t}_n) - z_{\tilde{R}_n}(0)| \\ &\geq \tilde{R}_n(M[u] - 5\epsilon) - 2R_0(\epsilon)M[u]\,. \end{aligned}$$

Dividing by $\tilde{t}_n$ and using that $\tilde{R}_n \geq R_n$ (assume $\epsilon \leq \frac{1}{5}M[u]$), we obtain

$$5\,\epsilon \geq \frac{R_n}{\tilde{t}_n}(M[u] - 5\epsilon) - \frac{2R_0(\epsilon)M[u]}{\tilde{t}_n}\,.$$

Since $R_n/\tilde{t}_n \geq \epsilon_0$, we have

$$5\,\epsilon \geq \epsilon_0(M[u] - 5\epsilon) - \frac{2R_0(\epsilon)M[u]}{\tilde{t}_n}\,.$$

Take $\epsilon = M[u]\epsilon_0/16$ (assume $\epsilon_0 \leq 1$), and then send $n \to +\infty$. Since $\tilde{t}_n \to +\infty$, we get a contradiction. □

6. Rigidity theorem

We now prove the following rigidity, or Liouville-type, theorem.

Theorem 6.1 (Rigidity)**.** *Suppose $u_0 \in H^1$ satisfies $P[u_0] = 0$,*

$$M[u_0]E[u_0] < M[Q]E[Q] \tag{6.1}$$

and

$$\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}. \tag{6.2}$$

Let u be the global H^1 solution of (1.1) *with initial data u_0 and suppose that*

$$K = \{\, u(\cdot - x(t), t) \,|\, t \in [0, +\infty) \,\} \quad \textit{is precompact in } H^1.$$

Then $u_0 = 0$.

Before beginning the proof, we recall in Lemma 6.2 below a few basic facts proved in [8]. These facts are consequences of the Gagliardo-Nirenberg inequality

$$\|u\|_{L^4}^4 \le c_{\mathrm{GN}}\|u\|_{L^2}\|\nabla u\|_{L^2}^3$$

with the sharp value of c_{GN} expressed as

$$c_{\mathrm{GN}} = \frac{4}{3\|Q\|_2\|\nabla Q\|_2}\,.$$

One also uses the relation

$$M[Q]E[Q] = \frac{1}{6}\|Q\|_{L^2}^2\|\nabla Q\|_{L^2}^2\,,$$

which is a consequence of the Pohozhaev identities.

Lemma 6.2. *If $M[u]E[u] < M[Q]E[Q]$ and $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$, then for all t,*

$$\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \le \omega\|Q\|_{L^2}\|\nabla Q\|_{L^2} \tag{6.3}$$

where $\omega = \left(\frac{M[u]E[u]}{M[Q]E[Q]}\right)^{1/2}$. We also have the bound, for all t

$$8\|\nabla u(t)\|^2 - 6\|u(t)\|_{L^4}^4 \ge 8(1-\omega)\|\nabla u(t)\|_{L^2}^2 \ge 16(1-\omega)E[u]. \tag{6.4}$$

We remark that under the hypotheses here, $E[u] > 0$ unless $u \equiv 0$. In fact, one has the bound $E[u] \ge \frac{1}{6}\,\|\nabla u_0\|_{L^2}^2$.

Proof of Theorem 6.1. In the proof below, all instances of a constant c refer to some absolute constant. Let $\varphi \in C_0^\infty$ be radial with

$$\varphi(x) = \begin{cases} |x|^2 & \text{for} \quad |x| \le 1 \\ 0 & \text{for} \quad |x| \ge 2 \end{cases}.$$

For $R > 0$, define

$$z_R(t) = \int R^2 \varphi\left(\frac{x}{R}\right) |u(x,t)|^2 \, dx \, .$$

Then, by direct calculation,

$$z_R'(t) = 2 \operatorname{Im} \int R \nabla\varphi\left(\frac{x}{R}\right) \cdot \nabla u(t) \, \bar{u}(t) \, dx$$

By the Hölder inequality,

$$|z_R'(t)| \le cR \int_{|x|\le 2R} |\nabla u(t)||u(t)| \, dx \le cR \|\nabla u(t)\|_{L^2} \|u(t)\|_{L^2} \tag{6.5}$$

Also by direct calculation, we have the local virial identity

$$z_R''(t) = 4 \sum_{j,k} \int \frac{\partial^2 \varphi}{\partial x_j \partial x_k}\left(\frac{x}{R}\right) \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_k} - \frac{1}{R^2} \int (\Delta^2 \varphi)\left(\frac{x}{R}\right) |u|^2 - \int (\Delta\varphi)\left(\frac{x}{R}\right) |u|^4.$$

Since φ is radial, we have

$$z_R''(t) = \left(8 \int |\nabla u|^2 - 6 \int |u|^4\right) + A_R(u(t)), \tag{6.6}$$

where

$$\begin{aligned} A_R(u(t)) = \; & 4 \sum_j \int \left((\partial_{x_j}^2 \varphi)\left(\frac{x}{R}\right) - 2\right) \left|\partial_{x_j} u\right|^2 + 4 \sum_{j\neq k} \int_{R \le |x| \le 2R} \frac{\partial^2 \varphi}{\partial x_j \partial x_k}\left(\frac{x}{R}\right) \frac{\partial u}{\partial x_j} \frac{\partial \bar{u}}{\partial x_k} \\ & - \frac{1}{R^2} \int (\Delta^2 \varphi)\left(\frac{x}{R}\right) |u|^2 - \int \left((\Delta\varphi)\left(\frac{x}{R}\right) - 6\right) |u|^4. \end{aligned}$$

From this expression, we obtain the bound

$$|A_R(u(t))| \le c \int_{|x|\ge R} \left(|\nabla u(t)|^2 + \frac{1}{R^2} |u(t)|^2 + |u(t)|^4 \right) dx. \tag{6.7}$$

We want to examine $z_R(t)$, for R chosen suitably large, over a suitably chosen time interval $[t_0, t_1]$, where $1 \ll t_0 \ll t_1 < \infty$. By (6.6) and (6.4), we have

$$|z_R''(t)| \ge 16(1-\omega) E[u] - |A_R(u(t))|. \tag{6.8}$$

Set $\epsilon = \frac{1-\omega}{c} E[u]$ in Corollary 3.3 to obtain $R_0 \ge 0$ such that $\forall\, t$,

$$\int_{|x+x(t)|\ge R_0} (|\nabla u|^2 + |u|^2 + |u|^4) \le \frac{(1-\omega)}{c} E[u]. \tag{6.9}$$

If we select $R \ge R_0 + \sup_{t_0 \le t \le t_1} |x(t)|$, then (6.8) combined with the bounds (6.7) and (6.9) will imply that, for all $t_0 \le t \le t_1$,

$$|z_R''(t)| \ge 8(1-\omega) E[u]. \tag{6.10}$$

By Lemma 5.1, there exists $t_0 \geq 0$ such that for all $t \geq t_0$, we have $|x(t)| \leq \eta t$, with $\eta > 0$ to be selected later. Thus, by taking $R = R_0 + \eta t_1$, we obtain that (6.10) holds for all $t_0 \leq t \leq t_1$. Integrating (6.10) over $[t_0, t_1]$, we obtain

$$|z_R'(t_1) - z_R'(t_0)| \geq 8(1-\omega)E[u](t_1 - t_0). \tag{6.11}$$

On the other hand, for all $t_0 \leq t \leq t_1$, by (6.5) and (6.3), we have

$$\begin{aligned} |z_R'(t)| &\leq cR\|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2} \leq cR\|Q\|_{L^2}\|\nabla Q\|_{L^2} \\ &\leq c\|Q\|_{L^2}\|\nabla Q\|_{L^2}(R_0 + \eta t_1). \end{aligned} \tag{6.12}$$

Combining (6.11) and (6.12), we obtain

$$8(1-\omega)E[u](t_1 - t_0) \leq 2c\|Q\|_{L^2}\|\nabla Q\|_{L^2}(R_0 + \eta t_1).$$

Recall that ω and R_0 are constants depending only upon $(M[u]E[u])/(M[Q]E[Q])$, while $\eta > 0$ is yet to be specified and $t_0 = t_0(\eta)$. Put $\eta = (1-\omega)E[u]/(c\|Q\|_2\|\nabla Q\|_2)$ and then send $t_1 \to +\infty$ to obtain a contradiction unless $E[u] = 0$ which implies $u \equiv 0$. □

To complete the proof of Theorem 1.1, we just apply Theorem 6.1 to u_{c} constructed in Proposition 3.1, which by Propositions 3.2 and 4.1, meets the hypotheses in Theorem 6.1. Thus $u_{\mathrm{c},0} = 0$, which contradicts the fact that $\|u_{\mathrm{c}}\|_{S(\dot{H}^{1/2})} = \infty$. We have thus obtained that if $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2} < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$ and $M[u]E[u] < M[Q]E[Q]$, then $\mathrm{SC}(u_0)$ holds, i.e. $\|u\|_{S(\dot{H}^{1/2})} < \infty$. By Proposition 2.2 [8], H^1 scattering holds.

7. Remarks on the defocusing equation

One may use the above arguments to show H^1-scattering of solutions of the defocusing equation

$$i\partial_t u + \Delta u - |u|^2 u = 0, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \tag{7.1}$$

$$u(x,0) = u_0 \in H^1(\mathbb{R}^3). \tag{7.2}$$

In this case, scattering is already known, as a consequence of Morawetz [6], or interaction Morawetz [4] inequalities.

We argue by contradiction. If scattering does not hold, there exists a critical solution u_c, which does not scatter, and such that $M[u_c]E[u_c]$ is minimal for non-scattering solutions of (7.1). As before, one shows that $P[u_c] = 0$, and that there exists $x(t)$ such that the set $K = \{u_c(t, \cdot - x(t)),\ t \in \mathbb{R}\}$ is precompact in H^1. Note that because of the defocusing sign of the non-linearity, we do not need to assume $M[u_c]E[u_c] < M[Q]E[Q]$ and $\|u_c(0)\|_{L^2}\,\|\nabla u_c(0)\|_{L^2} < \|Q\|_{L^2}\,\|\nabla Q\|_{L^2}$. The control of the spatial translation $x(t)$ works as in Section 5, and one concludes as in Section 6, by a localized virial argument, using that in the defocusing case, the second derivative of the localized variance $z_R(t)$ is

$$z_R''(t) = \left(8\int |\nabla u|^2 + 6\int |u|^4\right) + B_R(u(t)),$$

where B_R satisfies the bound

$$|B_R(u(t))| \le c\int_{|x|\ge R}\left(|\nabla u(t)|^2 + \frac{1}{R^2}|u(t)|^2 + |u(t)|^4\right)\,dx.$$

Note that the use of the virial identity is potentially more robust since one might be able to handle variants of the NLS equation (for example with a linear potential) that might be out of reach for Morawetz based proofs.

Appendix A. A lifting lemma

In this appendix, we discuss some basic analysis facts needed in the very beginning of the proof of Prop. 3.2.

Let $G \cong \mathbb{R}^3$ act on H^1 by translation, i.e., $(x_0 \cdot \phi)(x) = \phi(x - x_0)$. Write $G\backslash H^1$ for the quotient space endowed with the quotient topology. We represent elements of $G\backslash H^1$ (the equivalence classes) by $[\phi]$, and let $\pi : H^1 \to G\backslash H^1$ be the natural projection.

Lemma A.1. *$G\backslash H^1$ is metrizable with metric*

$$d([\phi],[\psi]) = \inf_{x_0\in\mathbb{R}^3} \|\phi(\cdot - x_0) - \psi\|_{H^1}.$$

With respect to this metric, $G\backslash H^1$ is complete. (Caution that $G\backslash H^1$ is not a vector space, however.)

Proof. First, we establish that the orbits of G are closed in H^1. The orbit of 0 is 0. Suppose $\phi \neq 0$, $\{x_n\} \subset \mathbb{R}^3$ and $\phi(\cdot - x_n)$ converges to ψ in H^1. Then we claim that x_n converges. Indeed, if not, then either x_n is unbounded and there is a subsequence x_n such that $|x_n| \to \infty$, or x_n is bounded and there are two subsequences $x_n \to x_0$ and $x_{n'} \to x_0'$. In the first case, we obtain that $\psi = 0$ (by examining, for fixed $R > 0$, the convergence on $B(0,R)$), which implies $\phi = 0$, a contradiction. In the second case, we obtain that $\phi(\cdot - x_0) = \phi(\cdot - x_0')$, only possible if $\phi = 0$, a contradiction.

Next, we verify that d is a metric. Suppose $d([\phi],[\psi]) = 0$. Then $\inf_{x_0\in\mathbb{R}^3} \|\phi(\cdot - x_0) - \psi\|_{H^1} = 0$, and thus ψ is a point of closure (in H^1) of the orbit of ϕ. But since the orbits are closed, ψ belongs to this orbit, and thus, $[\phi] = [\psi]$. The triangle inequality is a straightforward exercise dealing with infima, and symmetry is obvious.

Suppose $[\phi_n]$ is a Cauchy sequence; to show that it converges, it suffices to show that a subsequence converges. We can pass to a subsequence $[\phi_n]$ so that $d([\phi_n],[\phi_{n+1}]) \le 2^{-n}$. Take $x_1 = 0$. Construct a sequence x_n inductively as follows: given x_{n-1}, select x_n so that $\|\phi_{n-1}(\cdot - x_{n-1}) - \phi_n(\cdot - x_n)\|_{H^1} \le 2^{-n+1}$. Then $\phi_n(\cdot - x_n)$ is a Cauchy sequence in H^1, and hence, converges to some ϕ. It is then clear that $[\phi_n] \to [\phi]$ in $G\backslash H^1$.

It can be checked that for each $\phi \in H^1$ and $r > 0$, $\pi(B(\phi, r)) = B([\phi], r)$. Therefore, the topology induced by the metric d on $G\backslash H^1$ is the quotient topology. $\square$

The following two lemmas will reduce Prop. 3.2 to proving that the set $\pi(\{ u(\cdot, t) \mid t \in [0, +\infty) \})$ is precompact in $G\backslash H^1$.

Lemma A.2. *Let K be a precompact subset of $G\backslash H^1$. Assume*

$$\exists\, \eta > 0 \text{ such that } \forall\, \phi \in \pi^{-1}(K), \quad \eta \leq \|\phi\|_{H^1}. \tag{A.1}$$

Then there exists $\tilde{K}$ precompact in H^1 such that $\pi(\tilde{K}) = K$.

Proof. Let $B(0, 1)$ be the unit ball in $\mathbb{R}^3$. We first show by contradiction that there exists $\varepsilon > 0$ such that for all p in K, there exists $\psi = \psi(p) \in \pi^{-1}(p)$ such that

$$\|\psi(p)\|_{H^1(B(0,1))} \geq \varepsilon. \tag{A.2}$$

If not, there exists a sequence ϕ_n in $\pi^{-1}(K)$ such that

$$\sup_{x_0 \in \mathbb{R}^3} \|\phi_n(\cdot - x_0)\|_{H^1(B(0,1))} \leq \frac{1}{n}. \tag{A.3}$$

The precompactness of K implies, extracting a subsequence from ϕ_n if necessary, that there exists $\phi \in H^1$ such that $\pi(\phi_n) \to p$ in $G\backslash H^1$. In other words, if ϕ is fixed in $\pi^{-1}(p)$, $\inf_{x_0 \in \mathbb{R}^3} \|\phi_n(\cdot - x_0) - \phi\|_{H^1}$ tends to 0 as n tends to infinity. Thus, one may find a sequence x_n in $\mathbb{R}^3$ such that

$$\|\phi_n(\cdot - x_n) - \phi\|_{H^1} \underset{n \to +\infty}{\longrightarrow} 0. \tag{A.4}$$

Now, by (A.3), for all $x_0 \in \mathbb{R}^3$, $\|\phi_n(\cdot - x_0 - x_n)\|_{H^1(B(0,1))} \leq \frac{1}{n}$. Hence, by (A.4), for all x_0, ϕ vanishes on $B(x_0, 1)$. But then $\phi = 0$, which contradicts assumption (A.1), concluding the proof of the existence of $x(\phi)$.

Let $\tilde{K} = \{\psi(p) \mid p \in K\}$, where $\psi(p)$ satisfies (A.2). Of course, $\pi(\tilde{K}) = K$. By the definition of $x(\phi)$,

$$\forall\, \phi \in \pi^{-1}(K), \quad \|\phi\|_{H^1(B(0,1))} \geq \varepsilon. \tag{A.5}$$

Let us show that $\tilde{K}$ is precompact. Let ϕ_n be a sequence in $\tilde{K}$. Then by the precompactness of K, there exists (extracting subsequences) $\phi \in H^1$ and a sequence x_n of $\mathbb{R}^3$, such that

$$\lim_{n \to +\infty} \|\phi_n(\cdot - x_n) - \phi\|_{H^1} = 0. \tag{A.6}$$

Note that K being precompact, ϕ_n is bounded in H^1, thus, we may assume (extracting again)

$$\lim_{n \to +\infty} \|\phi_n\|_{H^1} = \ell \in (0, +\infty). \tag{A.7}$$

Let us show that x_n is bounded. If not, we may assume that $|x_n| \to +\infty$. By (A.5) and (A.7), we have

$$\limsup_{n\to\infty} \|\phi_n(\cdot - x_n)\|_{H^1(B(0,|x_n|-1))} \leq \ell - \varepsilon.$$

As $|x_n| \to \infty$, we conclude that $\|\phi\|_{H^1} \leq \ell - \varepsilon$, contradicting (A.7). Therefore, x_n is bounded. Extracting if necessary, we may assume that x_n converges, which shows by (A.6) that ϕ_n converges. This concludes the proof of the precompactness of $\tilde{K}$. □

Lemma A.3. *Let u be a global H^1 solution to* (1.1). *Suppose*

$$\pi(\{\, u(\cdot,t) \,|\, t \in [0,+\infty) \,\})$$

is precompact in $G\backslash H^1$. Then there exists $x(t)$, a continuous path in $\mathbb{R}^3$, such that

$$\{\, u(\cdot - x(t),t) \,|\, t \in [0,+\infty) \,\}$$

is precompact in H^1.

Proof. By taking $K = \pi(\{\, u(\cdot,t) \,|\, t \in [0,+\infty) \,\})$ in Lemma A.2, we obtain $\tilde{K}$ precompact in H^1 such that $\pi(\tilde{K}) = K$. For each N, the map $u : [N, N{+}1] \to H^1$ is uniformly continuous. Thus, for each N, there exists $\delta_N > 0$ such that if $t, t' \in [N, N+1]$ and $|t - t'| \leq \delta_N$, then $\|u(t,\cdot) - u(t',\cdot)\|_{H^1} \leq 1/N$. Let t_n be the increasing sequence of times $\to +\infty$ defined to include evenly spaced elements with density δ_N in $[N, N+1]$ for each N. (Thus, t_n is an increasing sequence with possibly more elements per unit interval as we move out to $+\infty$). For each n, select $x(t_n) \in \mathbb{R}^3$ such that $u(\cdot - x(t_n), t_n) \in \tilde{K}$. Now define $x(t)$ to be the continuous function that connects $x(t_n)$ to $x(t_{n+1})$ by a straight line in $\mathbb{R}^3$.

We claim that $\{\, u(\cdot - x(t),t) \,|\, t \in [0,+\infty) \,\}$ is precompact in H^1. Indeed, let s_k be a sequence in $[0,+\infty)$. Then there exists a subsequence (also labeled s_k) such that either s_k converges to some finite s_0 or $s_k \to +\infty$. In the first case, $u(\cdot - x(s_k), s_k) \to u(\cdot - x(s_0), s_0)$ by the continuity of $u(t)$ and $x(t)$. In the second case, for each k, obtain the unique index $n(k)$ such that $t_{n(k)-1} \leq s_k < t_{n(k)}$. By the precompactness of $\tilde{K}$, we can pass to a subsequence (in k) such that both $u(\cdot - x(t_{n(k)-1}), t_{n(k)-1})$ and $u(\cdot - x(t_{n(k)}), t_{n(k)})$ converge in H^1. By the density of the t_n sequence and uniform continuity of u, we obtain that $u(\cdot - x(t_{n(k)-1}), t_{n(k)})$ converges and that it suffices to show that $u(\cdot - x(s_k), t_{n(k)})$ has a convergent subsequence. But since both $u(\cdot - x(t_{n(k)-1}), t_{n(k)})$ and $u(\cdot - x(t_{n(k)}), t_{n(k)})$ converge, we have that $x(t_{n(k)-1}) - x(t_{n(k)})$ converges. Recall that $x(s_k)$ lies on the line segment joining $x(t_{n(k)-1})$ and $x(t_{n(k)})$, and thus, $x(s_k) - x(t_{n(k)-1})$ converges (after passing to a subsequence). Hence, $u(\cdot - x(s_k), t_{n(k)})$ converges in H^1. □

Thus, to prove Prop. 3.2, it suffices to prove that

$$\pi(\{\, u(\cdot,t) \,|\, t \in [0,+\infty) \,\}) \tag{A.8}$$

is precompact in $G\backslash H^1$. Since $G\backslash H^1$ is complete, if we assume that (A.8) is not precompact in $G\backslash H^1$, then there exists a sequence $\{[u(t_n)]\}$ in $G\backslash H^1$ and $\eta > 0$ such that $d([u(t_n)],[u(t_{n'})]) \geq \eta$, or equivalently, (3.4) in the proof of Prop 3.2 holds.

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