

PHASE-DRIVEN INTERACTION OF WIDELY SEPARATED NONLINEAR SCHRÖDINGER SOLITONS

JUSTIN HOLMER AND QUANHUI LIN

ABSTRACT. We show that, for the 1d cubic NLS equation, widely separated equal amplitude in-phase solitons attract and opposite-phase solitons repel. Our result gives an exact description of the evolution of the two solitons valid until the solitons have moved a distance comparable to the logarithm of the initial separation. Our method does not use the inverse scattering theory and should be applicable to nonintegrable equations with local nonlinearities that support solitons with exponentially decaying tails. The result is presented as a special case of a general framework which also addresses, for example, the dynamics of single solitons subject to external forces as in [7, 8].

1. INTRODUCTION

We consider the 1d nonlinear Schrödinger equation (NLS)

$$(1.1) \quad i\partial_t u + \frac{1}{2}\partial_x^2 u + |u|^2 u = 0.$$

It has a single soliton solution $u(x, t) = e^{it/2} \operatorname{sech} x$. The invariances of (1.1) can be applied to produce a whole family of solutions. To describe them, let ¹

$$(1.2) \quad \eta(x, \mu, a, \theta, v) = e^{i\theta} e^{i\mu^{-1}v(x-a)} \mu \operatorname{sech}(\mu(x-a))$$

for parameters $\theta, a, v \in \mathbb{R}, \mu > 0$. Then $u(x, t) = \eta(x, \mu(t), a(t), \theta(t), v(t))$ solves (1.1) provided

$$(1.3) \quad \begin{cases} \mu(t) = \mu_0 \\ a(t) = a_0 + tv_0\mu_0^{-1} \\ \theta(t) = \theta_0 + \frac{1}{2}t(\mu_0^2 + \mu_0^{-2}v_0^2) \\ v(t) = v_0 \end{cases}$$

In this paper, we study the evolution of initial data that is the sum of two widely separated solitons:

$$(1.4) \quad u_0(x) = \eta(x, \mu_{10}, a_{10}, \theta_{10}, v_{10}) + \eta(x, \mu_{20}, a_{20}, \theta_{20}, v_{20})$$

¹We order the parameters as (μ, a, θ, v) to mimic (q^1, q^2, p^1, p^2) as canonical coordinates for the four dimensional symplectic space with symplectic form $dp^1 \wedge dq^1 + dp^2 \wedge dq^2$.

where $|a_{20} - a_{10}| \gg 1$. In particular, we focus on two illustrative cases. In both cases, we consider identical mass solitons with zero initial velocity. In Case 0, we take the same initial phase, corresponding to an even superposition and in Case 1, we take opposite initial phase corresponding to an odd superposition.

$$(1.5) \quad u_0(x) = \begin{cases} \eta(x, 1, -a_0, 0, 0) + \eta(x, 1, a_0, 0, 0) & \text{Case } \sigma = 0 \\ \eta(x, 1, -a_0, \pi, 0) + \eta(x, 1, a_0, 0, 0) & \text{Case } \sigma = 1 \end{cases}$$

We find that in the same phase case (Case 0), the two solitons are drawn toward each other and in the opposite phase case (Case 1) they are pushed apart— see Fig. 1.1. In either case, the solution u to (1.1) is well-approximated by

$$(1.6) \quad u_z(x) = \eta(x, \mu_1, a_1, \theta_1, v_1) + \eta(x, \mu_2, a_2, \theta_2, v_2)$$

where z represents coordinates²

$$(1.7) \quad z = (z^1, z^2, z^3, z^4, z^5, z^6, z^7, z^8) = (\mu_1, a_1, \mu_2, a_2, \theta_1, v_1, \theta_2, v_2)$$

As parity is preserved by the flow (1.1), we have

$$(1.8) \quad \mu \stackrel{\text{def}}{=} \mu_1 = \mu_2, \quad a \stackrel{\text{def}}{=} -a_1 = a_2, \quad v \stackrel{\text{def}}{=} -v_1 = v_2,$$

and $\theta \stackrel{\text{def}}{=} \theta_1 = \theta_2$ in the same phase case (Case 0), while $\theta \stackrel{\text{def}}{=} \theta_1 - \pi = \theta_2$ in the opposite phase case (Case 1).

Theorem 1.1. *Suppose that $u(t)$ is the solution to (1.1) with initial data (1.5). Let $h = e^{-a_0} \ll 1$ (so $a_0 = \log h^{-1} \gg 1$). Let*

$$T \sim \begin{cases} h^{-1} & \text{Case } \sigma = 0 \\ h^{-1} \log h^{-1} & \text{Case } \sigma = 1 \end{cases}$$

Let $(a(t), v(t))$ solve

$$(1.9) \quad \begin{cases} \dot{a} = v \\ \dot{v} = -4(-1)^\sigma e^{-2a} \end{cases}$$

with initial data $(a_0, 0)$. Let μ solve

$$(1.10) \quad \dot{\mu} = (-1)^\sigma (8a - 4)v e^{-2a},$$

and then let θ solve

$$(1.11) \quad \dot{\theta} = \frac{1}{2}\mu^2 + \frac{1}{2}v^2\mu^{-2} + 18(-1)^\sigma e^{-2a}.$$

Then on $0 \leq t \leq T$, we have

$$\|u(t) - u_z\|_{H_x^1} \lesssim h^{2-},$$

²Superscripts are used on z to conform with geometric summation conventions used later in the paper.

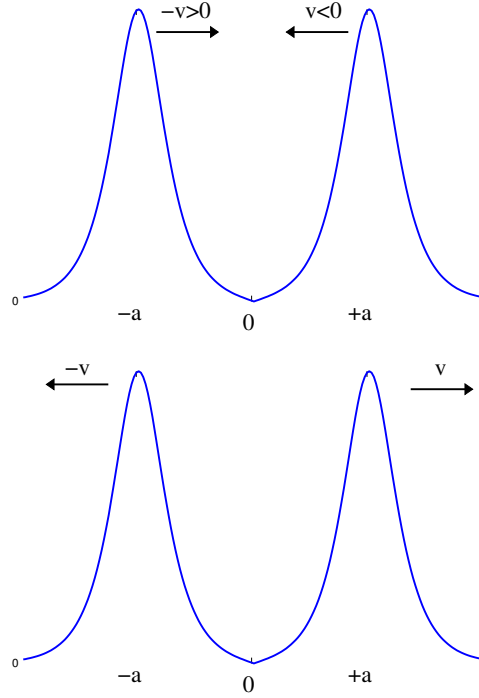


FIGURE 1.1. The top plot is a depiction of Case 0 (same phase; even solution), where the two solitons are pulled toward each other. The bottom plot depicts Case 1 (opposite phase; odd solution), where they repel. In each case, the solution is modeled in Theorem 1.1 as $u \approx u_z = \eta(\mu, -a, \theta + \sigma\pi, -v) + \eta(\mu, a, \theta, v)$ where (μ, a, θ, v) solve a specific ODE system.

where

$$(1.12) \quad u_z = \eta(\mu, -a, \theta + \sigma\pi, -v) + \eta(\mu, a, \theta, v)$$

Let us make some remarks on the ODE system (1.9). The energy associated to this system is

$$H_{\text{eff}} = v^2 - 4(-1)^\sigma e^{-2a} = -4(-1)^\sigma e^{-2a_0}$$

In the case $\sigma = 0$ (same phase), we have $v \leq 0$, $a \leq a_0$ and

$$\begin{cases} a(t) = a_0 - \log \sec(2ht) \\ v(t) = 2h \tan(2ht) \end{cases}$$

valid for $0 \leq t \lesssim e^{a_0} = h^{-1}$. In the case $\sigma = 1$ (opposite phase), we have $v \geq 0$, $a \geq a_0$ and

$$\begin{cases} a(t) = a_0 + \log \cosh(2ht) \\ v(t) = 2h \tanh(2ht) \end{cases}$$

valid for $0 \leq t \lesssim a_0 e^{a_0} = h^{-1} \log h^{-1}$. In either case, μ evolving according to (1.10) satisfies $|\mu - 1| \lesssim h^2$ and can thus be replaced by 1 in (1.12). However, the $O(h^2)$ behavior of μ is dynamically significant in that it yields $O(h^1)$ effects in θ through (1.11). It is evident from the explicit forms for $a(t)$ given above that, on the indicated time scale, the soliton has moved a distance comparable to $\log a_0$.

We remark that although (1.1) is completely integrable, we do not use the inverse scattering theory of Zakharov-Shabat [20]. We expect that one could compute the scattering data associated to our initial condition and conduct an analysis using inverse scattering theory that would describe the dynamics for all time. Our argument, however, has the merit of being relatively simple and should adapt to most nonintegrable nonlinearities that support stable solitons with exponentially decaying tails. An important example of such a nonintegrable equation is the 1d cubic-quintic NLS:

$$i\partial_t u + \frac{1}{2}\partial_x^2 u + |u|^2 u - \epsilon|u|^4 u = 0$$

Furthermore, our goal was not just to obtain Theorem 1.1 but to present it in the conceptual (yet rigorous) framework of symplectic restriction that illustrates its connection to previous work of the first author, Holmer-Zworski [7, 8].

We cite two papers from the physics literature as motivation for our problem. Stegeman-Segev [17] provide an overview of phase-driven two-soliton interaction in the context of optics, beginning with an account of the 1d case (1.1) that we study (see their Fig. 4) and proceeding to a discussion of two-soliton interaction in two dimensions in which the attractive forces between in-phase solitons can lead to spiraling structures – see their Fig. 6. The NLS equation also arises in a completely different physical setting, Bose-Einstein condensation. Strecker et.al. [18] describe an experiment producing multiple solitons, in which the model is (1.1) with a confining potential. A train of five solitons with successively opposite phases are produced and oscillate in a well. At the peak of the oscillations, the solitons bunch up but retain some separation; [18] explains this in terms of their phase differences.

We will now give an explanation of Theorem 1.1 and an overview of the proof. Consider $L^2(\mathbb{R}; \mathbb{C})$ as a manifold with metric

$$g_u(v_1, v_2) = \langle v_1, v_2 \rangle \stackrel{\text{def}}{=} \operatorname{Re} \int v_1 \bar{v}_2 \quad \text{for } u \in L^2, v_1, v_2 \in T_u L^2 \simeq L^2.$$

Introduce $J = -i$, viewed as an operator $T_u L^2 \rightarrow T_u L^2$. The corresponding symplectic form is

$$(1.13) \quad \omega_u(v_1, v_2) = \langle v_1, J^{-1}v_2 \rangle$$

Take as Hamiltonian the (densely defined, with domain³ $D = H^2$) function $H : L^2 \rightarrow \mathbb{R}$ given by

$$(1.14) \quad H(u) = \frac{1}{4} \int |u_x|^2 - \frac{1}{4} \int |u|^4$$

Then

$$H'(u) \in T_u^* L^2 \underset{\text{metric } g}{\simeq} T_u L^2$$

The corresponding flow is $\partial_t u = JH'(u)$ yielding (1.1).

Recalling that η is given by (1.2), consider the manifold of solitons

$$M = \{ \eta(\cdot, \mu, a, \theta, v) \mid \mu > 0, \theta \in \mathbb{R}, a \in \mathbb{R}, v \in \mathbb{R} \}.$$

Computations show that the restriction of the symplectic form ω to M is

$$i^* \omega = d\theta \wedge d\mu + dv \wedge da,$$

while the restriction of the Hamiltonian H to M is

$$H(\eta(\cdot, \mu, a, \theta, v)) = \frac{1}{2} \mu^{-1} v^2 - \frac{1}{6} \mu^3,$$

Note that the free single soliton flow (1.3) is just the solution to the Hamilton equations of motion for $H(\eta)$ with respect to $i^* \omega$:

$$\begin{cases} \dot{\mu} = \partial_\theta H(\eta) = 0 \\ \dot{a} = \partial_v H(\eta) = \mu^{-1} v \\ \dot{\theta} = -\partial_\mu H(\eta) = \frac{1}{2} \mu^{-2} v^2 + \frac{1}{2} \mu^2 \\ \dot{v} = -\partial_a H(\eta) = 0 \end{cases}$$

Turning to the double soliton problem, recall that we model the u in terms of u_z given by (1.6) where $z = (z^1, \dots, z^8)$ is given by (1.7). We introduce the shorthand notation

$$\eta_j \stackrel{\text{def}}{=} \eta(\cdot, \mu_j, a_j, \theta_j, v_j), \quad j = 1, 2.$$

Also recall that $h = e^{-a_0} \ll 1$, and the initial soliton separation is $2a_0 = 2 \log h^{-1} \gg 1$. Expanding the nonlinearity, we obtain

$$(1.15) \quad H_p(u_z) = H_p(\eta_1) + \langle H'_p(\eta_1), \eta_2 \rangle + H_p(\eta_2) + \langle H'_p(\eta_2), \eta_1 \rangle + O(h^4)$$

The last two terms are dominant near a_2 (on the effective support of η_2), so that the second soliton sees an “effective” Hamiltonian

$$(1.16) \quad H_{\text{eff}}(\mu_2, a_2, \theta_2, v_2) = H(\eta_2) + \langle H'_p(\eta_2), \eta_1 \rangle$$

³This domain is chosen so that $JH'(u) = -i(-\frac{1}{2}u_{xx} - |u|^2u) \in L^2$. Although we restrict to $u \in D = H^2$ here, we will prove estimates on the corresponding flow in H^1 . This parallels the situation in the theory of linear self-adjoint operators A , where a dense domain is specified but the flow associated to $-iA$ extends to a unitary operator on all of L^2 .

and thus its expected equations of motion are

$$(1.17) \quad \begin{cases} \dot{\mu}_2 = \partial_{\theta_2} H(\eta_2) + \partial_{\theta_2} \langle H'_p(\eta_2), \eta_1 \rangle \\ \dot{a}_2 = \partial_{v_2} H(\eta_2) + \partial_{v_2} \langle H'_p(\eta_2), \eta_1 \rangle \\ \dot{\theta}_2 = -\partial_{\mu_2} H(\eta_2) - \partial_{\mu_2} \langle H'_p(\eta_2), \eta_1 \rangle \\ \dot{v}_2 = -\partial_{a_2} H(\eta_2) - \partial_{a_2} \langle H'_p(\eta_2), \eta_1 \rangle \end{cases}$$

Likewise, the first two terms in (1.15) are dominant near a_1 so the first soliton sees an effective Hamiltonian

$$H_{\text{eff}}(\mu_1, a_1, \theta_1, v_1) = H(\eta_1) + \langle H'_p(\eta_1), \eta_2 \rangle$$

and thus its expected equations of motion are

$$(1.18) \quad \begin{cases} \dot{\mu}_1 = \partial_{\theta_1} H(\eta_1) + \partial_{\theta_1} \langle H'_p(\eta_1), \eta_2 \rangle \\ \dot{a}_1 = \partial_{v_1} H(\eta_1) + \partial_{v_1} \langle H'_p(\eta_1), \eta_2 \rangle \\ \dot{\theta}_1 = -\partial_{\mu_1} H(\eta_1) - \partial_{\mu_1} \langle H'_p(\eta_1), \eta_2 \rangle \\ \dot{v}_1 = -\partial_{a_1} H(\eta_1) - \partial_{a_1} \langle H'_p(\eta_1), \eta_2 \rangle \end{cases}$$

Pulling (1.17) and (1.18) together gives us a systems of eight equations in eight unknowns:

$$(1.19) \quad \begin{cases} \dot{\mu}_1 = \partial_{\theta_1} H(\eta_1) + \partial_{\theta_1} \langle H'_p(\eta_1), \eta_2 \rangle \\ \dot{a}_1 = \partial_{v_1} H(\eta_1) + \partial_{v_1} \langle H'_p(\eta_1), \eta_2 \rangle \\ \dot{\theta}_1 = -\partial_{\mu_1} H(\eta_1) - \partial_{\mu_1} \langle H'_p(\eta_1), \eta_2 \rangle \\ \dot{v}_1 = -\partial_{a_1} H(\eta_1) - \partial_{a_1} \langle H'_p(\eta_1), \eta_2 \rangle \\ \dot{\mu}_2 = \partial_{\theta_2} H(\eta_2) + \partial_{\theta_2} \langle H'_p(\eta_2), \eta_1 \rangle \\ \dot{a}_2 = \partial_{v_2} H(\eta_2) + \partial_{v_2} \langle H'_p(\eta_2), \eta_1 \rangle \\ \dot{\theta}_2 = -\partial_{\mu_2} H(\eta_2) - \partial_{\mu_2} \langle H'_p(\eta_2), \eta_1 \rangle \\ \dot{v}_2 = -\partial_{a_2} H(\eta_2) - \partial_{a_2} \langle H'_p(\eta_2), \eta_1 \rangle \end{cases}$$

After the even/odd symmetry assumption is imposed, one can distill from (1.19) the equations appearing in the statement of Theorem 1.1. ⁴

We find that the above argument yielding (1.19) is a little too vague to adapt to a rigorous proof. We now consider a different perspective that informally produces the

⁴In fact, the above heuristic argument does not invoke the even/odd symmetry assumption and thus we might expect the equations (1.19) even without this assumption. However, the equations (1.19) are only expected to be accurate to order $O(h^4)$. In the presence of the symmetry assumption the eight equations in (1.19) dramatically decouple as (1.9), (1.10), (1.11) which permits a direct analysis of these ODEs that shows that an $O(h^4)$ unknown can only have a limited $O(h^2)$ effect on the solution. In the general case, the eight equations in (1.19) are more interdependent and we are not certain as to the effect of $O(h^4)$ perturbations. This is not the only obstacle to removing the symmetry assumption; see comments below.

same set of equations (1.19) but adapts to yield a proof of Theorem 1.1 and in fact extends and unifies the results of [3, 6, 7, 8]. Recalling z defined in (1.7), consider now the eight-dimensional two-soliton manifold

$$M = \{ u_z = \eta_1 + \eta_2 = \eta(\cdot, \mu_1, a_1, \theta_1, v_1) + \eta(\cdot, \mu_2, a_2, \theta_2, v_2) \}$$

The symplectic form (1.13) restricted to M is

$$(1.20) \quad i^* \omega = \frac{1}{2} \sum_{\ell, m=1}^8 a_{\ell m}(z) dz^\ell \wedge dz^m$$

where

$$A(z) = (a_{\ell m}(z)), \quad a_{\ell m}(z) = \langle \partial_{z^\ell} u_z, J^{-1} \partial_{z^m} u_z \rangle$$

Let $H(u_z)$ denote the restriction to M of the Hamiltonian (1.14). The expected equations of motion for z^m are Hamilton's equations for $H(u_z)$ with respect to $i^* \omega$. These equations are:

$$(1.21) \quad \dot{z}^m = - \sum_{\ell=1}^8 \partial_{z^\ell} H(u_z) a^{\ell m}(z), \quad m = 1, \dots, 8$$

where $a^{\ell m}$ denotes the components of the inverse of the matrix $A = (a_{\ell m})$.

The matrix A contains $O(h^2)$ terms that result from the pairing of directions parallel to the first soliton with directions parallel to the second soliton. Moreover, $H(u_z)$ contains additional $O(h^2)$ terms arising from the quadratic part of (1.14) not represented in (1.19). It turns out that $O(h^2)$ terms in $a^{\ell m}$ and $O(h^2)$ terms in $H(u_z)$ each give rise to terms which cancel in (1.21). This hinges upon the fact that

$$(1.22) \quad \partial_{z^\ell} H(u_z) = - \sum_{j=1}^8 b^j a_{j\ell} + \partial_{z^\ell} \langle \eta_1, H'_p(\eta_2) \rangle + \partial_{z^\ell} \langle \eta_2, H'_p(\eta_1) \rangle$$

where

$$b_2 = \partial_{v_1} H(\eta_1), \quad b_4 = \partial_{v_2} H(\eta_2), \quad b_5 = -\partial_{\mu_1} H(\eta_1), \quad b_7 = -\partial_{\mu_2} H(\eta_2)$$

and all other $b_j = 0$. When this equation is substituted into (1.21), one can witness the simplification arising from the pairing of A and A^{-1} , and this shows that (1.21) is equivalent to (1.19). We elaborate upon this in Appendix A.

The merit in this point of view is that the equations (1.21) readily follow from the symplectic decomposition of the flow—that is, we select z (via the implicit function theorem) so that

$$(1.23) \quad u = u_z + w$$

where $w \in T_z M^\perp$ (the *symplectic* orthogonal complement to $T_z M$ in $T_{u_z} L^2$). In §3 (Lemma 3.1) a general argument is given showing that the equations (1.21) follow,

with errors of size $h^4 + \|w\|_{H^1}^2$. This argument exploits the fact that (1.21), with errors of size $h^4 + \|w\|_{H^1}^2$, is equivalent to

$$(1.24) \quad \partial_t u_z = \Pi_z JH'(u_z) + O(h^4 + \|w\|_{H^1}^2)$$

where $\Pi_z : T_{u_z} L^2 \rightarrow T_z M$ is the symplectic orthogonal projection operator given explicitly by

$$\Pi_z f = \sum_{\ell, m=1}^8 \langle f, J^{-1} \partial_{z^\ell} u_z \rangle a^{\ell m}(z) \partial_{z^m} u_z$$

The proof of Lemma 3.1 makes no reference to the specific meaning of H or u_z , and a similar result with nearly identical proof would yield the equations of motion in many other problems, including those studied in [6, 7, 8]. The fact that the equations of motion follow automatically but rigorously from the symplectic decomposition (1.23) is one of the main advantages of this geometric approach to our problem, as opposed to a more *ad hoc* approach based on the discussion surrounding (1.16).⁵

It then remains to show that $\|w(t)\|_{H_x^1} \lesssim h^2$ on the time scale $O(h^{-1})$, which we would like to prove using a suitable adaptation of the Lyapunov functional method initiated into the theory of orbital stability of single solitons by Weinstein [19]. Unfortunately, the presence of the Π_z projection in (1.24) corrupts this computation and only yields a bound $\|w(t)\|_{H_x^1} \lesssim h$. To eliminate this problem, we construct a function $\nu_z = O(h^2)$, whose only time dependence is through the parameter z , such that the distorted double-soliton function $\tilde{u}_z = u_z + \nu_z$ satisfies

$$(1.25) \quad \partial_t \tilde{u}_z = JH'(\tilde{u}_z) + O(h^4 + \|w\|_{H^1}^2)$$

which is just (1.24) without the Π_z projection. The construction of ν_z is carried out in §4.

We add this correction $\tilde{\nu}_z$ to our soliton manifold M and consider the distorted manifold $\tilde{M} = \{\tilde{u}_z\}$. The solution u to (1.1) now has a decomposition $u = \tilde{u}_z + \tilde{w}$ where \tilde{u}_z satisfies (1.25) and it suffices to prove that $\|\tilde{w}(t)\|_{H_x^1} \lesssim h^2$. In other words, we would like to show that the exact solution to (1.1) is approximately equal to the solution to the approximate equation (1.25). In §5, a Lyapunov functional is employed to obtain the needed control on \tilde{w} . The Lyapunov functional used is a superposition of two copies—one for each soliton—of the classical functional, built from energy, momentum and mass, employed by Weinstein [19] to prove orbital stability of single solitons. This superposition was previously used by Martel-Merle-Tsai [10]

⁵The idea that the equations of motions should be Hamilton's equations for the restricted Hamiltonian with respect to the restricted symplectic form was introduced in [7, 8] and supported informally with an argument involving Darboux's theorem. The equations of motion thus obtained were used as a guide in the analysis in [6, 7, 8] but the general rigorous connection between the symplectic decomposition of the flow and the equations of motion, as obtained in our Lemma 3.1, was not obtained in [6, 7, 8].

in their study of the orbital stability of spreading multiple solitons. Our presentation of this component of the argument is a little different from [19] or [10] and more in line with the abstract orbital stability theory developed by Grillakis-Shatah-Strauss [4, 5]. Roughly, we prove that if $W : L^2 \rightarrow \mathbb{R}$ is a (densely defined) functional such that the derivative is of order $O(h)$ on \tilde{M} , and if we define L to be the quadratic part of W above \tilde{M} , then $\partial_t L$ is essentially the quadratic part of the Poisson bracket $\{H, W\}(u)$ above \tilde{M} , which we show is of order $O(h^5)$.

Let us note that $h^{-\delta}$ losses occur in several estimates, which were not necessarily indicated in the above introduction, owing to the fact that in the attractive case $|v|$ can exceed h by a factor of $\log h^{-1}$ and a decreases below a_0 , as well as the presence of an x -multiplication factor in terms involving $\partial_{v_j} u_z$ in both the attractive and repulsive cases. We indicate the presence of such losses by writing, for example, h^{4-} . These losses are more carefully quantified in the concluding summary of the proof in §6.

We emphasize that the methods in §3 and §5, although stated only for the problem at hand, are fairly general and widely applicable to problems in orbital stability of single [19, 4, 5] and multiple [11] solitons and the dynamics of solitons in slowly varying potentials [3, 6, 8, 2, 13], weak rough potentials [1, 7, 15], and the interaction of two soliton tails, as considered here. The portion of the analysis most specific to the problem at hand appears in §4, where the approximate solution is constructed. In this section, we consider the two components of the double-soliton separately and exploit the group structure of each individual soliton to pull-back to a nearly-stationary problem, which can be solved by operator inversion. This method was introduced by Holmer-Zworski [8] to produce an improvement of the result by Fröhlich-Gustafson-Jonsson-Sigal [3] on the dynamics of single solitons in a slowly-varying potential, eliminating the uncontrollable errors in the ODEs appearing in [3].

Let us point out some related papers. Marzuola-Weinstein [12] consider the dynamics of symmetric and antisymmetric states in a double well-potential. Krieger-Martel-Raphaël [9] construct two-soliton solutions with separating components asymptotically as $t \rightarrow +\infty$ for the nonlinear Hartree equation, where the long-range effects of the nonlinearity complicate the analysis but also lead to nonnegligible perturbations of the asymptotic trajectories. Our analysis is similar in several ways to that of [9], although our priorities are different – we study the dynamics for a finite (but dynamically significant) time of an initial data that is close to a double-soliton, whereas they provide infinite time dynamics for an exact double-soliton solution. The problem of stability of nonintegrable NLS multiple solitons, with components that separate as $t \rightarrow \infty$, has been considered by Perelman [14], Rodnianski-Soffer-Schlag [16], and Martel-Merle-Tsai [10].

We now remark on where we rely upon the even/odd symmetry assumption on the solution. While the arguments in §3 yielding (1.21) apply in general, in §4, when constructing the solution \tilde{u}_z to the approximate equation (1.25), we do make use of the

symmetry assumption, although we have sketched an argument (not included in this paper) showing how one can adapt the argument to the general case. The symmetry assumption also greatly simplifies the computations carried out in Appendix A which ultimately yield the ODEs (1.9), (1.10), (1.11) in Theorem 1.1. The integrals in the general case appear very complicated, and we are less confident that we could control the propagation of $O(h^4)$ errors, as previously remarked. However, the one place where the symmetry assumption is used critically is in obtaining the *upper* bound on the Lyapunov function used in §5 to show the closeness of the true solution u and the solution \tilde{u}_z of the approximate equation (1.25). Our guess is that to resolve this issue, one would need to restructure the Martel-Merle-Tsai Lyapunov function in a substantial way. The lower bound on the Lyapunov function, however, carries through in general.

1.1. Acknowledgements. We thank Maciej Zworski and Galina Perelman for helpful discussion related to this paper. J.H. was partially supported by NSF Grant DMS-0901582 and a Sloan Research Fellowship.

2. BACKGROUND ON SOLITONS, HAMILTONIAN STRUCTURE, AND LYAPUNOV FUNCTIONALS

The NLS equation (1.1) can be put into Hamiltonian form as follows. Take as the ambient symplectic manifold $L^2 = L^2(\mathbb{R}; \mathbb{C})$ with metric $\langle v_1, v_2 \rangle_u = \operatorname{Re} \int v_1 \bar{v}_2$ for $u \in L^2$, $v_1, v_2 \in T_u L^2 = L^2$. Let $J = -i$, viewed as an operator $T_u L^2 \rightarrow T_u L^2$. The corresponding symplectic form is $\omega_u(v_1, v_2) = \langle v_1, J^{-1}v_2 \rangle_u$ (we henceforth drop the u -subscript). Define the (densely defined, with domain $D = H^2$) Hamiltonian $H : L^2 \rightarrow \mathbb{R}$ as

$$H(u) = \frac{1}{4} \int |u_x|^2 dx - \frac{1}{4} \int |u|^4.$$

Using the metric $\langle \cdot, \cdot \rangle$ defined above, $H'(u) \in T_u^* L^2$ is identified with an element of $T_u L^2$. The free NLS equation (1.1) is

$$(2.1) \quad \partial_t u = JH'(u)$$

Solutions to (1.1) also satisfy conservation of mass $M(u)$ and momentum $P(u)$, where

$$M(u) = \frac{1}{2} \int |u|^2, \quad P(u) = \frac{1}{2} \operatorname{Im} \int \bar{u} u_x.$$

Let $\phi(x) = \operatorname{sech} x$ and

$$\eta(x, \mu, a, \theta, v) = e^{i\theta} e^{i\mu^{-1}v(x-a)} \mu \phi(\mu(x-a))$$

Direct computation shows that $M(\eta) = \mu$ and $P(\eta) = v$. Consider the manifold of solitons

$$M = \{ \eta(\cdot, \mu, a, \theta, v) \mid \mu > 0, \theta \in \mathbb{R}, a \in \mathbb{R}, v \in \mathbb{R} \}.$$

The tangent space at $\eta = \eta(\cdot, \mu, a, \theta, v)$ is

$$T_{(\mu, a, \theta, v)}M = \text{span}\{\partial_\mu\eta, \partial_\theta\eta, \partial_a\eta, \partial_v\eta\}.$$

Note that $JH'(\eta) \in T_{(\mu, a, \theta, v)}M$, and thus the flow associated to (1.1) will remain on M if it is initially on M . Specifically, direct computation shows

$$(2.2) \quad JH'(\eta) = \left(\frac{1}{2}\mu^{-2}v^2 + \frac{1}{2}\mu^2\right)\partial_\theta\eta + \mu^{-1}v\partial_a\eta.$$

To gain a better understanding of (1.3) and (2.2), we can restrict ω to M to obtain

$$i^*\omega = d\theta \wedge d\mu + dv \wedge da,$$

where $i : M \rightarrow L^2$ denotes the inclusion and restrict H to M to obtain

$$H(\eta) = \frac{1}{2}\mu^{-1}v^2 - \frac{1}{6}\mu^3,$$

and then note that (1.3) is just the solution to the Hamilton equations of motion for $H(\eta)$ with respect to $i^*\omega$:

$$(2.3) \quad \begin{cases} \dot{\mu} = \partial_\theta H(\eta) = 0 \\ \dot{a} = \partial_v H(\eta) = \mu^{-1}v \\ \dot{\theta} = -\partial_\mu H(\eta) = \frac{1}{2}\mu^{-2}v^2 + \frac{1}{2}\mu^2 \\ \dot{v} = -\partial_a H(\eta) = 0 \end{cases}$$

Suppose we knew that $JH'(\eta) \in T_{(\mu, a, \theta, v)}M$ and wanted to recover the coefficients as in (2.2). This could be achieved by noting that

$$\begin{aligned} JH'(\eta) &= \partial_t\eta \\ &= \dot{\mu}\partial_\mu\eta + \dot{a}\partial_a\eta + \dot{\theta}\partial_\theta\eta + \dot{v}\partial_v\eta \\ &= \partial_v H(\eta)\partial_a\eta - \partial_\mu H(\eta)\partial_\theta\eta \end{aligned}$$

Moreover, the functionals M and P , considered as *auxiliary* Hamiltonians, have associated Hamilton vector fields

$$JM'(\eta) = -\partial_\theta\eta \quad JP'(\eta) = \partial_a\eta.$$

This enables us to write

$$(2.4) \quad JH'(\eta) = \partial_v H(\eta)JP'(\eta) + \partial_\mu H(\eta)JM'(\eta).$$

From this, we learn that $W'_{(\mu, a, \theta, v)}(\eta) = 0$, where

$$(2.5) \quad W_{(\mu, a, \theta, v)}(u) \stackrel{\text{def}}{=} -\partial_\mu H(\eta)M(u) - \partial_v H(\eta)P(u) + H(u).$$

The functional $L_{(\mu, a, \theta, v)}(u) = W_{(\mu, a, \theta, v)}(u) - W_{(\mu, a, \theta, v)}(\eta)$ is the Lyapunov functional used in the classical orbital stability theory for (1.1) due to Weinstein [19].

3. EFFECTIVE DYNAMICS

Now we turn to the double soliton problem and begin the proof of Theorem 1.1. Consider the two-soliton submanifold M of L^2 given by

$$M = \{u_z \stackrel{\text{def}}{=} \eta(\cdot, \mu_1, a_1, \theta_1, v_1) + \eta(\cdot, \mu_2, a_2, \theta_2, v_2)\}.$$

Note that M is just the linear superposition of two single solitons. We adopt the notation

$$z = (z^1, z^2, z^3, z^4, z^5, z^6, z^7, z^8) = (\mu_1, a_1, \mu_2, a_2, \theta_1, v_1, \theta_2, v_2),$$

for coordinates on this manifold M . Next, we give the form of the symplectic orthogonal projection operator

$$\Pi_z : T_{u_z}L^2 \rightarrow T_zM,$$

Note that $T_{u_z}L^2$ is naturally identified with L^2 . A consequence of the requirement that $\langle f - \Pi_z f, J^{-1}\partial_{z^\ell}u_z \rangle = 0$, $\ell = 1, \dots, 8$ is that

$$(3.1) \quad \Pi_z f = \sum_{\ell, m=1}^8 \langle f, J^{-1}\partial_{z^\ell}u_z \rangle a^{\ell m}(z) \partial_{z^m}u_z$$

where $A(z) = (a_{\ell m}(z))$ is the 8×8 matrix with components $a_{\ell m}(z) = \langle \partial_{z^\ell}u_z, J^{-1}\partial_{z^m}u_z \rangle$ and $A(z)^{-1} = (a^{\ell m}(z))$ is the inverse matrix.

Let $i : M \rightarrow L^2$ denote the inclusion. It follows from the definition of $A(z)$ that the restricted symplectic form $i^*(\omega)$ takes the form

$$(3.2) \quad i^*(\omega) = \frac{1}{2} \sum_{\ell, m=1}^8 a_{\ell m} dz^\ell \wedge dz^m.$$

It also follows by substitution into (3.1) that

$$\Pi_z JH'(u_z) = - \sum_{\ell, m=1}^8 \partial_{z^\ell}H(u_z) a^{\ell m}(z) \partial_{z^m}u_z$$

Consequently, the equation $\partial_t u_z = \Pi_z JH'(u_z)$ is equivalent to the system of equations

$$\dot{z}^m = - \sum_{\ell=1}^8 \partial_{z^\ell}H(u_z) a^{\ell m}(z) \quad m = 1, \dots, 8,$$

which are precisely Hamiltonian's equations of motion for the restricted (to M) Hamiltonian $z \mapsto H(u_z)$ with respect to the restricted (to M) symplectic form $i^*(\omega)$.

We propose to model the solution u to (2.1) by

$$(3.3) \quad u = u_z + w$$

where $u_z \in M$ is chosen so that the symplectic orthogonality conditions

$$(3.4) \quad \langle w, J^{-1}\partial_{z^\ell}u_z \rangle = 0, \quad \ell = 1, \dots, 8.$$

hold. The fact that such a z exists follows from the implicit function theorem and the assumed smallness of w . Note that if we assume $u(t)$ solves (2.1), this induces time dependence on the parameters $z \in \mathbb{R}^8$.⁶

Lemma 3.1 (effective dynamics). *Suppose that u evolves according to (2.1) and z, w are defined by (3.3) so that the orthogonality conditions (3.4) hold. Then*

$$(3.5) \quad \|\partial_t u_z - \Pi_z JH'(u_z)\|_{T_z M} \lesssim \|w\|_{H^1}^2 + \max_{1 \leq n \leq 8} \|J^{-1} \partial_{z^n} \Pi_z^\perp JH'(u_z)\|_{H^1}^2.$$

Equivalently, considering M as an 8-dimensional symplectic manifold equipped with the symplectic form $i^*(\omega)$ given in (3.2), the Hamilton's equations of motion for z induced by the restricted Hamiltonian $z \mapsto H(u_z)$ approximately hold as follows:

$$(3.6) \quad \left| \dot{z}^m + \sum_{\ell=1}^8 \partial_{z^\ell} H(u_z) a^{\ell m}(z) \right| \lesssim \|w\|_{H^1}^2 + \max_{1 \leq n \leq 8} \|J^{-1} \partial_{z^n} \Pi_z^\perp JH'(u_z)\|_{H^1}^2 \quad m = 1, \dots, 8$$

The norm $\|\cdot\|_{T_z M}$ is the one induced by the metric $\langle \cdot, \cdot \rangle_{u_z}$. As $T_z M$ is finite-dimensional, we have the norm-equivalence to

$$\left\| \sum_{\ell=1}^8 \gamma(z^\ell) \partial_{z^\ell} u_z \right\|_{T_z M} \sim \max_{1 \leq \ell \leq 8} |\gamma(z^\ell)|$$

Proof. Since u solves (2.1), we obtain from (3.3) the equation for w :

$$(3.7) \quad \partial_t w = -(\partial_t u_z - \Pi_z JH'(u_z)) + \Pi_z^\perp JH'(u_z) + JH''(u_z)w + O_{H^1}(\|w\|_{H^1}^2)$$

By applying ∂_t to (3.4), we obtain

$$0 = \langle \partial_t w, J^{-1} \partial_{z^n} u_z \rangle + \langle w, J^{-1} \partial_{z^n} \partial_t u_z \rangle$$

Here we have used that $\partial_t \partial_{z^n} = \partial_{z^n} \partial_t$, which holds provided we adopt the convention that $\partial_{z^\ell} \dot{z}^m = 0$ for all $1 \leq \ell, m \leq 8$. Substituting (3.7) and using that $\langle \Pi_z^\perp JH'(u_z), J^{-1} \partial_{z^n} u_z \rangle = 0$, we obtain

$$(3.8) \quad 0 = A + B + C + D$$

where

$$\begin{aligned} A &= -\langle \partial_t u_z - \Pi_z JH'(u_z), J^{-1} \partial_{z^n} u_z \rangle \\ B &= \langle JH''(u_z)w, J^{-1} \partial_{z^n} u_z \rangle \\ C &= \langle w, J^{-1} \partial_{z^n} \partial_t u_z \rangle \\ D &= \langle O(w^2), J^{-1} \partial_{z^n} u_z \rangle \end{aligned}$$

⁶Note that here w is properly understood as an element of $T_{u_z} L^2$ and in (3.3) we mean that, starting at u_z we take the flow-forward (by “time” 1) in the direction w . However, using the natural identification between $T_{u_z} L^2$ and L^2 , (3.3) makes sense as an equation involving functions in L^2 .

Since $J^*J^{-1} = -1$ and $H''(u_z)$ is self-adjoint,

$$B = -\langle w, H''(u_z)\partial_{z^n}u_z \rangle = -\langle w, \partial_{z^n}H'(u_z) \rangle = -\langle w, J^{-1}\partial_{z^n}JH'(u_z) \rangle$$

Hence

$$\begin{aligned} B + C &= \langle w, J^{-1}\partial_{z^n}(\partial_t u_z - JH'(u_z)) \rangle \\ &= \langle w, J^{-1}\partial_{z^n}(\partial_t u_z - \Pi_z JH'(u_z)) \rangle - \langle w, J^{-1}\partial_{z^n}\Pi_z^\perp JH'(u_z) \rangle \end{aligned}$$

Let $R = \partial_t u_z - \Pi_z JH'(u_z) \in T_z M$, and expand with respect to the basis of $T_z M$ as

$$R = \sum_{\ell=1}^8 \gamma_\ell(z) \partial_{z^\ell} u_z.$$

It follows from (3.8) that

$$(3.9) \quad \langle R, J^{-1}\partial_{z^n}u_z \rangle = \langle w, J^{-1}\partial_{z^n}R \rangle - \langle w, J^{-1}\partial_{z^n}\Pi_z^\perp JH'(u_z) \rangle + O(\|w\|_{H^1}^2).$$

We have

$$\partial_{z^n}R = \sum_{\ell=1}^8 \partial_{z^n}\gamma_\ell(z) \partial_{z^\ell}u_z + \sum_{\ell=1}^8 \gamma_\ell(z) \partial_{z^n}\partial_{z^\ell}u_z.$$

Since $w \in T_z M^\perp$,

$$\langle w, J^{-1}\partial_{z^n}R \rangle = \sum_{\ell=1}^8 \gamma_\ell(z) \langle w, J^{-1}\partial_{z^n}\partial_{z^\ell}u_z \rangle$$

and hence

$$(3.10) \quad |\langle w, J^{-1}\partial_{z^n}R \rangle| \lesssim \|w\|_{H^1} \|R\|_{T_z M}.$$

The lemma follows from (3.9), $\|R\|_{T_z M} = \max_{1 \leq n \leq 8} |\langle R, J^{-1}\partial_{z^n}u_z \rangle|$, (3.10), and Cauchy-Schwarz. \square

In our case we shall have

$$\|J^{-1}\partial_{z^n}\Pi_z^\perp JH'(u_z)\|_{H^1}^2 \lesssim h^{4-}.$$

We carry out computations of (3.5) in Appendix A and show that (3.5) is equivalent to (1.19), with error terms $O(h^{4-})$, even without the even/odd assumption on the solution. It is further shown in Appendix A that when the even/odd assumption is imposed and the integrals in (1.19) are explicitly computed, we obtain

$$\begin{cases} \dot{\mu} = (-1)^\sigma (8a - 4)ve^{-2a} + O(h^{4-}) \\ \dot{a} = \mu^{-1}v + (-1)^\sigma (-4a + \frac{2}{3}\pi^2)ve^{-2a} + O(h^{4-}) \\ \dot{\theta} = \frac{1}{2}\mu^2 + \frac{1}{2}v^2\mu^{-2} + 18(-1)^\sigma e^{-2a} + O(h^{4-}) \\ \dot{v} = -4(-1)^\sigma e^{-2a} + O(h^{4-}) \end{cases}$$

The solution (μ, a, θ, v) is adequately approximated by the ODEs appearing in the statement of Theorem 1.1.

4. APPROXIMATE SOLUTION

By Lemma 3.1 and (3.7), the equation for w is

$$\begin{cases} \partial_t w = JH''(u_z)w + \Pi_z^\perp JH'(u_z) + O_{H^1}(\|w\|_{H^1}^2 + h^{4-}) \\ w|_{t=0} = w_0 \end{cases}$$

The next step is to show that there exists a function $\nu_z(x)$ such that $\|\nu_z\|_{H^1} \lesssim h^2$, whose only time dependence occurs through the parameter z , such that

$$(4.1) \quad \partial_t \nu_z = JH''(u_z)\nu_z + \Pi_z^\perp JH'(u_z) + O_{H^1}(h^{4-} + \|w\|_{H^1}^2)$$

Here it is assumed that $z \in \mathbb{R}^8$ evolves according to Lemma 3.1, i.e.

$$(4.2) \quad \partial_t u_z = \Pi_z JH'(u_z) + O_{H^1}(h^{4-} + \|w\|_{H^1}^2)$$

The initial data $\nu_z|_{t=0}$ is *not* prescribed but our structural assumption on ν_z is fairly rigid. Note that given (4.2), the assertion that ν_z solve (4.1) is equivalent to the statement that $\tilde{u}_z \stackrel{\text{def}}{=} u_z + \nu_z$ solve

$$(4.3) \quad \partial_t \tilde{u}_z = JH'(\tilde{u}_z) + O_{H^1}(h^{4-} + \|w\|_{H^1}^2).$$

This is an approximate solution to (2.1) (that does not, in general, satisfy the specified initial data).

Let $g : L^2 \rightarrow L^2$ be the operator attached to the parameters (μ, a, θ, v) that acts on a function ρ as follows:

$$(4.4) \quad (g\rho)(x) = e^{i\theta} e^{i\mu^{-1}v(x-a)} \mu \rho(\mu(x-a)).$$

The inverse action is

$$g^{-1}\rho(x) = e^{-i\theta} e^{-i\mu^{-2}vx} \mu^{-1} \rho(\mu^{-1}x + a).$$

The adjoint action g^* with respect to $\langle \cdot, \cdot \rangle$ is

$$g^*\rho(x) = e^{-i\theta} e^{-i\mu^{-2}vx} \rho(\mu^{-1}x + a) = \mu g^{-1}\rho(x).$$

Denote $\phi(x) = \text{sech } x$. Then $u_z = g_1\phi + g_2\phi$. We look for a solution ν_z to (4.1) in the form

$$(4.5) \quad \nu_z = \sum_{j=1}^2 \alpha_j g_j \rho_j,$$

where $\alpha_j = \alpha_j(\mu_1, a_1, \theta_1, v_1, \mu_2, a_2, \theta_2, v_2)$ and g_j is the operator corresponding to $(\mu_j, a_j, \theta_j, v_j)$. That is, we assume ν_z can be decomposed into two pieces, each of which can be pulled back to a stationary equation and solved by operator inversion. The time dependence of ν_z occurs only through z .

The next step is to substitute (4.5) into (4.1). The resulting equation simplifies provided we assume that each ρ_j satisfies $|\rho_j(x)| \lesssim h^2 e^{(-1+)|x|}$ for $|x| \geq 1$ as certain cross terms become $O_{H^1}(h^{4-})$. In this case, (4.1) will be satisfied provided for both $j = 1, 2$, we have

$$\partial_t(\alpha_j g_j \rho_j) = JH''(g_j \phi)(\alpha_j g_j \rho_j) + \Pi_z^\perp J((g_j \phi)^2 \overline{g_{3-j} \phi} + 2|g_j \phi|^2 g_{3-j} \phi) + O_{H^1}(h^{4-} + \|w\|_{H^1}^2)$$

In the proof, we delete the j -subscripts, denote $\tilde{g} = g_{3-j}$ and moreover assume that $\dot{\alpha}_j = O(h^2)$. Then we aim to solve

$$\partial_t(g\rho) = JH''(g\phi)(g\rho) + \alpha^{-1} \Pi^\perp J((g\phi)^2 \overline{\tilde{g}\phi} + 2|g\phi|^2 \tilde{g}\phi) + O_{H^1}(h^{4-} + \|w\|_{H^1}^2)$$

The form of the operator Π^\perp can be simplified, since we only need to keep the $O(1)$ and $O(h)$ parts. This equation takes the form

$$(4.6) \quad \partial_t(g\rho) = JH''(g\phi)(g\rho) + \alpha^{-1} \Pi^\perp Jgf + O_{H^1}(h^{4-} + \|w\|_{H^1}^2)$$

where, in the case $j = 1$,

$$(4.7) \quad \begin{aligned} f &= g_1^{-1}[(g_1 \phi)^2 \overline{g_2 \phi}] + 2g_1^{-1}[|g_1 \phi|^2 g_2 \phi] \\ &= e^{i\omega_1} e^{i\omega_2 x} \mu_j^2 \mu_{3-j} \phi(\mu_{3-j} \mu_j^{-1} x + (a_j - a_{3-j})) \phi(x)^2 \\ &\quad + 2e^{i\omega_3} e^{i\omega_4 x} \mu_j^2 \mu_{3-j} \phi(\mu_{3-j} \mu_j^{-1} x + (a_j - a_{3-j})) \phi(x)^2 \end{aligned}$$

with

$$\begin{aligned} \omega_1 &\stackrel{\text{def}}{=} \theta_1 - \theta_2 - \mu_{3-j}^{-1} v_{3-j} (a_j - a_{3-j}) \\ \omega_2 &\stackrel{\text{def}}{=} \mu_j^{-2} v_j - \mu_{3-j}^{-1} \mu_j^{-1} v_{3-j} \\ \omega_3 &\stackrel{\text{def}}{=} \theta_2 - \theta_1 + \mu_{3-j}^{-1} v_{3-j} (a_j - a_{3-j}) \\ \omega_4 &\stackrel{\text{def}}{=} \mu_j^{-2} v_j + \mu_{3-j}^{-1} \mu_j^{-1} v_{3-j} \end{aligned}$$

In the case $j = 2$,

$$(4.8) \quad f = g_2^{-1}[(g_2 \phi)^2 \overline{g_1 \phi}] + 2g_2^{-1}[|g_2 \phi|^2 g_1 \phi].$$

with a similar expansion. The only important feature of these expressions is that $e^{(1-)\langle x \rangle} f = O(h^2)$ and $e^{(1-)\langle x \rangle} \partial_t f = O(h^3)$ when $\theta_1 - \theta_2$ is a constant (as in the case of the even/odd symmetry assumption in Theorem 1.1).

Now we begin the task of pulling back (4.6) – applying g^* to (4.6), we obtain

$$(4.9) \quad g^* \partial_t g\rho = g^* [JH''(g\phi)(g\rho)] + \alpha^{-1} g^* \Pi^\perp Jgf + O_{H^1}(h^{4-} + \|w\|_{H^1}^2).$$

First, we aim to simplify the term $g^* [JH''(g\phi)(g\rho)]$ in (4.9). Let $K_g(\phi) = H(g\phi)$. It follows that $K'_g(\phi) = g^* H'(g\phi)$ and $K''_g(\phi) = g^* [H''(g\phi)(g\bullet)]$. By direct substitution, we compute:

$$K_g(u) = \mu^3 H(u) + \mu v P(u) + \frac{1}{2} v^2 \mu^{-1} M(u)$$

Since $g^*J = Jg^*$, we have

$$(4.10) \quad \begin{aligned} g^*[JH''(g\phi)(g\bullet)] &= JK_g''(\phi) \\ &= \mu^3 JH''(\phi) + \mu v JP''(\phi) + \frac{1}{2}v^2\mu^{-1}JM''(\phi) \end{aligned}$$

Second, we seek to simplify the term $g^*\partial_t g\rho$ in (4.9). Define the operators

$$\begin{aligned} \tilde{\partial}_\mu &\stackrel{\text{def}}{=} \partial_\mu|_{(1,0,0,0)} = \partial_x x \\ \tilde{\partial}_a &\stackrel{\text{def}}{=} \partial_a|_{(1,0,0,0)} = -\partial_x \\ \tilde{\partial}_\theta &\stackrel{\text{def}}{=} \partial_\theta|_{(1,0,0,0)} = i \\ \tilde{\partial}_v &\stackrel{\text{def}}{=} \partial_v|_{(1,0,0,0)} = ix \end{aligned}$$

Let

$$(4.11) \quad \begin{aligned} \bar{\partial}_\mu &\stackrel{\text{def}}{=} g^*\partial_\mu g &= \partial_x x - i\mu^{-2}vx &= \tilde{\partial}_\mu - \mu^{-2}v\tilde{\partial}_v \\ \bar{\partial}_a &\stackrel{\text{def}}{=} g^*\partial_a g &= -\mu^2\partial_x - iv &= \mu^2\tilde{\partial}_a - v\tilde{\partial}_\theta \\ \bar{\partial}_\theta &\stackrel{\text{def}}{=} g^*\partial_\theta g &= i\mu &= \mu\tilde{\partial}_\theta \\ \bar{\partial}_v &\stackrel{\text{def}}{=} g^*\partial_v g &= i\mu^{-1}x &= \mu^{-1}\tilde{\partial}_v \end{aligned}$$

It follows from the chain rule that

$$g^*\partial_t g = \dot{\mu}\bar{\partial}_\mu + \dot{a}\bar{\partial}_a + \dot{\theta}\bar{\partial}_\theta + \dot{v}\bar{\partial}_v.$$

Using (3.5),

$$(4.12) \quad \begin{aligned} g^*\partial_t g &= v\mu^{-1}\bar{\partial}_a + \frac{1}{2}\mu^2\bar{\partial}_\theta + O_{H^1}(h^2) \\ &= v\mu\tilde{\partial}_a + \frac{1}{2}\mu^3\tilde{\partial}_\theta + O_{H^1}(h^2) \end{aligned}$$

Finally, we aim to simplify the term $g^*\Pi^\perp Jgf$ in (4.9). We will show that

$$(4.13) \quad g^*\Pi_{(\mu,a,\theta,v)} Jgf = \mu\Pi_{(1,0,0,0)} Jf$$

Using that $J^*J^{-1} = -1$ and (4.11), we obtain

$$\begin{aligned} g^*\Pi Jgf &= \langle Jgf, J^{-1}\partial_a g\phi \rangle g^*\partial_v g\phi - \langle Jgf, J^{-1}\partial_v g\phi \rangle g^*\partial_a g\phi \\ &\quad + \langle Jgf, J^{-1}\partial_\mu g\phi \rangle g^*\partial_\theta g\phi - \langle Jgf, J^{-1}\partial_\theta g\phi \rangle g^*\partial_\mu g\phi \\ &= -\langle f, \bar{\partial}_a \phi \rangle \bar{\partial}_v \phi + \langle f, \bar{\partial}_v \phi \rangle \bar{\partial}_a \phi - \langle f, \bar{\partial}_\mu \phi \rangle \bar{\partial}_\theta \phi + \langle f, \bar{\partial}_\theta \phi \rangle \bar{\partial}_\mu \phi \end{aligned}$$

Substituting (4.11), after a few cancelations we obtain

$$\begin{aligned} \mu^{-1}g^*\Pi Jgf &= -\langle f, \tilde{\partial}_a \phi \rangle \tilde{\partial}_v \phi + \langle f, \tilde{\partial}_v \phi \rangle \tilde{\partial}_a \phi - \langle f, \tilde{\partial}_\mu \phi \rangle \tilde{\partial}_\theta \phi + \langle f, \tilde{\partial}_\theta \phi \rangle \tilde{\partial}_\mu \phi \\ &= +\langle Jf, J^{-1}\tilde{\partial}_a \phi \rangle \tilde{\partial}_v \phi - \langle Jf, J^{-1}\tilde{\partial}_v \phi \rangle \tilde{\partial}_a \phi \\ &\quad + \langle Jf, J^{-1}\tilde{\partial}_\mu \phi \rangle \tilde{\partial}_\theta \phi - \langle Jf, J^{-1}\tilde{\partial}_\theta \phi \rangle \tilde{\partial}_\mu \phi \end{aligned}$$

which establishes (4.13).

Note that it follows from (4.13) that

$$g^* \Pi_{(\mu, \alpha, \theta, v)}^\perp g J f = \mu \Pi_{(1, 0, 0, 0)}^\perp J f$$

Using the expressions (4.10), (4.12), and (4.13), the equation (4.9) converts to

$$\begin{aligned} v \mu \tilde{\partial}_a \rho + \frac{1}{2} \mu^3 \tilde{\partial}_\theta \rho + \mu \partial_t \rho &= \mu^3 J H''(\phi)(\rho) + \mu v J P''(\phi)(\rho) \\ &+ \alpha^{-1} \mu \Pi_{(1, 0, 0, 0)}^\perp J f + O_{H^1}(h^{4^-} + \|w\|_{H^1}^2) \end{aligned}$$

Noting that $J P''(\phi) = \tilde{\partial}_a$ and $J M''(\phi) = -\tilde{\partial}_\theta$, the equation becomes

$$\frac{1}{2} \mu^3 J M''(\phi) \rho + \mu^3 J H''(\phi) \rho - \mu \partial_t \rho = -\alpha^{-1} \mu \Pi_{(1, 0, 0, 0)}^\perp J f + O_{H^1}(h^{4^-} + \|w\|_{H^1}^2)$$

Hence we see we should take $\alpha = \mu^{-2}$ so that the equation becomes

$$\frac{1}{2} J M''(\phi) \rho + J H''(\phi) \rho - \mu^{-2} \partial_t \rho = -\Pi_{(1, 0, 0, 0)}^\perp J f + O_{H^1}(h^{4^-} + \|w\|_{H^1}^2)$$

Now apply J^{-1} to obtain the equation

$$(4.14) \quad S \rho = J^{-1} \mu^{-2} \partial_t \rho - J^{-1} \Pi_{(1, 0, 0, 0)}^\perp J f + O(h^{4^-} + \|w\|_{H^1}^2)$$

where the operator

$$S(\rho) \stackrel{\text{def}}{=} \frac{1}{2} M''(\phi)(\rho) + H''(\phi)(\rho) = \frac{1}{2} \rho - \frac{1}{2} \partial_x^2 \rho - 2|\phi|^2 \rho - \phi^2 \bar{\rho}$$

is self-adjoint with respect to the inner product $\langle u, v \rangle = \text{Re} \int u \bar{v}$. The kernel is spanned by $\tilde{\partial}_a \phi$ and $\tilde{\partial}_\theta \phi$.

Lemma 4.1 (properties of S).

(1) For any $f \in H^1$, let $F = J^{-1} \Pi_{(1, 0, 0, 0)}^\perp J f$. Then F satisfies the orthogonality conditions

$$(4.15) \quad \langle F, \tilde{\partial}_\theta \phi \rangle = 0, \quad \langle F, \tilde{\partial}_a \phi \rangle = 0$$

$$(4.16) \quad \langle F, \tilde{\partial}_\mu \phi \rangle = 0, \quad \langle F, \tilde{\partial}_v \phi \rangle = 0$$

(2) For any F satisfying (4.15), $S^{-1} F$ is defined and satisfies the boundedness properties

$$(4.17) \quad \|S^{-1} F\|_{H^1} \lesssim \|F\|_{L^2},$$

$$(4.18) \quad \|e^{\sigma(x)} S^{-1} F\|_{H^2} \lesssim \|e^{\sigma(x)} F\|_{L^2}.$$

for $0 \leq \sigma < 1$.

(3) For any F satisfying (4.15) and (4.16), $S^{-1} F$ satisfies the orthogonality properties

$$(4.19) \quad \langle S^{-1} F, \tilde{\partial}_\theta \phi \rangle = 0, \quad \langle S^{-1} F, \tilde{\partial}_a \phi \rangle = 0$$

$$(4.20) \quad \langle J^{-1} S^{-1} F, \tilde{\partial}_\theta \phi \rangle = 0, \quad \langle J^{-1} S^{-1} F, \tilde{\partial}_a \phi \rangle = 0$$

Proof. Item (1) is immediate from the definition of $\Pi_{(1,0,0,0)}$. For item (2), we recall that $\ker S = \text{span}\{\tilde{\partial}_a\phi, \tilde{\partial}_\theta\phi\}$ and moreover, 0 is an isolated point in the spectrum of S . Thus $S^{-1} : (\ker S)^\perp \rightarrow (\ker S)^\perp$ is bounded as an operator on L^2 . The inequality (4.17) follows from this and elliptic regularity. To prove (4.18), it suffices to show that for any G and any $|\sigma| < 1$, we have

$$(4.21) \quad \|G\|_{H^2} \lesssim \|e^{\sigma x} S e^{-\sigma x} G\|_{L^2} + \|e^{-2|x|} G\|_{L^2}$$

Indeed, (4.18) follows by taking $G = e^{\sigma x} S^{-1} F$, appealing to (4.17), and separately considering $\sigma > 0$ and $\sigma < 0$ with $|\sigma| < 1$. To establish (4.21), we calculate

$$(4.22) \quad \begin{aligned} e^{\sigma x} S e^{-\sigma x} G &= (S + \sigma \partial_x - \frac{1}{2}\sigma^2)G \\ &= (\frac{1}{2}(1 - \sigma^2) + \sigma \partial_x - \frac{1}{2}\partial_x^2)G - 2\phi^2 G - \phi^2 \bar{G} \end{aligned}$$

and hence

$$(\frac{1}{2}(1 - \sigma^2) + \sigma \partial_x - \frac{1}{2}\partial_x^2)G = e^{\sigma x} S e^{-\sigma x} G + 2\phi^2 G + \phi^2 \bar{G}$$

On the left-hand side, we have an operator with symbol $\frac{1}{2}(1 - \sigma^2) + \sigma i\xi + \xi^2$, which dominates $\langle \xi \rangle^2$ under our assumption on σ . From this and the fact that $|\phi(x)|^2 \leq e^{-2|x|}$, we conclude (4.21).

For item (3), (4.19) follows from the fact that $S^{-1} : (\ker S)^\perp \rightarrow (\ker S)^\perp$. To establish (4.20), we note that by (4.16),

$$0 = \langle F, \tilde{\partial}_\mu \phi \rangle = \langle S^{-1} F, S \tilde{\partial}_\mu \phi \rangle$$

and similarly

$$0 = \langle F, \tilde{\partial}_v \phi \rangle = \langle S^{-1} F, S \tilde{\partial}_v \phi \rangle$$

and thus it suffices to establish that $S(\tilde{\partial}_v \phi) = J^{-1} \tilde{\partial}_a \phi$ and $S(\tilde{\partial}_\mu \phi) = J^{-1} \tilde{\partial}_\theta \phi$. To prove these equalities, recall

$$(4.23) \quad 0 = W'(\eta) = (\frac{1}{2}\mu^2 + \frac{1}{2}\mu^{-2}v^2)M'(\eta) - \mu^{-1}vP'(\eta) + H'(\eta)$$

Taking ∂_v and evaluating at $(\mu, a, \theta, v) = (1, 0, 0, 0)$ gives

$$S(\tilde{\partial}_v \phi) = P'(\phi) = J^{-1} \tilde{\partial}_a \phi.$$

Taking ∂_μ of (4.23) and evaluating at $(\mu, a, \theta, v) = (1, 0, 0, 0)$ gives

$$S(\tilde{\partial}_\mu \phi) = -M'(\phi) = J^{-1} \tilde{\partial}_\theta \mu$$

□

Recall that the task is to solve (4.14) where f is either (4.7) or (4.8). At this point, we impose the even/odd solution assumption as in Theorem 1.1 which implies that

$\theta_1 - \theta_2$ is constant. The other time dependent parameters in (4.7), (4.8) are all slowly varying so that $\partial_t f_j = O(h^3)$. Thus, we can solve (4.14) by iteration. Let⁷

$$(4.24) \quad \rho^1 = -S^{-1}J^{-1}\Pi_{(1,0,0,0)}^\perp Jf$$

By Lemma 4.1(1)(2), this is well-defined with $\rho^1 = O(h^{2^-})$ and satisfying all the needed regularity properties. With ρ^2 as yet undefined, we plug $\rho^1 + \rho^2$ into (4.14) to obtain

$$S\rho^2 = J^{-1}\mu^{-2}\partial_t\rho^1 + J^{-1}\mu^{-2}\partial_t\rho^2 + O(h^{4^-} + \|w\|_{H^1}^2)$$

As mentioned previously, $\partial_t f = O(h^{3^-})$ and thus

$$J^{-1}\mu^{-2}\partial_t\rho^1 = -J^{-1}\mu^{-2}S^{-1}J^{-1}\Pi_{(1,0,0,0)}^\perp J\partial_t f$$

is also $O(h^{3^-})$. By Lemma 4.1(3), in particular (4.20), with $F = J^{-1}\Pi_{(1,0,0,0)}^\perp J\partial_t f$, we have that

$$\tilde{F} \stackrel{\text{def}}{=} J^{-1}\mu^{-2}\partial_t\rho^1 = -J^{-1}\mu^{-2}S^{-1}F$$

satisfies the condition (4.15), and hence we can apply Lemma 4.1(2) with F replaced by \tilde{F} . That is, the function

$$(4.25) \quad \rho^2 \stackrel{\text{def}}{=} S^{-1}\tilde{F} = -S^{-1}J^{-1}\mu^{-2}S^{-1}J^{-1}\Pi_{(1,0,0,0)}^\perp J\partial_t f$$

satisfies all the needed regularity properties. Note further that $\partial_t\rho^2 = O(h^{4^-})$. Upon substituting $\rho^1 + \rho^2$ into (4.14) with ρ^1 defined by (4.24) and ρ^2 defined by (4.25), we find that equality holds with $O(h^{4^-})$ error.

Thus we have successfully constructed a solution to the approximate equation (4.1). We summarize our conclusions in the next lemma.

Lemma 4.2 (approximate solution). *Recall the operator g_j associated to $(\mu_j, a_j, \theta_j, v_j)$ defined in (4.4) and f_j defined in (4.7) ($j = 1$) or (4.8) ($j = 2$). Let ρ_j^1 be given by (4.24) and then let ρ_j^2 be given by (4.25). Then ρ_j^k for $1 \leq j, k \leq 2$ satisfy*

$$\|e^{(1^-)(x)}\rho_j^k\|_{H^2} \lesssim h^{1+k^-},$$

Let

$$\nu_z = \mu_1^2 g_1(\rho_1^1 + \rho_1^2) + \mu_2^2 g_2(\rho_2^1 + \rho_2^2)$$

Suppose that the parameter $z \in \mathbb{R}^8$ evolves according to the ODEs obtained from Lemma 3.1 (in the same phase or opposite phase case). Then $\nu_z(x)$ solves (4.1).

⁷As indicated earlier, ρ can stand for either ρ_1 or ρ_2 . The superscript introduced here is different and meant to indicate part of an asymptotic expansion for either function. In other words, we have $\rho_j = \rho_j^1 + \rho_j^2$ for both $j = 1, 2$.

5. LYAPUNOV FUNCTIONAL

The final step is to show that the true solution u to (2.1) is approximately the approximate solution $\tilde{u}_z = u_z + \nu_z$. For this purpose, we introduce a Lyapunov functional. First, some general considerations. We consider the ‘‘perturbed’’ 8-dimensional manifold

$$\tilde{M} = \{ \tilde{u}_z \mid z \in \mathbb{R}^8 \}$$

Introduce the notation $\tilde{w} = u - \tilde{u}_z$ (so that $w = \tilde{w} + \nu_z$). Now it follows from (4.1) that

$$(5.1) \quad \partial_t \tilde{u}_z = JH'(\tilde{u}_z) + F.$$

where $F = O_{H^1}(h^{4-} + \|\tilde{w}\|_{H^1}^2)$.

Suppose that $W_z : L^2 \rightarrow \mathbb{R}$ is a densely defined functional. We write $\partial_{z^\ell} W_z : L^2 \rightarrow \mathbb{R}$ to indicate partial derivatives with respect to z and

$$W'_z(u) \in T_u^* L^2 \underset{\text{metric } g}{\simeq} T_u L^2 \simeq L^2$$

to indicate partial derivatives with respect to u (ignoring the interdependence between z and u given by (3.3), (3.4)).

Suppose that W_z can be extended to a differentiable functional $H^1 \rightarrow \mathbb{R}$; then for each $u \in H^1$, we have a bounded linear map $W'_z(u) : H^1 \rightarrow \mathbb{R}$ which, under the aforementioned identification, becomes a function belonging to H^{-1} . In fact, our choice of W_z is differentiable at all orders as a map $H^1 \rightarrow \mathbb{R}$, which is to say that $W_z^{(k)}(u) : \underbrace{H^1 \times \cdots \times H^1}_{k \text{ copies}} \rightarrow \mathbb{R}$ is a bounded k -multilinear map.

We further assume that $\partial_{z^\ell} W_z(u) = 0$ unless $|\dot{z}^\ell| \lesssim h$. Let

$$(5.2) \quad L_z(u) = W_z(u) - W_z(\tilde{u}_z) - \langle W'_z(\tilde{u}_z), \tilde{w} \rangle.$$

That is, $L_z(u)$ is the quadratic part of $W_z(u)$ above the base manifold \tilde{M} . Now viewing $u = u(t)$ and $z = z(t)$ in accordance with (3.3), (3.4) (and thus reinstating the interdependence between z and u), we have, for any functional $G_z : L^2 \rightarrow \mathbb{R}$,

$$\partial_t G_z(u) = \langle G'_z(u), \partial_t u \rangle + \sum_{k=1}^8 [\partial_{z^k} G](u) \dot{z}^k.$$

This leads to:

Lemma 5.1. *Suppose that u solves (2.1) and z evolves so that \tilde{u}_z solves (5.1), and that $L_z(u)$ is given by (5.2), the quadratic part of $W_z(u)$ above \tilde{M} . Then*

$$(5.3) \quad \partial_t L_z(u) = \{H, W_z\}(u) - \{H, W_z\}(\tilde{u}_z) - \langle \{H, W_z\}'(\tilde{u}_z), \tilde{w} \rangle - E_1 + E_2,$$

where

$$E_1 \stackrel{\text{def}}{=} \langle W''(\tilde{u}_z)F, \tilde{w} \rangle + \langle W'(\tilde{u}_z), [JH'(u) - JH'(\tilde{u}_z) - JH''(\tilde{u}_z)\tilde{w}] \rangle$$

and

$$E_2 \stackrel{\text{def}}{=} \sum_{k=1}^8 ([\partial_{z^k} W_z](u) - [\partial_{z^k} W_z](\tilde{u}_z) - \langle [\partial_{z^k} W'_z](\tilde{u}_z), \tilde{w} \rangle) \dot{z}^k$$

In other words, $\partial_t L_z(u)$ is, up to error E_1 and E_2 , the quadratic part of $\{H, W_z\}(u)$ above \tilde{M} . Note that E_2 just involves the quadratic part of $[\partial_{z^k} W](u)$ above \tilde{M} .

In the typical application of this lemma (as for our W_z , defined below), we have bounded operators $W''_z(\tilde{u}_z) : H^1 \rightarrow H^{-1}$ and $W'''_z(\tilde{u}_z) : H^1 \times H^1 \rightarrow L^2$ which implies the bound

$$|E_1| \lesssim \|F\|_{H^1} \|\tilde{w}\|_{H^1} + \|W'(\tilde{u}_z)\|_{H^1} \|\tilde{w}\|_{H^1}^2$$

Thus, one just needs $\|F\|_{H^1} \lesssim h^3$ and $\|W'(\tilde{u}_z)\|_{H^1} \lesssim h$; in our case we in fact have the stronger statements $\|F\|_{H^1} \lesssim h^{4-}$ and $\|W'(\tilde{u}_z)\|_{H^1} \lesssim h^{2-}$. Moreover, in our case we will have

$$|E_2| \lesssim h^{2-} \|\tilde{w}\|_{H^1}^2$$

since $\dot{\mu}, \dot{v} = O(h^{2-})$.

Proof. By (5.2),

$$\partial_t L_z(u) = \partial_t W_z(u) - \partial_t W_z(\tilde{u}_z) - \partial_t \langle W'_z(\tilde{u}_z), \tilde{w} \rangle$$

We compute each of the three terms on the right-hand side separately.

$$\begin{aligned} \partial_t W_z(u) &= \langle W'_z(u), \partial_t u \rangle + \sum_{k=1}^8 [\partial_{z^k} W_z](u) \dot{z}^k \\ (5.4) \qquad &= \langle W'_z(u), JH'(u) \rangle + \sum_{k=1}^8 [\partial_{z^k} W_z](u) \dot{z}^k \end{aligned}$$

where we invoked (2.1). Second, we compute

$$\begin{aligned} \partial_t W_z(\tilde{u}_z) &= \langle W'_z(\tilde{u}_z), \partial_t \tilde{u}_z \rangle + \sum_{k=1}^8 [\partial_{z^k} W_z](\tilde{u}_z) \dot{z}^k \\ (5.5) \qquad &= \langle W'_z(\tilde{u}_z), JH'(\tilde{u}_z) \rangle + \langle W'_z(\tilde{u}_z), F \rangle + \sum_{k=1}^8 [\partial_{z^k} W_z](\tilde{u}_z) \dot{z}^k \end{aligned}$$

where we invoked (5.1). Finally, we compute

$$\begin{aligned}
 & \partial_t \langle W'_z(\tilde{u}_z), u - \tilde{u}_z \rangle \\
 &= \langle W''_z(\tilde{u}_z) \partial_t \tilde{u}_z, u - \tilde{u}_z \rangle + \langle W'_z(\tilde{u}_z), \partial_t u - \partial_t \tilde{u}_z \rangle \\
 & \quad + \sum_{k=1}^8 \langle [\partial_{z^k} W]_z'(\tilde{u}_z), u - \tilde{u}_z \rangle \dot{z}^k \\
 (5.6) \quad &= \langle W''_z(\tilde{u}_z) JH'(\tilde{u}_z), \tilde{w} \rangle + \langle W''_z(\tilde{u}_z) F, \tilde{w} \rangle + \langle W'_z(\tilde{u}_z), JH'(u) - JH'(\tilde{u}_z) \rangle \\
 & \quad - \langle W'(\tilde{u}_z), F \rangle + \langle \sum_{k=1}^8 [\partial_{z^k} W]_z'(\tilde{u}_z), \tilde{w} \rangle \dot{z}^k
 \end{aligned}$$

Taking (5.4) minus (5.5) minus (5.6), noting the cancelation of $+\langle W'_z(\tilde{u}_z), F \rangle$ in (5.5) with $-\langle W'(\tilde{u}_z), F \rangle$ in (5.6), we obtain (5.3). \square

To produce $W_z(u)$, we use an idea of Martel-Merle-Tsai [10]. Let

$$\Psi(x) = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{if } x \leq -1 \end{cases}$$

such that $(\Psi'(x))^2 \lesssim \min(\Psi(x), 1 - \Psi(x))$. Set $\delta = 4/(\log h^{-1}) = 4/a_0$, so $0 < \delta \ll 1$. Introduce the localizations $\psi_2(x) = \Psi(\delta x)$ and $\psi_1(x) = 1 - \Psi(\delta x)$, and set $M_j(u) = M(\psi_j^{1/2} u)$ and $P_j(u) = P(\psi_j^{1/2} u)$. Define

$$\begin{aligned}
 (5.7) \quad W_z(u) &\stackrel{\text{def}}{=} - \sum_{j=1}^2 \frac{\partial H(\eta_j)}{\partial \mu_j} M_j(u) - \sum_{j=1}^2 \frac{\partial H(\eta_j)}{\partial v_j} P_j(u) + H(u) \\
 &= \frac{1}{2} \sum_{j=1}^2 (\mu_j^2 + \mu_j^{-2} v_j^2) M_j(u) - \sum_{j=1}^2 \mu_j^{-1} v_j P_j(u) + H(u)
 \end{aligned}$$

The Lyapunov functional $L_z(u)$ we use is then defined as in (5.2).

Lemma 5.1 facilitates the computation of $\partial_t L_z(u)$, since $W_z(u)$ is built from “nearly conserved” quantities. Indeed, we have the following Poisson brackets:

$$\begin{aligned}
 \{H, M_j\}(u) &= \frac{1}{2} \text{Im} \int \psi_j' \bar{u} u_x \\
 \{H, P_j\}(u) &= \int \psi_j' \left(\frac{1}{2} |u_x|^2 - \frac{1}{4} |u|^4 \right) - \frac{1}{8} \int \psi_j''' |u|^2
 \end{aligned}$$

It thus follows from Lemma 5.1 that

$$\begin{aligned}
 (5.8) \quad \partial_t L_z(u) &= \frac{1}{2} \sum_{j=1}^2 (\mu_j^2 + \mu_j^{-2} v_j^2) \langle \{H, M_j\}''(\tilde{u}_z) \tilde{w}, \tilde{w} \rangle - \sum_{j=1}^2 \mu_j^{-1} v_j \langle \{H, P_j\}''(\tilde{u}_z) \tilde{w}, \tilde{w} \rangle \\
 & \quad + O(\|w\|_{H^1}^3) - E_1 + E_2
 \end{aligned}$$

For our choice of $W_z(u)$, as remarked earlier, we have suitable bounds for E_1 and E_2 . Moreover, once one imposes the even/odd solution assumption of Theorem 1.1, we have $\mu_1 = \mu_2$ and $v_1^2 = v_2^2$, so the first term in (5.8) disappears.⁸ Hence

$$(5.9) \quad |\partial_t L_z(u)| \lesssim ((|v_1| + |v_2|)\delta + h)\|\tilde{w}\|_{H^1}^2 + h^3\|\tilde{w}\|_{H^1} + \|\tilde{w}\|_{H^1}^3$$

Since $|v_j| \lesssim h^{-1} \log h^{-1}$ and $\delta \sim (\log h^{-1})^{-1}$, the term $(|v_1| + |v_2|)\delta \lesssim h$.

Now we turn to the matter of obtaining a lower bound for $L_z(u)$. First note that

$$\langle W_z''(\tilde{u}_z)\tilde{w}, \tilde{w} \rangle = L_z(u) + O(\|\tilde{w}\|_{H^1}^3).$$

Given that $\|\tilde{v}_z\|_{H^1} \lesssim h^{2-}$, we have

$$(5.10) \quad \langle W_z''(u_z)w, w \rangle = L_z(u) + O(h^{4-}) + O(h^{0+})\|w\|_{H^1}^2.$$

The needed lower bound for the left-hand side will be established below in Lemma 5.2.

For the single-soliton case, we have coercivity for the classical functional from Weinstein [19], which we now recall. Taking $\eta = \eta(\cdot, \mu, a, \theta, v)$ and

$$\begin{aligned} R_{(\mu, a, \theta, v)}(u) &\stackrel{\text{def}}{=} -\frac{\partial H(\eta)}{\partial \mu} M(u) - \frac{\partial H(\eta)}{\partial v} P(u) + H(u) \\ &= \frac{1}{2}(\mu^2 + \mu^{-2}v^2)M(u) - \mu^{-1}vP(u) + H(u) \end{aligned}$$

then

$$(5.11) \quad \|w\|_{H^1}^2 \lesssim \langle R''(\eta)w, w \rangle,$$

provided we assume the orthogonality conditions

$$\begin{aligned} \langle w, J^{-1}\partial_\mu\eta \rangle &= 0, & \langle w, J^{-1}\partial_a\eta \rangle &= 0, \\ \langle w, J^{-1}\partial_\theta\eta \rangle &= 0, & \langle w, J^{-1}\partial_v\eta \rangle &= 0. \end{aligned}$$

A direct proof of (5.11) is possible; see [7, Prop. 4.1].

We now prove a similar argument for the double-soliton functional $W_z(u)$ defined in (5.7). Before proceeding, we record the formulae

$$(5.12) \quad \begin{aligned} M_j''(u) &= \psi_j \\ P_j''(u) &= -i\psi_j^{1/2}\partial_x\psi_j^{1/2} = -\frac{1}{2}i\psi_j' - i\psi_j\partial_x \\ H''(u) &= -\frac{1}{2}\partial_x^2 - 2|u|^2 - u^2C \end{aligned}$$

where C denotes the operator of complex conjugation.

⁸In fact, this is more easily seen by observing that once $\mu_1 = \mu_2$ and $v_1^2 = v_2^2$, we have that the first term in (5.7) becomes $M(u)$, whose Poisson bracket vanishes. We included the localization in this term to illustrate the difficulty in treating the asymmetric case – one would not have that the first term in (5.8) is $O(h^5)$.

Lemma 5.2. *Suppose w satisfies the orthogonality conditions (3.4). Then*

$$(5.13) \quad \|w\|_{H^1}^2 \lesssim \langle W_z''(u_z)w, w \rangle.$$

Proof. Denote $w_j = \psi_j^{1/2}w$, $j = 1, 2$. Note that $w_1 + w_2 \neq w$, although $1 = \psi_1 + \psi_2 \leq \psi_1^{1/2} + \psi_2^{1/2} \leq 2$. Define functionals

$$\begin{aligned} W_j(u) &= -\frac{\partial H(\eta_j)}{\partial \mu_j} M(u) - \frac{\partial H(\eta_j)}{\partial v_j} P(u) + H(u) \\ &= \frac{1}{2}(\mu_j^2 + \mu_j^{-2}v_j^2)M(u) - \mu_j^{-1}v_j P(u) + H(u) \end{aligned}$$

We claim that

$$(5.14) \quad \left| \langle W''(u_z)w, w \rangle - \sum_{j=1}^2 \langle W_j''(\eta_j)w_j, w_j \rangle \right| \lesssim \delta^2 \|w\|_{H^1}^2$$

and

$$(5.15) \quad \left| \|w\|_{H^1}^2 - \sum_{j=1}^2 \|w_j\|_{H^1}^2 \right| \lesssim \delta^2 \|w\|_{H^1}^2$$

We now establish (5.14). Note that (see (5.12))

$$\begin{aligned} &\langle W''(u_z)w, w \rangle - \sum_{j=1}^2 \langle W_j''(\eta_j)w_j, w_j \rangle \\ &= \langle (H''(u_z) - \psi_1^{1/2}H''(\eta_1)\psi_1^{1/2} - \psi_2^{1/2}H''(\eta_2)\psi_2^{1/2})w, w \rangle \end{aligned}$$

The operator appearing on the right-hand side can be decomposed into $A_1 + A_2 + A_3$ where

$$\begin{aligned} A_1 &\stackrel{\text{def}}{=} -\frac{1}{2}(\partial_x^2 - \psi_1^{1/2}\partial_x^2\psi_1^{1/2} - \psi_2^{1/2}\partial_x^2\psi_2^{1/2}) \\ A_2 &\stackrel{\text{def}}{=} -2(|u_z|^2 - \psi_1|\eta_1|^2 - \psi_2|\eta_2|^2) \\ A_3 &\stackrel{\text{def}}{=} -(u_z^2 - \psi_1\eta_1^2 - \psi_2\eta_2^2)C \end{aligned}$$

We compute A_1 explicitly:

$$A_1 = \sum_{j=1}^2 \left(-\frac{1}{2}\psi_j'\partial_x - \frac{1}{4}\psi_j^{-1}(\psi_j')^2 - \frac{1}{2}\psi_j'' \right) = -\frac{1}{4} \sum_{j=1}^2 \psi_j^{-1}(\psi_j')^2$$

where we have used that $\psi_1 + \psi_2 = 1$ in the second equality. We have $(\psi_j')^2 \lesssim \delta^2 \psi_j$ by the corresponding property of Ψ and thus A_1 is a multiplication operator with symbol bounded by δ^2 . By the support properties of ψ_1, ψ_2 , we obtain that the multiplication operators A_2, A_3 have symbols bounded by h . This completes the proof of (5.14), and the proof of (5.15) is similar.

By the orthogonality conditions (3.4),

$$\langle w_1, J^{-1} \partial_{z^j} u_z \rangle = -\langle (1 - \psi_1^{1/2}) w, J^{-1} \partial_{z^j} u_z \rangle,$$

and we have, for example

$$\langle w_1, J^{-1} \partial_{\mu_1} u_z \rangle = -\langle (1 - \psi_1^{1/2}) w, J^{-1} \partial_{\mu_1} u_z \rangle = -\langle (1 - \psi_1^{1/2}) w, J^{-1} \partial_{\mu_1} \eta_1 \rangle \lesssim h^{1/2} \|w\|_{L^2},$$

due to the fact that $\|(1 - \psi_1^{1/2}) J^{-1} \partial_{\mu_1} \eta_1\|_{L^2} \lesssim h^{1/2}$. Hence, by the coercivity of the classical Lyapunov functional (see discussion surrounding (5.11)), we have that

$$\sum_{j=1}^2 \langle W_j''(\eta_j) w_j, w_j \rangle + h \|w\|_{L^2}^2 \gtrsim \sum_{j=1}^2 \|w_j\|_{H^1}^2,$$

From this and (5.14), (5.15), we obtain (5.13). \square

6. CONCLUSION OF PROOF

In this section, we conclude the proof of Theorem 1.1.

Recall that $h = e^{-a_0}$ which implies that $a_0 = \log h^{-1}$, and that we are in the even/odd solution setting with (1.7), (1.8) in place.

We introduce $0 < \delta \ll 1$. The constant δ is absolute and is chosen sufficiently small in terms of the accumulation of numerous other absolute constants appearing in several estimates. In our argument, c will represent a large absolute constant that may change (typically enlarge) from one line to the next. At the conclusion of the argument, we can finally declare that δ should be taken small enough that $c\delta < \frac{1}{2}$. This does not constitute circular reasoning since one could tally up all of the absolute constants (the c 's) in each estimate in advance of executing the argument and suitably define δ *a priori* but this is not a practical manner of exposition.

Recall that we started by defining

$$w(t) = u(t) - u_{z(t)}$$

where z was selected by the implicit function theorem so that orthogonality conditions (3.4) hold. By continuity of the flow in H^1 , this is possible at least up to some small positive time. Let T be the supremum of all times $0 < T \leq h^{-1-\delta}$ for which

$$(6.1) \quad \|w\|_{L_{[0,T]}^\infty H_x^1} \leq h^{3/2}$$

$$(6.2) \quad |v| \leq h^{1-\delta}$$

$$(6.3) \quad a \geq a_0^{1-\delta}$$

Note that the requirement (6.1) implies

$$\|w\|_{H^1}^3 \leq h \|w\|_{H^1}^2 + h^3 \|w\|_{H^1},$$

and enables us to discard cubic error terms in w in our estimates.

In the course of the argument that follows, we work on the time interval $[0, T]$. At the conclusion of the argument, we are able to assert that either $T = \delta h^{-1} \log h^{-1}$ or that (6.2) or (6.3) fail to hold at $t = T$.

It follows from the decomposition (1.22) and the bootstrap assumptions (6.2), (6.3) above that (see Appendix A)

$$\sup_{1 \leq n \leq 8} \|J^{-1} \partial_{z^n} \Pi_z^\perp JH'(u_z)\|_{H_x^1} \leq h^{2-c\delta}$$

Let

$$\epsilon \stackrel{\text{def}}{=} h^{4-c\delta} + \|w\|_{L_{[0,T]}^\infty H_x^1}^2$$

By Lemma 3.1 and the computations in Appendix A, the ODEs

$$\begin{cases} \dot{\mu} = (-1)^\sigma (8a - 4) v e^{-2a} + O(\epsilon) \\ \dot{a} = \mu^{-1} v + (-1)^\sigma (-4a + \frac{2}{3} \pi^2) v e^{-2a} + O(\epsilon) \\ \dot{\theta} = \frac{1}{2} \mu^2 + \frac{1}{2} v^2 \mu^{-2} + 18(-1)^\sigma e^{-2a} + O(\epsilon) \\ \dot{v} = -4(-1)^\sigma e^{-2a} + O(\epsilon) \end{cases}$$

hold on $[0, T]$. By the first of these equations and (6.1), (6.2), (6.3), we have $|\mu - 1| \leq h^{2-c\delta}$. From the above ODEs and (6.1), (6.2), (6.3), we can deduce bounds on $\dot{\mu}$, \dot{a} , $\dot{\theta}$, and \dot{v} that justify the estimates involved in the construction of ν_z in §4 summarized in Lemma 4.2. The result is that

$$(6.4) \quad \|\nu_z\|_{H_x^1} \lesssim h^{2-c\delta}$$

and (5.1) holds with

$$\|F\|_{H_x^1} \lesssim \epsilon.$$

By (5.9),

$$(6.5) \quad |\partial_t L_z(u)| \lesssim h \|\tilde{w}\|_{H^1}^2 + h^3 \|\tilde{w}\|_{H^1} + \|\tilde{w}\|_{H^1}^3$$

where we recall that $\tilde{w} = w - \nu_z$. By (6.4), $\|\tilde{w}\|_{H^1} \lesssim \|w\|_{H^1} + h^{2-c\delta}$, we obtain from (6.5) that

$$(6.6) \quad |\partial_t L_z(u)| \lesssim h \|w\|_{H^1}^2 + h^{5-c\delta}.$$

From (5.13) and (5.10), the bound

$$(6.7) \quad \|w\|_{H^1}^2 \lesssim L_z(u) + h^{4-c\delta}$$

holds. Combining (6.6) and (6.7), we obtain the bound

$$|\partial_t L_z(u)| \lesssim h L_z(u) + h^{5-c\delta}$$

By Gronwall's inequality, it follows that

$$L_z(u) \lesssim e^{cth} h^{4-c\delta}.$$

Provided we restrict to $t \lesssim \delta h^{-1} \log h^{-1}$, this implies

$$L_z(u) \lesssim h^{4-c\delta}$$

Reapplying (6.7), we obtain

$$(6.8) \quad \|w\|_{H^1} \lesssim h^{2-c\delta}.$$

At this point, we can declare that δ should have been taken sufficiently small so that $c\delta < \frac{1}{2}$, where c is as it appears in (6.8). It follows that (6.1), (6.2), (6.3) can only break down provided $T \gtrsim \delta h^{-1} \log h^{-1}$ or if either (6.2) or (6.3) fails at $t = T$.

We will see from the following ODE analysis that (6.3) always holds; in the same phase (even solution, attractive) case, the assumption (6.2) first fails at $T \sim h^{-1}$, and in the opposite phase (odd solution, repulsive) case, (6.2) remains valid and we can reach $T \sim h^{-1} \log h^{-1}$.

Since we now restrict to $t \lesssim \delta h^{-1} \log h^{-1}$, we can assume that (6.8) holds and thus $\epsilon \lesssim h^{4-c\delta}$.

Let $\tilde{z} = (\tilde{\mu}, \tilde{a}, \tilde{\theta}, \tilde{v})$ solve

$$\begin{cases} \dot{\tilde{\mu}} = (-1)^\sigma (8\tilde{a} - 4)\tilde{v}e^{-2\tilde{a}} \\ \dot{\tilde{a}} = \tilde{v} \\ \dot{\tilde{\theta}} = \frac{1}{2}\tilde{\mu}^2 + \frac{1}{2}\tilde{v}^2\tilde{\mu}^{-2} + 18(-1)^\sigma e^{-2\tilde{a}} \\ \dot{\tilde{v}} = -4(-1)^\sigma e^{-2\tilde{a}} \end{cases}$$

These tilde equations appear in the statement of Theorem 1.1 without tildes. Note that the $\dot{\tilde{a}}$ and $\dot{\tilde{v}}$ equations can be solved separately as discussed in §1. Let $\bar{a} = \mu a - \tilde{a}$ and $\bar{v} = v - \tilde{v}$. Then we get the system

$$\begin{cases} \dot{\bar{a}} = \bar{v} + O(h^{3-c\delta}) \\ \dot{\bar{v}} = -2(-1)^\sigma e^{-2\bar{a}}\bar{a} + O(h^{4-c\delta}) \end{cases}$$

Let $\gamma = (\bar{a})^2 + h^{-2}\bar{v}^2$. Then, substituting

$$\dot{\gamma} \lesssim h\bar{a}(h^{-1}\bar{v}) + (h^{1/2}\bar{a})h^{\frac{5}{2}-c\delta} + (h^{-1/2}\bar{v})(h^{\frac{5}{2}-c\delta})$$

By the inequality $\alpha\beta \leq \alpha^2 + \beta^2$, we obtain

$$\dot{\gamma} \lesssim h\gamma + h^{5-c\delta}$$

By Gronwall's inequality,

$$\gamma \lesssim e^{cht}(\gamma_0 + h^{4-c\delta})$$

It follows that

$$|\bar{a}| \lesssim h^{2-\delta}, \quad |\bar{v}| \lesssim h^{3-c\delta}$$

These errors only affect the $\dot{\mu}$ equation at order $h^{4-c\delta}$ so μ is only affected at order $h^{3-c\delta}$. Given this, the $\dot{\theta}$ equation is only affected at order $h^{3-c\delta}$. Thus, the impact on θ is of size $h^{2-c\delta}$. In conclusion

$$|\bar{\theta}| \lesssim h^{2-c\delta} \quad |\bar{\mu}| \lesssim h^{3-c\delta}$$

Thus

$$\|u_z - u_{\bar{z}}\|_{H^1} \lesssim h^{2-c\delta}$$

Since u_z in Theorem 1.1 in fact means $u_{\bar{z}}$, this completes the proof of Theorem 1.1.

APPENDIX A. COMPUTATIONS

We shall carry out the computations of the ODEs appearing in (3.5) in Lemma 3.1 and show that they are equivalent to (1.19), with errors of size $O(h^{4-})$. This is carried out without making the symmetry assumption on the solution. When the even/odd symmetry assumption is imposed, we will carry out the integrals appearing in (1.19) and show that the ODEs claimed in the statement of Theorem 1.1 hold.

Denote $u_z = \eta_1 + \eta_2$. Let $L = \{1, 2, 5, 6\}$ denote the indices that refer to the left soliton and $R = \{3, 4, 7, 8\}$ denote the indices that refer to the right soliton. The coefficient matrix of the symplectic form is

$$(a_{\ell m}) = A = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} + O(h^{2-})$$

where the $O(h^{2-})$ contributions come from $a_{\ell m}$ with $\ell \in L$ and $m \in R$ (and vice-versa, but of course $a_{\ell m} = -a_{m\ell}$). Fortunately, we do not need to compute these terms. Note that

$$(a^{\ell m}) = A^{-1} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} + O(h^{2-})$$

In fact, we can substantially reduce the complexity of computation in applying Lemma 3.1 by observing that $JH'(u_z)$ decomposes into terms parallel to M plus other terms which are $O(h^{2-})$. To this end, we expand:

$$H'(u_z) = H'(\eta_1) + H'(\eta_2) + H_p''(\eta_1)\eta_2 + H_p''(\eta_2)\eta_1 + O(h^4),$$

where

$$H_p(u) = -\frac{1}{4} \int |u|^4.$$

Moreover, we have

$$JH'(\eta) = \partial_v H(\eta) \partial_a \eta - \partial_\mu H(\eta) \partial_\theta \eta.$$

Hence,

$$(A.1) \quad H'(u_z) = \sum_{j=1}^8 b^j J^{-1} \partial_{z^j} u_z + H_p''(\eta_1)\eta_2 + H_p''(\eta_2)\eta_1$$

where

$$b_2 = \partial_{v_1} H(\eta_1), \quad b_4 = \partial_{v_2} H(\eta_2), \quad b_5 = -\partial_{\mu_1} H(\eta_1), \quad b_7 = -\partial_{\mu_2} H(\eta_2)$$

and all other $b_j = 0$. Observe that $\langle H_p''(\eta_1)\eta_2, \partial_{z^\ell} u_z \rangle = O(h^{4-})$ for any $\ell \in R$ and $\langle H_p''(\eta_2)\eta_1, \partial_{z^\ell} u_z \rangle = O(h^{4-})$ for any $\ell \in L$. Note further that for $\ell \in L$ (and hence $\partial_{z^\ell} u_z = \partial_{z^\ell} \eta_1$) we have

$$(A.2) \quad \langle H_p''(\eta_1)\eta_2, \partial_{z^\ell} u_z \rangle = \langle \eta_2, H_p''(\eta_1)\partial_{z^\ell} \eta_1 \rangle = \partial_{z^\ell} \langle \eta_2, H_p'(\eta_1) \rangle$$

Similarly, for $\ell \in R$ and (and hence $\partial_{z^\ell} u_z = \partial_{z^\ell} \eta_2$) we have

$$(A.3) \quad \langle H_p''(\eta_2)\eta_1, \partial_{z^\ell} u_z \rangle = \langle \eta_1, H_p''(\eta_2)\partial_{z^\ell} \eta_2 \rangle = \partial_{z^\ell} \langle \eta_1, H_p'(\eta_2) \rangle$$

From (A.1),(A.2), and (A.3), we obtain

$$\partial_{z^\ell} H(u_z) = - \sum_{j=1}^8 b^j a_{j\ell} + \partial_{z^\ell} \langle \eta_1, H_p'(\eta_2) \rangle + \partial_{z^\ell} \langle \eta_2, H_p'(\eta_1) \rangle$$

It follows that the equations (3.6) reduce to

$$\dot{z}^m = b^m - \sum_{\ell \in L} \partial_{z^\ell} \langle H_p'(\eta_1), \eta_2 \rangle a^{\ell m} - \sum_{\ell \in R} \partial_{z^\ell} \langle H_p'(\eta_2), \eta_1 \rangle a^{\ell m} + O(h^{4-})$$

It suffices in this sum to discard $O(h^{2-})$ terms in $a^{\ell m}$. Thus we obtain the equations

$$\left\{ \begin{array}{l} \dot{z}^1 = b^1 + \partial_{z^5} \langle H_p'(\eta_1), \eta_2 \rangle + O(h^{4-}) \\ \dot{z}^2 = b^2 + \partial_{z^6} \langle H_p'(\eta_1), \eta_2 \rangle + O(h^{4-}) \\ \dot{z}^3 = b^3 + \partial_{z^7} \langle H_p'(\eta_2), \eta_1 \rangle + O(h^{4-}) \\ \dot{z}^4 = b^4 + \partial_{z^8} \langle H_p'(\eta_2), \eta_1 \rangle + O(h^{4-}) \\ \dot{z}^5 = b^5 - \partial_{z^1} \langle H_p'(\eta_1), \eta_2 \rangle + O(h^{4-}) \\ \dot{z}^6 = b^6 - \partial_{z^2} \langle H_p'(\eta_1), \eta_2 \rangle + O(h^{4-}) \\ \dot{z}^7 = b^7 - \partial_{z^3} \langle H_p'(\eta_2), \eta_1 \rangle + O(h^{4-}) \\ \dot{z}^8 = b^8 - \partial_{z^4} \langle H_p'(\eta_2), \eta_1 \rangle + O(h^{4-}) \end{array} \right.$$

In more direct language, these equations are

$$\begin{aligned}
 \dot{\mu}_1 &= && + \partial_{\theta_1} \langle H'_p(\eta_1), \eta_2 \rangle + O(h^{4-}) \\
 \dot{a}_1 &= + \partial_{v_1} H(\eta_1) + \partial_{v_1} \langle H'_p(\eta_1), \eta_2 \rangle + O(h^{4-}) \\
 \dot{\mu}_2 &= && + \partial_{\theta_2} \langle H'_p(\eta_2), \eta_1 \rangle + O(h^{4-}) \\
 \dot{a}_2 &= + \partial_{v_2} H(\eta_2) + \partial_{v_2} \langle H'_p(\eta_2), \eta_1 \rangle + O(h^{4-}) \\
 \dot{\theta}_1 &= - \partial_{\mu_1} H(\eta_1) - \partial_{\mu_1} \langle H'_p(\eta_1), \eta_2 \rangle + O(h^{4-}) \\
 \dot{v}_1 &= && - \partial_{a_1} \langle H'_p(\eta_1), \eta_2 \rangle + O(h^{4-}) \\
 \dot{\theta}_2 &= - \partial_{\mu_2} H(\eta_2) - \partial_{\mu_2} \langle H'_p(\eta_2), \eta_1 \rangle + O(h^{4-}) \\
 \dot{v}_2 &= && - \partial_{a_2} \langle H'_p(\eta_2), \eta_1 \rangle + O(h^{4-})
 \end{aligned}$$

We note that these equations hold in general, without assuming that the solution is even or odd.

The next step is then to compute $\langle H'_p(\eta_1), \eta_2 \rangle$ and $\langle H'_p(\eta_2), \eta_1 \rangle$. Let $\phi(x) = \operatorname{sech} x$. We have

$$\begin{aligned}
 \langle H'_p(\eta_2), \eta_1 \rangle &= - \operatorname{Re} \left(e^{i(\theta_2 - \theta_1)} e^{i(\mu_1^{-1} v_1 a_1 - \mu_2^{-1} v_2 a_2)} \mu_2^3 \mu_1 \right. \\
 \text{(A.4)} \quad &\quad \times \left. \int e^{i(\mu_2^{-1} v_2 - \mu_1^{-1} v_1)x} \phi^3(\mu_2(x - a_2)) \phi(\mu_1(x - a_1)) dx \right)
 \end{aligned}$$

At this point we will make the even/odd assumption. In the even case, we may set

$$\text{(A.5)} \quad (\mu, a, \theta, v) \stackrel{\text{def}}{=} (\mu_1, -a_1, \theta_1, -v_1) = (\mu_2, a_2, \theta_2, v_2)$$

Then $\theta_1 - \theta_2 = 0$. In the odd case, we may set

$$\text{(A.6)} \quad (\mu, a, \theta, v) \stackrel{\text{def}}{=} (\mu_1, -a_1, \theta_1 - \pi, -v_1) = (\mu_2, a_2, \theta_2, v_2)$$

Then $\theta_1 - \theta_2 = \pi$.

In either the even or odd case, we find that $\dot{\mu} = \dot{\mu}_1 = \dot{\mu}_2 = O(h^{3-})$, from which it follows that

$$\text{(A.7)} \quad \mu = \mu_1 = \mu_2 = 1 + O(h^{2-})$$

Take $\sigma = 0$ in the even case and $\sigma = 1$ in the odd case. We compute the equations for $\dot{\mu}_2, \dot{a}_2, \dot{\theta}_2, \dot{v}_2$ by carrying out the appropriate derivative of (A.4), and then evaluating the resulting expression using (A.7), (A.5), (A.6). By residue calculus computations and asymptotic expansion,

$$\begin{aligned}
 \alpha(\xi, a) &\stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} e^{-ix\xi} \phi^3(x - a) \phi(x + a) dx \\
 &= e^{-2a} \left[4 + (2 - 4a)i\xi + \left(-\frac{\pi^2}{6} + 2a - 2a^2\right)\xi^2 + O(\xi^3) \right] + O(e^{-4a})
 \end{aligned}$$

and

$$\begin{aligned}\beta(\xi, a) &\stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} e^{-ix\xi} [\phi^3]'(x-a) \phi(x+a) dx \\ &= e^{-2a} [4 + (6-4a)i\xi] + O(h^{4-})\end{aligned}$$

We find that

$$\begin{cases} \dot{\mu} = (-1)^\sigma \operatorname{Re}[-i\alpha] + O(h^{4-}) \\ \dot{a} = \mu^{-1}v + (-1)^\sigma \operatorname{Re}[+ia\alpha + \partial_\xi\alpha] + O(h^{4-}) \\ \dot{\theta} = \frac{1}{2}\mu^2 + \frac{1}{2}v^2\mu^{-2} + (-1)^\sigma \operatorname{Re}[(iva + 3)\alpha + v(\partial_\xi\alpha)] + \operatorname{Re}(a\beta - i\partial_\xi\beta) + O(h^{4-}) \\ \dot{v} = (-1)^\sigma \operatorname{Re}[-iv\alpha - \beta] + O(h^{4-}) \end{cases}$$

where α , $\partial_\xi\alpha$, and β are evaluated at $\xi = -2v$.

Substituting, we obtain

$$\begin{cases} \dot{\mu} = (-1)^\sigma (8a-4)ve^{-2a} + O(h^{4-}) \\ \dot{a} = \mu^{-1}v + (-1)^\sigma (-4a + \frac{2}{3}\pi^2)ve^{-2a} + O(h^{4-}) \\ \dot{\theta} = \frac{1}{2}\mu^2 + \frac{1}{2}v^2\mu^{-2} + 18(-1)^\sigma e^{-2a} + O(h^{4-}) \\ \dot{v} = -4(-1)^\sigma e^{-2a} + O(h^{4-}) \end{cases}$$

The system $(\mu a, v)$ can be solved with error $O(h^{2-})$; from which (a, v) can be recovered with error $O(h^{2-})$. At this accuracy the dynamics are comparable to

$$\begin{cases} \dot{a} = v \\ \dot{v} = -4(-1)^\sigma e^{-2a} \end{cases}$$

Then μ can be solved with “explicit” order h^2 term coming from the order h^3 term in the equation for $\dot{\mu}$, and then θ can be obtained with error of size h^2 .

REFERENCES

- [1] W.K. Abou Salem, *Effective dynamics of solitons in the presence of rough nonlinear perturbations*, Nonlinearity 22 (2009), no. 4, pp. 747–763.
- [2] K. Datchev and I. Ventura, *Solitary waves for the Hartree equation with a slowly varying potential*, Pacific J. Math. 248 (2010), no. 1, pp. 63–90.
- [3] J. Fröhlich, S. Gustafson, B.L.G. Jonsson, and I.M. Sigal, *Solitary wave dynamics in an external potential*, Comm. Math. Phys. 250 (2004), no. 3, pp. 613–642.
- [4] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. 74 (1987), no. 1, pp. 160–197.
- [5] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. II*, J. Funct. Anal. 94 (1990), no. 2, pp. 308–348.
- [6] J. Holmer, G. Perelman, and M. Zworski, *Effective dynamics of double solitons for perturbed mKdV*, Comm. Math. Phys. 305 (2011), no. 2, pp. 363–425.

- [7] J. Holmer and M. Zworski, *Slow soliton interaction with delta impurities*, J. Mod. Dyn. 1 (2007), no. 4, pp. 689–718.
- [8] J. Holmer and M. Zworski, *Soliton interaction with slowly varying potentials*, Int. Math. Res. Not. IMRN 2008, no. 10, Art. ID rnn026, 36 pp.
- [9] J. Krieger, Y. Martel, and P. Raphaël, *Two-soliton solutions to the three-dimensional gravitational Hartree equation*, Comm. Pure Appl. Math. 62 (2009), no. 11, pp. 1501–1550.
- [10] Y. Martel, F. Merle, and T.-P. Tsai, *Stability in H^1 of the sum of K solitary waves for some nonlinear Schrödinger equations*, Duke Math. J. 133 (2006), no. 3, pp. 405–466.
- [11] J.H. Maddocks and R.L. Sachs, *On the stability of KdV multi-solitons*, Comm. Pure Appl. Math. 46 (1993), no. 6, pp. 867–901.
- [12] J.L. Marzuola and M.I. Weinstein, *Long time dynamics near the symmetry breaking bifurcation for nonlinear Schrödinger/Gross-Pitaevskii equations*, Discrete Contin. Dyn. Syst. 28 (2010), no. 4, pp. 1505–1554.
- [13] C. Muñoz, *Dynamics of soliton-like solutions for slowly varying, generalized gKdV equations: refraction vs. reflection*, arxiv.org preprint [arXiv:1009.4905](https://arxiv.org/abs/1009.4905).
- [14] G. Perelman, *Asymptotic stability of multi-soliton solutions for nonlinear Schrödinger equations*, Comm. Partial Differential Equations 29 (2004), no. 7-8, pp. 1051–1095.
- [15] O. Pocovnicu, *Soliton interaction with small Toeplitz potentials for the cubic Szegő equation on the real line*, preprint, available at <http://www.math.u-psud.fr/~pocovnicu/>.
- [16] I. Rodnianski, W. Schlag, A. Soffer, *Asymptotic stability of N -soliton states of NLS*, unpublished manuscript available at [arXiv:math/0309114](https://arxiv.org/abs/math/0309114).
- [17] G.I. Stegeman and M. Segev, *Optical Spatial Solitons and Their Interactions: Universality and Diversity*, Science 286, no. 5444 (19 November 1999), pp. 1518–1523.
- [18] K.E. Strecker, G.B. Partridge, A.G. Truscott, and R.G. Hulet, *Formation and propagation of matter-wave soliton trains*, Nature 417 (9 May 2002), pp. 150–153
- [19] M.I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. 39 (1986), no. 1, pp. 51–67.
- [20] V.E. Zakharov and A.B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Soviet Physics JETP 34 (1972), no. 1, pp. 62–69.