THE UNIVERSITY OF CHICAGO

UNIFORM ESTIMATES FOR THE ZAKHAROV SYSTEM AND THE INITIAL-BOUNDARY VALUE PROBLEM FOR THE KORTEWEG-DE VRIES AND NONLINEAR SCHRÖDINGER EQUATIONS

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BY
JUSTIN ALEXANDER HOLMER

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To my wife, Susan

ABSTRACT

In Chapter 1, we consider the 1D Zakharov system

$$\begin{cases} \partial_t u_\epsilon = i\partial_x^2 u_\epsilon \mp i n_\epsilon u_\epsilon \\ \epsilon^2 \partial_t^2 n_\epsilon - \partial_x^2 n_\epsilon = \partial_x^2 |u_\epsilon|^2 \\ u_\epsilon\big|_{t=0} = u_0 \\ n_\epsilon\big|_{t=0} = n_0 \\ \partial_t n_\epsilon\big|_{t=0} = n_1 \end{cases}$$

where $u = u_\epsilon : \mathbb{R} \times [0,T] \to \mathbb{C}$, and $n = n_\epsilon : \mathbb{R} \times [0,T] \to \mathbb{R}$. We prove the estimate

$$\|u\|_{L_T^\infty H_x^k} + \|\langle x \rangle^{-1} D_x^{1/2} \partial_x^k u\|_{L_x^2 L_T^2} \leq c \exp(c\|\langle x \rangle u_0\|_{H^1}) \|u_0\|_{H_x^k} \tag{1}$$

for given initial data $(u_0, n_0, n_1) \in H^k \cap H^1(\langle x \rangle^2 dx) \times H^{k-\frac{1}{2}} \times H^{k-\frac{3}{2}}$, which is uniform in ϵ as $\epsilon \to 0$, by the technique of introducing a suitable pseudodifferential operator change of variable. This method has been previously applied to derivative nonlinear Schrödinger (NLS) equations by [Chi96], [KPV98], and the primary new obstacle here is commuting the pseudodifferential operator past the inverse wave operator. Applications of (1) to the convergence of u_ϵ to the solution of the cubic NLS equation corresponding to initial data u_0 are also explored.

In Chapter 2, we consider the initial-boundary value problem for the Korteweg-de Vries (KdV) equation on the right half-line

$$\begin{cases} \partial_t u + \partial_x^3 u + u \partial_x u = 0 & \text{for } (x,t) \in (0,+\infty) \times (0,T) \\ u(0,t) = f(t) & \text{for } t \in (0,T) \\ u(x,0) = \phi(x) & \text{for } x \in (0,+\infty) \end{cases}$$

and prove local well-posedness for $(\phi, f) \in H^s(\mathbb{R}^+) \times H^{\frac{s+1}{3}}(\mathbb{R}^+)$, $-\frac{3}{4} < s < \frac{3}{2}$, $s \neq \frac{1}{2}$, with the compatibility condition $\phi(0) = f(0)$ for $\frac{1}{2} < s < \frac{3}{2}$. On the left half-line, the corresponding problem is

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & \text{for } (x,t) \in (-\infty, 0) \times (0,T) \\ u(0,t) = f(t) & \text{for } t \in (0,T) \\ \partial_x u(0,t) = g(t) & \text{for } t \in (0,T) \\ u(x,0) = \phi(x) & \text{for } x \in (-\infty, 0) \end{cases}$$

and we prove local well-posedness for $(\phi, f, g) \in H^s(\mathbb{R}^-) \times H^{\frac{s+1}{3}}(\mathbb{R}^+) \times H^{\frac{s}{3}}(\mathbb{R}^+)$, $-\frac{3}{4} < s < \frac{3}{2}$, $s \neq \frac{1}{2}$, with the compatibility condition $\phi(0) = f(0)$ for $\frac{1}{2} < s < \frac{3}{2}$. A finite-length interval analogue is also addressed. These results are obtained through modifications of the technique of [CK02], where a boundary forcing operator is introduced to set the boundary conditions.

In Chapter 3, we consider the initial-boundary value problem for the nonlinear Schrödinger equation (NLS) on the right-half line

$$\begin{cases} i\partial_t u + \partial_x^2 u + \lambda u|u|^{\alpha-1} = 0 & \text{for } (x,t) \in (0,+\infty) \times (0,T) \\ u(0,t) = f(t) & \text{for } t \in (0,T) \\ u(x,0) = \phi(x) & \text{for } x \in (0,+\infty) \end{cases}$$

and prove local well-posedness for $(\phi, f) \in H^s(\mathbb{R}^+) \times H^{\frac{2s+1}{4}}(\mathbb{R}^+)$, in the case $s = 0$, $1 < \alpha \leq 5$, and in the case $s = 1$, $1 < \alpha < +\infty$ with the compatibility condition $\phi(0) = f(0)$. The left half-line problem is actually the same problem since $u(x,t)$ solves the left-hand problem for $\phi(x)$ and $f(t)$ iff $u(-x,t)$ solves the right-hand problem for $\phi(-x)$ and $f(t)$. These results are also obtained through modifications of the technique of [CK02].

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CHAPTER 1
ESTIMATES FOR THE 1D ZAKHAROV SYSTEM UNIFORM IN ION SOUND SPEED

1.1 Introduction and summary

Let $u_0 : \mathbb{R}^d \to \mathbb{C}$, $n_0,\ n_1 : \mathbb{R}^d \to \mathbb{R}$ be given, and consider the Zakharov system in $\mathbb{R}^d$:

$$\mathrm{ZS}_\epsilon = \begin{cases} \partial_t u_\epsilon = i\Delta u_\epsilon \mp i n_\epsilon u_\epsilon & (1.1) \\ \epsilon^2 \partial_t^2 n_\epsilon - \Delta n_\epsilon = \Delta |u_\epsilon|^2 & (1.2) \\ u_\epsilon\big|_{t=0} = u_0 \\ n_\epsilon\big|_{t=0} = n_0 \\ n_\epsilon\big|_{t=0} = n_1 \end{cases}$$

where $u_\epsilon : \mathbb{R}^d \times [0,T] \to \mathbb{C}$ and $n_\epsilon : \mathbb{R}^d \times [0,T] \to \mathbb{R}$. [1] The case of $-$ in (1.1) is called "focusing" (FZS_ϵ), and the case of $+$ in (1.1) is called "defocusing" (DZS_ϵ). In this chapter, we will concentrate on the case $d = 1$, where ZS_ϵ takes the form:

$$\text{1D } \mathrm{ZS}_\epsilon = \begin{cases} \partial_t u_\epsilon = i\partial_x^2 u_\epsilon \mp i n_\epsilon u_\epsilon & (1.3) \\ \epsilon^2 \partial_t^2 n_\epsilon - \partial_x^2 n_\epsilon = \partial_x^2 |u_\epsilon|^2 & (1.4) \\ u_\epsilon\big|_{t=0} = u_0 \\ n_\epsilon\big|_{t=0} = n_0 \\ \partial_t n_\epsilon\big|_{t=0} = n_1 \end{cases}$$

where $u_\epsilon : \mathbb{R} \times [0,T] \to \mathbb{C}$, and $n_\epsilon : \mathbb{R} \times [0,T] \to \mathbb{R}$. The strongest local well-posedness results available for ZS_1 ($\epsilon = 1$) were obtained by [BC96] for $d = 2,3$ and $(u_0, n_0, n_1) \in H^1 \times L^2 \times \dot{H}^{-1}$, under a smallness assumption on u_0, and then

1. We will often suppress the subscript ϵ, and simply write u and n instead of u_ϵ and n_ϵ.

extended by [GTV97] to general d for $(u_0, n_0, n_1) \in H^k \times H^l \times H^{l-1}$ for a range of k and l values, without a smallness assumption on u_0. Here, $\dot{H}^{-1} = \{ f \in \mathcal{S}'(\mathbb{R}^d) \mid |\xi|^{-1}\hat{f}(\xi) \in L^2(\mathbb{R}^d) \}$. [GTV97] avoid introducing the assumption $n_1 \in \dot{H}^{-1}$, which appears in many papers on this topic, e.g. [SS79], [AA88], [OT92b]. As [OT92b] point out, this assumption is especially limiting for $d = 1$, where $\mathcal{S} \not\subseteq \dot{H}^{-1}$, since nice functions like e^{-x^2} fail to have an antiderivative in L^2.

By a scaling argument, the [GTV97] result extends to arbitrary ϵ and gives local well-posedness of ZS$_\epsilon$ on a time interval $[0, T_\epsilon]$ whose length depends on ϵ. The case of $l = k - 1$ for $d = 1$ appears as part of Prop. 1.2 of [GTV97], and we now state this result since we will appeal to it later.

Proposition 1 (special case of [GTV97], Prop. 1.2). *If $k \geq \frac{1}{2}$, and $(u_0, n_0, n_1) \in H^k \times H^{k-1} \times H^{k-2}$, then there exists a unique pair (u_ϵ, n_ϵ), solving* 1D ZS$_\epsilon$ *on a time interval $[0, T_\epsilon]$, such that $(u_\epsilon, n_\epsilon, \partial_t n_\epsilon) \in C([0, T_\epsilon];\ H^k \times H^{k-1} \times H^{k-2})$.*

Uniqueness holds with the auxiliary condition that u and n belong to the Bourgain spaces used in the [GTV97] argument (see [BSZ04] for a discussion of this matter).

As a formal exercise, if we send $\epsilon \to 0$ in ZS$_\epsilon$, and assume that $u_\epsilon \to v$ for some v, then by setting $\epsilon = 0$ in (1.2), we expect that v solves the cubic nonlinear Schrödinger equation

$$\text{NLS} = \begin{cases} \partial_t v = i\Delta v \pm i|v|^2 v \\ v\big|_{t=0} = u_0 \end{cases} \tag{1.5}$$

The case of $+$ in (1.5) is called "focusing" (FNLS), and the case of $-$ in (1.5) is called "defocusing" (DNLS). In the $d = 1$ case, we expect u_ϵ solving 1D ZS$_\epsilon$ to converge to v solving

$$\text{1D NLS} = \begin{cases} \partial_t v = i\partial_x^2 v \pm iv|v|^2 \\ v\big|_{t=0} = u_0 \end{cases} \tag{1.6}$$

In order to prove rigorous results concerning the convergence $u_\epsilon \to v$ as $\epsilon \to 0$, uniform in ϵ bounds on u_ϵ are needed. Some such uniform estimates were obtained by [AA88] from energy identities, and were applied by [AA88] and [OT92b] to obtain results on the aforementioned convergence. The purpose of the present chapter is to obtain improved uniform bounds in the $d = 1$ case in order to enhance the convergence results of [OT92b] in this context. Two methods will be explored for obtaining such uniform estimates.

The first method, via local smoothing properties for the Schrödinger group $e^{it\partial_x^2}$ and uniform estimates for the inverse reduced wave operators $(\epsilon\partial_t \pm \partial_x)^{-1}$, is summarized in §1.1.1 and detailed in §1.3. Earlier work in this direction was done by [KPV95]. The main result of this chapter (Prop. 4) falls into this category, and is an extension of the [KPV95] result to large initial data. The main advantages of the local smoothing approach are that (1) it applies to both DZS_ϵ and FZS_ϵ, and potentially even other, more general nonlinearities; and (2) it yields uniform bounds on a smoothing estimate, which has applications to a sharpening of the convergence $u_\epsilon \to v$ results obtained by [OT92b].

The second method, via energy identities, is summarized in §1.1.2 and carried out in detail in §1.5. The method is very similar to that appearing in [AA88], although there uniform bounds are obtained on $\|u(t)\|_{H_x^k}$, $\|n(t)\|_{H_x^{k-1}}$ under the assumption $(u_0, n_0, n_1) \in H^{k+2} \times H^{k+1} \times (H^k \cap \dot{H}^{-1})$, whereas we shall want to avoid assuming extra regularity on the initial data, and to this end carry out bounds on $\|u(t)\|_{H_x^k}$, $\|n(t)\|_{H_x^{k-1}}$ under the assumption $(u_0, n_0, n_1) \in H^k \times H^{k-1} \times H^{k-2}$. Moreover, we drop the assumption $n_1 \in \dot{H}^{-1}$. The main advantages of the energy method are that (1) u_0 is not required to belong to a weighted L^2 space; and (2) n_0 and n_1 are required to belong only to H^{k-1} and H^{k-2}, respectively, rather than $H^{k-\frac{1}{2}}$ and $H^{k-\frac{3}{2}}$ as in the local smoothing approach. Its main disadvantage is that it only applies to FZS_ϵ, and not to DZS_ϵ.

The applications to the convergence $u_\epsilon \to v$ are summarized in §1.1.3, and detailed in §1.4. Whether or not the method of [GTV97] itself can be modified to yield uniform in ϵ estimates remains a topic for future investigation.

1.1.1 Uniform estimates via local smoothing

[KPV95] prove a uniform in ϵ estimate for the solution to the inhomogeneous wave equation

$$\sup_{\alpha\in\mathbb{Z}^d} \|\nabla_x \Box_\epsilon^{-1} F\|_{L^2(Q_\alpha\times[0,T])} \le c\sum_\alpha \|F\|_{L^2(Q_\alpha\times[0,T])} \tag{1.7}$$

where Q_α is the unit cube centered at the lattice point $\alpha\in\mathbb{Z}^d$, and c is independent of ϵ. They also use the smoothing for solutions to the inhomogeneous Schrödinger equation

$$\sup_{\alpha\in\mathbb{Z}^d} \left\| \nabla_x \int_0^t e^{i(t-t')\Delta} f(\cdot,t)\,dt' \right\|_{L^2(Q_\alpha\times[0,T])} \le c\sum_\alpha \|f\|_{L^2(Q_\alpha\times[0,T])}$$

to obtain

Proposition 2 ([KPV95]). *Let $d\ge 1$. Then $\exists\ k>0, m\in\mathbb{Z}^+, \delta>0$ such that for any*

$$(u_0,n_0,n_1)\in H^k\cap H^{k_0}(|x|^m dx)\times H^{k-\frac12}\times H^{k-\frac32}\equiv X_{s,m},\quad k_0=\left[\frac{k+3}{2}\right]$$

with $\|(u_0,n_0,n_1)\|_{X_{s,m}}\le\delta$, then $\exists\ T=T(\|(u_0,n_0,n_1)\|_{X_{s,m}})>0$ independent of $\epsilon\le 1$, and a unique solution (u_ϵ,n_ϵ) of ZS_ϵ satisfying the uniform in ϵ bounds

$$\|u_\epsilon\|_{L_T^\infty H_x^k} + \||x|^m u_\epsilon\|_{L_T^\infty H_x^{k_0}} + \sup_\alpha \|D_x^{k+\frac12}u_\epsilon\|_{L^2(Q_\alpha\times[0,T])}\le c$$

Let us analyze the 1D case more carefully. Using the inverse directional derivative operators

$$P_\pm z(x,t)=\int_{s=0}^{t/\epsilon} z(x\mp s, t-\epsilon s)\,ds$$

we have

$$n=-\tfrac12 P_+\partial_x(u\bar u)+\tfrac12 P_-\partial_x(u\bar u)+\tfrac12 n_0(x-\tfrac12)+\tfrac12 n_0(x+\tfrac{t}{\epsilon})+\tfrac12\epsilon\int_{x-\frac{t}{\epsilon}}^{x+\frac{t}{\epsilon}} n_1(s)\,ds$$

Then replace the uniform in ϵ estimate (1.7) with the 1D uniform in ϵ estimates

$$\|P_\pm z\|_{L^\infty_x L^2_T} \le \|z\|_{L^1_x L^2_T}$$

Also use the 1D sharp Schrödinger smoothing estimates

$$\|D_x^{1/2} e^{it\partial_x^2} u_0\|_{L^\infty_x L^2_T} \le c\|u_0\|_{L^2_x}$$

$$\left\| D_x \int_0^t e^{i(t-t')\partial_x^2} f(x,t') \right\|_{L^\infty_x L^2_T} \le c\|f\|_{L^1_x L^2_T}$$

Following the [KPV95] method, we can prove

Proposition 3 (Modified [KPV95]). *Let $k \ge 4$, $(u_0, n_0, n_1) \in H^k \cap H^1(\langle x\rangle^2 dx) \times H^{k-\frac{1}{2}} \times H^{k-\frac{3}{2}}$, and assume $\|\langle x\rangle u_0\|_{L^2_x} \le \frac{1}{10}$. Then there is*

$$T = T(\|u_0\|_{H^k}, \|n_0\|_{H^{k-\frac{1}{2}}}, \|n_1\|_{H^{k-\frac{3}{2}}}) > 0$$

(independent of ϵ) and a solution (u,n) to 1D ZS_ϵ *on $[0,T]$ such that $\forall\ \epsilon,\ 0 < \epsilon \le 1$,*

$$\|u\|_{L^\infty_T H^k_x} + \|D_x^{1/2}\partial_x^k u\|_{L^\infty_x L^2_T} + \|n\|_{L^\infty_T H^{k-1}_x} \le c$$

with c independent of ϵ, but depending on the norms $\|u_0\|_{H^k}$, $\|n_0\|_{H^{k-\frac{1}{2}}}$, $\|n_1\|_{H^{k-\frac{3}{2}}}$.

The main result of this paper is the same conclusion without the smallness assumption $\|\langle x\rangle u_0\|_{L^2_x} \le \frac{1}{10}$. To prove this, we adapt a method previously developed by [Chi96], [KPV98] to treat NLS equations having an order 1 nonlinearity. We introduce a pseudodifferential operator B with symbol $b(x,\xi) \in S^0$ depending on a constant M and satisfying

$$e^{-M} \le b(x,\xi) \le e^M$$

and apply it to the k-th derivative of (1.3) in the form

$$\partial_t u = i\partial_x^2 u \pm \tfrac{1}{2} iuP_\pm \partial_x(u\bar{u}) - ifu \tag{1.8}$$

where

$$f(x,t) = \tfrac{1}{2}n_0(x+\tfrac{t}{\epsilon}) + \tfrac{1}{2}n_0(x-\tfrac{t}{\epsilon}) + \tfrac{1}{2}\epsilon\int_{x-\frac{t}{\epsilon}}^{x+\frac{t}{\epsilon}} n_1(y)\,dy$$

The commutator $[B, i\partial_x^2]$ generates a first order term that is negative and whose size can be controlled by the constant M. In fact, by selecting $M = c\|\langle x\rangle u_0\|_{L_x^2}$, this commutator is sufficiently negative to absorb the first order terms $B\partial_x^k(\pm\frac{1}{2}iuP_\pm\partial_x(u\bar{u}))$. The key obstacle in showing this is that $[B, P_\pm]$ is not of lower order in x (nor can it be made small by any other device). It would instead suffice if the composition $BP_\pm B^{-1}$ were bounded independently of M; however, this turns out to be false as well. This problem is resolved by observing that $BP_\pm B^{-1}$ is in fact bounded independently of M if we restrict to certain spatial frequency ranges, and that $BP_\pm^* B^{-1}$ is bounded independently of M on the complementary spatial frequency ranges. The "error terms" obtained by replacing $P_\pm$ by $-P_\pm^*$ are handled using positivity properties of the operators $U_\pm = P_\pm + P_\pm^*$, and using once again that u solves (1.8). Specifically, we make use of an "extra smoothing" property of solutions of (1.8). Since $\partial_t \sim i\partial_x^2$ in the Schrödinger component and $\epsilon\partial_t \sim \pm\partial_x$ in the reduced wave components, we expect that $\epsilon\partial_x^2 \sim \mp i\partial_x$, or in other words, by absorbing an ϵ, we can convert a second-order term to a first-order term. It turns out that in the needed spatial frequency zones, the second order term we consider is equivalent to a *negative* first-order term that can, therefore, be dropped. To make this a bit more precise, suppose z solves $\partial_t z = i\partial_x^2 z +$ first-order terms, and K is an order 0 operator (in x). Let $\widehat{L_- z}(\xi,t) = \chi_{\xi\leq 0}\hat{z}(\xi,t)$. Then, with $\approx$ meaning "modulo order 0 terms or order 1 terms with an ϵ coefficient," and $\langle z_1, z_2\rangle = \int_0^T\int_x z_1\bar{z}_2\,dx\,dt$,

$$\begin{aligned}
&\epsilon\langle L_-U_+\partial_x Kz, \partial_x Kz\rangle \\
&\qquad \approx -\epsilon\langle L_-U_+K\partial_x^2 z, Kz\rangle \\
&\qquad \approx +\epsilon\langle L_-U_+Ki\partial_t z, Kz\rangle \\
&\qquad = i\langle L_-U_+(\epsilon\partial_t)Kz, Kz\rangle \\
&\qquad = -i\langle L_-U_+\partial_x Kz, Kz\rangle \\
&\qquad = -\langle L_-U_+D_x^{1/2}Kz, D_x^{1/2}Kz\rangle
\end{aligned}$$

We started with the positive second-order term $\epsilon\langle L_-U_+\partial_x Kz, \partial_x Kz\rangle$, and ended with the negative first-order term $-\langle L_-U_+D_x^{1/2}Kz, D_x^{1/2}Kz\rangle$, and thus the original term is (effectively) zero. Similarly, with $\widehat{L_+z}(\xi,t) = \chi_{\xi\geq 0}\hat{z}(\xi,t)$,

$$\epsilon\langle L_+U_-\partial_x Kz, \partial_x Kz\rangle \approx -\langle L_+U_-D_x^{1/2}Kz, D_x^{1/2}Kz\rangle$$

The result is

Proposition 4 (Uniform local smoothing for FZS$_\epsilon$ and DZS$_\epsilon$). *Let* $k \geq 4$, $(u_0, n_0, n_1) \in H^k \cap H^1(\langle x\rangle^2 dx) \times H^{k-\frac{1}{2}} \times H^{k-\frac{3}{2}}$. *Then* $\exists\, T > 0$ *with*

$$T \sim \left(\exp(\|\langle x\rangle u_0\|_{H_x^1}) + \|u_0\|_{H_x^k} + \|n_0\|_{H_x^{k-\frac{1}{2}}} + \|n_1\|_{H_x^{k-\frac{3}{2}}}\right)^{-N}$$

(independent of ϵ*) and a solution* (u,n) *to* 1D ZS$_\epsilon$ *on* $[0,T]$ *such that* $\forall\, \epsilon,\ 0 < \epsilon \leq 1$,

$$\|u\|_{L_T^\infty H_x^k} + \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2} \leq c(\exp c\|\langle x\rangle u_0\|_{H_x^1})\|u_0\|_{H_x^k} \tag{1.9}$$

with c *independent of* ϵ.

We remark that the proof of this result appearing in §1.3 can probably be adapted to yield a bound for any given time $T > 0$, provided $0 < \epsilon \leq \epsilon_0$, where

$$\epsilon_0 = \epsilon_0(T, \|u_0\|_{H_x^k}, \|\langle x\rangle u_0\|_{H_x^1}, \|n\|_{H_x^{k-\frac{1}{2}}}, \|n_1\|_{H_x^{k-\frac{3}{2}}})$$

1.1.2 Uniform estimates via energy identities

We shall assume that $0 < \epsilon \leq 1$. The conservation of energy identity [SS79] is

$$\partial_t \int_x |\nabla u|^2 \pm \tfrac{1}{2}n^2 \pm n|u|^2 \pm \tfrac{1}{2}\epsilon^2|\nu|^2 = 0 \tag{1.10}$$

where $\nu : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ [2] is such that

$$\partial_t n = \nabla \cdot \nu \qquad \epsilon^2 \partial_t \nu = \nabla(n + |u|^2)$$
$$n\big|_{t=0} = n_0 \qquad \nabla \cdot \nu\big|_{t=0} = n_1$$

The top row of signs applies to FZS$_\epsilon$ and the bottom row to DZS$_\epsilon$. The introduction of the variable ν requires the assumption $n_1 \in \dot{H}^{-1}$. By the Gagliardo-Nirenberg inequality applied to the term $\int_x n|u|^2$, we obtain constant in time, independent of ϵ, bounds on $\|u(t)\|_{H^1_x}$, $\|n(t)\|_{L^2_x}$ for (u,n) solving FZS$_\epsilon$ with initial data $(u_0, n_0, n_1) \in H^1 \times L^2 \times \dot{H}^{-1}$, in the following cases:

- $d = 1$ (with no further assumptions)
- $d = 2$ and $\|u_0\|_{L^2_x}$ small.
- $d = 3$, with $\|u_0\|_{H^1}$ small, $\|n_0\|_{L^2_x}$ small, and $\|n_1\|_{\dot{H}^{-1}}$ small.

[GM94] give a partially conserved quantity that avoids the need to assume $n_1 \in \dot{H}^{-1}$. Decompose n_1 into low and high frequencies as $n_1 = n_{1L} + n_{1H}$.

$$\partial_t \int_x |\nabla u|^2 \pm \tfrac{1}{2} n^2 \pm n|u|^2 \pm \tfrac{1}{2}\epsilon^2 |\nu|^2 = \int_x n_{1L}(n + u\bar{u}) \tag{1.11}$$

where $\nu : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ is defined by

$$\partial_t n = \nabla \cdot \nu + n_{1L} \qquad \epsilon^2 \partial_t \nu = \nabla(n + |u|^2)$$
$$n\big|_{t=0} = n_0 \qquad \nabla \cdot \nu\big|_{t=0} = n_{1H}$$

Equivalently,

$$\partial_t \int_x |\nabla u|^2 \pm \tfrac{1}{2}(n + |u|^2)^2 \mp \tfrac{1}{2}|u|^4 + \epsilon^2 |\nu|^2 = \int_x n_{1L}(n + u\bar{u})$$

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2. Note that $\nu = \nu_\epsilon$ does depend on ϵ, although we are following our convention to suppress this subscript.

In this form, the identity should be compared with the conservation of energy identity for NLS:

$$\partial_t \int_x |\nabla u|^2 \mp \tfrac{1}{2}|u|^4 = 0$$

where the top sign is for FNLS and the bottom sign for DNLS. By the Gagliardo-Nirenberg inequality applied to the term $\int_x |u|^4$, we obtain the bounds $\|u(t)\|_{H^1_x} \leq c\langle t\rangle$, $\|n(t)\|_{L^2_x} \leq c\langle t\rangle$, with c independent of ϵ, for (u,n) solving FZS$_\epsilon$ with initial data $(u_0, n_0, n_1) \in H^1 \times L^2 \times H^{-1}$, in the following cases:

- $d = 1$ (with no further assumptions)
- $d = 2$ and $\|u_0\|_{L^2_x}$ small.

We remark that due to the $-$ signs in (1.10) and (1.11), we are not able to obtain any information for solutions to DZS$_\epsilon$.

For $d = 1$, we prefer to work in reduced wave variables. As in [GM94], decompose $n_1 = n_{1H} + n_{1L}$ and let

$$\hat{\nu}(\xi) = \frac{\hat{n}_{1H}(\xi)}{i\xi}$$

Then n_+ and n_- are defined (see §1.5) so that

$$(\epsilon\partial_t \pm \partial_x)n_\pm = \mp\tfrac{1}{2}\partial_x(u\bar{u})$$

and $n = n + n_- + f$, where

$$f(x,t) = \tfrac{1}{2}\epsilon \int_{y=x-\frac{t}{\epsilon}}^{y=x+\frac{t}{\epsilon}} n_{1L}(y)\, dy$$

In these variables, we derive the identity, for 1D FZS$_\epsilon$

$$\partial_t \int |\partial_x u|^2 + (n_+ + \tfrac{1}{2}u\bar{u})^2 + (n_- + \tfrac{1}{2}u\bar{u})^2 - \tfrac{1}{2}|u|^4 + fu\bar{u} = \tfrac{1}{2}\int [n_{1L}(x+\tfrac{t}{\epsilon}) + n_{1L}(x-\tfrac{t}{\epsilon})]u\bar{u}$$

from which it follows that $\|u(t)\|_{H^1_x} \leq c\langle t\rangle^{1/2}$ and $\|n(t)\|_{L^2_x} \leq c\langle t\rangle$, thus improving the bound on $\|u(t)\|_{H^1_x}$ over that obtained from (1.11). In dealing with the $(u_0, n_0, n_1) \in$

$H^k \times H^{k-1} \times H^{k-2}$ setting, it appears necessary (if you want to preserve uniformity in ϵ) to apply Gronwall's inequality to treat the terms

$$\operatorname{Re} i \int_0^t \int_x \partial_x^{k-1}(n_\pm + \tfrac{1}{2}u\bar{u})\, \partial_x u\, \partial_x^k \bar{u} \tag{1.12}$$

and thus we are only able to obtain exponential in time bounds. This result is:

Proposition 5 (Energy estimates). *Suppose $k \geq 3$, $(u_0, n_0, n_1) \in H^k \times H^{k-1} \times H^{k-2}$. Then $\forall\ T > 0$, there exists $\epsilon_0 = \epsilon_0(T, \|u_0\|_{H_x^k}, \|n_0\|_{H_x^{k-1}}, \|n_1\|_{H_x^{k-2}})$ such that $\forall\ \epsilon,\ 0 < \epsilon \leq \epsilon_0$ there exists (u, n) solving* 1D FZS$_\epsilon$ *on $[0, T]$ such that*

$$\|u(t)\|_{H_x^k} + \|n(t)\|_{H_x^{k-1}} \leq ch(t)$$

where c depends on the norms $\|u_0\|_{H^k}$, $\|n_0\|_{H^{k-1}}$, and $\|n_1\|_{H^{k-2}}$ and is independent of ϵ, and $h(t)$ has exponential growth in t.

However, ϵ dependent bounds can be achieved with a $\langle t \rangle^m$ bound, for some $m \in \mathbb{R}^+$. This is obtained by using integrability of the term

$$\operatorname{Re} \int_x \partial_x^k (u\bar{u}u) \partial_x^k \bar{u} \tag{1.13}$$

and reducing the order of (1.12) by introducing a factor $\frac{1}{\epsilon}$. The integrability of (1.13) is limited to the case $d = 1$. For $d = 2$, [CS02] obtain a similar result, using a method developed for Hamiltonian systems by [Bou96] and subsequently refined by [Sta97]. It remains unanswered, even for $d = 1$, whether or not a $\langle t \rangle^m$ bound holds uniformly in ϵ, for $k \geq 3$.

1.1.3 Convergence to NLS

In §1.4, we shall apply the uniform local smoothing estimates obtained in §1.3 to prove results on the convergence $u_\epsilon \to v$ as $\epsilon \to 0$, following the method of [OT92a]. We begin here by summarizing some earlier results.

[AA88] obtained results for $d = 1, 2, 3$, and we quote their result in the case $d = 1$.

Proposition 6 (Case $d = 1$ of [AA88]). *If $(u_0, n_0, n_1) \in H^{k+2} \times H^{k+1} \times (H^k \cap \dot{H}^{-1})$, for $k \geq 3$, then $\exists\, h(t) \in L^\infty_{loc}(\mathbb{R}^+)$ such that u_ϵ solving* 1D FZS$_\epsilon$ *satisfies*

$$\|u_\epsilon - v\|_{H^k_x} \leq h(t)\epsilon^{1/2}$$

Their method is to derive an energy identity in $u_\epsilon - v$ from the equation solved by $u_\epsilon - v$, obtained by taking the difference of (1.3) and (1.6):

$$\partial_t(u - v) = i\partial_x^2(u - v) - i(n + u\bar{u})u + i(|u|^2u - |v|^2v) \tag{1.14}$$

Thus, [AA88] obtain convergence at rate $\epsilon^{1/2}$ with two derivatives of separation between the space H^{k+2} in which the initial data u_0 resides and the space H^k in which the convergence $u_\epsilon \to v$ takes place. They do not assume that the initial data belongs to any weighted spaces. By introducing the modifications of the type appearing in §1.5, we could remove the assumption $n_1 \in \dot{H}^{-1}$ in Prop. 6.

[OT92b] build on the method of [AA88] and obtain, by introducing weights on the initial data, results for $d = 1, 2, 3$. The [OT92b] result in the case $d = 1$ is:

Proposition 7 (Case $d = 1$ of [OT92b]). *If $(u_0, n_0, n_1) \in \mathcal{S} \times \mathcal{S} \times \mathcal{S} \cap \dot{H}^{-1}$, then $\forall\, T > 0$, $\exists\, \epsilon_0 > 0$ and $c = c(T) > 0$ such that if $0 < \epsilon \leq \epsilon_0$, then u_ϵ solving* 1D FZS$_\epsilon$ *satisfies*

$$\|u_\epsilon - v\|_{L^\infty_T H^k_x} \leq c\epsilon \tag{1.15}$$

in the noncompatible case $n_0 + u_0\bar{u}_0 \neq 0$ and

$$\|u_\epsilon - v\|_{L^\infty_T H^k_x} \leq c\epsilon^2 \tag{1.16}$$

in the compatible case $n_0 + u_0\bar{u}_0 = 0$.

Once again, by introducing the modifications of the type appearing in §1.5, we could remove the assumption $n_1 \in \dot{H}^{-1}$, but only for (1.15) and not (1.16). [OT92b] borrow the uniform bounds on u provided by [AA88] but carry out a separate analysis of (1.14). By refining their argument in parts and introducing some power tools of harmonic analysis [KPV93b] to obtain maximal function estimates, we can reduce the assumptions on the initial data to

$$(u_0, n_0, n_1) \in (H^{k+2} \cap H^1(\langle x \rangle^2 dx)) \times (H^{k+1} \cap L^1) \times (H^k \cap L^1)$$

for (1.15) and

$$\begin{gathered}(u_0, n_0, n_1) \in (H^{k+3} \cap H^1(\langle x \rangle^2 dx)) \times H^{k+2} \times H^{k+1} \\ \text{and } \exists\, \nu \in L^1 \text{ such that } \partial_x \nu = n_1\end{gathered}$$

for (1.16). If, instead of invoking the uniform bounds provided by [AA88], we instead use the local smoothing estimate Prop. 4, then we can reduce the assumptions on the initial data further to

$$(u_0, n_0, n_1) \in (H^{k+1} \cap H^1(\langle x \rangle^2 dx)) \times (H^{k+\frac{1}{2}} \cap L^1) \times (H^{k-\frac{1}{2}} \cap L^1) \tag{1.17}$$

for (1.15) and

$$\begin{gathered}(u_0, n_0, n_1) \in (H^{k+2} \cap H^1(\langle x \rangle^2 dx)) \times H^{k+\frac{3}{2}} \times H^{k+\frac{1}{2}} \\ \text{and } \exists\, \nu \in L^1 \text{ such that } \partial_x \nu = n_1\end{gathered} \tag{1.18}$$

for (1.16). Thus, in the noncompatible case, we obtain convergence at rate ϵ with one derivative of separation between the space H^{k+1} in which the initial data u_0 resides and the space H^k in which the convergence $u_\epsilon \to v$ takes place; and in the compatible case, we obtain convergence at rate ϵ^2 with two derivatives of separation between the space H^{k+2} in which the initial data u_0 resides and the space H^k in which the convergence $u_\epsilon \to v$ takes place. [OT92b] show further that the rates ϵ and ϵ^2 in the noncompatible and compatible cases, respectively, are optimal.

An additional bonus of invoking the uniform bound Prop. 4 in place of the bounds in [AA88] is that convergence can be obtained in the defocusing case as well. That is, we obtain the convergence (1.15) and (1.16) for u_ϵ solving 1D DZS$_\epsilon$ and v solving 1D DNLS, under the assumptions (1.17) and (1.18). To summarize, the result is:

Proposition 8 (Convergence of 1D ZS$_\epsilon$ to 1D NLS). *For given* (u_0, n_0, n_1)*, let* u_ϵ *be the solution to* 1D ZS$_\epsilon$ *with initial data* (u_0, n_0, n_1) *on the time interval* $[0,T]$ *given by Prop. 4, and let* v *be the (global) solution to* 1D NLS *with initial data* u_0*. In the* noncompatible case $(n_0 + u_0\bar{u}_0) \neq 0$*, if*

$$(u_0, n_0, n_1) \in (H^{k+1} \cap H^1(\langle x\rangle^2 dx)) \times (H^{k+\frac{1}{2}} \cap L^1) \times (H^{k-\frac{1}{2}} \cap L^1) \tag{1.19}$$

then

$$\|\langle x\rangle^{-1} D_x^{1/2} \partial_x^k (u_\epsilon - v)\|_{L^2_x L^2_T} + \|u_\epsilon - v\|_{L^\infty_T H^k_x} \leq c\epsilon$$

where c *depends on the norms of the spaces in* (1.19)*. In the* compatible case $(n_0 + u_0\bar{u}_0) = 0$*, if*

$$\begin{gathered}(u_0, n_0, n_1) \in (H^{k+2} \cap H^1(\langle x\rangle^2 dx)) \times H^{k+\frac{3}{2}} \times H^{k+\frac{1}{2}} \\ \text{and } \exists\, \nu \in L^1 \text{ such that } \partial_x \nu = n_1\end{gathered} \tag{1.20}$$

then

$$\|\langle x\rangle^{-1} D_x^{1/2} \partial_x^k (u_\epsilon - v)\|_{L^2_x L^2_T} + \|u_\epsilon - v\|_{L^\infty_T H^k_x} \leq c\epsilon^2$$

where c *depends on the norms of the spaces in* (1.20)*.*

1.2 Definitions and basic properties of operators

All statements with regard to adjoints and positivity of operators (by which we mean nonnegativity) are meant with respect to the inner product

$$\langle z_1, z_2\rangle = \int_{t=0}^{T} \int_x z_1(x,t)\overline{z_2(x,t)}\, dx\, dt$$

For a given $T > 0$ and $\epsilon > 0$, define

$$P_\pm z(x,t) = \int_{s=0}^{t/\epsilon} z(x \mp s, t - \epsilon s)\, ds \quad U_\pm z(x,t) = \int_{s=(t-T)/\epsilon}^{t/\epsilon} z(x \mp s, t - \epsilon s)\, ds \tag{1.21}$$

These operators have the properties

$$(\epsilon\partial_t \pm \partial_x)P_\pm z(x,t) = z(x,t) \qquad P_\pm z(x,0) = 0 \qquad \partial_t P_\pm z(x,0) = \frac{1}{\epsilon} z(x,0)$$

A brief calculation shows that $P_\pm^* = -P_\pm + U_\pm$, $U_\pm^* = U_\pm$, and

$$P_\pm^* z(x,t) = \int_{\frac{t-T}{\epsilon}}^{0} z(x \mp s, t - \epsilon s)\, ds$$

Let B be the order 0 in x ψ-do with symbol

$$b(x,\xi) = \exp\left[-M\left(\int_0^x \langle s\rangle^{-2}\, ds\right) \operatorname{sgn} \xi\right] \tag{1.22}$$

where $M > 0$ is a constant to be selected. The symbol $b(x,\xi)$ satisfies

$$e^{-M} \le b(x,\xi) \le e^{+M}$$

Define $L_\pm$ as operators in x by $\widehat{L_+ f}(\xi) = \chi_{\xi \ge 0} \hat{f}(\xi)$ and $\widehat{L_- f}(\xi) = \chi_{\xi \le 0} \hat{f}(\xi)$.

Lemma 1. *(a)* $\|P_\pm z\|_{L_x^\infty L_T^2} \le \|z\|_{L_x^1 L_T^2}$*, and similarly for* $P_\pm^*$ *and* $U_\pm$*.*

(b) We have the bounds

$$\begin{aligned}
\|BL_+P_+B^{-1}z\|_{L_x^\infty L_T^2} &\le c\|L_+z\|_{L_x^1 L_T^2} \\
\|BL_-P_+^*B^{-1}z\|_{L_x^\infty L_T^2} &\le c\|L_-z\|_{L_x^1 L_T^2} \\
\|BL_-P_-B^{-1}z\|_{L_x^\infty L_T^2} &\le c\|L_-z\|_{L_x^1 L_T^2} \\
\|BL_+P_-^*B^{-1}z\|_{L_x^\infty L_T^2} &\le c\|L_+z\|_{L_x^1 L_T^2}
\end{aligned} \tag{1.23}$$

up to lower order terms in x*, where* c *is independent of* M*.*

(c) $L_\pm U_+ = U_+ L_\pm$*,* $L_\pm P_+ = P_+ L_\pm$*,* $L_\pm P_+^* = P_+^* L_\pm$*, and similarly for* U_-*,* P_-*, and* P_-^**.*

(d) $U_\pm = V_\pm \chi_{0\le t\le T}$*, where*

$$V_\pm z(x,t) = \int_{s=-\infty}^{s=+\infty} z(x \mp s, t - \epsilon s)\, ds$$

and $\widehat{V_\pm z}(\xi,\tau) = \delta(\epsilon\tau \pm \xi)\hat{z}(\xi,\tau)$*. Hence* $\langle U_\pm z, z\rangle \ge 0$ *and*

$$|\langle U_\pm z_1, z_2\rangle| \le \langle U_\pm z_1, z_1\rangle^{1/2}\langle U_\pm z_2, z_2\rangle^{1/2} \tag{1.24}$$

Also,

$$\begin{aligned} |\langle L_\pm U_+ z_1, z_2\rangle| &\le \langle [L_\pm] U_+ z_1, z_1\rangle^{1/2}\langle [L_\pm] U_+ z_2, z_2\rangle^{1/2} \\ |\langle L_\pm U_- z_1, z_2\rangle| &\le \langle [L_\pm] U_- z_1, z_1\rangle^{1/2}\langle [L_\pm] U_- z_2, z_2\rangle^{1/2} \end{aligned} \tag{1.25}$$

where $[L_\pm]$ *indicates that the operator* $L_\pm$ *can be either included or omitted in that position. Finally,*

$$\begin{aligned} |\langle L_\pm U_+ z_1, z_2\rangle| &\le \|z_1\|_{L^1_x L^2_T}\|z_2\|_{L^1_x L^2_T} \\ |\langle L_\pm U_- z_1, z_2\rangle| &\le \|z_1\|_{L^1_x L^2_T}\|z_2\|_{L^1_x L^2_T} \end{aligned} \tag{1.26}$$

Proof. (a) follows from Minkowskii's integral inequality and formulas (1.21), keeping in mind

$$\begin{aligned} 0 \le s \le \tfrac{t}{\epsilon} &\iff 0 \le t - \epsilon s \le t \\ \tfrac{t-T}{\epsilon} \le s \le \tfrac{t}{\epsilon} &\iff 0 \le t - \epsilon s \le T \\ \tfrac{t-T}{\epsilon} \le s \le 0 &\iff t \le t - \epsilon s \le T \end{aligned}$$

(b)

$$BP_+B^{-1}z(x,t) = \int_\xi e^{ix\xi} b(x,\xi)\widehat{P_+B^{-1}}z(\xi,t)\, d\xi \tag{1.27}$$

By Fubini:

$$\widehat{P_+B^{-1}}z(\xi,t) = \int_{s=0}^{t/\epsilon} e^{-is\xi}\widehat{B^{-1}z}(\xi, t-\epsilon s)\, ds$$

Substituting into (1.27) and using Fubini:

$$BP_+B^{-1}z(x,t) = \int_{s=0}^{t/\epsilon}\int_\xi e^{i(x-s)\xi} b(x,\xi)\widehat{B^{-1}z}(\xi, t-\epsilon s)\, d\xi\, ds$$

Let $b^s(x,\xi) = b(x+s,\xi)$ and B^s be the corresponding operator. Then

$$\begin{aligned} BP_+B^{-1}z(x,t) &= \int_{s=0}^{t/\epsilon}\int_\xi e^{i(x-s)\xi} b^s(x-s,\xi)\widehat{B^{-1}z}(\xi, t-\epsilon s)\, d\xi\, ds \\ &= \int_{s=0}^{t/\epsilon} (B^s B^{-1})z(x-s, t-\epsilon s)\, ds \end{aligned}$$

Similarly,

$$BP_+^* B^{-1}z(x,t) = \int_{s=\frac{t-T}{\epsilon}}^{0} (B^s B^{-1})z(x-s, t-\epsilon s)\, ds$$

B^sB^{-1} has symbol, modulo lower order,

$$b(x+s,\xi)b^{-1}(x,\xi) = \exp\left[M\int_{x+s}^{x} \langle r\rangle^{-2}\, dr\, \theta^2(\xi)\,\mathrm{sgn}\,\xi\right]$$

If $s \geq 0$ and $\xi \geq 0$, then $[\cdots] \leq 0$ and $\exp[\cdots] \leq 1$. If $s \leq 0$ and $\xi \leq 0$, then $[\cdots] \leq 0$ and $\exp[\cdots] \leq 1$.

A similar calculation shows

$$BP_-B^{-1}z(x,t) = \int_{s=0}^{t/\epsilon} (B_s B^{-1})(x+s, t-\epsilon s)\, ds$$

and

$$BP_-^* B^{-1}z(x,t) = \int_{s=\frac{t-T}{\epsilon}}^{0} (B_s B^{-1})(x+s, t-\epsilon s)\, ds$$

where B_s has symbol $b_s(x,\xi) = b(x-s,\xi)$. B_sB^{-1} has symbol, modulo lower order,

$$b(x-s,\xi)b^{-1}(x,\xi) = \exp\left[M\int_{x-s}^{x} \langle r\rangle^{-2}\, dr\, \theta^2(\xi)\,\mathrm{sgn}\,\xi\right]$$

If $s \geq 0$ and $\xi \leq 0$, then $[\cdots] \leq 0$ and $\exp[\cdots] \leq 1$. If $s \leq 0$ and $\xi \geq 0$, then $[\cdots] \leq 0$ and $\exp[\cdots] \leq 1$.

(c) By Fubini, if L is any multiplier operator with symbol $m(\xi)$, then

$$\begin{aligned}
LP_+z &= \int_\xi e^{ix\xi} m(\xi) \left[\int_{s=0}^{t/\epsilon} z(x-s, t-\epsilon s)\, ds \right]^\frown (\xi)\, d\xi \\
&= \int_{s=0}^{t/\epsilon} \int_\xi e^{i(x-s)\xi} m(\xi)\, \hat{z}(\xi, t-\epsilon s)\, d\xi\, ds \\
&= \int_{s=0}^{t/\epsilon} (Lz)(x-s, t-\epsilon s)\, ds = P_+Lz
\end{aligned}$$

(d) Since $\frac{t-T}{\epsilon} \le s \le \frac{t}{\epsilon} \iff 0 \le t - \epsilon s \le T$, we have $U_\pm = V_\pm \chi_{0\le t\le T}$. By Fubini,

$$\widehat{V_+ z}(\xi, t) = \int_\xi e^{-is\xi} \hat{z}(\xi, t-\epsilon s)\, ds$$

Again, by Fubini,

$$\widehat{V_+ z}(\xi, \tau) = \int_\xi e^{-is(\epsilon\tau+\xi)} \hat{z}(\xi, \tau)\, ds = \delta(\epsilon\tau + \xi)\hat{z}(\xi, \tau)$$

We now prove (1.24). For given z_1, z_2, set $w_j(x,t) = \chi_{0\le t\le T} z_j(x,t)$. Then

$$\begin{aligned}
\langle U_+ z_1, z_2 \rangle &= \int_t \int_x V_+ w_1(x,t)\, \overline{w_2(x,t)}\, dx\, dt \\
&= \int_\tau \int_\xi \widehat{V_+ w_1}(\xi, \tau)\, \overline{\hat{w}_2(\xi, \tau)}\, d\xi\, d\tau \\
&= \int_\xi \hat{w}_1 \left(\xi, -\frac{\xi}{\epsilon} \right) \overline{\hat{w}_2 \left(\xi, -\frac{\xi}{\epsilon} \right)}\, d\xi
\end{aligned}$$

Apply Cauchy-Schwarz,

$$\begin{aligned}
|\langle U_+ z_1, z_2 \rangle| &\le \prod_{j=1}^{2} \left(\int_\xi \hat{w}_j \left(\xi, -\frac{\xi}{\epsilon} \right) \overline{\hat{w}_j \left(\xi, -\frac{\xi}{\epsilon} \right)}\, d\xi \right)^{1/2} \\
&= \langle U_+ z_1, z_1 \rangle^{1/2} \langle U_+ z_2, z_2 \rangle^{1/2}
\end{aligned}$$

Now we prove (1.25). From Lemma 1c and (1.24),

$$|\langle L_\pm U_+ z_1, z_2\rangle| \leq \langle L_\pm U_+ z_1, z_1\rangle^{1/2}\langle L_\pm U_+ z_2, z_2\rangle^{1/2}$$

But, by (1.24) again

$$\langle L_\pm U_+ z_1, z_1\rangle = \langle U_+(L_\pm z_1), z_1\rangle \leq \langle L_\pm U_+ z_1, z_1\rangle^{1/2}\langle U_+ z_1, z_1\rangle^{1/2}$$

and hence

$$\langle L_\pm U_+ z_1, z_1\rangle \leq \langle U_+ z_1, z_1\rangle$$

(1.26) is a corollary of (1.25) and Lemma 1a. □

1.3 Uniform estimates via local smoothing

We start with the solution given by [GTV97], cited in Prop. 1, and prove uniform bounds for it.

The goal of this section is to prove Prop. 4.

Proof. We shall assume $T \leq 1$. For convenience, we shall only write out the computation for 1D FZS$_\epsilon$, although it will be evident that it also applies to 1D DZS$_\epsilon$. We can write 1D FZS$_\epsilon$ (1.3)-(1.4) as

$$\partial_t u = i\partial_x^2 u - in_\pm u - ifu \tag{1.28}$$

where

$$n_\pm = \mp\tfrac{1}{2}\partial_x P_\pm(u\bar{u})$$

and

$$f(x,t) = \tfrac{1}{2}n_0(x+\tfrac{t}{\epsilon}) + \tfrac{1}{2}n_0(x-\tfrac{t}{\epsilon}) + \tfrac{1}{2}\epsilon\int_{x-\frac{t}{\epsilon}}^{x+\frac{t}{\epsilon}} n_1(y)\,dy$$

Note that $n = n_+ + n_- + f$.

1.3.1 Preliminary estimates

In this subsection, we pause to establish some elementary bounds that will be needed in the proof of Prop. 4.

Lemma 2 (Weighted norm estimates).

$$\begin{aligned}\|\langle x\rangle u\|^2_{L^\infty_T H^1_x} \leq\ & \|\langle x\rangle u_0\|^2_{H^1_x} + cT^2\|u\|^2_{L^\infty_T H^2_x} \\ & + cT^4\|n\|^4_{L^\infty_T H^1_x}\left(\|xu_0\|^2_{H^1_x} + cT^2\|u\|^2_{L^\infty_T H^1_x}\right)\end{aligned} \tag{1.29}$$

Proof. The equation $\partial_t u = i\partial_x^2 u - inu$ implies

$$\partial_t(xu) = i\partial_x^2(xu) - 2i\partial_x u - in(xu)$$

Estimating by the "energy method,"

$$\begin{aligned}\|xu(T)\|^2_{L^2_x} &= \|xu_0\|^2_{L^2_x} + 2\text{Re }\langle -2i\partial_x u - in(xu), xu\rangle \\ &= \|xu_0\|^2_{L^2_x} - 4\text{Re }i\langle \partial_x u, xu\rangle \\ &\leq \|xu_0\|^2_{L^2_x} + cT\|\partial_x u\|_{L^\infty_T L^2_x}\|xu\|_{L^\infty_T L^2_x}\end{aligned}$$

Replace T by T' and take the supremum over $0 \leq T' \leq T$,

$$\|xu\|^2_{L^\infty_T L^2_x} \leq \|xu_0\|^2_{L^2_x} + cT^2\|\partial_x u\|^2_{L^\infty_T L^2_x} \tag{1.30}$$

Observe that $\partial_t u = i\partial_x^2 u - inu$ implies

$$\partial_t(x\partial_x u) = i\partial_x^2(x\partial_x u) - 2i\partial_x^2 u - i(\partial_x n)(xu) - in(x\partial_x u)$$

Estimating by the "energy method,"

$$\begin{aligned}\|x\partial_x u(T)\|_{L^2_x}^2 &= \|x\partial_x u_0\|_{L^2_x}^2 + 2\text{Re }\langle -2i\partial_x^2 u - i(\partial_x n)(xu) - in(x\partial_x u), x\partial_x u\rangle \\ &\leq \|x\partial_x u_0\|_{L^2_x}^2 + cT\|\partial_x^2 u\|_{L^\infty_T L^2_x}\|x\partial_x u\|_{L^\infty_T L^2_x} \\ &\quad + cT\|\partial_x n\|_{L^\infty_T L^2_x}\|xu\|_{L^\infty_T L^\infty_x}\|x\partial_x u\|_{L^\infty_T L^2_x}\end{aligned}$$

Use the interpolation estimate

$$\|xu\|_{L^\infty_T L^\infty_x} \leq \|xu\|_{L^\infty_T L^2_x}^{1/2}\|\partial_x(xu)\|_{L^\infty_T L^2_x}^{1/2}$$

to obtain

$$\|x\partial_x u\|_{L^\infty_T L^2_x}^2 \leq \|x\partial_x u_0\|_{L^2_x}^2 + \|u_0\|_{L^2_x}^2 + cT^2\|\partial_x^2 u\|_{L^\infty_T L^2_x}^2 + cT^4\|\partial_x n\|_{L^\infty_T L^2_x}^4\|xu\|_{L^\infty_T L^2_x}^2 \tag{1.31}$$

(1.30) and (1.31) together give (1.29). □

Lemma 3 (An estimate for the maximal function).

$$\sum_{j=0}^{k-2}\|\partial_x^j u\|_{L^2_x L^\infty_T} \leq c\langle T\rangle^{1/2}\|u_0\|_{H^{k-1}_x} + cT\langle T\rangle^{1/2}\|n\|_{L^\infty_T H^{k-1}_x}\|u\|_{L^\infty_T H^k_x} \tag{1.32}$$

$$\|\langle x\rangle u\|_{L^2_x L^\infty_T} \leq c\langle T\rangle^{1/2}\|\langle x\rangle u_0\|_{H^1_x} + cT\langle T\rangle^{1/2}(\|u\|_{L^\infty_T H^2_x} + \|n\|_{L^\infty_T H^1_x}\|\langle x\rangle u\|_{L^\infty_T H^1_x}) \tag{1.33}$$

Proof of claim. Theorem 3.1 of [KPV93b] provides the estimate

$$\|e^{it\partial_x^2}u_0\|_{L^2_x L^\infty_T} \leq c\langle T\rangle^{1/2}\|u_0\|_{H^1_x} \tag{1.34}$$

and consequently

$$\left\|\int_0^t e^{i(t-t')\partial_x^2}h(x,t')\,dt'\right\|_{L^2_x L^\infty_T} \leq c\langle T\rangle^{1/2}\|h\|_{L^1_T H^1_x} \tag{1.35}$$

The integral representation for ZS_ϵ is

$$u = e^{it\partial_x^2} u_0 - i \int_0^t e^{i(t-t')\partial_x^2}(nu)(t')\, dt' \tag{1.36}$$

For $0 \leq j \leq k-2$, apply ∂_x^j to (1.36) and estimate with (1.34) and (1.35) to obtain

$$\begin{aligned} \|\partial_x^j u\|_{L_x^2 L_T^\infty} &\leq c\langle T\rangle^{1/2} \|\partial_x^j u_0\|_{H_x^1} + c\langle T\rangle^{1/2}\|\partial_x^j (nu)\|_{L_T^1 H_x^1} \\ &\leq c\langle T\rangle^{1/2}\|u_0\|_{H_x^{j+1}} + c\langle T\rangle^{1/2} T\|n\|_{L_T^\infty H_x^{j+1}} \|u\|_{L_T^\infty H_x^{j+1}} \end{aligned}$$

To prove the second estimate,

$$\partial_t(\langle x\rangle u) = i\partial_x^2(\langle x\rangle u) - 2i(\partial_x\langle x\rangle)\partial_x u - i(\partial_x^2\langle x\rangle)u - in\langle x\rangle u$$

which has integral equation form

$$\langle x\rangle u = e^{it\partial_x^2}\langle x\rangle u_0 + \int_0^t e^{i(t-t')\partial_x^2}[-2i(\partial_x\langle x\rangle)\partial_x u - i(\partial_x^2\langle x\rangle)u - in\langle x\rangle u](t')\, dt'$$

Applying the estimates (1.34) and (1.35)

$$\begin{aligned} \|\langle x\rangle u\|_{L_x^2 L_T^\infty} \leq\ & c\langle T\rangle^{1/2}\|\langle x\rangle u_0\|_{H_x^1} + cT\langle T\rangle^{1/2}\|(\partial_x\langle x\rangle)\partial_x u\|_{L_T^\infty H_x^1} \\ & + cT\langle T\rangle^{1/2}\|(\partial_x^2\langle x\rangle)u\|_{L_T^\infty H_x^1} + cT\langle T\rangle^{1/2}\|n\langle x\rangle u\|_{L_T^\infty H_x^1} \end{aligned}$$

□

Lemma 4 (Bound for n). *For $k \geq 3$,*

$$\begin{aligned}\|\partial_x^{k-1} n_\pm\|_{L_T^\infty L_x^2} \leq\ & c\|\langle x\rangle u\|_{L_x^2 L_T^\infty} \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2} \\ & + c\|u_0\|_{L_x^2}\|u\|_{L_T^\infty H_x^k} + cT^{1/2}\|u\|_{L_T^\infty H_x^k}^2 \\ & + cT^{1/2}\|u\|_{L_T^\infty H_x^k} \sum_{j=0}^{k-2} \|\partial_x^j u\|_{L_x^2 L_T^\infty}\end{aligned} \tag{1.37}$$

$$\|\partial_x^{k-1} f\|_{L_T^\infty L_x^2} \leq 2\|\partial_x^{k-1} n_0\|_{L_x^2} + \epsilon \|\partial_x^{k-2} n_1\|_{L_x^2}$$

and hence, for $0 < \epsilon \leq 1$,

$$\begin{aligned}\|n\|_{L_T^\infty H_x^{k-1}} \leq\ & 2\|n_0\|_{H_x^{k-1}} + 2\langle T\rangle \|n_1\|_{H_x^{k-2}} + c\|\langle x\rangle u\|_{L_x^2 L_T^\infty} \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2} \\ & + c\langle T\rangle \|u_0\|_{L_x^2}^{\frac{2k+1}{2k}} \|u\|_{L_T^\infty H_x^k}^{\frac{2k-1}{2k}} + cT^{1/2}\langle T\rangle^{1/2}\|u\|_{L_T^\infty H_x^k}^2 \\ & + cT^{1/2}\|u\|_{L_T^\infty H_x^k} \sum_{j=0}^{k-2} \|\partial_x^j u\|_{L_x^2 L_T^\infty}\end{aligned} \tag{1.38}$$

Proof.

$$\|\partial_x^{k-1} n_+(T)\|_{L_x^2}^2 = \tfrac{1}{2}\langle \partial_x^{k-1} n_+, \partial_t \partial_x^{k-1} n_+\rangle$$

Since $\partial_t \partial_x^{k-1} n_+ = -\epsilon^{-1}\partial_x^k n_+ + \frac{1}{2}\epsilon^{-1}\partial_x^k(u\bar{u})$,

$$\begin{aligned}\|\partial_x^{k-1} n_+(T)\|_{L_x^2}^2 &= \frac{1}{2\epsilon}\langle \partial_x^{k-1} n_+, -\partial_x^k n_+ + \tfrac{1}{2}\partial_x^k(u\bar{u})\rangle \\ &= \frac{1}{4\epsilon}\langle \partial_x^{k-1} n_+, \partial_x^k(u\bar{u})\rangle\end{aligned}$$

Since $\partial_x^{k-1} n_+ = \frac{1}{2}\partial_x^k P_+(u\bar{u})$,

$$\|\partial_x^{k-1} n_+(T)\|_{L_x^2}^2 = \frac{1}{8\epsilon}\langle P_+ \partial_x^k(u\bar{u}), \partial_x^k(u\bar{u})\rangle$$

Writing $P_+ = (P_+ - \frac{1}{2}U_+) + \frac{1}{2}U_+$ and using that $P_+ - \frac{1}{2}U_+$ is skew-adjoint,

$$\|\partial_x^{k-1} n_+(T)\|_{L_x^2}^2 = \frac{1}{16\epsilon}\langle U_+\partial_x^k(u\bar{u}), \partial_x^k(u\bar{u})\rangle$$

Proceeding,

$$\begin{aligned}\|\partial_x^{k-1} n_+(T)\|_{L_x^2}^2 &= \frac{1}{\epsilon}\langle \partial_x U_+ \partial_x^{k-1}(u\bar{u}), \partial_x^k(u\bar{u})\rangle \\ &= -\langle \partial_t U_+ \partial_x^{k-1}(u\bar{u}), \partial_x^k(u\bar{u})\rangle \\ &= \int_x U_+\partial_x^{k-1}(u\bar{u})\big|_{t=0}\partial_x^k(u\bar{u})\big|_{t=0}\,dx \\ &\quad - \int_x U_+\partial_x^{k-1}(u\bar{u})\big|_{t=T}\partial_x^k(u\bar{u})\big|_{t=T}\,dx \\ &\quad + \langle U_+\partial_x^{k-1}(u\bar{u}), \partial_t\partial_x^k(u\bar{u})\rangle\end{aligned}$$

Now

$$\begin{aligned}U_+\partial_x^{k-1}(u\bar{u})(x,t) = &- U_+\epsilon\partial_t\partial_x^{k-2}(u\bar{u})(x,t) - \partial_x^{k-2}(u\bar{u})(x-\tfrac{t}{\epsilon},0) \\ &+ \partial_x^{k-2}(u\bar{u})(x-\tfrac{t}{\epsilon}+\tfrac{T}{\epsilon},T)\end{aligned}$$

Hence,

$$\|\partial_x^{k-1} n_+(T)\|_{L_x^2}^2 = \mathrm{I} + \mathrm{II} + \mathrm{III} \tag{1.39}$$

where

$$\mathrm{I} = \langle U_+\partial_x^{k-1}(u\bar{u}), \partial_t\partial_x^k(u\bar{u})\rangle$$

$$\begin{aligned}\mathrm{II} = &-\epsilon\int_x U_+\partial_t\partial_x^{k-2}(u\bar{u})\big|_{t=0}\partial_x^k(u\bar{u})\big|_{t=0}\,dx \\ &+\epsilon\int_x U_+\partial_t\partial_x^{k-2}(u\bar{u})\big|_{t=T}\partial_x^k(u\bar{u})\big|_{t=T}\,dx\end{aligned}$$

$$\begin{aligned}\mathrm{III} = &-\int_x \partial_x^{k-2}(u\bar{u})(x,0)\partial_x^k(u\bar{u})(x,0)\,dx \\ &+\int_x \partial_x^{k-2}(u\bar{u})(x+\tfrac{T}{\epsilon},T)\partial_x^k(u\bar{u})(x,0)\,dx \\ &+\int_x \partial_x^{k-2}(u\bar{u})(x-\tfrac{T}{\epsilon},0)\partial_x^k(u\bar{u})(x,T)\,dx \\ &-\int_x \partial_x^{k-2}(u\bar{u})(x,T)\partial_x^k(u\bar{u})(x,T)\,dx\end{aligned}$$

To address Term I, use $\partial_t(u\bar{u}) = i\bar{u}\partial_x^2 u - iu\partial_x^2\bar{u}$, $(\widehat{Hf}(\xi) = i(\text{sgn }\xi)\hat{f}(\xi))$,

$$|\text{I}| \leq |\langle U_+(\bar{u}\partial_x^k u + u\partial_x^k\bar{u}), HD_x(\bar{u}\partial_x^k u - u\partial_x^k\bar{u})\rangle| + cT\left(\sum_{j=0}^{k-2}\|\partial_x^j u\|_{L_x^2L_T^\infty}\right)^2\|u\|_{L_T^\infty H_x^k}^2$$

$$\leq \|D_x^{1/2}(\bar{u}\partial_x^k u)\|_{L_x^1L_T^2}^2 + cT\left(\sum_{j=0}^{k-2}\|\partial_x^j u\|_{L_x^2L_T^\infty}\right)^2\|u\|_{L_T^\infty H_x^k}^2$$

by Lemma 1d, (1.26)

$$\leq c\|\langle x\rangle u\|_{L_x^2L_T^\infty}^2\|\langle x\rangle^{-1}D_x^{1/2}\partial_x^k u\|_{L_x^2L_T^2}^2 + cT\|u\|_{L_T^\infty H_x^k}^2\left(\sum_{j=0}^{k-2}\|\partial_x^j u\|_{L_x^2L_T^\infty}\right)^2$$

where above, we have used the following Leibniz rule for fractional derivatives

$$\|D_x^{1/2}(\phi\, D_x^{1/2}f) - \phi\, D_x f\|_{L_x^1L_T^2} \leq c\left(\sum_{j=0}^{1}\|\partial_x^j\phi\|_{L_x^2L_T^\infty}\right)(\|f\|_{L_x^2L_T^2} + \|D_x^{1/2}f\|_{L_x^2L_T^2})$$

with $f = HD_x^{1/2}\partial_x^{k-1}u$ and $\phi = \bar{u}$. Turning to Term II,

$$\begin{aligned}|\text{II}| &\leq \epsilon\|U_+\partial_t\partial_x^{k-2}(u\bar{u})\|_{L_T^\infty L_x^2}\|\partial_x^k(u\bar{u})\|_{L_T^\infty L_x^2}\\ &\leq T\|\partial_t\partial_x^{k-2}(u\bar{u})\|_{L_T^\infty L_x^2}\|\partial_x^k(u\bar{u})\|_{L_T^\infty L_x^2}\\ &\leq cT\|u\|_{L_T^\infty H_x^k}^4\end{aligned}$$

Turning to Term III,

$$\begin{aligned}|\text{III}| \leq\ & \|\partial_x^{k-1}(u\bar{u})\big|_{t=0}\|_{L_x^2}^2 + \|\partial_x^{k-1}(u\bar{u})\big|_{t=T}\|_{L_x^2}^2\\ &+ 2\|\partial_x^{k-1}(u\bar{u})\big|_{t=T}\|_{L_x^2}\|\partial_x^{k-1}(u\bar{u})\big|_{t=0}\|_{L_x^2}\\ \leq\ & 2\|\partial_x^{k-1}(u\bar{u})\big|_{t=0}\|_{L_x^2}^2 + 2\|\partial_x^{k-1}(u\bar{u})\big|_{t=T}\|_{L_x^2}^2\\ \leq\ & c\|u_0\|_{L_x^2}^{\frac{2k+1}{k}}\|u\|_{L_T^\infty H_x^k}^{\frac{2k-1}{k}}\end{aligned}$$

by interpolation and L^2 conservation. Substituting into (1.39), we obtain

$$\|\partial_x^{k-1}n_+(T)\|_{L^2_x}^2 \le c\|\langle x\rangle u\|_{L^2_xL^\infty_T}^2\|\langle x\rangle^{-1}D_x^{1/2}\partial_x^k u\|_{L^2_xL^2_T}^2 + cT\|u\|_{L^\infty_T H^k_x}^4 + cT\|u\|_{L^\infty_T H^k_x}^2\left(\sum_{j=0}^{k-2}\|\partial_x^j u\|_{L^2_xL^\infty_T}\right)^2 + c\|u_0\|_{L^2_x}^{\frac{2k+1}{k}}\|u\|_{L^\infty_T H^k_x}^{\frac{2k-1}{k}}$$

There is a similar bound for $\|\partial_x^{k-1}n_-(T)\|_{L^2_x}^2$. It is evident from the definition of f that

$$\|\partial_x^{k-1}f(T)\|_{L^2_x} \le 2\|\partial_x^{k-1}n_0\|_{L^2_x} + \epsilon\|\partial_x^{k-2}n_1\|_{L^2_x}$$

It remains to bound $\|n\|_{L^2_x}$. We use a reformulation of 1D FZS$_\epsilon$ derived in §1.5,

$$n+u\bar{u} = \tfrac{1}{2}\epsilon P_+\partial_t(u\bar{u}) + \tfrac{1}{2}\epsilon P_-\partial_t(u\bar{u}) + \tfrac{1}{2}(n_0+u_0\bar{u}_0)(x-\tfrac{t}{\epsilon}) + \tfrac{1}{2}(n_0+u_0\bar{u}_0)(x+\tfrac{t}{\epsilon}) + \tfrac{1}{2}\epsilon\int_{x-\frac{t}{\epsilon}}^{x+\frac{t}{\epsilon}} n_1(y)\,dy \tag{1.40}$$

Upon applying the estimate $\epsilon\|P_+w(T)\|_{L^2_x} \le \|w\|_{L^1_TL^2_x}$, this gives

$$\|(n+u\bar{u})(T)\|_{L^2_x} \le \|\partial_t(u\bar{u})\|_{L^1_TL^2_x} + \|n_0+u_0\bar{u}_0\|_{L^2_x} + T\|n_1\|_{L^2_x}$$

We complete the estimate by using $\partial_t(u\bar{u}) = i\bar{u}\,\partial_x^2 u - iu\,\partial_x^2\bar{u}$. □

Now we will collect the results of this subsection and put them in a form that we can apply directly, later. We shall use the notation $p(\cdots)$ to represent a polynomial expression in the listed quantities. We assume $0 < T \le 1$. Plug (1.29) into (1.33) to obtain

$$\|\langle x\rangle u\|_{L^2_xL^\infty_T} \le c\|\langle x\rangle u_0\|_{H^1_x} + Tp(\|u\|_{L^\infty_T H^2_x}, \|n\|_{L^\infty_T H^1_x}) \tag{1.41}$$

Plug the (easily obtained from (1.40)) estimate

$$\|n\|_{L^\infty_T H^1_x} \le \|n_0\|_{H^1_x} + \|n_1\|_{L^2_x} + \|u\|_{L^\infty_T H^3_x}^2$$

into (1.41) to get

$$\|\langle x\rangle u\|_{L_x^2 L_T^\infty} \leq c\|\langle x\rangle u_0\|_{H_x^1} + Tp\left(\|n_0\|_{H_x^1}, \|n_1\|_{L_x^2}, \|u\|_{L_T^\infty H_x^3}\right) \tag{1.42}$$

Now plug (1.42) into (1.38) to obtain

$$\begin{aligned}\|n\|_{L_T^\infty H_x^{k-1}} \leq p\Bigg(&\|\langle x\rangle u_0\|_{H_x^1}, \|n_0\|_{H_x^{k-1}}, \|n_1\|_{H_x^{k-2}}, \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2}, \\ &\|u\|_{L_T^\infty H_x^k}, \sum_{j=0}^{k-2} \|\partial_x^j u\|_{L_x^2 L_T^\infty}\Bigg)\end{aligned} \tag{1.43}$$

Plug (1.43) into (1.32) to get

$$\sum_{j=0}^{k-2} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \leq c\|u_0\|_{H_x^k} + Tp(\cdots) \tag{1.44}$$

where p has the same dependence as in (1.43). We shall apply (1.42), (1.43), and (1.44) in the argument below.

1.3.2 Diagonalization

The aim of this subsection is to prove (1.65) below. We shall use the notation $p(\cdots)$ to represent a polynomial expression in the listed quantities. Beginning with the representation (1.28) for 1D FZS$_\epsilon$, namely,

$$\partial_t u = i\partial_x^2 u \pm \tfrac{1}{2} iu\partial_x P_\pm(u\bar{u}) - iuf \tag{1.45}$$

apply ∂_x^k, and let $y = \partial_x^k u$. Then

$$\partial_t y = i\partial_x^2 y \pm \tfrac{1}{2} iu\partial_x P_\pm \bar{u} y \pm \tfrac{1}{2} iu\partial_x P_\pm u\bar{y} - i\partial_x^k(uf) + g_1 \tag{1.46}$$

where

$$g_1 = -i \sum_{\substack{j_1+j_2=k \\ j_1 \le k-1}} c_j \partial_x^{j_1} n_\pm \partial_x^{j_2} u \pm \tfrac{1}{2} i \sum_{\substack{j_1+j_2=k+1 \\ j_1 \le k \\ j_2 \le k}} c_j u P_\pm (\partial_x^{j_1} u \partial_x^{j_2} \bar{u})$$

Note

$$\|g_1\|_{L_T^2 L_x^2} \le cT^{1/2} \|n_\pm\|_{L_T^\infty H_x^{k-1}} \|u\|_{L_T^\infty H_x^k} + cT^{1/2} \left(\sum_{j=0}^{k-2} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \right)^2 \|u\|_{L_T^\infty H_x^k}$$

The goal of the diagonalization is to eliminate the $\pm\frac{1}{2} i u \partial_x P_\pm u \bar{y}$ term from (1.46). Let us drop the P_- terms for convenience. Define the operator

$$V_+ z(x,t) = z(x - \tfrac{t}{\epsilon}, 0)$$

and also the operators $(\widehat{\partial_{xR}^{-2} f}(\xi) = -\xi^{-2} \chi_{|\xi| \ge R} \hat{f}(\xi))$

$$S_+ \bar{y} = -\tfrac{1}{4} u \partial_x P_+ u \partial_{xR}^{-2} \bar{y}$$

$$Q_+ \bar{y} = -\tfrac{1}{4} u V_+ u \partial_{xR}^{-2} \bar{y}$$

Note that S_+ is of order -1 and Q_+ is of order -2. Let

$$z = y - S_+ \bar{y} - Q_+ \bar{y}$$

We then compute

$$\begin{aligned} \partial_t S_+ \bar{y} = \; & -\tfrac{1}{4} (\partial_t u) \partial_x P_+ u \partial_{xR}^{-2} \bar{y} - \tfrac{1}{4} \epsilon^{-1} u \partial_x V_+ u \partial_{xR}^{-2} \bar{y} - \tfrac{1}{4} u \partial_x P_+ (\partial_t u) \partial_{xR}^{-2} \bar{y} \\ & -\tfrac{1}{4} u \partial_x P_+ u \partial_{xR}^{-2} (\partial_t \bar{y}) \end{aligned}$$

$$\partial_t Q_+ \bar{y} = -\tfrac{1}{4} (\partial_t u) V_+ u \partial_{xR}^{-2} \bar{y} + \tfrac{1}{4} \epsilon^{-1} u \partial_x V_+ u \partial_{xR}^{-2} \bar{y}$$

Thus:

$$\begin{aligned}\partial_t z &= \partial_t y - \partial_t S_+ \bar{y} - \partial_t Q_+ \bar{y} \\ &= \partial_t y + \tfrac{1}{4} u \partial_x P_+ u \partial_{xR}^{-2} \partial_t \bar{y} + g_2 \end{aligned} \tag{1.47}$$

where

$$g_2 = +\tfrac{1}{4}(\partial_t u)\partial_x P_+ u \partial_{xR}^{-2}\bar{y} + \tfrac{1}{4} u \partial_x P_+ (\partial_t u)\partial_{xR}^{-2}\bar{y} + \tfrac{1}{4}(\partial_t u) V_+ u \partial_{xR}^{-2}\bar{y}$$

We have the estimate:

$$\begin{aligned}\|g_2\|_{L_T^2 L_x^2} \leq\ & cT^{1/2}\|\partial_t u\|_{L_x^2 L_T^\infty}\left(\sum_{j=0}^{1}\|\partial_x^j u\|_{L_x^2 L_T^\infty}\right)\|u\|_{L_T^\infty H_x^{k-1}} \\ & + cT^{1/2}\|u\|_{L_x^2 L_T^\infty}\left(\sum_{j=0}^{1}\|\partial_x^j \partial_t u\|_{L_x^2 L_T^\infty}\right)\|u\|_{L_T^\infty H_x^{k-1}} \\ & + cT^{1/2}\|\partial_t u\|_{L_T^\infty L_x^\infty}\|u\|_{L_T^\infty L_x^\infty}\|u\|_{L_T^\infty H_x^{k-1}} \\ \leq\ & cT^{1/2} p\left(\|n\|_{H_x^{k-1}}, \sum_{j=0}^{k-2}\|\partial_x^j u\|_{L_x^2 L_T^\infty}, \|u\|_{L_T^\infty H_x^k}\right)\end{aligned}$$

Substituting (1.46) into (1.47),

$$\begin{aligned}\partial_t z =\ & i\partial_x^2 y + \tfrac{1}{2} i u \partial_x P_+ \bar{u} y + \tfrac{1}{2} i u \partial_x P_+ u \bar{y} - i\partial_x^k (uf) \\ & + \tfrac{1}{4} u \partial_x P_+ u \partial_{xR}^{-2}(-i\partial_x^2 \bar{y} - \tfrac{1}{2} i \bar{u}\partial_x P_+ u \bar{y} - \tfrac{1}{2} i \bar{u} \partial_x P_+ \bar{u} y + i\partial_x^k(\bar{u}f) + \bar{g}_1) \\ & + g_1 + g_2 \\ =\ & i\partial_x^2 z - \tfrac{1}{4} i \partial_x^2 u \partial_x P_+ u \partial_{xR}^{-2}\bar{y} - \tfrac{1}{4} i \partial_x^2 u V_+ u \partial_{xR}^{-2}\bar{y} + \tfrac{1}{2} i u \partial_x P_+ \bar{u} y + \tfrac{1}{2} i u \partial_x P_+ u \bar{y} \\ & + \tfrac{1}{4} u \partial_x P_+ u \partial_{xR}^{-2}(-i\partial_x^2 \bar{y} - \tfrac{1}{2} i \bar{u}\partial_x P_+ u \bar{y} - \tfrac{1}{2} i \bar{u} \partial_x P_+ \bar{u} y + i\partial_x^k(\bar{u}f) + \bar{g}_1) \\ & - i\partial_x^k(uf) + g_1 + g_2 \\ =\ & i\partial_x^2 z + \tfrac{1}{2} i u \partial_x P_+ \bar{u} y - i\partial_x^k(uf) + g_3\end{aligned}$$

where $g_3 = g_4 + g_5$ and

$$g_4 = -\tfrac{1}{4} i\partial_x^2 u\partial_x P_+ u\partial_{xR}^{-2}\bar{y} - \tfrac{1}{4} iu\partial_x P_+ u\partial_{xR}^{-2}\partial_x^2\bar{y} + \tfrac{1}{2} iu\partial_x P_+ u\bar{y}$$

$$\begin{aligned} g_5 = & -\tfrac{1}{4} i\partial_x^2 uV_+ u\partial_{xR}^{-2}\bar{y} - \tfrac{1}{8} iu\partial_x P_+ u\partial_{xR}^{-2}\bar{u}\partial_x P_+ u\bar{y} - \tfrac{1}{8} iu\partial_x P_+ u\partial_{xR}^{-2}\bar{u}\partial_x P_+ \bar{u}y \\ & + \tfrac{1}{4} iu\partial_x P_+ u\partial_{xR}^{-2}\partial_x^k(\bar{u}f) + \tfrac{1}{4} u\partial_x P_+ u\partial_{xR}^{-2}\bar{g}_1 + g_1 + g_2 \end{aligned}$$

Note that g_4 is the combination of three first order terms that sum to an order 0 term. In fact, since

$$-\tfrac{1}{4} iu\partial_x P_+ u\partial_{xR}^2\partial_{xR}^{-2}\bar{y} - \tfrac{1}{4} iu\partial_x P_+ u\partial_{xR}^{-2}\partial_x^2\bar{y} + \tfrac{1}{2} iu\partial_x P_+ u\bar{y} = -\tfrac{1}{4} iu\partial_x P_+ u(\chi_{|\xi|\le R})\bar{y}$$

we have

$$\|g_4\|_{L_T^2 L_x^2} \le cT^{1/2} R\|u\|^2_{L_x^2 L_T^\infty}\|u\|_{L_T^\infty H_x^k} + cT^{1/2}\left(\sum_{j=0}^{3}\|\partial_x^j u\|_{L_x^2 L_T^\infty}\right)^2 \|u\|_{L_T^\infty H_x^k} \tag{1.48}$$

For g_5, we have the estimate

$$\begin{aligned} \|g_5\|_{L_T^2 L_x^2} \le \; & + cT^{1/2}\|u\|_{L_T^\infty H_x^3}\|u\|_{L_T^\infty H_x^k} + cT^{1/2}\left(\sum_{j=0}^{1}\|\partial_x^j u\|_{L_x^2 L_T^\infty}\right)^4 \|u\|_{L_T^\infty H_x^k} \\ & + cT^{1/2}\left(\sum_{j=0}^{1}\|\partial_x^j u\|_{L_x^2 L_T^\infty}\right)^2 \|u\|_{L_T^\infty H_x^{k-1}}\|f\|_{L_T^\infty H_x^{k-1}} \\ & + c\left[\left(\sum_{j=0}^{1}\|\partial_x^j u\|_{L_x^2 L_T^\infty}\right)^2 + 1\right]\|g_1\|_{L_x^2 L_T^2} + \|g_2\|_{L_T^2 L_x^2} \\ \le \; & cT^{1/2} p\left(\sum_{j=0}^{k-2}\|\partial_x^j u\|_{L_x^2 L_T^\infty}, \|u\|_{L_T^\infty H_x^k}, \|n\|_{L_T^\infty H_x^{k-1}}, \|f\|_{L_T^\infty H_x^{k-1}}\right) \end{aligned} \tag{1.49}$$

For temporary convenience, we shall set $G = \frac{1}{2}iu\partial_x P_+ \bar{u}y - i\partial_x^k(uf)$ and thus

$$\partial_t z = i\partial_x^2 z + G + g_3 \tag{1.50}$$

Applying B to (1.50), we obtain

$$\partial_t Bz = iB\partial_x^2 z + BG + Bg_3 \tag{1.51}$$

Direct computation from the symbol $b(x,\xi)$ for B shows

$$B\partial_x^2 z = \partial_x^2(Bz) + 2iM\langle x\rangle^{-2}BD_x z + Ez \tag{1.52}$$

where E is an order 0 symbol. Substituting (1.52) into (1.51):

$$\partial_t Bz = i\partial_x^2(Bz) - 2M\langle x\rangle^{-2}BD_x z + BG + Ez + Bg_3 \tag{1.53}$$

By the fundamental theorem of calculus:

$$\|Bz(T)\|_{L_x^2}^2 = \|Bz(0)\|_{L_x^2}^2 + 2\mathrm{Re}\ \langle \partial_t Bz, Bz\rangle$$

Substituting (1.53),

$$\|Bz(T)\|_{L_x^2}^2 = \|Bz(0)\|_{L_x^2}^2 + 2\mathrm{Re}\ \langle i\partial_x^2(Bz) - 2M\langle x\rangle^{-2}BD_x z + BG + Ez + Bg_3, Bz\rangle$$

and therefore

$$\begin{aligned}\|Bz(T)\|_{L_x^2}^2 =& \|Bz(0)\|_{L_x^2}^2 - 4M\mathrm{Re}\ \langle\langle x\rangle^{-2}BD_x z, Bz\rangle + 2\mathrm{Re}\ \langle BG, Bz\rangle \\ &+ c(M)\|g_3\|_{L_T^2L_x^2}\|z\|_{L_T^2L_x^2} + c(M)\|z\|_{L_T^2L_x^2}^2\end{aligned} \tag{1.54}$$

We now convert (1.54) to a similar statement with y in place of z. Using that $z = y - S_+\bar{y} - Q_+\bar{y}$, we find:

$$\begin{aligned} &-4M\mathrm{Re}\,\langle\langle x\rangle^{-2}BD_x z, Bz\rangle \\ &\quad = \; -4M\mathrm{Re}\,\langle\langle x\rangle^{-2}BD_x y, By\rangle \\ &\qquad + 4M\mathrm{Re}\,\langle\langle x\rangle^{-2}BD_x y, S_+\bar{y} + Q_+\bar{y}\rangle \\ &\qquad + 4M\mathrm{Re}\,\langle\langle x\rangle^{-2}BD_x(S_+\bar{y} + Q_+\bar{y}), B(y - S_+\bar{y} - Q_+\bar{y})\rangle \end{aligned} \tag{1.55}$$

Using that:

$$\begin{aligned} \sum_{j=0}^{1} \|\partial_x^j S_+\bar{y}\|_{L_x^2 L_T^2} &\leq cT^{1/2} \left(\sum_{j=0}^{2} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \right)^2 \|u\|_{L_T^\infty H_x^k} \\ \sum_{j=0}^{1} \|\partial_x^j Q_+\bar{y}\|_{L_x^2 L_T^2} &\leq cT^{1/2} \|u\|_{L_T^\infty H_x^2}^2 \|u\|_{L_T^\infty H_x^k} \end{aligned} \tag{1.56}$$

we get:

$$M|\langle\langle x\rangle^{-2}BD_x y, S_+\bar{y} + Q_+\bar{y}\rangle| \leq c(M)T \left(\|u\|_{L_T^\infty H_x^2} + \sum_{j=0}^{2} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \right)^2 \|u\|_{L_T^\infty H_x^k}^2 \tag{1.57}$$

and

$$\begin{aligned} &M|\langle\langle x\rangle^{-2}BD_x(S_+\bar{y} + Q_+\bar{y}), B(y - S_+\bar{y} - Q_+\bar{y})\rangle| \\ &\quad \leq c(M)T \left(1 + \|u\|_{L_T^\infty H_x^2} + \sum_{j=0}^{2} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \right)^4 \|u\|_{L_T^\infty H_x^k}^2 \end{aligned} \tag{1.58}$$

Applying (1.57) and (1.58) to (1.55):

$$\begin{aligned}&-4M\text{Re }\langle\langle x\rangle^{-2}BD_xz,Bz\rangle\\&\qquad\le -4M\text{Re }\langle\langle x\rangle^{-2}BD_xy,By\rangle+c(M)Tp\left(\sum_{j=0}^{k-2}\|\partial_x^j u\|_{L_x^2L_T^\infty},\|u\|_{L_T^\infty H_x^k}\right)\end{aligned}\tag{1.59}$$

We also have

$$2\text{Re }\langle BG,Bz\rangle=2\text{Re }\langle BG,By\rangle-2\text{Re }\langle BG,S_+\bar{y}+Q_+\bar{y}\rangle$$

We handle the second term in two pieces:

$$\begin{aligned}&|\langle Bu\partial_xP_+\bar{u}y,B(S_+\bar{y}+Q_+\bar{y})\rangle|\\&\qquad\le c(M)T^{1/2}\|u\|_{L_x^2L_T^\infty}\left(\sum_{j=0}^{1}\|\partial_x^j u\|_{L_x^2L_T^\infty}\right)\|u\|_{L_T^\infty H_x^k}\sum_{j=0}^{1}\|\partial_x^j(S_+\bar{y}+Q_+\bar{y})\|_{L_x^2L_T^2}\end{aligned}$$

and

$$\begin{aligned}&|\langle B\partial_x^k(uf),B(S_+\bar{y}+Q_+\bar{y})\rangle|\\&\qquad\le c(M)T^{1/2}\|u\|_{L_T^\infty H_x^{k-1}}\|f\|_{L_T^\infty H_x^{k-1}}\sum_{j=0}^{1}\|\partial_x^j(S_+\bar{y}+Q_+\bar{y})\|_{L_T^2L_x^2}\end{aligned}$$

and therefore:

$$2\text{Re }\langle BG,Bz\rangle\le 2\text{Re }\langle BG,By\rangle+c(M)Tp\left(\sum_{j=0}^{k-2}\|\partial_x^j u\|_{L_x^2L_T^\infty},\|f\|_{L_T^\infty H_x^{k-1}},\|u\|_{L_T^\infty H_x^k}\right)\tag{1.60}$$

We now claim, for $0\le t\le T$,

$$\|Bz(t)-By(t)\|_{L_x^2}\le c(M)\left(\frac{1}{R^2}+T\right)p(\|u\|_{L_T^\infty H_x^k},\|n\|_{L_T^\infty H_x^{k-1}})\tag{1.61}$$

Noting that $Bz(t) - By(t) = -BS_+\bar{y}(t) - BQ_+\bar{y}(t)$, to prove (1.61), rewrite:

$$\begin{aligned} BS_+\bar{y}(t) &= -\tfrac{1}{4}Bu\partial_x P_+ u\partial_{xR}^{-2}\bar{y}(t) \\ &= -\tfrac{1}{4}Bu(\epsilon\partial_t + \partial_x)P_+ u\partial_{xR}^{-2}\bar{y}(t) + \tfrac{1}{4}\epsilon Bu\partial_t P_+ u\partial_{xR}^{-2}\bar{y}(t) \\ &= -\tfrac{1}{4}Buu\partial_{xR}^{-2}\bar{y}(t) + \tfrac{1}{4}BuV_+ u\partial_{xR}^{-2}\bar{y}(t) + \tfrac{1}{4}\epsilon BuP_+\partial_t(u\partial_{xR}^{-2}\bar{y}(t)) \end{aligned}$$

Using $\epsilon\|P_+y(t)\|_{L_x^2} \le \|y\|_{L_T^1 L_x^2} \le T\|y\|_{L_T^\infty L_x^2}$ to address the third term, we obtain

$$\begin{aligned} \|BS_+\bar{y}(t)\|_{L_x^2} \le\ & \frac{c(M)}{R^2}\|u\|_{L_T^\infty L_x^\infty}^2 \|y\|_{L_T^\infty L_x^2} \\ & + c(M)T\|u\|_{L_T^\infty L_x^\infty}(\|u\|_{L_T^\infty L_x^\infty} + \|\partial_t u\|_{L_T^\infty L_x^\infty}) \\ & \quad \times (\|\partial_t\partial_{xR}^{-2}y\|_{L_T^\infty L_x^2} + \|\partial_{xR}^{-2}y\|_{L_T^\infty L_x^2}) \\ \le\ & \frac{c(M)}{R^2}\|u\|_{L_T^\infty L_x^\infty}^2 \|u\|_{L_T^\infty H_x^k} \\ & + c(M)T\|u\|_{L_T^\infty H_x^1}(\|u\|_{L_T^\infty H_x^k} + \|n\|_{L_T^\infty H_x^{k-2}}\|u\|_{L_T^\infty H_x^{k-2}})^2 \end{aligned}$$

Direct estimates give

$$\|BQ_+\bar{y}(t)\|_{L_x^2} \le \frac{c(M)}{R^2}\|u\|_{L_T^\infty L_x^\infty}^2 \|u\|_{L_T^\infty H_x^k}$$

Combining yields (1.61). Applying (1.59), (1.60), (1.61) to (1.54), (1.54) becomes:

$$\begin{aligned} \|By(T)\|_{L_x^2}^2 \le & \|By(0)\|_{L_x^2}^2 - 4M\mathrm{Re}\ \langle\langle x\rangle^{-2}BD_x y, By\rangle + 2\mathrm{Re}\ \langle BG, By\rangle \\ & + c(M)\|g_3\|_{L_T^2 L_x^2}\|y - S_+\bar{y} - Q_+\bar{y}\|_{L_T^2 L_x^2} \\ & + c(M)\|y - S_+\bar{y} - Q_+\bar{y}\|_{L_T^2 L_x^2}^2 \\ & + c(M)Tp\left(\sum_{j=0}^{2}\|\partial_x^j u\|_{L_x^2 L_T^\infty}, \|f\|_{L_T^\infty H_x^{k-1}}, \|u\|_{L_T^\infty H_x^k}\right) \\ & + c(M)\left(\frac{1}{R^2} + T\right)^2 p(\|u\|_{L_T^\infty H_x^k}, \|n\|_{L_T^\infty H_x^{k-1}}) \end{aligned} \tag{1.62}$$

Applying (1.48), (1.49),

$$\begin{aligned}\|By(T)\|_{L_x^2}^2 \le &\|By(0)\|_{L_x^2}^2 - 4M\text{Re } \langle \langle x\rangle^{-2} BD_x y, By\rangle + 2\text{Re } \langle BG, By\rangle \\ &+ c(M)\left[T + TR + \left(\frac{1}{R^2} + T\right)^2\right] p\end{aligned} \tag{1.63}$$

where

$$p = p\left(\sum_{j=0}^{k-2} \|\partial_x^j u\|_{L_x^2 L_T^\infty}, \|u\|_{L_T^\infty H_x^k}, \|n\|_{L_T^\infty H_x^{k-1}}, \|f\|_{L_T^\infty H_x^{k-1}}\right) \tag{1.64}$$

As we explain in more detail in §1.3.5,

$$-4M\text{Re } \langle \langle x\rangle^{-2} BD_x y, By\rangle \le -M\|\langle x\rangle^{-1} BD_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2}^2 + c(M)T\|u\|_{H_x^k}^2$$

Take $R \sim T^{-1/5} >> 1$, so that $TR + \left(R^{-2} + T\right)^2 \sim T^{4/5}(1 + T^{3/5})^2$. Replace $y = \partial_x^k u$. Then, with

$$G = \tfrac{1}{2} iu\partial_x P_+ \bar{u}\partial_x^k u - \tfrac{1}{2} iu\partial_x P_- \bar{u}\partial_x^k u - i\partial_x^k(uf)$$

we have

$$\begin{aligned}&\|B\partial_x^k u(T)\|_{L_x^2}^2 + M\|\langle x\rangle^{-1} BD_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2}^2 \\ &\quad\le \|B\partial_x^k u_0\|_{L_x^2}^2 + |\langle BG, B\partial_x^k u\rangle| + c(M)T^{4/5}\langle T\rangle^2 p\end{aligned} \tag{1.65}$$

with p a polynomial in the quantities (1.64).

1.3.3 Conclusion

The aim of this subsection is to complete the argument assuming:

Main Claim. With $G = \frac{1}{2} iu\partial_x P_+ \bar{u}\partial_x^k u - \frac{1}{2} iu\partial_x P_- \bar{u}\partial_x^k u - i\partial_x^k(uf)$, we have

$$|\langle BG, B\partial_x^k u\rangle| \le c\|\langle x\rangle u\|_{L_x^2 L_T^\infty}^2 \|\langle x\rangle^{-1} BD_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2}^2 + c(M)T^{1/2} p \tag{1.66}$$

with

$$p = p\Big(\|u\|_{L_T^\infty H_x^k}, \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2}, \|\langle x\rangle u\|_{L_x^2}, \|n\|_{L_T^\infty H_x^{k-1}},$$
$$\|f\|_{L_T^\infty H_x^{k-\frac{1}{2}}}, \sum_{j=0}^{k-2} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \Big)$$

and c independent of M.

This claim will be proved later in §1.3.4, although now we shall assume it and complete the remainder of the argument. As a consequence of (1.66) and (1.42), (1.43),

$$|\langle BG, B\partial_x^k u\rangle| \leq c\|\langle x\rangle u_0\|_{H_x^1}^2 \|\langle x\rangle^{-1} BD_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2}^2 + c(M)T^{1/2}p \tag{1.67}$$

where p is a polynomial expression

$$p\Big(\|\langle x\rangle u_0\|_{H_x^1}, \|n_0\|_{H_x^{k-\frac{1}{2}}}, \|n_1\|_{H_x^{k-\frac{3}{2}}}, \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2},$$
$$\|u\|_{L_T^\infty H_x^k}, \sum_{j=0}^{k-2} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \Big) \tag{1.68}$$

Setting $M \sim \|\langle x\rangle u_0\|_{H_x^1}$, applying (1.67) to (1.65), and applying (1.43) to (1.64), (1.65) reduces to

$$\|B\partial_x^k u(T)\|_{L_x^2}^2 + \|\langle x\rangle^{-1} BD_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2}^2 \leq \|B\partial_x^k u_0\|_{L_x^2}^2 + c(M)T^{1/2}p \tag{1.69}$$

with p as in (1.68). By the strict positivity of $b(x,\xi)$, namely $b(x,\xi) \geq e^{-M}$, and the sharp scalar Gårding inequality, we have

$$e^{-2M}\|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2}^2 \leq \|\langle x\rangle^{-1} BD_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2}^2$$

Since, in addition, B and B^{-1} are bounded operators on L^2 with norm e^M, (1.69) implies

$$\|\partial_x^k u(T)\|_{L_x^2}^2 + \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2}^2 \leq e^{4M}\|\partial_x^k u_0\|_{L_x^2}^2 + \langle M\rangle^2 e^{4M} T^{1/2} p \tag{1.70}$$

Replacing T by T' and taking the supremum over $0 \leq T' \leq T$, and adding the bound (1.44), and also L^2 conservation of u, we obtain (1.9). □

1.3.4 Proof of Main Claim

Our goal is to estimate $|\langle BG, B\partial_x^k u\rangle|$, where $G = \frac{1}{2} iu\partial_x P_+ \bar{u}\partial_x^k u - \frac{1}{2} iu\partial_x P_- \bar{u}\partial_x^k u - i\partial_x^k(uf)$. We shall separately estimate

$$|\langle Bu\partial_x P_+ \bar{u}\partial_x^k u, B\partial_x^k u\rangle| \tag{1.71}$$

$$|\langle Bu\partial_x P_- \bar{u}\partial_x^k u, B\partial_x^k u\rangle| \tag{1.72}$$

$$|\langle B\partial_x^k (uf), B\partial_x^k u\rangle| \tag{1.73}$$

beginning with (1.71). Decompose

$$u\partial_x P_+ \bar{u}\partial_x^k u = u\partial_x L_+ P_+ \bar{u}\partial_x^k u - u\partial_x L_- P_+^* \bar{u}\partial_x^k u + u\partial_x L_- U_+ \bar{u}\partial_x^k u$$

and consider the corresponding decomposition

$$\begin{aligned}\langle Bu\partial_x P_+ \bar{u}\partial_x^k u, B\partial_x^k u\rangle = &\ \langle Bu\partial_x L_+ P_+ \bar{u}\partial_x^k u, B\partial_x^k u\rangle - \langle Bu\partial_x L_- P_+^* \bar{u}\partial_x^k u, B\partial_x^k u\rangle \\ &+ \langle Bu\partial_x L_- U_+ \bar{u}\partial_x^k u, B\partial_x^k u\rangle \\ = &\ \mathrm{I} + \mathrm{II} + \mathrm{III}\end{aligned} \tag{1.74}$$

We first address Term I in (1.74). We use $\lesssim$ and $\approx$ to mean "modulo lower order terms" arising from commutators. Rigorous treatment of commutators is deferred to

§1.3.5.

$$
\begin{aligned}
|\mathrm{I}| &\approx |\langle u[BL_+P_+B^{-1}]\bar{u}BD_x^{1/2}\partial_x^k u, BD_x^{1/2}\partial_x^k u\rangle| \\
&\lesssim \|L_+\bar{u}BD_x^{1/2}\partial_x^k u\|_{L_x^1L_T^2}\|\bar{u}BD_x^{1/2}\partial_x^k u\|_{L_x^1L_T^2} \\
&\lesssim c\|\langle x\rangle u\|_{L_x^2L_T^\infty}^2\|\langle x\rangle^{-1}BD_x^{1/2}\partial_x^k u\|_{L_x^2L_T^2}
\end{aligned} \tag{1.75}
$$

where c is independent of M, by Lemma 1b, (1.23). We also have

$$
|\mathrm{II}| \leq c\|\langle x\rangle u\|_{L_x^2L_T^\infty}^2\|\langle x\rangle^{-1}BD_x^{1/2}\partial_x^k u\|_{L_x^2L_T^2}^2 \tag{1.76}
$$

by Lemma 1b (1.23), and similar manipulations. We now consider term III in (1.74). Let $A = B^*B$. **Note:** In the calculations below, we shall often use the property $\partial_x(Aw) = A\partial_x w + A'w$, but omit the term $A'w$, since these terms are simpler to address than those already present.

We have

$$
\begin{aligned}
\mathrm{III} &\approx \langle L_-\partial_x U_+\bar{u}\partial_x^k u, \bar{u}A\partial_x^k u\rangle \\
&= \langle L_-\partial_x U_+\partial_x(\bar{u}\partial_x^{k-1}u), \bar{u}A\partial_x^k u\rangle \\
&\quad - \langle L_-\partial_x U_+(\partial_x\bar{u})\partial_x^{k-1}u, \bar{u}A\partial_x^k u\rangle \\
&= -\langle L_-U_+\partial_x(\bar{u}\partial_x^{k-1}u), \bar{u}A\partial_x^{k+1}u\rangle \\
&\quad - \langle L_-U_+\partial_x(\bar{u}\partial_x^{k-1}u), (\partial_x\bar{u})A\partial_x^k u\rangle \\
&\quad - \langle L_-U_+\partial_x((\partial_x\bar{u})\partial_x^{k-1}u), \bar{u}A\partial_x^k u\rangle
\end{aligned}
$$

and thus, by Lemma 1d, (1.25) and (1.26),

$$
|\mathrm{III}| \leq J^{1/2}K^{1/2} + cT\left(\sum_{j=0}^{2}\|\partial_x^j u\|_{L_x^2L_T^\infty}\right)^2\|u\|_{L_T^\infty H_x^k}^2 \tag{1.77}
$$

where

$$J = \frac{1}{\epsilon}\langle U_+\partial_x(\bar{u}\partial_x^{k-1}u), \partial_x(\bar{u}\partial_x^{k-1}u)\rangle$$
$$K = \epsilon\langle L_-U_+\bar{u}A\partial_x^{k+1}u, \bar{u}A\partial_x^{k+1}u\rangle$$

The term J is treated by a calculation similiar to that in Lemma 4. The result is

$$\begin{aligned} J \leq\ & c\|\langle x\rangle u\|^2_{L^2_x L^\infty_T}\|\langle x\rangle^{-1}D_x^{1/2}\partial_x^k u\|^2_{L^2_x L^2_T} \\ & + p\left(\|u\|_{L^\infty_T H^k_x}, \sum_{j=0}^{k-2}\|\partial_x^j u\|_{L^2_x L^\infty_T}, \|n\|_{L^\infty_T H^{k-1}_x}\right) \end{aligned} \tag{1.78}$$

Now we analyze K. Using (1.28),

$$\begin{aligned} K =\ & -\operatorname{Re}\, i\epsilon\langle L_-U_+\bar{u}A\partial_t\partial_x^{k-1}u, \bar{u}A\partial_x^{k+1}u\rangle \\ & + \operatorname{Re}\,\epsilon\langle L_-U_+\bar{u}A\partial_x^{k-1}(nu), \bar{u}A\partial_x^{k+1}u\rangle \\ =\ & -\operatorname{Re}\, i\langle L_-U_+\epsilon\partial_t(\bar{u}A\partial_x^{k-1}u), \bar{u}A\partial_x^{k+1}u\rangle \\ & + \operatorname{Re}\, i\epsilon\langle L_-U_+(\partial_t\bar{u})(A\partial_x^{k-1}u), \bar{u}A\partial_x^{k+1}u\rangle \\ & + \operatorname{Re}\,\epsilon\langle L_-U_+\bar{u}A\partial_x^{k-1}(nu), \bar{u}A\partial_x^{k+1}u\rangle \end{aligned}$$

Apply Lemma 1d (1.25) and Cauchy-Schwarz to the second and third terms to obtain

$$\begin{aligned} K \leq\ & -2\operatorname{Re}\, i\langle L_-U_+\epsilon\partial_t(\bar{u}A\partial_x^{k-1}u), \bar{u}A\partial_x^{k+1}u\rangle \\ & + \epsilon\langle U_+(\partial_t\bar{u})A\partial_x^{k-1}u, (\partial_t\bar{u})A\partial_x^{k-1}u\rangle \\ & + \epsilon\langle U_+\bar{u}A\partial_x^{k-1}(nu), \bar{u}A\partial_x^{k-1}(nu)\rangle \\ =\ & -2\operatorname{Re}\, i\langle L_-U_+\epsilon\partial_t(\bar{u}A\partial_x^{k-1}u), \bar{u}A\partial_x^{k+1}u\rangle + K_1 \end{aligned}$$

where

$$K_1 \leq \epsilon T c(M)\left(\|\partial_t u\|^2_{L^2_x L^\infty_T} + \|u\|^2_{L^2_x L^\infty_T}\|n\|^2_{L^\infty_T H^{k-1}_x}\right)\|u\|^2_{L^\infty_T H^k_x}$$

Continuing, we use that $U_+(\epsilon\partial_t + \partial_x) = -z(x - \frac{t}{\epsilon}, 0) + z(x - \frac{t}{\epsilon} + \frac{T}{\epsilon}, T)$ to get

$$\begin{aligned} K \leq\ & 2\text{Re } i\langle L_- U_+ \partial_x(\bar{u}A\partial_x^{k-1}u), \bar{u}A\partial_x^{k+1}u\rangle \\ & - 2\text{Re } i\langle (L_-\bar{u}A\partial_x^{k-1}u)(x - \tfrac{t}{\epsilon} + \tfrac{T}{\epsilon}, T), \bar{u}A\partial_x^{k+1}u\rangle \\ & + 2\text{Re } i\langle (L_-\bar{u}A\partial_x^{k-1}u)(x - \tfrac{t}{\epsilon}, 0), \bar{u}A\partial_x^{k+1}u\rangle + K_1 \\ =\ & 2\text{Re } i\langle L_- U_+ \partial_x(\bar{u}A\partial_x^{k-1}u), \bar{u}A\partial_x^{k+1}u\rangle + K_2 \end{aligned}$$

where

$$K_2 \leq K_1 + c(M)T\|u\|^2_{L_T^\infty H_x^2}\|u\|^2_{L_T^\infty H_x^k}$$

Continuing,

$$\begin{aligned} K \leq\ & 2\text{Re } i\langle L_- U_+ \bar{u}A\partial_x^k u, \bar{u}A\partial_x^{k+1}u\rangle \\ & + 2\text{Re } i\langle L_- U_+ (\partial_x\bar{u})A\partial_x^{k-1}u, \bar{u}A\partial_x^{k+1}u\rangle + K_2 \\ =\ & 2\text{Re } i\langle L_- U_+ \bar{u}A\partial_x^k u, \partial_x(\bar{u}A\partial_x^k u)\rangle \\ & - 2\text{Re } i\langle L_- U_+ \bar{u}A\partial_x^k u, (\partial_x\bar{u})A\partial_x^k u\rangle \\ & - 2\text{Re } i\langle L_- U_+ (\partial_x\bar{u})A\partial_x^{k-1}u, (\partial_x\bar{u})A\partial_x^k u\rangle \\ & - 2\text{Re } i\langle L_- U_+ \partial_x((\partial_x\bar{u})A\partial_x^{k-1}u), \bar{u}A\partial_x^k u\rangle + K_2 \\ =\ & 2\text{Re } i\langle L_- U_+ \bar{u}A\partial_x^k u, \partial_x(\bar{u}A\partial_x^k u)\rangle + K_3 \end{aligned} \tag{1.79}$$

where, by Lemma 1d (1.26),

$$K_3 \leq K_2 + c(M)T\left(\sum_{j=0}^{2}\|\partial_x^j u\|_{L_x^2 L_T^\infty}\right)^2 \|u\|^2_{L_T^\infty H_x^k}$$

Remarkably, the first term in (1.79) is negative, i.e.

$$K \leq -2\text{Re } \langle U_+ L_- D_x^{1/2}(\bar{u}A\partial_x^k u), L_- D_x^{1/2}(\bar{u}A\partial_x^k u)\rangle + K_3 \leq K_3$$

That is,

$$K \le \epsilon T c(M)\left(\|\partial_x^2 u\|^2_{L^2_x L^\infty_T} + \|u\|^2_{L^2_x L^\infty_T}\|n\|^2_{L^\infty_T H^{k-1}_x}\right)\|u\|^2_{L^\infty_T H^k_x}$$
$$+ c(M)T\|u\|^2_{L^\infty_T H^2_x}\|u\|^2_{L^\infty_T H^k_x} + c(M)T\left(\sum_{j=0}^{2}\|\partial_x^j u\|_{L^2_x L^\infty_T}\right)^2 \|u\|^2_{L^\infty_T H^k_x}$$
$$\le T c(M) p\left(\|u\|_{L^\infty_T H^k_x}, \sum_{j=0}^{2}\|\partial_x^j u\|_{L^2_x L^\infty_T}, \|n\|_{L^\infty_T H^{k-1}_x}\right) \tag{1.80}$$

Substituting bounds (1.78), (1.80) into (1.77), we see that Term III in (1.74) is bounded as:

$$|\mathrm{III}| \le T^{1/2} c(M) p\left(\|u\|_{L^\infty_T H^k_x}, \sum_{j=0}^{2}\|\partial_x^j u\|_{L^2_x L^\infty_T}, \|n\|_{L^\infty_T H^{k-1}_x}, \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k u\|_{L^2_x L^2_T}\right) \tag{1.81}$$

Collecting (1.75), (1.76), and (1.81) as bounds for each term in (1.74), we have

$$(1.71) \le c\|\langle x\rangle u\|^2_{L^2_x L^\infty_T}\|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k u\|^2_{L^2_x L^2_T} + T^{1/2}\langle T\rangle^{1/2} p$$

with c independent of M. (1.72) is handled similar to (1.71). Finally, we estimate (1.73). Decompose

$$B\partial_x^k(fu) = B(u\,\partial_x^k f) + \sum_{\substack{j_1+j_2=k\\ j_2\le k-1}} c_j B(\partial_x^{j_1} u\,\partial_x^{j_2} f)$$

and thus:

$$\langle B\partial_x^k(uf), B\partial_x^k u\rangle = \langle Bu\partial_x^k f, B\partial_x^k u\rangle + \sum_{\substack{j_1+j_2=k\\ j_2\le k-1}} c_j \langle B(\partial_x^{j_1} u\,\partial_x^{j_2} f), B\partial_x^k u\rangle$$
$$= \mathrm{I} + \mathrm{II}$$

Below, $\lesssim$ means "mod lower order terms". Rigorous treatment of commutators is deferred to §1.3.5.

$$\begin{aligned}
|\mathrm{I}| &= |\langle B(u\,\partial_x^k f), B\partial_x^k u\rangle| \\
&\lesssim |\langle H_x D_x^{1/2}\partial_x^{k-1} f, \langle x\rangle \bar{u}\, B^*\langle x\rangle^{-1} B D_x^{1/2}\partial_x^k u\rangle| \\
&\leq \|H_x D_x^{1/2}\partial_x^{k-1} f\|_{L_x^\infty L_T^2} \|\langle x\rangle u\|_{L_x^2 L_T^\infty} \|B^*\langle x\rangle^{-1} B D_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^2} \\
&\leq c(M)\|H_x D_x^{1/2}\partial_x^{k-1} f\|^2_{L_x^\infty L_T^2} + \|\langle x\rangle u\|^2_{L_x^2 L_T^\infty} \|\langle x\rangle^{-1} B D_x^{1/2}\partial_x^k u\|^2_{L_x^2 L_T^2} \\
&\leq c(M)(\epsilon\|n_0\|^2_{H_x^{k-\frac{1}{2}}} + \epsilon^2\|n_1\|^2_{H_x^{k-\frac{3}{2}}}) + \|\langle x\rangle u\|^2_{L_x^2 L_T^\infty} \|\langle x\rangle^{-1} B D_x^{1/2}\partial_x^k u\|^2_{L_x^2 L_T^2}
\end{aligned}$$

$$\begin{aligned}
|\mathrm{II}| &\leq \langle B\partial_x^{j_1} u\partial_x^{j_2} f, B\partial_x^k u\rangle \\
&\leq c(M)T\|\partial_x^{j_1} u\partial_x^{j_2} f\|_{L_T^\infty L_x^2} \|\partial_x^k u\|_{L_T^\infty L_x^2} \\
&\leq c(M)T\|f\|_{L_T^\infty H_x^{k-1}} \|u\|^2_{L_T^\infty H_x^k}
\end{aligned}$$

Hence

$$\begin{aligned}
(1.73) \leq\ & c(M)(\epsilon\|n_0\|^2_{H_x^{k-\frac{1}{2}}} + \epsilon^2\|n_1\|^2_{H_x^{k-\frac{3}{2}}}) + \|\langle x\rangle u\|^2_{L_x^2 L_T^\infty} \|\langle x\rangle^{-1} B D_x^{1/2}\partial_x^k u\|^2_{L_x^2 L_T^2} \\
& + c(M)T\|f\|_{L_T^\infty H_x^{k-1}} \|u\|^2_{L_T^\infty H_x^k}
\end{aligned} \tag{1.82}$$

1.3.5 Rigorous treatment of commutators

We shall need the following Leibniz estimates for fractional derivatives: if $y = \partial_x^k u$, $\partial_x^j \phi_l(x,t) \in L_x^2 L_T^\infty$ for $j = 0,1,2$, $l = 1,2$, and $\partial_x^j b(x) \in L_x^\infty$ for $j = 0,1,2$, and $\varphi(x) \in H_x^2$, then

$$\|L_+\phi_1\partial_x P_+\phi_2 y - iD_x^{1/2}\phi_1 P_+\phi_2 L_+ D_x^{1/2} y\|_{L_T^2 L_x^2}$$
$$\leq c\left(\prod_{l=1,2}\sum_{j=0}^{2}\|\partial_x^j\phi_l\|_{L_x^2 L_T^\infty}\right)\|u\|_{L_T^2 H_x^k} \tag{1.83}$$

$$\|L_-\phi_1\partial_x P_+^*\phi_2 y + iD_x^{1/2}\phi_1 P_+^*\phi_2 L_- D_x^{1/2} y\|_{L_T^2 L_x^2}$$
$$\leq c\left(\prod_{l=1,2}\sum_{j=0}^{2}\|\partial_x^j\phi_l\|_{L_x^2 L_T^\infty}\right)\|u\|_{L_T^2 H_x^k} \tag{1.84}$$

$$\|L_-\phi_1\partial_x U_+\phi_2 y - \phi_1\partial_x L_- U_+\phi_2 y\|_{L_T^2 L_x^2}$$
$$\leq c\left(\prod_{l=1,2}\sum_{j=0}^{2}\|\partial_x^j\phi_l\|_{L_x^2 L_T^\infty}\right)\|u\|_{L_T^2 H_x^k} \tag{1.85}$$

$$\|D_x^{1/2}(bf) - bD_x^{1/2} f\|_{L_T^2 L_x^2} \leq c(\|b\|_{L_x^\infty} + \|\partial_x b\|_{L_x^\infty})\|f\|_{L_T^2 L_x^2} \tag{1.86}$$

$$\|L_-[\varphi L_+ f]\|_{L_x^2 L_T^2} \leq c\|\varphi\|_{H_x^2}\|f\|_{L_T^2 H_x^{-1}} \tag{1.87}$$

Indeed, assuming (1.83)–(1.87) hold, we explain how to rigorously carry out the commutator argument in the proof of the Main Claim. Let

$$b_\pm(x,\xi) = \exp\left[\mp M\int_0^x \langle s\rangle^{-2}\, ds\right]$$

so that $BL_+ = b_+L_+$ and $BL_- = b_-L_-$. Then

$$\begin{aligned} Bu\partial_x P_+\bar{u}y &= BL_+u\partial_x P_+\bar{u}y - BL_-u\partial_x P_+^*\bar{u}y + BL_-u\partial_x U_+\bar{u}y \\ &= b_+L_+u\partial_x P_+\bar{u}y - b_-L_-u\partial_x P_+^*\bar{u}y + BL_-u\partial_x U_+\bar{u}y \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III} \end{aligned}$$

We shall address Term I.

$$
\begin{aligned}
\mathrm{I} &\approx b_+ L_+ u \partial_x P_+ \bar{u} y \\
&\approx i b_+ D_x^{1/2} u P_+ \bar{u} L_+ D_x^{1/2} y && \text{by (1.83)} \\
&\approx i D_x^{1/2} b_+ u P_+ \bar{u} L_+ D_x^{1/2} y && \text{by (1.86)} \\
&= i D_x^{1/2} u [b_+ P_+ b_+^{-1}] \bar{u} B L_+ D_x^{1/2} y && (1.88)
\end{aligned}
$$

By the proof of Lemma 1b (1.23),

$$
\|b_+ P_+ b_+^{-1} z\|_{L_x^\infty L_T^2} \leq c \|z\|_{L_x^1 L_T^2}
$$

with c independent of M. Pairing (1.88) with By,

$$
\begin{aligned}
&\langle D_x^{1/2} u [b_+ P_+ b_+^{-1}] \bar{u} B L_+ D_x^{1/2} y, By \rangle \\
&\quad = \langle u [b_+ P_+ b_+^{-1}] \bar{u} B L_+ D_x^{1/2} y, D_x^{1/2} By \rangle \\
&\quad = \langle u [b_+ P_+ b_+^{-1}] \bar{u} B L_+ D_x^{1/2} y, D_x^{1/2} b_+ L_+ y \rangle \\
&\qquad + \langle u [b_+ P_+ b_+^{-1}] \bar{u} B L_+ D_x^{1/2} y, D_x^{1/2} b_- L_- y \rangle \\
&\quad \approx \langle u [b_+ P_+ b_+^{-1}] \bar{u} B L_+ D_x^{1/2} y, b_+ D_x^{1/2} L_+ y \rangle && \text{by (1.86)} \\
&\qquad + \langle u [b_+ P_+ b_+^{-1}] \bar{u} B L_+ D_x^{1/2} y, b_- D_x^{1/2} L_- y \rangle \\
&\quad = \langle u [b_+ P_+ b_+^{-1}] \bar{u} B L_+ D_x^{1/2} y, B D_x^{1/2} y \rangle \\
&\quad \leq \|\langle x \rangle u\|_{L_x^2 L_T^2}^2 \|\langle x \rangle^{-1} B L_+ D_x^{1/2} y\|_{L_x^2 L_T^2} \|\langle x \rangle^{-1} B D_x^{1/2} y\|_{L_x^2 L_T^2} && (1.89)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\langle x \rangle^{-1} B L_+ D_x^{1/2} y &= \langle x \rangle^{-1} b_+ L_+ D_x^{1/2} y \\
&= L_+ \langle x \rangle^{-1} b_+ L_+ D_x^{1/2} y + L_- \langle x \rangle^{-1} b_+ L_+ D_x^{1/2} y \\
&\approx L_+ \langle x \rangle^{-1} b_+ L_+ D_x^{1/2} y && \text{by (1.87)} \\
&\approx L_+ \langle x \rangle^{-1} b_+ L_+ D_x^{1/2} y + L_+ \langle x \rangle^{-1} b_- L_- D_x^{1/2} y && \text{by (1.87)} \\
&= L_+ \langle x \rangle^{-1} B D_x^{1/2} y
\end{aligned}
$$

and therefore, into (1.89), we can substitute

$$\|\langle x\rangle^{-1} B L_+ D_x^{1/2} y\|_{L^2_x L^2_T} \lesssim \|\langle x\rangle^{-1} B D_x^{1/2} y\|_{L^2_x L^2_T}$$

Term II is handled similarly. For Term III,

$$\begin{aligned}
\langle B L_- u \partial_x U_+ \bar{u} y, By\rangle &= \langle L_- u \partial_x U_+ \bar{u} y, B^* B y\rangle \\
&\approx \langle u L_- \partial_x U_+ \bar{u} y, B^* B y\rangle && \text{by (1.85)} \\
&= \langle L_- U_+ \bar{u} y, \partial_x(\bar{u} B^* B y)\rangle
\end{aligned}$$

and then proceed with the argument in §1.3.4. Next, we show that

$$-4M\langle \langle x\rangle^{-2} B D_x y, By\rangle \le -M\|\langle x\rangle^{-1} B D_x^{1/2} y\|^2_{L^2_T L^2_x} + c(M)\|y\|^2_{L^2_T L^2_x} \tag{1.90}$$

Indeed,

$$\begin{aligned}
B^*\langle x\rangle^{-2} B D_x y &= (L_+ b_+ + L_- b_-)\langle x\rangle^{-2}(b_+ L_+ + b_- L_-) D_x y \\
&\approx L_+ D_x^{1/2} b_+ \langle x\rangle^{-2} b_+ L_+ D_x^{1/2} y && \text{by (1.86)} \\
&\quad + L_- D_x^{1/2} b_- \langle x\rangle^{-2} b_+ L_+ D_x^{1/2} y \\
&\quad + L_+ D_x^{1/2} b_+ \langle x\rangle^{-2} b_- L_- D_x^{1/2} y \\
&\quad + L_- D_x^{1/2} b_- \langle x\rangle^{-2} b_- L_- D_x^{1/2} y \\
&= D_x^{1/2} B^* \langle x\rangle^{-2} B D_x^{1/2} y
\end{aligned}$$

establishing (1.90).

1.4 Convergence to NLS

1.4.1 The noncompatible case $(n_0 + u_0\bar{u}_0) \neq 0$

Here, we suppose that $k \geq 4$,

$$(u_0, n_0, n_1) \in (H^{k+1} \cap H^1(\langle x\rangle^2 dx)) \times (H^{k+\frac{1}{2}} \cap L^1) \times (H^{k-\frac{1}{2}} \cap L^1)$$

We then have the uniform bound on $\|u\|_{L_T^\infty H_x^{k+1}}$, $\|\langle x\rangle^{-1} D_x^{1/2} \partial_x^{k+1} u\|_{L_x^2 L_T^2}$ furnished by Prop. 4. We shall establish convergence at the optimal rate ϵ (optimal in the noncompatible case $n_0 + u_0\bar{u}_0 \neq 0$) of the solution u_ϵ of 1D ZS$_\epsilon$ to the solution v of 1D NLS in H^k as $\epsilon \to 0$. Again, for convenience in exposition, we restrict to 1D FZS$_\epsilon$, although the result applies just as well to 1D DZS$_\epsilon$. It will be shown in §1.5 that 1D FZS$_\epsilon$ can be reformulated as

$$\partial_t u = i\partial_x^2 u - \tfrac{1}{2} i\epsilon u P_+ \partial_t(u\bar{u}) - \tfrac{1}{2} i\epsilon u P_- \partial_t(u\bar{u}) - if_1 u - i\epsilon f_2 u + iu|u|^2 \tag{1.91}$$

where

$$f_1(x,t) = \tfrac{1}{2}(n_0 + u_0\bar{u}_0)(x + \tfrac{t}{\epsilon}) + \tfrac{1}{2}(n_0 + u_0\bar{u}_0)(x - \tfrac{t}{\epsilon})$$

$$f_2(x,t) = \tfrac{1}{2}\int_{x-\frac{t}{\epsilon}}^{x+\frac{t}{\epsilon}} n_1(y)\, dy$$

Subtract (1.6) from (1.91):

$$\partial_t(u-v) = i\partial_x^2(u-v) + iu|u|^2 - iv|v|^2 - \tfrac{1}{2} i\epsilon u P_+ \partial_t(u\bar{u}) - \tfrac{1}{2} i\epsilon u P_- \partial_t(u\bar{u}) - if_1 u - i\epsilon f_2 u \tag{1.92}$$

Apply $B\partial_x^k$ (here, we may take $M = 1$ in the symbol $b(x,\xi)$), taking $y = \partial_x^k(u-v)$:

$$B\partial_t y = iB\partial_x^2 y + iB\partial_x^k(u|u|^2 - v|v|^2) - \tfrac{1}{2} i\epsilon B\partial_x^k[uP_\pm \partial_t(u\bar{u})] - iB\partial_x^k(f_1 u) - i\epsilon B\partial_x^k(f_2 u) \tag{1.93}$$

Direct computation from the definition of B gives

$$B\partial_x^2 y = \partial_x^2(By) + 2i\langle x\rangle^{-2} BD_x y + Ey \tag{1.94}$$

where E is an order 0 operator. By the fundamental theorem of calculus and (1.93), (1.94),

$$\begin{aligned}\|By(T)\|_{L_x^2}^2 = & -\text{Re } 2\langle\langle x\rangle^{-2}BD_xy, By\rangle + \text{Re } \langle Ey, By\rangle \\ & - \text{Re } i\langle B\partial_x^k(u|u|^2 - v|v|^2), By\rangle - \text{Re } i\epsilon\langle B\partial_x^k[uP_\pm\partial_t(u\bar{u})], By\rangle \\ & - \text{Re } i\langle B\partial_x^k(f_1u), By\rangle - \text{Re } i\epsilon\langle B\partial_x^k(f_2u), By\rangle \\ = & -\text{Re } 2\langle\langle x\rangle^{-2}BD_xy, By\rangle + \text{Re } \langle Ey, By\rangle + \text{I} + \text{II} + \text{III} + \text{IV} \end{aligned} \tag{1.95}$$

We begin by addressing Term I. Since

$$u u\bar{u} - v v\bar{v} = (u-v)u\bar{u} + v(u-v)\bar{u} + vv(\overline{u-v}) \tag{1.96}$$

we have

$$|\text{I}| \le c(\|u\|_{L_T^\infty H_x^k}^2 + \|v\|_{L_T^\infty H_x^k}^2)\int_0^T \|u(t)-v(t)\|_{H_x^k}^2\, dt$$

Now we address Term II. Using $\partial_t(u\bar{u}) = i\bar{u}\,\partial_x^2 u - iu\,\partial_x^2\bar{u}$,

$$\begin{aligned}\text{II} = & \text{ Re } \epsilon\langle Bu\partial_x P_\pm(\bar{u}\,\partial_x^{k+1}u - u\,\partial_x^{k+1}\bar{u}), By\rangle \\ & + \text{Re} \sum_{\substack{j_1+j_2+j_3=k+2 \\ j_2,j_3\le k+1 \\ j_1\le k}} c_j\epsilon\langle B(\partial_x^{j_1}u)P_\pm(\partial_x^{j_2}u\,\partial_x^{j_3}\bar{u}), By\rangle \\ = & \text{ II}_a + \text{II}_b\end{aligned}$$

Note:

$$\begin{aligned}|\text{II}_b| &\le \epsilon T\left(\sum_{j=0}^{k-1}\|\partial_x^j u\|_{L_x^2L_T^\infty}\right)^2 \|u\|_{L_T^\infty H_x^{k+1}}\|y\|_{L_T^\infty H_x^k} \\ &\le c\epsilon^2T^2\left(\sum_{j=0}^{k-1}\|\partial_x^j u\|_{L_x^2L_T^\infty}\right)^4 \|u\|_{L_T^\infty H_x^{k+1}}^2 + \tfrac{1}{10}\|y\|_{L_T^\infty H_x^k}^2\end{aligned}$$

Also

$$\text{II}_a = \epsilon\langle iHD_x^{1/2}P_\pm(\bar{u}\,\partial_x^{k+1}u - u\,\partial_x^{k+1}\bar{u}), D_x^{1/2}(\bar{u}\,Ay)\rangle$$

and thus, ignoring commutators,

$$\begin{aligned}|\mathrm{II}_a| &\le c\epsilon \|\langle x\rangle u\|_{L^2_x L^\infty_T}^2 \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^{k+1} u\|_{L^2_x L^2_T} \|\langle x\rangle^{-1} D_x^{1/2} y\|_{L^2_x L^2_T} \\ &\le c\epsilon^2 \|\langle x\rangle u\|_{L^2_x L^\infty_T}^4 \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^{k+1} u\|_{L^2_x L^2_T}^2 + \tfrac{1}{10}\|\langle x\rangle^{-1} D_x^{1/2} y\|_{L^2_x L^2_T}^2\end{aligned}$$

Term III is

$$\begin{aligned}\mathrm{III} = &\; -\operatorname{Re} i\langle B[f_1(\partial_x^k u)], By\rangle - \operatorname{Re} i \sum_{\substack{j_1+j_2\le k \\ 1\le j_1\le k \\ 0\le j_2\le k-1}} \langle B(\partial_x^{j_1} f_1\, \partial_x^{j_2} u), By\rangle \\ = &\; \mathrm{III}_a + \mathrm{III}_b\end{aligned}$$

The treatment of Term III_a is subtle, and is deferred to §1.4.2. We now address Term III_b. Using that $\partial_x(n_0 + u_0\bar{u}_0)(x \pm \frac{t}{\epsilon}) = \pm\epsilon\partial_t(n_0+u_0\bar{u}_0)(x\pm\frac{t}{\epsilon})$, we have (with $A = B^*B$)

$$\begin{aligned}\mathrm{III}_{b,j} &= -\operatorname{Re} i\epsilon \int_0^T \int_x \partial_t \partial_x^{j_1-1} f_1\, \partial_x^{j_2} u\, \overline{Ay} \\ &= \; +\operatorname{Re} i\epsilon \int_0^T \int_x A(\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u)\, \partial_t \bar{y} \\ &\quad +\operatorname{Re} i\epsilon \int_0^T \int_x \partial_x^{j_1-1} f_1\, \partial_t(\partial_x^{j_2} u)\, \overline{Ay} \\ &\quad -\operatorname{Re} i\epsilon \int_x \partial_x^{j_1-1} f_1(T)\, \partial_x^{j_2} u(T)\, \overline{Ay(T)}\, dx\end{aligned}$$

The last two terms are treated by Cauchy-Schwarz, and therefore, we drop these terms, leaving only the first term. Into this term, we substitute

$$\partial_t y = i\partial_x^2 y + i\partial_x^k(u|u|^2 - v|v|^2) - \tfrac{1}{2} i\epsilon\partial_x^k[uP_\pm \partial_t(u\bar{u})] - i\partial_x^k(f_1 u) - i\epsilon\partial_x^k(f_2 u)$$

which gives five terms for us to consider:

$$\begin{aligned}
\mathrm{III}_{b,j} \approx\ & \mathrm{Re}\ \epsilon \int_0^T \int_x A(\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u)\, \partial_x^2 \bar{y} \\
& + \mathrm{Re}\ \epsilon \int_0^T \int_x A(\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u)\, \partial_x^k (\bar{u}|u|^2 - \bar{v}|v|^2) \\
& - \tfrac{1}{2}\mathrm{Re}\ \epsilon^2 \int_0^T \int_x A(\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u)\, \partial_x^k [\bar{u}\, P_\pm \partial_t (u\bar{u})] \\
& - \mathrm{Re}\ \epsilon \int_0^T \int_x A(\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u)\, \partial_x^k (f_1 \bar{u}) \\
& - \mathrm{Re}\ \epsilon^2 \int_0^T \int_x A(\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u)\, \partial_x^k (f_2 \bar{u}) \\
=\ & \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} + \mathrm{E}
\end{aligned}$$

We first consider Term A. Suppose $j_1 \leq k-1$. Then, after two integrations by parts

$$\mathrm{A} = \mathrm{Re}\ \epsilon \int_0^T \int_x [\partial_x^2 A(\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u)]\, \bar{y}$$

Since $j_1 - 1 \leq k-2$ and $j_2 \leq k-1$, we can bound this term by Cauchy-Schwarz. Suppose now that $j_1 = k$.

$$\begin{aligned}
\mathrm{A} =\ & - \mathrm{Re}\ \epsilon \int_0^T \int_x A[\partial_x^k f_1\, u]\, \partial_x \bar{y} \\
& + \mathrm{Re}\ \epsilon \int_0^T \int_x \partial_x A'[\partial_x^{k-1} f_1\, u]\, \bar{y} \\
& + \mathrm{Re}\ \epsilon \int_0^t \int_x \partial_x A[\partial_x^{k-1} f_1\, \partial_x u]\, \bar{y}
\end{aligned}$$

The last two terms are bounded by Cauchy-Schwarz, leaving us with the first term. In the first term, we split the derivative on $\bar{y}$ as $iHD_x^{1/2}D_x^{1/2}$ and bound as (ignoring commutators)

$$\begin{aligned}
A &\leq \epsilon \| (D_x^{1/2} \partial_x^k f_1)\, \langle x \rangle u \|_{L_x^2 L_T^2} \| \langle x \rangle^{-1} D_x^{1/2} y \|_{L_x^2 L_T^2} \\
&\leq \epsilon T^{1/2} \| n_0 + u_0 \bar{u}_0 \|_{H_x^{k+\frac{1}{2}}} \| \langle x \rangle u \|_{L_T^\infty L_x^\infty} \| \langle x \rangle^{-1} D_x^{1/2} y \|_{L_x^2 L_T^2}
\end{aligned}$$

and then apply Cauchy-Schwarz. Term B can also be addressed by Cauchy-Schwarz and (1.96). After an integration by parts

$$\mathrm{C} = +\tfrac{1}{2}\mathrm{Re}\ \epsilon^2 \int_0^T \int_x \partial_x A(\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u)\, \partial_x^{k-1}[uP_\pm \partial_t(u\bar{u})]$$

To address Term D, we expand by Leibniz:

$$\mathrm{D} = -\mathrm{Re}\ \epsilon \sum_{j_3+j_4=k} c_j \int_0^T \int_x A(\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u)\, \partial_x^{j_3} f_1\, \partial_x^{j_4} \bar{u}$$

and then bound as:

$$\begin{aligned}
|\mathrm{D}_j| &\le \epsilon \|A(\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u)\|_{L_T^2 L_x^2} \|\partial_x^{j_3} f_1\, \partial_x^{j_4} u\|_{L_x^2 L_T^2} \\
&\le \epsilon \|\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u\|_{L_T^2 L_x^2} \|\partial_x^{j_3} f_1\, \partial_x^{j_4} u\|_{L_x^2 L_T^2} \\
&\le \epsilon \|\partial_x^{j_1-1} f_1\|_{L_x^\infty L_T^2} \|\partial_x^{j_2} u\|_{L_x^2 L_T^\infty} \|\partial_x^{j_3} f_1\|_{L_x^\infty L_T^2} \|\partial_x^{j_4} u\|_{L_x^2 L_T^\infty}
\end{aligned}$$

and use that $\|\partial_x^j f_1\|_{L_x^\infty L_T^2} \le \epsilon^{1/2} \|\partial_x^j (n_0 + u_0 \bar{u}_0)\|_{L_x^2}$. To address Term E, we also expand by Leibniz:

$$\mathrm{E} = -\mathrm{Re}\ \epsilon^2 \sum_{j_3+j_4=k} c_j \int_0^T \int_x A(\partial_x^{j_1-1} f_1\, \partial_x^{j_2} u)\, \partial_x^{j_3} f_2\, \partial_x^{j_4} \bar{u}$$

The terms E_j are bounded in the same way as D_j above:

$$|\mathrm{E}_j| \le \epsilon^2 \|\partial_x^{j_1-1} f_1\|_{L_x^\infty L_T^2} \|\partial_x^{j_2} u\|_{L_x^2 L_T^\infty} \|\partial_x^{j_3} f_2\|_{L_x^\infty L_T^2} \|\partial_x^{j_4} u\|_{L_x^2 L_T^\infty}$$

When $j_3 \ge 1$, we have

$$\partial_x^{j_3} f_2(x,t) = \tfrac{1}{2} \partial_x^{j_3-1} n_1(x + \tfrac{t}{\epsilon}) - \tfrac{1}{2} \partial_x^{j_3-1} n_1(x - \tfrac{t}{\epsilon})$$

and thus

$$\|\partial_x^{j_3} f_2\|_{L_x^\infty L_T^2} \le \epsilon^{1/2} \|\partial_x^{j_3-1} n_1\|_{L_x^2}$$

When $j_3 = 0$, we use

$$|f_2(x,t)| \le \int_{x-\frac{T}{\epsilon}}^{x+\frac{T}{\epsilon}} |n_1(s)|\, ds \le T^{1/2}\epsilon^{-1/2}\|n_1\|_{L^2} \tag{1.97}$$

and thus $\|f_2\|_{L_x^\infty L_T^2} \le T\epsilon^{-1/2}\|n_1\|_{L_x^2}$. The $\epsilon^{-1/2}$ factor is canceled by the $\epsilon^{1/2}$ factor in the bound $\|\partial_x^{j_1-1} f_1\|_{L_x^\infty L_T^2} \le \epsilon^{1/2}\|\partial_x^{j_1-1}(n_0 + u_0\bar{u}_0)\|_{L_x^2}$. Now we turn to Term IV. By a Leibniz expansion

$$\text{IV} = -\text{Re } i\epsilon \sum_{j_1+j_2=k} c_j\langle B[\partial_x^{j_1} f_2\, \partial_x^{j_2} u], By\rangle$$

If $1 \le j_1 \le k$, then $\partial_x^{j_1} f_2 = \partial_x^{j_1-1} n_1(x + \frac{t}{\epsilon}) - \partial_x^{j_1-1} n_1(x - \frac{t}{\epsilon})$ and bound as

$$\begin{aligned} |\text{IV}_j| &\le \epsilon T\|\partial_x^{j_1} f_2\|_{L_T^\infty L_x^2}\|\partial_x^{j_2} u\|_{L_T^\infty L_x^\infty}\|y\|_{L_T^\infty L_x^2} \\ &\le \epsilon T\|n_1\|_{H_x^{k-1}}\|u\|_{L_T^\infty H_x^k}\|y\|_{L_T^\infty L_x^2} \end{aligned}$$

If $j_1 = 0$, then use $\|f_2\|_{L_T^\infty L_x^\infty} \le \|n_1\|_{L_x^1}$ and bound as

$$|\text{IV}_j| \le \epsilon T\|f_2\|_{L_T^\infty L_x^\infty}\|\partial_x^k u\|_{L_T^\infty L_x^2}\|y\|_{L_T^\infty L_x^2}$$

It is this term that forces the requirement $n_1 \in L^1$. Combining the above bounds for Term I, II, III, and IV, we have from (1.95)

$$\begin{aligned} &\|\partial_x^k (u-v)(T)\|_{L_x^2}^2 + \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k (u-v)\|_{L_x^2 L_T^2}^2 \\ &\quad \le c\epsilon^2 + \tfrac{1}{2}\|u-v\|_{L_T^\infty H_x^k}^2 + c\int_0^T \|(u-v)(t)\|_{H_x^k}^2\, dt \end{aligned}$$

Replacing T by T', and then taking the supremum over $0 \le T' \le T$, we have

$$\begin{aligned} &\|\partial_x^k(u-v)\|^2_{L^\infty_T L^2_x} + \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k(u-v)\|^2_{L^2_x L^2_T} \\ &\quad \le c\epsilon^2 + \tfrac{1}{2}\|u-v\|^2_{L^\infty_T H^k_x} + c\int_0^T \|(u-v)(t)\|^2_{H^k_x}\,dt \end{aligned} \tag{1.98}$$

To (1.98) we need to add an L^2_x estimate of $u-v$. This is obtained by performing an energy estimate on (1.92) directly, which gives

$$\begin{aligned} \|(u-v)(T)\|^2_{L^2_x} &= \text{Re } i\langle u|u|^2 - iv|v|^2, u-v\rangle - \tfrac{1}{2}\epsilon\text{Re } i\langle uP_\pm\partial_t(u\bar{u}), u-v\rangle \\ &\qquad - \text{Re } i\langle f_1 u, u-v\rangle - \text{Re } i\epsilon\langle f_2 u, u-v\rangle \\ &= \text{I} + \text{II} + \text{III} + \text{IV} \end{aligned} \tag{1.99}$$

The estimates for I and II are straightforward.

$$|\text{III}| \le \|f_1\|_{L^\infty_x L^1_T}\|u\|_{L^1_x L^\infty_T}\|u-v\|_{L^\infty_x L^\infty_T} \le \epsilon\|n_0+u_0\bar{u}_0\|_{L^1_x}\|\langle x\rangle u\|_{L^2_x L^\infty_T}\|u-v\|_{L^\infty_T H^1_x}$$

Similarly,

$$|\text{IV}| \le \epsilon T^{1/2}\|f_2\|_{L^\infty_x L^\infty_T}\|u\|_{L^2_T L^2_x}\|u-v\|_{L^\infty_T L^2_x}$$

and then use the estimate $\|f_2\|_{L^\infty_x L^\infty_T} \le \|n_1\|_{L^1}$. Applying the bounds for I, II, III, and IV to (1.99), we have

$$\|u-v\|^2_{L^\infty_T L^2_x} \le c\epsilon^2 + c\int_0^T \|(u-v)(t)\|^2_{L^2_x}\,dt + \tfrac{1}{2}\|u-v\|^2_{L^\infty_T H^1_x} \tag{1.100}$$

Adding (1.98) and (1.99), and applying Gronwall's inequality,

$$\|u-v\|^2_{L^\infty_T H^k_x} + \|\langle x\rangle^{-1} D_x^{1/2}\partial_x^k(u-v)\|^2_{L^2_x L^2_T} \le c\epsilon^2$$

1.4.2 Maximal function estimates

Here, we treat Term III_a in §1.4.1. Let $\eta(\xi) \in C^\infty(\mathbb{R})$, $\eta(\xi) = 1$ for $-\frac{1}{2} \le |\xi| \le 2$ and supp $\eta(\xi) \subset \{\,\xi \mid \frac{1}{4} \le |\xi| \le 4\,\}$. For $j \ge 1$, define $\widehat{Q_j f}(\xi) = \eta(2^{-j}\xi)\hat{f}(\xi)$, set

$\widehat{Q_0 f}(\xi) = \sum_{j=-\infty}^{0} \eta(2^{-j})\hat{f}(\xi)$, and for $j < 0$, set $Q_j f = 0$. Put $\Psi = \sum_{j \le -3} Q_j$. This notation will be used a bit loosely, with Q_j in one line defined in terms of a different η (sharing, of the course, the same essential properties) than in the previous line.

$$\begin{aligned} \mathrm{III}_a &= \langle B[f_1\, \partial_x^k u], By \rangle \\ &= \sum_j \left\langle B \left[Q_j f_1\, \Psi_j \partial_x^k u + \Psi_j f_1\, Q_j \partial_x^k u + \sum_{|\alpha| \le 2} Q_j f_1\, Q_{j-\alpha} \partial_x^k \right], B\partial_x^k (u-v) \right\rangle \end{aligned} \tag{1.101}$$

The first piece and third piece of (1.101) are treated similarly. Of these two, we only address the first piece. Recall $A = B^* B$.

$$\begin{aligned} &\sum_j \langle B[Q_j f_1\, \Psi_j \partial_x^k u], B\partial_x^k (u-v) \rangle \\ &= \sum_j \langle \partial_x^2 A[Q_j f_1\, \Psi_j \partial_x^k u], \partial_x^{k-2}(u-v) \rangle \\ &= \sum_{\alpha=0}^{2} \sum_j \langle A^{(2-\alpha)}[Q_j^1 \partial_x^{\alpha+1} f_1\; Q_j^2 \partial_x^{k-1} u], \partial_x^{k-2}(u-v) \rangle \end{aligned}$$

where Q_j^1 is defined in terms of $\dfrac{\eta(\xi)}{i\xi}$, Q_j^2 is defined in terms of $\psi(\xi) i\xi$, and $A^{(2-\alpha)}$ means the operator with $\partial_x^{2-\alpha}$ applied to the symbol of A. Now, we use that $\partial_x f_1 =$

$\epsilon \partial_t f_1$, and integrate by parts in t to obtain

$$
\begin{aligned}
= & \sum_{\alpha=0}^{2} \sum_j \epsilon \int_x Q_j^1 \partial_x^\alpha f_1(T)\; Q_j^2 \partial_x^{k-1} u(T) \overline{A^{(2-\alpha)} \partial_x^{k-2}(u-v)(T)}\, dx \\
& - \sum_{\alpha=0}^{2} \sum_j \epsilon \int_x Q_j^1 \partial_x^\alpha f_1(0)\; Q_j^2 \partial_x^{k-1} u(0) \overline{A^{(2-\alpha)} \partial_x^{k-2}(u-v)(0)}\, dx \\
& - \sum_{\alpha=0}^{2} \sum_j \epsilon \langle Q_j^1 \partial_x^\alpha f_1\; \partial_t Q_j^2 \partial_x^{k-1} u, A^{(2-\alpha)} \partial_x^{k-2}(u-v) \rangle \\
& - \sum_{\alpha=0}^{2} \sum_j \epsilon \langle Q_j^1 \partial_x^\alpha f_1\; Q_j^2 \partial_x^{k-1} u, A^{(2-\alpha)} \partial_t \partial_x^{k-2}(u-v) \rangle \\
= & \,\mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D}
\end{aligned}
$$

Estimating:

$$
\begin{aligned}
|\mathrm{A}| + |\mathrm{B}| &\leq \sum_{\alpha=0}^{2} \sum_j \epsilon \| Q_j^1 \partial_x^\alpha f_1 \|_{L_T^\infty L_x^\infty} \| Q_j^2 \partial_x^{k-1} u \|_{L_T^\infty L_x^2} \| \partial_x^{k-2}(u-v) \|_{L_T^\infty L_x^2} \\
&\leq \epsilon \| f_1 \|_{H_x^3} \| \partial_x^{k-1} u \|_{L_T^\infty L_x^2} \| \partial_x^{k-2}(u-v) \|_{L_T^\infty L_x^2} \\
&\leq c\epsilon^2 \| f_1 \|_{H_x^3}^2 \| \partial_x^{k-1} u \|_{L_T^\infty L_x^2}^2 + \tfrac{1}{10} \| \partial_x^{k-2}(u-v) \|_{L_T^\infty L_x^2}^2
\end{aligned}
$$

Similarly,

$$
|\mathrm{C}| \leq cT^2 \epsilon^2 \| f_1 \|_{H_x^3}^2 \| \partial_t \partial_x^{k-1} u \|_{L_T^\infty L_x^2}^2 + \tfrac{1}{10} \| \partial_x^{k-2}(u-v) \|_{L_T^\infty L_x^2}^2
$$

For Term D, transfer one x-derivative off $\partial_t \partial_x^{k-2}(u-v)$ and (if $\alpha = 0$) one x-derivative off $\partial_x^{k-1} u$ onto $\partial_x^\alpha f_1$. Convert $\partial_x f_1 = \epsilon \partial_t f_1$, and proceed by integration by parts as above. This will furnish an ϵ^2 coefficient, and a typical term is

$$
\epsilon^2 \sum_j \langle Q_j^1 \partial_x^2 f_1\; Q_j^2 \partial_x^{k-1} u, A \partial_t^2 \partial_x^{k-3}(u-v) \rangle
$$

We then use the bound

$$\|\partial_t^2\partial_x^{k-3}(u-v)\|_{L_T^\infty L_x^2} \le \|\partial_t^2\partial_x^{k-3}u\|_{L_T^\infty L_x^2} + \|\partial_t^2\partial_x^{k-3}u\|_{L_T^\infty L_x^2} \le c$$

Now we consider the second piece of (1.101). We may assume $j \ge 1$.

$$\begin{aligned}
&= \sum_j \langle B[\Psi_j f_1\ Q_j\partial_x^k u], B\partial_x^k(u-v)\rangle \\
&\approx \sum_j \langle \Psi_j f_1\ Q_j\partial_x^k u, D_x^{1/2}AH_xD_x^{1/2}\partial_x^{k-1}(u-v)\rangle \\
&= \sum_j \langle D_x^{1/2}[\Psi_j f_1\ Q_j\partial_x^k u], Q_jAH_xD_x^{1/2}\partial_x^{k-1}(u-v)\rangle \\
&= \sum_j \|D_x^{1/2}[\Psi_j f_1\ Q_j\partial_x^k u]\|_{L_x^2L_T^1}\|Q_jAH_xD_x^{1/2}\partial_x^{k-1}(u-v)\|_{L_x^2L_T^\infty}
\end{aligned}$$

AH_x is a composition of Hilbert transforms and multiplication operators, and after some work (because of the presence of Q_j, the symbol of the Hilbert transform can be smoothed at the origin), we have essentially,

$$\begin{aligned}
&\lesssim \sum_j \|\Psi_j f_1\|_{L_x^\infty L_T^1}\|Q_jD_x^{1/2}\partial_x^k u\|_{L_x^2L_T^\infty}\|Q_jD_x^{1/2}\partial_x^{k-1}(u-v)\|_{L_x^2L_T^\infty} \\
&\le \epsilon\|n_0+u_0\bar{u}_0\|_{L_x^1}\left(\sum_j \|Q_jD_x^{1/2}\partial_x^k u\|_{L_x^2L_T^\infty}^2\right)^{1/2} \\
&\quad \times \left(\sum_j \|Q_jD_x^{1/2}\partial_x^{k-1}(u-v)\|_{L_x^2L_T^\infty}^2\right)^{1/2}
\end{aligned}$$

where we have used that $\|f_1\|_{L_x^\infty L_T^1} \le \epsilon\|n_0+u_0\bar{u}_0\|_{L_x^1}$. Now we appeal to Claims 1 and 2 below.

The needed estimates for the Schrödinger group $S(t) = e^{it\partial_x^2}$ appear in [KPV93b].

For convenience, define the operator $\mathcal{I}$ on a function $h(x,t)$ as

$$\mathcal{I}h(t) = \int_0^t S(t-t')h(t')\,dt'$$

Lemma 5.

$$\|Q_j S(t)\phi\|_{L^2_x L^\infty_T} \le c2^{j/2}\|Q_j\phi\|_{L^2_x} \tag{1.102}$$

$$\|Q_j\mathcal{I}h\|_{L^2_x L^\infty_T} \le \begin{cases} c\|Q_j h\|_{L^1_x L^2_T} & (1.103) \\ c2^{j/2}\|Q_j h\|_{L^1_T L^2_x} & (1.104) \\ c2^{j}\|Q_j h\|_{L^2_x L^1_T} & (1.105) \end{cases}$$

Claim 1. *Assume* $(u_0, n_0, n_1) \in (H^{k+1} \cap H^1(\langle x\rangle^2\,dx)) \times H^{k+\frac{1}{2}} \times H^{k-\frac{1}{2}}$. *Then, assuming the uniform bounds furnished by Prop. 4,*

$$\left(\sum_{j=0}^{+\infty} \|Q_j D_x^{1/2}\partial_x^k u\|^2_{L^2_x L^\infty_T}\right)^{1/2} \le c$$

Proof of Claim 1. With $y = \partial_x^k u$, u solving ZS_ϵ, we can write

$$\partial_t y = i\partial_x^2 y \pm \tfrac{1}{2} iuP_\pm \bar{u}\partial_x^{k+1}u - ig_1 - i\partial_x^k(uf_1) - i\epsilon\partial_x^k(uf_2)$$

where g_1 is a sum of terms of the form $(\partial_x^{j_1}u)P_+(\partial_x^{j_2}\bar{u}\,\partial_x^{j_3}u)$, $j_1, j_2, j_3 \le k$,

$$f_1(x,t) = n_0(x - \tfrac{t}{\epsilon}) + n_0(x + \tfrac{t}{\epsilon})$$

$$f_2(x,t) = \int_{x-\frac{t}{\epsilon}}^{x+\frac{t}{\epsilon}} n_1(y)\,dy$$

Equivalently,

$$y = S(t)\partial_x^k u_0 \pm \tfrac{1}{2} i\mathcal{I}(uP_\pm \bar{u}\partial_x y) - i\mathcal{I}(g_1) - i\mathcal{I}\partial_x^k(uf_1) - i\epsilon\mathcal{I}\partial_x^k(uf_2) = \text{A}+\text{B}+\text{C}+\text{D}+\text{E} \tag{1.106}$$

We now apply $\sum_{j=0}^{+\infty} \|Q_j D_x^{1/2}(\cdots)\|^2_{L^2_x L^\infty_T}$ to each term. The case Q_0 is straightforward, and thus below we assume $j \geq 1$.

Term A. By (1.102),

$$\sum_j \|Q_j D_x^{1/2} S(t)\partial_x^k u_0\|^2_{L^2_x L^\infty_T} \leq c \sum_j \|Q_j \partial_x^{k+1} u_0\|^2_{L^2_x} \leq c\|u_0\|^2_{H^{k+1}_x}$$

Term B. See treatment of Term A in the proof of Claim 2 below (without the ϵ-coefficient).

Term C. By (1.104),

$$\sum_{j=0}^{+\infty} \|Q_j D_x^{1/2} \mathcal{I}(g_1)\|^2_{L^2_x L^\infty_T} \leq cT \sum_{j=0}^{+\infty} 2^j \|\tilde{Q}_j g_1\|^2_{L^2_T L^2_x} \leq cT(\|g_1\|_{L^2_T L^2_x} + \|\partial_x g_1\|_{L^2_T L^2_x})^2$$

Term D.

$$\begin{aligned}
Q_j D_x^{1/2} \partial_x^k (u f_1) &= \sum_{\alpha=1}^{k} c_\alpha Q_j D_x^{1/2} (\partial_x^\alpha u \, \partial_x^{k-\alpha} f_1) + Q_j D_x^{1/2} (u \, \partial_x^k f_1) \\
&= \sum_{\alpha=1}^{k} c_\alpha Q_j D_x^{1/2} (\partial_x^\alpha u \, \partial_x^{k-\alpha} f_1) + Q_j D_x^{1/2} (\Psi_j u \, Q_j \partial_x^k f_1) \\
&\quad + Q_j D_x^{1/2} (Q_j u \, \Psi_j \partial_x^k f_1) + \sum_{\alpha=j-3}^{+\infty} \sum_{|\beta|\leq 2} Q_j D_x^{1/2} (Q_\alpha u \, Q_{\alpha-\beta} \partial_x^k f_1) \\
&= \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{D}_3 + \mathrm{D}_4
\end{aligned}$$

D_2 is treated using (1.103).

$$\begin{aligned}
&\sum_j \|Q_j D_x^{1/2} (\Psi_j u \, Q_j \partial_x^k f_1)\|^2_{L^1_x L^2_T} \\
&\qquad \leq c \sup_j \|\Psi_j u\|^2_{L^1_x L^\infty_T} \sum_j \|Q_j D_x^{1/2} \partial_x^k f_1\|^2_{L^\infty_x L^2_T} \\
&\qquad \leq c \|u\|^2_{L^1_x L^\infty_T} \|D_x^{1/2} \partial_x^k f_1\|^2_{L^\infty_x L^2_T}
\end{aligned}$$

For the terms D_1, D_3, and D_4, we instead use (1.104).

Term E. Similar to Term D. $\square$

Claim 2. *Assume* $(u_0, n_0, n_1) \in (H^{k+1} \cap H^1(\langle x\rangle^2\, dx)) \times (H^{k+\frac{1}{2}} \cap L^1) \times (H^{k-\frac{1}{2}} \cap L^1)$. *Then, assuming the uniform bounds furnished by Prop. 4,*

$$\sum_{j=0}^{+\infty} \left(\|Q_j D_x^{1/2} \partial_x^{k-1}(u-v)\|_{L_x^2 L_T^\infty}^2 \right)^{1/2} \leq c\epsilon + cT\|u-v\|_{L_T^\infty H_x^k}$$

Proof of Claim 2. Set $y = \partial_x^{k-1}(u-v)$. Then, with $f_1(x,t) = (n_0 + u_0\bar{u}_0)(x \pm \frac{t}{\epsilon})$,

$$\begin{aligned} y &= -\tfrac{1}{2} i\epsilon\mathcal{I}\partial_x^{k-1}(uP_\pm \partial_t(u\bar{u})) - \mathcal{I}\partial_x^{k-1}(u|u|^2 - v|v|^2) - i\mathcal{I}\partial_x^{k-1}(uf_1) - i\epsilon\mathcal{I}\partial_x^{k-1}(uf_2) \\ &= \mathrm{A} + \mathrm{B} + \mathrm{C} + \mathrm{D} \end{aligned} \tag{1.107}$$

Term A. Using $\partial_t(u\bar{u}) = i\partial_x^2 u\,\bar{u} - iu\,\partial_x^2\bar{u}$ and expanding by the Leibniz rule, we obtain a sum of terms, with a representative difficult component being $uP_+\bar{u}\partial_x^{k+1}u$.

$$\begin{aligned} &\epsilon D_x^{1/2} Q_j \mathcal{I} uP_+\bar{u}\partial_x^{k+1}u \\ &= \epsilon\mathcal{I} D_x^{1/2} Q_j \int_{s=0}^{t/\epsilon} u(x,t)\ \langle x-s\rangle \bar{u}(x-s,t-\epsilon s)\ \langle x-s\rangle^{-1}\partial_x^{k+1}u(x-s,t-\epsilon s)\, ds \\ &\approx \epsilon\mathcal{I} Q_j^1 \int_{s=0}^{t/\epsilon} \Psi_j[u(x,t)\ \langle x-s\rangle\bar{u}(x-s,t-\epsilon s)] \\ &\qquad\qquad \times D_x^{1/2} Q_j^2[\langle x-s\rangle^{-1}\partial_x^{k+1}u(x-s,t-\epsilon s)]\, ds \end{aligned}$$

where we have kept only the most difficult component. Now apply $\sum_j \|(\cdots)\|_{L_x^2 L_T^\infty}^2$, and apply (1.103).

$$\begin{aligned}
&\le \epsilon^2 \sum_j \left\| \int_{s=0}^{t/\epsilon} \Psi_j[u(x,t)\ \langle x-s\rangle \bar{u}(x-s,t-\epsilon s)] \right. \\
&\qquad\qquad \left. \times D_x^{1/2} Q_j^2 [\langle x-s\rangle^{-1} \partial_x^{k+1} u(x-s,t-\epsilon s)]\, ds \right\|_{L_x^1 L_T^2}^2 \\
&\le \epsilon^2 \sum_j \|\Psi_j[u(x,t)\ \langle x-s\rangle \bar{u}(x-s,t-\epsilon s)]\|_{L_x^1 L_s^2 L_T^\infty}^2 \qquad (1.108) \\
&\qquad \times \|D_x^{1/2} Q_j^2 [\langle x-s\rangle^{-1} \partial_x^{k+1} u(x-s,t-\epsilon s)]\|_{L_s^2 L_T^2}^2
\end{aligned}$$

by Minkowskii. Use that

$$\begin{aligned}
\sup_j &\|\Psi_j[u(x,t)\ \langle x-s\rangle \bar{u}(x-s,t-\epsilon s)]\|_{L_x^1 L_s^2 L_T^\infty}^2 \\
&\le \|u(x,t)\langle x-s\rangle u(x-s,t-\epsilon s)\|_{L_x^1 L_s^2 L_T^\infty}^2 \\
&= \|u(x,t)\|_{L_x^1 L_T^\infty}^2 \|\langle x-s\rangle u(x-s,t-\epsilon s)\|_{L_s^2 L_T^\infty}^2 \\
&\le \|\langle x\rangle u\|_{L_x^2 L_T^\infty}^4
\end{aligned}$$

and

$$\sum_j \|Q_j D_x^{1/2} [\langle x-s\rangle^{-1} \partial_x^{k+1} u(x-s,t-\epsilon s)]\|_{L_s^2 L_T^2}^2 \le c\|\langle x\rangle^{-1} D_x^{1/2} \partial_x^{k+1} u\|_{L_x^2 L_T^2}^2$$

Substituting into (1.108),

$$(1.108) \le c\epsilon^2 \|\langle x\rangle u\|_{L_x^2 l_T^\infty}^4 \|\langle x\rangle^{-1} D_x^{1/2} \partial_x^{k+1} u\|_{L_x^2 L_T^2}^2$$

Term B. By (1.104),

$$\begin{aligned}
&\sum_j \|D_x^{1/2} Q_j \mathcal{I}\partial_x^{k-1}(u|u|^2 - v|v|^2)\|^2_{L^2_x L^\infty_T} \\
&= \sum_j \|D_x^1 Q_j \partial_x^{k-1}(u|u|^2 - v|v|^2)\|^2_{L^1_T L^2_x} \\
&\leq cT\|u-v\|^2_{L^\infty_T H^k_x}
\end{aligned}$$

Term C.

$$Q_j D_x^{1/2}\mathcal{I}\partial_x^{k-1}(uf_1) = Q_j D_x^{1/2}\sum_{\alpha=1}^{k}\mathcal{I}(\partial_x^{k-\alpha-1}u\,\partial_x^\alpha f_1) + Q_j D_x^{1/2}\mathcal{I}(\partial_x^{k-1}u\,f_1) = \mathrm{C}_1 + \mathrm{C}_2$$

We further decompose the product in C_2 in frequencies:

$$\begin{aligned}
\mathrm{C}_2 = \ & Q_j D_x^{1/2}\mathcal{I}(Q_j\partial_x^{k-1}u\,\Psi_j f_1) + Q_j D_x^{1/2}\mathcal{I}(\Psi_j\partial_x^{k-1}u\,Q_j f_1) \\
&+ \sum_{|\beta|\leq 2}\sum_{\alpha=j-3}^{+\infty} D_x^{1/2}\mathcal{I}(Q_\alpha\partial_x^{k-1}u\,Q_{\alpha-\beta}f_1) \\
= \ & \mathrm{C}_{2,1} + \mathrm{C}_{2,2} + \mathrm{C}_{2,3}
\end{aligned}$$

In terms $\mathrm{C}_{2,2}$ and $\mathrm{C}_{2,3}$, x-derivatives can be transfered to f_1 and converted to $\epsilon\partial_t$. We will in a similar manner address term C_1. But first, let us address term $\mathrm{C}_{2,1}$. By (1.105),

$$\begin{aligned}
\sum_j \|\mathrm{C}_{2,1}\|^2_{L^2_x L^\infty_T} &\leq c\sum_j \|Q_j\partial_x D_x^{1/2}(Q_j\partial_x^{k-1}u\,\Psi_j f_1)\|^2_{L^2_x L^1_T} \\
&\leq \sum_j \|Q_j^1(Q_j^2 D_x^{1/2}\partial_x^k u\,\Psi_j f_1)\|^2_{L^2_x L^1_T} \\
&\leq c\sup_j \|\Psi_j f_1\|^2_{L^\infty_x L^1_T}\sum_j \|Q_j^2 D_x^{1/2}\partial_x^k u\|^2_{L^2_x L^\infty_T} \\
&\leq c\epsilon^2\|n_0 + u_0\bar{u}_0\|^2_{L^1_x}\sum_j \|Q_j^2 D_x^{1/2}\partial_x^k u\|^2_{L^2_x L^\infty_T}
\end{aligned}$$

We appeal to Claim 1 for a bound on the second term. Turning to C_1, we use the identity (integration by parts in the definition of $\mathcal{I}$)

$$\mathcal{I}[\partial_{t'} h(t')](t) = i\partial_x^2 \mathcal{I} h(t) + h(t) - S(t)h(0) \tag{1.109}$$

Recalling that $\alpha \geq 1$,

$$\begin{aligned} C_{1,\alpha} &= Q_j D_x^{1/2} \mathcal{I}(\partial_x^{k-\alpha-1} u \, \partial_x^{\alpha} f_1) \\ &= Q_j D_x^{1/2} \mathcal{I}(\partial_x^{k-\alpha-1} u \, \epsilon \partial_t \partial_x^{\alpha-1} f_1) \\ &= \epsilon Q_j D_x^{1/2} \mathcal{I} \partial_t (\partial_x^{k-\alpha-1} u \, \partial_x^{\alpha-1} f_1) - \epsilon Q_j D_x^{1/2} \mathcal{I}[(\partial_t \partial_x^{k-\alpha-1} u) \, \partial_x^{\alpha-1} f_1] \end{aligned}$$

Invoking (1.109),

$$\begin{aligned} C_{1,\alpha} = &\; i\epsilon Q_j D_x^{1/2} \mathcal{I} \partial_x^2 (\partial_x^{k-\alpha-1} u \; \partial_x^{\alpha-1} f_1) \\ &+ \epsilon Q_j D_x^{1/2} (\partial_x^{k-\alpha-1} u(t) \; \partial_x^{\alpha-1} f_1(t)) \\ &- \epsilon Q_j D_x^{1/2} S(t) (\partial_x^{k-\alpha-1} u(0) \; \partial_x^{\alpha-1} f_1(0)) \\ &- \epsilon Q_j D_x^{1/2} \mathcal{I}[(\partial_t \partial_x^{k-\alpha-1} u) \; \partial_x^{\alpha-1} f_1] \\ = &\; \mathrm{C}_{1,\alpha,1} + \mathrm{C}_{1,\alpha,2} + \mathrm{C}_{1,\alpha,3} + \mathrm{C}_{1,\alpha,4} \end{aligned} \tag{1.110}$$

For $\alpha = k-1$, we further decompose $\mathrm{C}_{1,\alpha,1}$

$$i\epsilon Q_j \mathcal{I} D_x^{1/2} (\partial_x^2 u \, \partial_x^{k-2} f_1) + 2i\epsilon Q_j \mathcal{I} D_x^{1/2} (\partial_x u \, \partial_x^{k-1} f_1) + i\epsilon Q_j \mathcal{I} D_x^{1/2} (u \, \partial_x^k f_1) \tag{1.111}$$

and the last of these three terms we decompose in frequencies:

$$\begin{aligned} i\epsilon Q_j \mathcal{I} D_x^{1/2} (u \, \partial_x^k f_1) = &\; i\epsilon Q_j \mathcal{I} D_x^{1/2} (\Psi_j u \, Q_j \partial_x^k f_1) \\ &+ i\epsilon Q_j \mathcal{I} D_x^{1/2} (Q_j u \, \Psi_j \partial_x^k f_1) \\ &+ i\epsilon Q_j \mathcal{I} \sum_{\gamma \geq j-3} \sum_{|\beta| \leq 2} D_x^{1/2} (Q_\gamma u \, Q_{\gamma-\beta} \partial_x^k f_1) \end{aligned} \tag{1.112}$$

For the first piece in (1.112), we use (1.103), while for all other components in (1.112) and (1.111) and all $\alpha \le k-2$, we use (1.104).

$$\sum_j \sum_{\alpha=1}^{k-1} \|\mathrm{C}_{1,\alpha,1}\|^2_{L^2_x L^\infty_T} \le c\epsilon^2 T \|u\|^2_{L^\infty_T H^{k+1}_x} \|f_1\|^2_{L^\infty_T H^k_x} + c\epsilon^2 \sum_j \|Q_j D^{1/2}_x (\Psi_j u\, Q_j \partial^k_x f_1)\|^2_{L^1_x L^2_T}$$
$$\le c\epsilon^2 T \|u\|^2_{L^\infty_T H^{k+1}_x} \|f_1\|^2_{L^\infty_T H^k_x} + c\epsilon^3 \|u\|^2_{L^1_x L^\infty_T} \|D^{1/2}_x \partial^k_x f_1\|^2_{L^2_x}$$

Returning to the other components in (1.110),

$$\sum_j \|\mathrm{C}_{1,\alpha,2}\|^2_{L^2_x L^\infty_T} \le c\epsilon^2 \sum_j \|Q_j D^{1/2}_x (\partial^{k-\alpha-1}_x u(t)\, \partial^{\alpha-1}_x f_1(t))\|^2_{L^2_x L^\infty_T}$$
$$\le c\epsilon^2 \sum_j 2^{-j/2} \|Q^1_j \partial_x (\partial^{k-\alpha-1}_x u(t)\ \partial^{\alpha-1}_x f_1(t)\|^2_{L^2_x L^\infty_T}$$
$$\le c\epsilon^2 \sum_{\alpha=0}^{k-1} \|\partial^\alpha_x u\|^2_{L^\infty_x L^\infty_T} \sum_{\alpha=0}^{k-1} \|\partial^\alpha_x f_1\|^2_{L^2_x L^\infty_T}$$
$$\le c\epsilon^2 \|u\|^2_{L^\infty_T H^k_x} \|f_1\|^2_{H^k_x}$$

By (1.104),

$$\sum_j \|\mathrm{C}_{1,\alpha,3}\|^2_{L^2_x L^\infty_T} \le c\epsilon^2 \sum_j \|Q_j \partial_x (\partial^{k-\alpha-1}_x u(0)\, \partial^{\alpha-1}_x f_1(0))\|^2_{L^1_T L^2_x}$$
$$\le c\epsilon^2 T \|u\|^2_{L^\infty_T H^{k-1}_x} \|f_1\|^2_{L^\infty_T H^{k-1}_x}$$

By (1.104),

$$\sum_j \|\mathrm{C}_{1,\alpha,4}\|^2_{L^2_x L^\infty_T} \le c\epsilon^2 \sum_j \|Q_j \partial_x (\partial_{t'} \partial^{k-\alpha-1}_x u\, \partial^{\alpha-1}_x f_1)\|^2_{L^1_T L^2_x}$$
$$\le c\epsilon^2 \|\partial_t u\|^2_{L^\infty_T H^{k-1}_x} \|f_1\|^2_{H^{k-1}_x}$$

Term D. The techniques used to address Term C apply here, although this term is a bit easier in parts, due to the ϵ-coefficient. □

1.4.3 The compatible case $(n_0 + u_0\bar{u}_0) = 0$

Here, we suppose that $k \geq 4$, $(u_0, n_0, n_1) \in H^{k+2} \cap H^1(\langle x\rangle^2 dx)) \times H^{k+\frac{3}{2}} \times H^{k+\frac{1}{2}}$, $\exists\ \nu \in W^{1,k+1}$ such that $\partial_x \nu = n_1$, and $n_0 + u_0\bar{u}_0 = 0$. We then have the uniform bound on $\|u\|_{L_T^\infty H_x^{k+2}}$, $\|\langle x\rangle^{-1} D_x^{1/2} \partial_x^{k+2} u\|_{L_x^2 L_T^2}$ furnished by Prop. 4. We shall establish convergence at the optimal rate ϵ^2 of the solution u_ϵ of 1D ZS$_\epsilon$ to the solution v of 1D NLS in H^k as $\epsilon \to 0$.

We begin as in §1.4.1, where now $f_1 = 0$ and (1.95) takes the form (recall $y = \partial_x^k(u - v)$)

$$\|By(T)\|_{L_x^2}^2 = -\text{Re } 2\langle \langle x\rangle^{-2} B D_x y, By\rangle + \text{Re } \langle Ey, By\rangle + \text{I} + \text{II} + \text{IV}$$

and

$$\begin{aligned} \text{I} &= -\text{Re } i\langle B\partial_x^k(u|u|^2 - v|v|^2), By\rangle \\ \text{II} &= -\text{Re } i\epsilon\langle B\partial_x^k[uP_\pm \partial_t(u\bar{u})], By\rangle \\ \text{IV} &= -\text{Re } i\epsilon\langle B\partial_x^k(f_2 u), By\rangle \end{aligned}$$

Term III of (1.95) is 0, since $n_0 + u_0\bar{u}_0 = 0$, making this case a bit easier than §1.4.1. Term I is treated exactly as in §1.4.1. We now address Term II, where we need to generate an extra factor of ϵ. Since $\partial_t(u\bar{u}) = i\partial_x(\partial_x u\,\bar{u} - u\partial_x\bar{u})$,

$$P_+\partial_t(u\bar{u}) + P_-\partial_t(u\bar{u}) = \pm i(\partial_x u\,\bar{u} - u\,\partial_x\bar{u})(x \pm \tfrac{t}{\epsilon}, 0) \mp i\epsilon P_\pm \partial_t(\partial_x u\,\bar{u} - u\partial_x\bar{u})$$

and therefore (taking $w(x,t) = \pm(\partial_x u\,\bar{u} - u\partial_x\bar{u})(x \pm \frac{t}{\epsilon}, 0)$)

$$\begin{aligned} \text{II} = &\ \mp\epsilon^2 \text{Re } \langle B\partial_x^k[uP_\pm\partial_t(\partial_x u\,\bar{u} - u\,\partial_x\bar{u})], B\partial_x^k(u - v)\rangle \\ &+ \text{Re } \epsilon\langle B\partial_x^k(uw), B\partial_x^k(u - v)\rangle \\ = &\ \text{II}_a + \text{II}_b \end{aligned}$$

$$|\mathrm{II}_a| \leq c\epsilon^2 \|\langle x\rangle u\|^2_{L^2_x L^\infty_T} \|\langle x\rangle^{-1} D_x^{1/2} \partial_x^{k+2} u\|_{L^2_x L^2_T} \|\langle x\rangle^{-1} D_x^{1/2} \partial_x^k (u-v)\|_{L^2_x L^2_T}$$
$$+ cT\epsilon^2 \left(\sum_{j=0}^k \|\partial_x^j u\|^2_{L^2_x L^\infty_T} \right) \|u\|_{L^\infty_T H^{k+2}_x} \|u-v\|_{L^\infty_T H^k_x}$$

$$\begin{aligned} |\mathrm{II}_b| &\leq \epsilon \|\partial_x^{k+1}(uw)\|_{L^2_x L^1_T} \|\partial_x^{k-1}(u-v)\|_{L^2_x L^\infty_T} \\ &\leq \epsilon \left(\sum_{j=0}^{k+1} \|\partial_x^j u\|_{L^2_x L^\infty_T} \right) \left(\sum_{j=0}^{k+1} \|\partial_x^j w\|_{L^\infty_x L^1_T} \right) \|\partial_x^{k-1}(u-v)\|_{L^2_x L^\infty_T} \\ &\leq \epsilon^2 \|u_0\|^2_{H^{k+2}_x} \left(\sum_{j=0}^{k+1} \|\partial_x^j u\|_{L^2_x L^\infty_T} \right) \|\partial_x^{k-1}(u-v)\|_{L^2_x L^\infty_T} \end{aligned}$$

Turning to Term IV,

$$\begin{aligned} \mathrm{IV} &\approx \mp \mathrm{Re}\ i\epsilon \langle B\partial_x^{k+1}[\nu(x \pm \tfrac{t}{\epsilon})u(x,t)], B\partial_x^{k-1}(u-v)\rangle \\ &\leq c\epsilon \sum_{j=0}^{k+1} \|\partial_x^j \nu(x \pm \tfrac{t}{\epsilon})\|_{L^\infty_x L^1_T} \sum_{j=0}^{k+1} \|\partial_x^j u\|_{L^2_x L^\infty_T} \|\partial_x^{k-1}(u-v)\|_{L^2_x L^\infty_T} \\ &\leq c\epsilon^2 \|\nu\|_{W^{1,k+1}} \left(\sum_{j=0}^{k+1} \|\partial_x^j u\|_{L^2_x L^\infty_T} \right) \|\partial_x^{k-1}(u-v)\|_{L^2_x L^\infty_T} \end{aligned}$$

To obtain the estimate at the L^2 level, we mimick the computation in §1.4.1, and in place of (1.98), obtain

$$\|(u-v)(T)\|^2_{L^2_x} = \mathrm{I} + \mathrm{II} + \mathrm{IV}$$

(III of (1.98) is 0 since $f_1 = 0$), where

$$\begin{aligned} \mathrm{I} &= \mathrm{Re}\ i\langle u|u|^2 - v|v|^2, u-v\rangle \\ \mathrm{II} &= -\tfrac{1}{2}\mathrm{Re}\ i\epsilon\langle uP_\pm \partial_t(u\bar{u}), u-v\rangle \\ \mathrm{IV} &= -\mathrm{Re}\ i\epsilon\langle f_2 u, u-v\rangle \end{aligned}$$

For II, we use $\partial_t(u\bar{u}) = i\partial_x(\partial_x u\,\bar{u} - u\partial_x\bar{u})$, as above, and obtain

$$\begin{aligned}\mathrm{II} &= \mp\epsilon^2 \mathrm{Re}\,\langle uP_\pm \partial_t(\partial_x u\,\bar{u} - u\,\partial_x\bar{u}), u-v\rangle + \mathrm{Re}\,\epsilon\langle uw, u-v\rangle \\ &= \mathrm{II}_a + \mathrm{II}_b\end{aligned}$$

with $w(x,t) = \pm(\partial_x u\,\bar{u} - u\,\partial_x\bar{u})(x \pm \frac{t}{\epsilon}, 0)$. Thus

$$|\mathrm{II}_a| \le c\epsilon^2 T \left(\sum_{j=0}^{1} \|\partial_x^j u\|_{L_x^2 L_T^\infty}\right)^2 \|u\|_{L_T^\infty H_x^3} \|u-v\|_{L_T^\infty L_x^2}$$

$$\begin{aligned}|\mathrm{II}_b| &\le \epsilon\|u\|_{L_x^1 L_T^\infty}\|w\|_{L_x^\infty L_T^1}\|u-v\|_{L_x^\infty L_T^\infty} \\ &\le c\epsilon^2\|u_0\|_{H_x^1}^2\|\langle x\rangle u\|_{L_x^2 L_T^\infty}\|u-v\|_{L_T^\infty H_x^1}\end{aligned}$$

$$\begin{aligned}|\mathrm{IV}| &\le c\epsilon\|f_2\|_{L_x^\infty L_T^1}\|u\|_{L_x^1 L_T^\infty}\|u-v\|_{L_x^\infty L_T^\infty} \\ &\le c\epsilon^2\|\nu\|_{L_x^1}\|\langle x\rangle u\|_{L_x^2 L_T^\infty}\|u-v\|_{L_T^\infty H_x^1}\end{aligned}$$

By the methods of §1.4.2, we can show

Claim 3. *Let* $(u_0, n_0, n_1) \in (H^{k+1} \cap H^1(\langle x\rangle^2\,dx)) \times H^{k+\frac{1}{2}} \times H^{k-\frac{1}{2}}$. *Assuming the uniform bounds on* $[0,T]$ *provided by Prop. 4, then*

$$\left(\sum_{j=0}^{+\infty} \|Q_j D_x^{1/2}\partial_x^k u\|_{L_x^2 L_T^\infty}^2\right)^{1/2} \le c$$

Claim 4. *Let* $(u_0, n_0, n_1) \in H^{k+2} \cap H^1(\langle x\rangle^2\,dx) \times H^{k+\frac{3}{2}} \times H^{k+\frac{1}{2}}$ *and suppose* $\exists\,\nu \in L^1$ *such that* $\partial_x \nu = n_1$. *Assuming the uniform bounds on* $[0,T]$ *provided by Prop. 4, then*

$$\left(\sum_{j=0}^{+\infty} \|Q_j D_x^{1/2}\partial_x^{k-1}(u-v)\|_{L_x^2 L_T^\infty}^2\right)^{1/2} \le c\epsilon^2 + cT\|u-v\|_{L_T^\infty H_x^k}$$

1.5 Uniform estimates via energy identities

Just as in §1.3, we start with the solution given by [GTV97], cited in Prop. 1, and prove uniform bounds for it.

Given a solution (u,n) to 1D FZS$_\epsilon$ corresponding to initial data (u_0,n_0,n_1), split n_1 into low and high frequency parts as $n_1 = n_{1L} + n_{1H}$, and put

$$\hat{\nu}(\xi) = \frac{\hat{n}_{1H}(\xi)}{i\xi}$$

Note that $\|\nu\|_{L^2_x} \le c\|n_1\|_{H^{-1}}$. Define

$$\begin{cases} n_+ = -\frac{1}{2}\partial_x P_+(u\bar{u}) + \frac{1}{2}n_0(x-\frac{t}{\epsilon}) - \frac{1}{2}\epsilon\nu(x-\frac{t}{\epsilon}) \\ n_- = \frac{1}{2}\partial_x P_-(u\bar{u}) + \frac{1}{2}n_0(x+\frac{t}{\epsilon}) + \frac{1}{2}\epsilon\nu(x+\frac{t}{\epsilon}) \end{cases}$$

These definitions lead to

$$\begin{cases} (\epsilon\partial_t + \partial_x)n_+ = -\frac{1}{2}\partial_x(u\bar{u}) \\ (\epsilon\partial_t - \partial_x)n_- = +\frac{1}{2}\partial_x(u\bar{u}) \end{cases}$$

so that $n = n_+ + n_- + f$, where where

$$f(x,t) = \tfrac{1}{2}\epsilon\int_{y=x-t/\epsilon}^{y=x+t/\epsilon} n_{1L}(y)\,dy$$

and n satisfies

$$\begin{cases} \epsilon^2\partial_t^2 n - \partial_x^2 n = \partial_x^2(u\bar{u}) \\ n(x,0) = n_0(x) \\ \partial_t n(x,0) = n_1(x) \end{cases}$$

Reexpress

$$-\tfrac{1}{2}\partial_x P_+(u\bar{u}) = -\tfrac{1}{2}u\bar{u} + \tfrac{1}{2}(u_0\bar{u}_0)(x-\tfrac{t}{\epsilon}) + \tfrac{1}{2}\epsilon P_+(\partial_t[u\bar{u}])$$
$$\tfrac{1}{2}\partial_x P_-(u\bar{u}) = -\tfrac{1}{2}u\bar{u} + \tfrac{1}{2}(u_0\bar{u}_0)(x+\tfrac{t}{\epsilon}) + \tfrac{1}{2}\epsilon P_-(\partial_t[u\bar{u}])$$

to obtain, with $q_\pm = n_\pm + \frac{1}{2}u\bar{u}$,

$$\begin{cases} q_+ = \frac{1}{2}\epsilon P_+(\partial_t[u\bar{u}]) + \frac{1}{2}(n_0 + u_0\bar{u}_0)(x - \frac{t}{\epsilon}) - \frac{1}{2}\epsilon\nu(x - \frac{t}{\epsilon}) \\ \quad = \frac{1}{2}i\epsilon P_+(\bar{u}\partial_x^2 u - u\partial_x^2\bar{u}) + \frac{1}{2}(n_0 + u_0\bar{u}_0)(x - \frac{t}{\epsilon}) - \frac{1}{2}\epsilon\nu(x - \frac{t}{\epsilon}) \\ q_- = \frac{1}{2}\epsilon P_-(\partial_t[u\bar{u}]) + \frac{1}{2}(n_0 + u_0\bar{u}_0)(x + \frac{t}{\epsilon}) + \frac{1}{2}\epsilon\nu(x + \frac{t}{\epsilon}) \\ \quad = \frac{1}{2}i\epsilon P_-(\bar{u}\partial_x^2 u - u\partial_x^2\bar{u}) + \frac{1}{2}(n_0 + u_0\bar{u}_0)(x + \frac{t}{\epsilon}) + \frac{1}{2}\epsilon\nu(x + \frac{t}{\epsilon}) \end{cases}$$

Then ZS_ϵ becomes

$$\begin{cases} \partial_t u = i\partial_x^2 u - iq_+ u - iq_- u + i|u|^2 u - ifu \\ \epsilon\partial_t q_\pm = \mp\partial_x q_\pm + \frac{1}{2}\epsilon\partial_t(u\bar{u}) \\ \qquad = \mp\partial_x q_\pm + \frac{1}{2}i\epsilon(\bar{u}\partial_x^2 u - u\partial_x^2\bar{u}) \end{cases}$$

Proposition 9 (H^1 bound). *Assume $(u_0, n_0, n_1) \in H^1 \times L^2 \times H^{-1}$, and split n_1 into low and high frequencies as $n_1 = n_{1L} + n_{1H}$. Then we have the identity*

$$\partial_t \int |\partial_x u|^2 + q_+^2 + q_-^2 - \frac{1}{2}|u|^4 + fu\bar{u} = \frac{1}{2}\int [n_{1L}(x + \frac{t}{\epsilon}) + n_{1L}(x - \frac{t}{\epsilon})]u\bar{u}$$

from which it follows that

$$\|u\|_{L_T^\infty H_x^1} \le c\langle T\rangle^{1/2}$$
$$\|n\|_{L_T^\infty L_x^2} \le c\langle T\rangle$$

with c depending only on the norms $\|u_0\|_{H_x^1}$, $\|n_0\|_{L_x^2}$, and $\|n_1\|_{H_x^{-1}}$.

Proof.

$$\partial_t \int_x |\partial_x u|^2 = i\int_x q_\pm(u\partial_x^2\bar{u} - \bar{u}\partial_x^2 u) + 2\text{Re }i\int_x \partial_x(u\bar{u}u)\partial_x\bar{u} - 2\text{Re }i\int_x \partial_x fu\partial_x\bar{u}$$

and

$$\partial_t \int_x q_\pm^2 = i\int q_\pm(\bar{u}\partial_x^2 u - u\partial_x^2\bar{u})$$

Also,

$$\partial_t \int |u|^4 = 2i \int \partial_x(u\bar{u}u)\,\partial_x\bar{u} - 2i \int \partial_x(\bar{u}u\bar{u})\,\partial_x u$$

Combining,

$$\partial_t \int |\partial_x u|^2 + q_+^2 + q_-^2 - \tfrac{1}{2}|u|^4 = -2\text{Re } i \int \partial_x f u \partial_x \bar{u}$$

But

$$-2\text{Re } i \int \partial_x f u \partial_x \bar{u} = i \int_x f(u\partial_x^2\bar{u} - \bar{u}\partial_x^2 u) = -\int_x f\partial_t(u\bar{u}) = -\partial_t \int_x f u\bar{u} + \int_x \partial_t f\, u\bar{u}$$
$$= -\partial_t \int f u\bar{u} + \tfrac{1}{2}\int [n_{1L}(x+\tfrac{t}{\epsilon}) + n_{1L}(x-\tfrac{t}{\epsilon})]u\bar{u}$$

So

$$\partial_t \int |\partial_x u|^2 + q_+^2 + q_-^2 - \tfrac{1}{2}|u|^4 + f u\bar{u} = \tfrac{1}{2}\int [n_{1L}(x+\tfrac{t}{\epsilon}) + n_{1L}(x-\tfrac{t}{\epsilon})]u\bar{u}$$

By Gagliardo-Nirenberg,

$$\int_x |u|^4 \le \|u_0\|_{L_x^2}^3 \|\partial_x u\|_{L_x^2}$$

Also,

$$\int_x f u\bar{u} \le \|f\|_{L_x^\infty}\|u_0\|_{L_x^2}^2 \le \epsilon^{1/2}t^{1/2}\|n_{1L}\|_{L_x^2}\|u_0\|_{L_x^2}^2$$

and

$$\tfrac{1}{2}\int_0^t \int_x [n_{1L}(x+\tfrac{t}{\epsilon}) + n_{1L}(x-\tfrac{t}{\epsilon})]u\bar{u} \le T\|n_{1L}\|_{L_T^\infty L_x^\infty}\|u\|_{L_T^\infty L_x^2}^2 \le T\|n_{1L}\|_{L_x^2}\|u_0\|_{L_x^2}^2$$

Thus,

$$\|u\|_{L_T^\infty H_x^1} + \|q_\pm\|_{L_T^\infty L_x^2}$$
$$\le c(\|u_0\|_{H_x^1} + \|u_0\|_{L_x^2}^3 + \|n_0 + u_0\bar{u}_0\|_{L_x^2} + \|n_1\|_{H_x^{-1}} + \langle T\rangle^{1/2}\|n_1\|_{H_x^{-1}}^{1/2}\|u_0\|_{L_x^2})$$

Using that $n + u\bar{u} = q_+ + q_- + f$, we obtain

$$\begin{aligned}\|n + u\bar{u}\|_{L^2_x} &\leq \|q_\pm\|_{L^\infty_T L^2_x} + cT\|n_1\|_{H^{-1}_x} \\ &\leq c(\|u_0\|_{H^1_x} + \|u_0\|^3_{L^2_x} + \|n_0 + u_0\bar{u}_0\|_{L^2_x} + \langle T\rangle \|n_1\|_{H^{-1}_x})\end{aligned}$$

□

Proposition 10 (Energy estimates). *Suppose $k \geq 3$, $(u_0, n_0, n_1) \in H^k \times H^{k-1} \times H^{k-2}$. Then $\forall\ T > 0$, there exists $\epsilon_0 = \epsilon_0(T, \|u_0\|_{H^k_x}, \|n_0\|_{H^{k-1}_x}, \|n_1\|_{H^{k-2}_x})$ such that $\forall\ \epsilon,\ 0 < \epsilon \leq \epsilon_0$ there exists (u, n) solving 1D FZS$_\epsilon$ on $[0, T]$ such that*

$$\|u(t)\|_{H^k_x} + \|n(t)\|_{H^{k-1}_x} \leq ch(t)$$

where c depends on the norms $\|u_0\|_{H^k}$, $\|n_0\|_{H^{k-1}}$, and $\|n_1\|_{H^{k-2}}$ and is independent of ϵ, and $h(t)$ has exponential growth in t.

We need a preliminary estimate in order to prove this.

Claim 5 (H^2 control). *For $0 < \epsilon \leq 1$,*

$$\|u\|^2_{L^\infty_T H^2_x} + \|q_\pm\|^2_{L^\infty_T H^1_x} \leq c\langle T\rangle^6 + c\epsilon T \left(\sum_{j=0}^{1} \|\partial^j_x u\|_{L^2_x L^\infty_T} \right)^2 \|u\|^2_{L^\infty_T H^3_x}$$

where c depends on the norms $\|u_0\|_{H^2}$, $\|n_0\|_{H^1}$, and $\|n_1\|_{L^2}$.

Proof of claim.

$$\begin{aligned}\partial_t \int_x |\partial^2_x u|^2 =& -i \int_x \partial^2_x q_\pm (u\partial^2_x \bar{u} - \bar{u}\partial^2_x u) - 4\mathrm{Re}\ i \int_x \partial_x q_\pm \partial_x u \partial^2_x \bar{u} \\ &+ 2\mathrm{Re}\ i \int_x \partial^2_x (u\bar{u}u)\partial^2_x \bar{u} - 2\mathrm{Re}\ i \int_x \partial^2_x (uf)\, \partial^2_x \bar{u} \\ \partial_t \int_x |\partial_x q_\pm|^2 =& -i \int_x (\partial^2_x u\bar{u} - u\partial^2_x \bar{u})\partial^2_x q_\pm\end{aligned}$$

and thus

$$\partial_t \int_x |\partial_x^2 u|^2 + |\partial_x q_\pm|^2 = -4\text{Re } i \int_x \partial_x q_\pm \partial_x u \partial_x^2 \bar{u} + 2\text{Re } i \int_x \partial_x^2(u\bar{u}u)\partial_x^2\bar{u} - 2\text{Re } i \int_x \partial_x^2(uf)\,\partial_x^2\bar{u}$$

Integrating in time, and adding the L^2 conservation for u and the H^1 estimate Prop. 9,

$$\begin{aligned} &\|u(t)\|_{H_x^2}^2 + \|q_\pm(t)\|_{H_x^1}^2 \\ &\quad \le c\langle T\rangle^{1/2} - 4\text{Re } i \int_0^t \int_x \partial_x q_\pm \partial_x u \partial_x^2 \bar{u} \\ &\qquad + 2\text{Re } i \int_0^t \int_x \partial_x^2(u\bar{u}u)\partial_x^2\bar{u} - 2\text{Re } i \int_0^t \int_x \partial_x^2(uf)\,\partial_x^2\bar{u} \\ &\quad = c\langle T\rangle^{1/2} + \text{I} + \text{II} + \text{III} \end{aligned} \tag{1.113}$$

Let $q_\pm^0 = \frac{1}{2} i\epsilon P_\pm(\partial_x^2 u\,\bar{u} - u\partial_x^2\bar{u})$, $f_\pm^0 = \frac{1}{2}(n_0 + u_0\bar{u}_0)(x \mp \frac{t}{\epsilon}) \mp \frac{1}{2}\epsilon\nu(x \mp \frac{t}{\epsilon})$, so that $q_\pm = q_\pm^0 + f_\pm^0$. Term I:

$$-4\text{Re } i \int_0^t \int_x \partial_x q_\pm^0 \partial_x u \partial_x^2 \bar{u} \le T^{1/2} \|\partial_x q_\pm^0\|_{L_x^\infty L_T^2} \|\partial_x u\|_{L_x^2 L_T^\infty} \|\partial_x^2 u\|_{L_T^\infty L_x^2}$$

Since $q_\pm^0 = \frac{1}{2} i\epsilon P_\pm(\partial_x^2 u\bar{u} - u\partial_x^2\bar{u})$, we have

$$\|\partial_x q_\pm^0\|_{L_x^\infty L_T^2} \le \epsilon T^{1/2} \|u\|_{L_x^2 L_T^\infty} \|\partial_x^3 u\|_{L_T^\infty L_x^2} + \epsilon T^{1/2} \|\partial_x u\|_{L_x^2 L_T^\infty} \|\partial_x^2 u\|_{L_T^\infty L_x^2}$$

and thus

$$-4\text{Re } i \int_0^t \int_x \partial_x q_\pm^0 \partial_x u \partial_x^2 \bar{u} \le \epsilon T \left(\sum_{j=0}^{1} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \right)^2 \|u\|_{L_T^\infty H_x^3}^2 \tag{1.114}$$

Also,

$$-4\text{Re } i\int_0^t\int_x \partial_x f_\pm^0 \partial_x u \partial_x^2 \bar{u} \leq T\|\partial_x f_\pm^0\|_{L_T^\infty L_x^2}\|\partial_x u\|_{L_T^\infty L_x^\infty}\|\partial_x^2 u\|_{L_T^\infty L_x^2}$$
$$\leq T(\|\partial_x(n_0+u_0\bar{u}_0)\|_{L_x^2}+\epsilon\|n_{1H}\|_{L_x^2})\|u\|_{L_T^\infty H_x^1}^{1/2}\|u\|_{L_T^\infty H_x^2}^{3/2} \tag{1.115}$$

by Gagliardo-Nirenberg. Combining (1.114) and (1.115),

$$\text{I} \leq \epsilon T\left(\sum_{j=0}^{1}\|\partial_x^j u\|_{L_x^2 L_T^\infty}\right)^2\|u\|_{L_T^\infty H_x^3}^2 \tag{1.116}$$
$$+T(\|\partial_x(n_0+u_0\bar{u}_0)\|_{L_x^2}+\epsilon\|n_{1H}\|_{L_x^2})\|u\|_{L_T^\infty H_x^1}^{1/2}\|u\|_{L_T^\infty H_x^2}^{3/2}$$

Term II:

$$\text{II} = 2\text{Re } i\int_0^t\int_x uu\partial_x^2\bar{u}\partial_x^2\bar{u} + 8\text{Re } i\int_0^t\int_x \partial_x u\partial_x\bar{u}u\partial_x^2\bar{u} + 8\text{Re } i\int_0^t\int_x \partial_x u\,\bar{u}\,\partial_x u\,\partial_x^2\bar{u}$$

On the other hand, by integration by parts

$$\partial_t\text{Re }\int_x uu\partial_x\bar{u}\partial_x\bar{u} = 2\text{Re } i\int_x uu\partial_x^2\bar{u}\partial_x^2\bar{u} + 2\text{Re } i\int_x \partial_x nuu\bar{u}\partial_x\bar{u}$$

and

$$\partial_t\int_x u\bar{u}\partial_x u\partial_x\bar{u} = 4\text{Re } i\int_x u\partial_x u\partial_x\bar{u}\partial_x^2\bar{u} - 2\text{Re } i\int_x \partial_x nuu\bar{u}\partial_x\bar{u}$$

Thus:

$$-\text{Re }\int_x uu\partial_x\bar{u}\partial_x\bar{u} - 4\int_x u\bar{u}\partial_x u\partial_x\bar{u}$$
$$= -\text{Re }\int_x u_0u_0\partial_x\bar{u}_0\partial_x\bar{u}_0 - 4\int_x u_0\bar{u}_0\partial_x u_0\partial_x\bar{u}_0 - 2\text{Re } i\int_0^t\int_x uu\partial_x^2\bar{u}\partial_x^2\bar{u}$$
$$-16\text{Re } i\int_0^t\int_x u\partial_x u\partial_x\bar{u}\partial_x^2\bar{u} + 8\text{Re } i\int_0^t\int_x \partial_x nuu\bar{u}\partial_x\bar{u}$$

Hence,

$$\mathrm{II} \leq c\|u\|^4_{L^\infty_T H^1_x} + cT\|n\|_{L^\infty_T H^1_x}\|u\|^4_{L^\infty_T H^1_x} \tag{1.117}$$

Term III:

$$\begin{aligned} -2\mathrm{Re}\ i\int_0^t\int_x \partial_x^2 f u \partial_x^2 \bar{u} &\leq T\|\partial_x^2 f\|_{L^\infty_T L^2_x}\|u\|_{L^\infty_T L^\infty_x}\|\partial_x^2 u\|_{L^\infty_T L^2_x} \\ &\leq T\epsilon\|n_{1L}\|_{H^1_x}\|u\|_{L^\infty_T H^1_x}\|\partial_x^2 u\|_{L^\infty_T L^2_x} \end{aligned}$$

and

$$\begin{aligned} -4\mathrm{Re}\ i\int_0^t\int_x \partial_x f \partial_x u \partial_x^2 \bar{u} &\leq T\|\partial_x f\|_{L^\infty_T L^\infty_x}\|\partial_x u\|_{L^\infty_T L^2_x}\|\partial_x^2 u\|_{L^\infty_T L^2_x} \\ &\leq T\epsilon\|n_{1L}\|_{H^1_x}\|u\|_{L^\infty_T H^1_x}\|\partial_x^2 u\|_{L^\infty_T L^2_x} \end{aligned}$$

giving

$$\mathrm{III} \leq T\epsilon\|n_{1L}\|_{H^1_x}\|u\|_{L^\infty_T H^1_x}\|u\|_{L^\infty_T H^2_x} \tag{1.118}$$

Note that since $n = q_+ + q_- + f - u\bar{u}$,

$$\|n\|^2_{L^\infty_T H^1_x} \leq \|q_\pm\|^2_{L^\infty_T H^1_x} + \langle T\rangle\|n_1\|^2_{L^2_x} + \|u\|^4_{H^1_x}$$

Combining the above bounds for I, II, and III, and applying them to (1.113), we obtain

$$\begin{aligned} &\|u\|^2_{L^\infty_T H^2_x} + \|q_\pm\|^2_{L^\infty_T H^1_x} \\ &\quad\leq c\langle T\rangle + cT^4(\|n_0 + u_0\bar{u}_0\|_{H^1_x} + \|n_1\|_{L^2_x})^4\|u\|^2_{L^\infty_T H^1_x} + c\|u\|^4_{L^\infty_T H^1_x} \\ &\qquad + cT^2\|u\|^8_{L^\infty_T H^1_x} + cT^2\|n_1\|^2_{L^2_x}\|u\|^2_{L^\infty_T H^1_x} \\ &\qquad + cT\epsilon\left(\sum_{j=0}^{1}\|\partial_x^j u\|_{L^2_x L^\infty_T}\right)^2\|u\|^2_{L^\infty_T H^3_x} \\ &\quad\leq c\langle T\rangle^6 + cT\epsilon\left(\sum_{j=0}^{1}\|\partial_x^j u\|_{L^2_x L^\infty_T}\right)^2\|u\|^2_{L^\infty_T H^3_x} \end{aligned}$$

□

Now we can carry out the proof of Prop. 10. A Leibniz expansion:

$$\partial_x^k(q_\pm u) = \partial_x^k q_\pm\, u + k\,\partial_x^{k-1} q_\pm\,\partial_x u + \sum_{\substack{j_1+j_2=k\\ j_1\le k-2}} c_j\,\partial_x^{j_1} q_\pm\,\partial_x^{j_2} u$$

which gives, by integration by parts,

$$\begin{aligned} 2\mathrm{Re}\, i\int \partial_x^k(q_\pm u)\,\partial_x^k\bar u = \; & -i\int \partial_x^{k-1} q_\pm\,(u\,\partial_x^{k+1}\bar u - \bar u\,\partial_x^{k+1} u) \\ & + 2(k-1)\mathrm{Re}\, i\int \partial_x^{k-1} q_\pm\,\partial_x u\,\partial_x^k \bar u \\ & + \sum_{\substack{j_1+j_2=k\\ j_1\le k-2}} c_j \mathrm{Re}\, i\int \partial_x^{j_1} q_\pm\,\partial_x^{j_2} u\,\partial_x^k\bar u \end{aligned} \tag{1.119}$$

By another Leibniz expansion:

$$\begin{aligned} & i\int \partial_x^{k-1} q_\pm\,\partial_x^{k-1}(u\,\partial_x^2\bar u - \bar u\,\partial_x^2 u) \\ & \quad = i\int \partial_x^{k-1} q_\pm\,(u\,\partial_x^{k+1}\bar u - \bar u\,\partial_x^{k+1} u) + 2(k-1)\mathrm{Re}\, i\int \partial_x^{k-1} q_\pm\,\partial_x u\,\partial_x^k\bar u \\ & \qquad + \sum_{\substack{j_1+j_2=k-1\\ j_2\le k-3}} c_j \mathrm{Re}\, i\int \partial_x^{k-1} q_\pm\,\partial_x^{j_1} u\,\partial_x^{2+j_2}\bar u \\ & \quad = i\int \partial_x^{k-1} q_\pm\,(u\,\partial_x^{k+1}\bar u - \bar u\,\partial_x^{k+1} u) + 2(k-1)\mathrm{Re}\, i\int \partial_x^{k-1} q_\pm\,\partial_x u\,\partial_x^k\bar u \\ & \qquad + \sum_{\substack{j_1+j_2=k+1\\ j_1\le k,\, j_2\le k}} c_j \mathrm{Re}\, i\int \partial_x^{k-2} q_\pm\,\partial_x^{j_1} u\,\partial_x^{j_2}\bar u \end{aligned}$$

When substituted into (1.119), (1.119) becomes

$$\begin{aligned} 2\mathrm{Re}\, i \int \partial_x^k(q_\pm u)\,\partial_x^k\bar{u} = & -i\int \partial_x^{k-1}q_\pm\,\partial_x^{k-1}(u\,\partial_x^2\bar{u} - \bar{u}\,\partial_x^2 u) \\ & + 4(k-1)\mathrm{Re}\, i\int \partial_x^{k-1}q_\pm\,\partial_x u\,\partial_x^k\bar{u} \\ & + \sum c_j \mathrm{Re}\, i\int \partial_x^{j_1}q_\pm\,\partial_x^{j_2}u\,\partial_x^{j_3}\bar{u} \end{aligned} \tag{1.120}$$

where now c_j may be positive, negative, or zero, and the constraint is that either $[j_1 = k-2$ and $j_2 + j_3 = k+1$, $j_2 \le k$, $j_3 \le k]$ or $[j_1 + j_2 = k$, $j_1 \le k-2$, and $j_3 = k]$. Now, from the equation, we have

$$\partial_t \int_x |\partial_x^k u|^2 = -2\mathrm{Re}\, i\int_x \partial_x^k(q_\pm u)\partial_x^k\bar{u} - 2\mathrm{Re}\, i\int_x \partial_x^k(u\bar{u}u)\partial_x^k\bar{u} - 2\mathrm{Re}\, i\int_x \partial_x^k(fu)\partial_x^k\bar{u} \tag{1.121}$$

$$\partial_t \int_x |\partial_x^{k-1}q_\pm|^2 = i\int_x \partial_x^{k-1}(\partial_x^2 u\bar{u} - u\partial_x^2\bar{u})\partial_x^{k-1}q_\pm \tag{1.122}$$

Substituting (1.120) into (1.121) and adding (1.122),

$$\begin{aligned} \partial_t \int_x |\partial_x^k u|^2 + |\partial_x^{k-1}q_\pm|^2 = & -4(k-1)\mathrm{Re}\, i\int_x \partial_x^{k-1}q_\pm\,\partial_x u\,\partial_x^k\bar{u} \\ & + \sum c_j\mathrm{Re}\, i\int_x \partial_x^{j_1}q_\pm\,\partial_x^{j_2}u\,\partial_x^{j_3}\bar{u} \\ & - 2\mathrm{Re}\, i\int_x \partial_x^k(u\bar{u}u)\partial_x^k\bar{u} - 2\mathrm{Re}\, i\int_x \partial_x^k(fu)\partial_x^k\bar{u} \end{aligned}$$

Integrating in time, and adding the L^2 conservation for u and the H^1 bound (Proposition 9),

$$
\begin{aligned}
&\|u(T)\|_{H_x^k}^2 + \|\partial_x^{k-1} q_\pm(T)\|_{H_x^{k-1}}^2 \\
&\quad \le c\langle T\rangle - 4(k-1)\mathrm{Re}\, i \int_0^t \int_x \partial_x^{k-1} q_\pm \, \partial_x u \, \partial_x^k \bar{u} \\
&\qquad + \sum c_j \mathrm{Re}\, i \int_0^t \int_x \partial_x^{j_1} q_\pm \, \partial_x^{j_2} u \, \partial_x^{j_3} \bar{u} \\
&\qquad - 2\mathrm{Re}\, i \int_0^t \int_x \partial_x^k (u\bar{u}u) \partial_x^k \bar{u} - 2\mathrm{Re}\, i \int_0^t \int_x \partial_x^k (fu) \partial_x^k \bar{u} \\
&\quad = c\langle T\rangle + \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}
\end{aligned} \tag{1.123}
$$

We bound Term I, in the case $k = 3$ by Claim 5, as:

$$
\begin{aligned}
\mathrm{I} &\le \int_0^T \|\partial_x^2 q_\pm(t)\|_{L_x^2} \|\partial_x^3 u(t)\|_{L_x^2} \|u(t)\|_{H_x^2}\, dt \\
&\le c\langle T\rangle^4 \int_0^T \|q_\pm(t)\|_{H_x^2} \|u(t)\|_{H_x^3}\, dt \\
&\quad + c\epsilon T^2 \|q_\pm\|_{L_T^\infty H_x^2} \|u\|_{L_T^\infty H_x^3} \left(\sum_{j=0}^{1} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \right)^2 \|u\|_{L_T^\infty H_x^3}^2
\end{aligned} \tag{1.124}
$$

In the case $k \ge 4$, just bound as

$$
\mathrm{I} \le \|u\|_{L_T^\infty H_x^3} \int_0^T \|\partial_x^{k-1} q_\pm(t)\|_{L_x^2} \|\partial_x^k u(t)\|_{L_x^2}\, dt \tag{1.125}
$$

Term II can be bounded by

$$
\begin{aligned}
\mathrm{II}_j &\le T \|\partial_x^{j_1} q_\pm\|_{L_T^\infty L_x^\infty} \|\partial_x^{j_2} u\|_{L_T^\infty L_x^2} \|\partial_x^{j_3} u\|_{L_T^\infty L_x^2} \\
&\le T \|\partial_x^{j_1} q_\pm\|_{L_T^\infty L_x^2}^{1/2} \|\partial_x^{j_1+1} q_\pm\|_{L_T^\infty L_x^2}^{1/2} \|\partial_x^{j_2} u\|_{L_T^\infty L_x^2} \|\partial_x^{j_3} u\|_{L_T^\infty L_x^2}
\end{aligned}
$$

In the case when $j_1 = k-2$, $j_2 + j_3 = k+1$, $j_2 \le k$, $j_3 \le k$, then we have

- If $j_2 = k$, then $j_3 = 1$
- If $j_3 = k$, then $j_2 = 1$

- Otherwise, both j_2, j_3 are $\leq k-1$.

Hence, in this case,

$$\begin{aligned}\mathrm{II}_j &\leq T\|q_\pm\|^{1/2}_{L^\infty_T H^{k-2}_x}\|q_\pm\|^{1/2}_{L^\infty_T H^{k-1}_x}\|u\|_{L^\infty_T H^{k-1}_x}\|u\|_{L^\infty_T H^k_x} \\ &\leq T^4\|q_\pm\|^2_{L^\infty_T H^{k-2}_x}\|u\|^4_{L^\infty_T H^{k-1}_x} + \tfrac{1}{10}\|q_\pm\|^{2/3}_{L^\infty_T H^{k-1}_x}\|u\|^{4/3}_{L^\infty_T H^k_x} \end{aligned} \tag{1.126}$$

In the case $j_1 + j_2 = k$, $j_1 \leq k-2$, $j_3 = k$, then we have

- If $j_2 = k$, then $\mathrm{II}_j = 0$
- Otherwise $j_2 \leq k-1$

Hence, we have the same bound (1.126) above. In the case $k = 3$, we apply Claim 5,

$$\mathrm{II} \leq cT^4\langle T\rangle^{18} + c\epsilon^3 T^7\left(\sum_{j=0}^{1}\|\partial_x^j u\|_{L^2_x L^\infty_T}\right)^6\|u\|^6_{L^\infty_T H^3_x} + \tfrac{1}{10}\|q_\pm\|^{2/3}_{L^\infty_T H^2_x}\|u\|^{4/3}_{L^\infty_T H^3_x} \tag{1.127}$$

Term III is treated as:

$$\begin{aligned}\mathrm{III} = &\ -2\mathrm{Re}\, i\int_0^T\int_x u^2(\partial_x^k\bar{u})^2 \\ &+ \sum_{\substack{j_1+j_2+j_3=k \\ j_1\leq k-1, j_2\leq k-1, j_3\leq k-1}} c_j\mathrm{Re}\, i\int_0^T\int_x \partial_x^{j_1}u\,\partial_x^{j_2}\bar{u}\,\partial_x^{j_3}u\,\partial_x^k\bar{u} \\ \leq &\ \|u\|^2_{L^\infty_T H^1_x}\int_0^T\|\partial_x^k u(t)\|^2_{L^2_x}\,dt \\ &+ cT\sum\left(\prod_{i=1}^{2}\|\partial_x^{j_i+1}u\|^{1/2}_{L^\infty_T L^2_x}\|\partial_x^{j_i}u\|^{1/2}_{L^\infty_T L^2_x}\right)\|u\|_{L^\infty_T H^{k-1}_x}\|u\|_{L^\infty_T H^k_x}\end{aligned}$$

where, we have assumed w.l.o.g. that j_3 is the largest of j_1, j_2, and j_3, forcing $j_1, j_2 \leq k-2$. We then have

$$\mathrm{III} \leq c\langle T\rangle\int_0^T\|\partial_x^k u(t)\|^2_{L^2_x}\,dt + cT^2\|u\|^6_{L^\infty_T H^{k-1}_x} + \tfrac{1}{10}\|u\|^2_{L^\infty_T H^k_x} \tag{1.128}$$

If $k = 3$, use Claim 5 to bound $\|u\|_{L_T^\infty H_x^{k-1}}$, giving

$$\begin{aligned} \text{III} \leq\ & c\langle T\rangle \int_0^T \|\partial_x^3 u(t)\|_{L_x^2}^2\, dt + cT^2\langle T\rangle^{18} + \tfrac{1}{10}\|u\|_{L_T^\infty H_x^3}^2 \\ & + c\epsilon^3 T^3 \left(\sum_{j=0}^{1} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \right)^6 \|u\|_{L_T^\infty H_x^3}^6 \end{aligned} \tag{1.129}$$

Term IV: When all k derivatives land on u, $\text{IV}_j = 0$.

$$\begin{aligned} \text{IV} &\leq \sum_{j_1+j_2=k} cT\|\partial_x^{j_1} f\|_{L_T^\infty L_x^\infty} \|\partial_x^{j_2} u\|_{L_T^\infty L_x^2} \|\partial_x^k u\|_{L_T^\infty L_x^2} \\ &\leq T\|f\|_{L_T^\infty H_x^{k+1}} \|u\|_{L_T^\infty H_x^{k-1}} \|u\|_{L_T^\infty H_x^k} \\ &\leq \langle T\rangle^2 \|n_1\|_{H_x^{k-2}}^2 \|u\|_{L_T^\infty H_x^{k-1}}^2 + \tfrac{1}{10}\|u\|_{L_T^\infty H_x^k}^2 \end{aligned} \tag{1.130}$$

In the case $k = 3$, we apply Claim 5,

$$\text{IV} \leq c\langle T\rangle^8 \|n_1\|_{H_x^1}^2 + c\epsilon\langle T\rangle^3 \|n_1\|_{H_x^1}^2 \left(\sum_{j=0}^{1} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \right)^2 \|u\|_{L_T^\infty H_x^3}^2 + \tfrac{1}{10}\|u\|_{L_T^\infty H_x^3}^2 \tag{1.131}$$

In the case $k = 3$, we apply the bounds (1.124) for I, (1.127) for II, (1.129) for III, (1.131) for IV and substitute into (1.123) to obtain a bound of the following type. Let

$$f(T) = \|u\|_{L_T^\infty H_x^3}^2 + \|q_\pm\|_{L_T^\infty H_x^2}^2$$

Using Lemma 3 for $k = 3$,

$$\sum_{j=0}^{1} \|\partial_x^j u\|_{L_x^2 L_T^\infty} \leq c\langle T\rangle^{1/2} \|u_0\|_{H_x^2} + cT\langle T\rangle^{1/2} \|n\|_{L_T^\infty H_x^2} \|u\|_{L_T^\infty H_x^3}$$

Then

$$f(t) \leq c\langle t\rangle^n + c\langle t\rangle^n \int_0^t f(s)\, ds + c\epsilon\langle t\rangle^n \langle f(t)\rangle^n$$

By Gronwall's inequality, for ϵ sufficiently small, we obtain the desired bound. In the case $k \geq 4$, we assume inductively that we have a bound on $\|u\|_{L_T^\infty H_x^{k-1}}$. Then, applying the bounds (1.125) for I, (1.126) for II, (1.128) for III, (1.130) for IV, with

$$f(T) = \|u\|_{L_T^\infty H_x^3}^2 + \|q_\pm\|_{L_T^\infty H_x^2}^2$$

we get

$$f(t) \leq c\langle t\rangle^n + c\langle t\rangle^n \int_0^t f(s)\,ds$$

By Gronwall's inequality, we have the desired bound.

CHAPTER 2
THE INITIAL-BOUNDARY VALUE PROBLEM FOR KDV ON THE HALF-LINE AND LINE SEGMENT

2.1 Introduction

Consider the initial-boundary value problem for the (bilinear) KdV equation on the right half-line: For $f(t) \in H^{\frac{s+1}{3}}(\mathbb{R}_t^+)$, $\phi(x) \in H^s(\mathbb{R}^+)$, find u solving

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & \text{for } (x,t) \in (0,+\infty) \times (0,T) \\ u(0,t) = f(t) & \text{for } t \in (0,T) \\ u(x,0) = \phi(x) & \text{for } x \in (0,+\infty) \end{cases} \tag{2.1}$$

[CK02] introduce a new powerful method for treating problems of this type. In one section of their paper, they treat (2.1) in the case $s = 0$. They introduce a Duhamel forcing operator

$$\begin{aligned} \mathcal{L}^0(h)(x,t) &= \int_0^t S(t-t')\delta_0(x)\mathcal{I}_{-2/3}h(t')dt' \\ &= \int_0^t A\left(\frac{x}{(t-t')^{1/3}}\right)\frac{\mathcal{I}_{-2/3}h(t')}{(t-t')^{1/3}}\,dt' \end{aligned} \tag{2.2}$$

where $A(x) = \frac{1}{2\pi}\int_\xi e^{ix\xi}e^{i\xi^3}\,d\xi$ denotes the Airy function, $S(t)$ denotes the linear solution group for $(\partial_t + \partial_x^3)u = 0$, and $\mathcal{I}_\alpha$ is the Riemann-Liouville fractional integral. (2.2) defines a continuous function in x and $\mathcal{L}^0(h)(0,t) = A(0)\Gamma(\frac{2}{3})h(t) = \frac{1}{3}h(t)$. [1] Moreover, $(\partial_t + \partial_x^3)\mathcal{L}^0(h)(x,t) = \delta_0(x)\mathcal{I}_{-2/3}h(t)$. Thus, by setting $u = 3\mathcal{L}^0(f)$, we

1. [Tay96], p. 462 gives $A(0) = \dfrac{1}{3\Gamma(\frac{2}{3})}$.

then have a solution to

$$\begin{cases} \partial_t u + \partial_x^3 u = 0 & \text{for } x \neq 0 \\ u(0,t) = f(t) \\ u(x,0) = 0 \end{cases} \tag{2.3}$$

In order to carry out the standard contraction scheme to obtain a solution to the nonlinear problem (2.1), [CK02] introduce a Bourgain space

$$\begin{aligned} \|u\|_{X_{s,b}\cap D_\alpha} = & \left(\iint |\hat{u}(\xi,\tau)|^2 \langle \xi \rangle^{2s} \langle \tau - \xi^3 \rangle^{2b} \, d\xi \, d\tau \right)^{1/2} \\ & + \left(\iint_{|\xi|\leq 1} |\hat{u}(\xi,\tau)|^2 \langle \tau \rangle^{2\alpha} \, d\xi \, d\tau \right)^{1/2} \end{aligned}$$

The estimate

$$\|\theta(t)\mathcal{L}^0(h)(x,t)\|_{X_{s,b}\cap D_\alpha} \leq c\|h\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \qquad -\tfrac{1}{2} < s \leq 1 \tag{2.4}$$

introduces the requirement that $b < \frac{1}{2}$. They carry out the bilinear estimate (for $b < \frac{1}{2}$, $\alpha > \frac{1}{2}$)

$$\|\partial_x(uv)\|_{X_{s,-b}} \leq c\|u\|_{X_{s,b}\cap D_\alpha}\|v\|_{X_{s,b}\cap D_\alpha}$$

in the case $s = 0$ using the techniques of [KPV93a]. A straightforward modification of their argument enables one to treat the problem (2.1) for $-\frac{1}{2} < s < \frac{1}{2}$. In order to treat the case $-\frac{3}{4} < s < -\frac{1}{2}$, or extend to $s > \frac{1}{2}$, some difficulties arise, particularly, those estimates that involve an exchange of t-derivatives for x-derivatives in the ratio 1 : 3, or vice versa, have a limited range of applicability. For example,

$$\sup_{t\in\mathbb{R}} \|\mathcal{L}^0(h)(x,t)\|_{H^s(\mathbb{R}_x)} \leq c\|h\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \qquad -\tfrac{1}{2} < s < \tfrac{5}{2} \tag{2.5}$$

$$\sup_{x\in\mathbb{R}} \left\| \theta(t) \int_0^t S(t-t')w(x,t')\,dt' \right\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \leq c\|w\|_{X_{s,-b}} \qquad -1 \leq s \leq \tfrac{1}{2} \tag{2.6}$$

and finally, (2.4). The case of (2.6) can be remedied by introducing a *time-adapted* Bourgain space

$$\|u\|_{Y_{s,b}} = \left(\iint |\hat{u}(\xi,\tau)|^2 \langle \tau \rangle^{2s/3} \langle \tau - \xi^3 \rangle^{2b} \, d\xi \, d\tau \right)^{1/2}$$

We have

$$\sup_{x\in\mathbb{R}} \left\| \theta(t) \int_0^t S(t-t') w(x,t') \, dt' \right\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \leq c(\|w\|_{X_{s,-b}} + \|w\|_{Y_{s,-b}}) \qquad \text{all } s \quad (2.7)$$

and also the bilinear estimate, obtained by the techniques of [KPV96], (see §2.10)

$$\|\partial_x(uv)\|_{X_{s,-b}} + \|\partial_x(uv)\|_{Y_{s,-b}} \leq c\|u\|_{X_{s,b}\cap D_\alpha} \|v\|_{X_{s,b}\cap D_\alpha} \qquad -\tfrac{3}{4} < s < 3 \qquad (2.8)$$

and (2.7) and (2.8) together suffice. By integration by parts in the definition (2.2) of $\mathcal{L}^0(h)(x,t)$, we obtain

$$\mathcal{L}^0(h)(x,t) = -\partial_x^3 \mathcal{L}^0(\mathcal{I}_1 h)(x,t) + \delta_0(x)\mathcal{I}_{1/3}h(t) \qquad (2.9)$$

Thus, if we replace $\mathcal{L}^0(h)(x,t)$ by

$$\mathcal{L}^{-3}(h)(x,t) = -\mathcal{L}^0(h)(x,t) + \delta_0(x)\mathcal{I}_{1/3}h(t) = \partial_x^3 \mathcal{L}^0(\mathcal{I}_1 h)(x,t) \qquad (2.10)$$

we have an operator with similar solution properties to $\mathcal{L}^0(h)(x,t)$ and the analogue of (2.4) is

$$\|\theta(t)\mathcal{L}^{-3}(h)(x,t)\|_{X_{s,b}\cap D_\alpha} \leq c\|h\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \qquad -\tfrac{7}{2} < s \leq -2$$

and the analogue of (2.5) is

$$\sup_{t\in\mathbb{R}} \|\theta(t)\mathcal{L}^{-3}(h)(x,t)\|_{H^s(\mathbb{R}_x)} \leq c\|h\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \qquad -\tfrac{7}{2} < s < -\tfrac{1}{2}$$

But this leaves a gap in s values ($-2 < s \leq -\frac{1}{2}$) and we need to "interpolate" between the operators $\mathcal{L}^{-3}$ and $\mathcal{L}^{0}$. From the second representation for $\mathcal{L}^{-3}$ given by (2.10), it is clear how to define $\mathcal{L}^{-1}$, $\mathcal{L}^{-2}$. The operator

$$\begin{aligned} \mathcal{L}^{-1}(h_2)(x,t) &= \partial_x \int_0^t S(t-t')\delta_0(x)\mathcal{I}_{-1/3}(h_2)(t')\,dt' \\ &= \int_0^t A'\left(\frac{x}{(t-t')^{1/3}}\right)\frac{\mathcal{I}_{-1/3}h_2(t')}{(t-t')^{2/3}}\,dt' \end{aligned} \tag{2.11}$$

is continuous in x and $\mathcal{L}^{-1}(h_2)(0,t) = A'(0)\Gamma(\frac{1}{3})h_2(t) = -\frac{1}{3}h_2(t)$. [2] Moreover, $(\partial_t + \partial_x^3)\mathcal{L}^{-1}(h)(x,t) = \delta_0'(x)\mathcal{I}_{-1/3}h(t)$. Thus, putting $u = -3\mathcal{L}^{-1}f$ solves

$$\begin{cases} \partial_t u + \partial_x^3 u = 0 & \text{for } x \neq 0 \\ u(0,t) = f(t) & \\ u(x,0) = 0 & \end{cases} \tag{2.12}$$

The estimate parallel to (2.4) is

$$\|\theta(t)\mathcal{L}^{-1}(h)(x,t)\|_{X_{s,b}\cap D_\alpha} \leq c\|h\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} \qquad -\tfrac{3}{2} < s \leq 0 \tag{2.13}$$

The estimate parallel to (2.5) is valid for $-\frac{3}{2} < s < \frac{3}{2}$.

This approach of introducing modified forcing operators has the additional advantage that it can also be adapted to treat the left-hand problem

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & \text{for } (x,t) \in (-\infty,0)\times(0,T) \\ u(0,t) = f(t) & \text{for } t \in (0,T) \\ \partial_x u(0,t) = g(t) & \text{for } t \in (0,T) \\ u(x,0) = \phi(x) & \text{for } x \in (-\infty,0) \end{cases} \tag{2.14}$$

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2. [Tay96] p. 462 gives the value $A'(0) = -\dfrac{1}{3\Gamma(\frac{1}{3})}$

We first consider the linear problem

$$\begin{cases} \partial_t u + \partial_x^3 u = 0 & \text{for } x \neq 0 \\ \lim_{x \to 0^-} u(x,t) = f(t) \\ \lim_{x \to 0^-} \partial_x u(x,t) = g(t) \\ u(x,0) = 0 \end{cases}$$

As a first attempt to solve this problem, let us consider

$$u = \mathcal{L}^0(h_1) + \mathcal{L}^{-1}(h_2) \tag{2.15}$$

which satisfies $(\partial_t + \partial_x^3)u(x,t) = \frac{1}{3}\delta_0(x)\mathcal{I}_{-2/3}h_1(t) - \frac{1}{3}\delta_0'(x)\mathcal{I}_{-1/3}h_2(t)$, and

$$u(0,t) = \tfrac{1}{3}h_1(t) - \tfrac{1}{3}h_2(t) \tag{2.16}$$

Now we examine $\partial_x u(x,t) = \partial_x \mathcal{L}^0(h_1)(x,t) + \partial_x \mathcal{L}^{-1}(h_2)(x,t)$ at $x = 0$. The function $\partial_x \mathcal{L}^0(h_1)(x,t)$ is continuous in x at $x = 0$ with

$$\partial_x \mathcal{L}^0(h_1)(0,t) = -\tfrac{1}{3}\mathcal{I}_{-1/3}h_1(t) \tag{2.17}$$

By integration by parts

$$\partial_x^2 \mathcal{L}^{-1}(h_2)(x,t) = \delta_0(x)\mathcal{I}_{-1/3}h_2(t) - \mathcal{L}^0(\mathcal{I}_{-2/3}h_2)(x,t)$$

and therefore $\partial_x \mathcal{L}^{-1}(h_2)(x,t)$ has a step discontinuity at $x = 0$. One can compute

$$\begin{aligned} \lim_{x \to 0^+} \partial_x \mathcal{L}^{-1}(h_2)(x,t) &= \tfrac{1}{3}\mathcal{I}_{-1/3}h_2(t) \\ \lim_{x \to 0^-} \partial_x \mathcal{L}^{-1}(h_2)(x,t) &= -\tfrac{2}{3}\mathcal{I}_{-1/3}h_2(t) \end{aligned} \tag{2.18}$$

Using (2.17) and (2.18), we find

$$\mathcal{I}_{1/3}\left[\lim_{x\to 0^+}\partial_x u(x,t)\right] = -\tfrac{1}{3}h_1(t)+\tfrac{1}{3}h_2(t) \tag{2.19}$$

$$\mathcal{I}_{1/3}\left[\lim_{x\to 0^-}\partial_x u(x,t)\right] = -\tfrac{1}{3}h_1(t)-\tfrac{2}{3}h_2(t) \tag{2.20}$$

The system consisting of (2.16) and (2.20) can be uniquely solved for $h_1(t)$ and $h_2(t)$ given $u(0,t)$ and $\lim_{x\to 0^-}\partial_x u(x,t)$. On the other hand, the system consisting of (2.16) and (2.19) is redundant, and only has a solution when $\lim_{x\to 0^+}\partial_x u(x,t) = -\mathcal{I}_{-1/3}u(0,t)$. This why one must specify both the value of $u(0,t)$ and $\partial_x u(0,t)$ for the left half-line problem but cannot do so for the right half-line problem.

To solve the nonlinear problem, we need a series of estimates in order to carry out the contraction argument, among them (2.4), valid for $-\frac{1}{2}<s\leq 1$ and (2.13), valid for $-\frac{3}{2}<s\leq 0$, and so together they only hold for $-\frac{1}{2}<s\leq 0$. Were we instead to use the two operators $\mathcal{L}^{-1}$ and $\mathcal{L}^{-2}$, the valid range would be $-\frac{3}{2}<s\leq -1$. This leaves the gap $-1<s\leq -\frac{1}{2}$, and for this purpose, we introduce an analytic family of operators $\mathcal{L}_\pm^\lambda$, extending $\mathcal{L}^0$, $\mathcal{L}^{-1}$, and $\mathcal{L}^{-2}$, as follows. The distribution $\dfrac{x_+^{\lambda-1}}{\Gamma(\lambda)}$ is defined as a locally integrable function for $\operatorname{Re}\lambda>0$ and by analytic continuation for all $\lambda\in\mathbb{C}$ (see [Fri98], pp. 20-22.) For $\lambda\in\mathbb{C}$, let $\mathcal{J}_+^\lambda$ be convolution with the distribution $\dfrac{x_+^{\lambda-1}}{\Gamma(\lambda)}$, and for $h\in C_0^\infty(\mathbb{R}^+)$, define

$$\mathcal{L}_-^\lambda(h)(x,t) = \mathcal{J}_+^\lambda[\mathcal{L}^0(\mathcal{I}_{-\frac{\lambda}{3}}h)(-,t)](x)$$

We prove, using the Mellin transform of the left side of the Airy function, that

$$\lim_{x\to 0-}\mathcal{L}_-^\lambda(h)(x,t) = \tfrac{2}{3}\sin(\tfrac{\pi}{3}\lambda+\tfrac{\pi}{6})h(t)$$

and, actually, it is continuous at $x=0$ for $\lambda>-2$. Note that

$$(\partial_t+\partial_x^3)\mathcal{L}_-^\lambda(h)(x,t) = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-\frac{\lambda}{3}-\frac{2}{3}}h(t)$$

Given $f \in H_0^{\frac{s+1}{3}}(\mathbb{R}^+)$, $g \in H_0^{\frac{s}{3}}(\mathbb{R}^+)$, to solve the forced problem

$$\begin{cases} (\partial_t + \partial_x^3)u = 0 & \text{for } x < 0 \\ \lim_{x \to 0^-} u(x,t) = f(t) \\ \lim_{x \to 0^-} \partial_x u(x,t) = g(t) \\ u(x,0) = 0 \end{cases} \tag{2.21}$$

we shall set

$$u = \mathcal{L}_-^{\lambda_1}(h_1) + \mathcal{L}_-^{\lambda_2}(h_2) \tag{2.22}$$

Then

$$\begin{aligned} f(t) &= \tfrac{2}{3}\sin(\tfrac{\pi}{3}\lambda_1 + \tfrac{\pi}{6})h_1(t) + \tfrac{2}{3}\sin(\tfrac{\pi}{3}\lambda_2 + \tfrac{\pi}{6})h_2(t) \\ \mathcal{I}_{1/3}g(t) &= \tfrac{2}{3}\sin(\tfrac{\pi}{3}\lambda_1 - \tfrac{\pi}{6})h_1(t) + \tfrac{2}{3}\sin(\tfrac{\pi}{3}\lambda_2 - \tfrac{\pi}{6})h_2(t) \end{aligned} \tag{2.23}$$

Using standard trigonometric identities, we compute the determinant

$$\begin{vmatrix} \sin\alpha_1 & \sin\alpha_2 \\ \sin(\alpha_1 - \alpha) & \sin(\alpha_2 - \alpha) \end{vmatrix} = \sin(\alpha_2 - \alpha_1)\sin\alpha$$

Therefore, the determinant of the coefficient matrix to the system (2.23) is

$$\frac{2\sqrt{3}}{9}\sin\left[\frac{\pi}{3}(\lambda_2 - \lambda_1)\right]$$

which is $\neq 0$ provided $\lambda_2 - \lambda_1 \neq 3n$, $n \in \mathbb{Z}$. Thus, given $-\frac{3}{4} < s < \frac{3}{2}$, we shall select λ_1 and λ_2 so that the relevant estimates in §2.5 are valid, invert the system (2.23) to obtain h_1 and h_2, and then define u by (2.22) to obtain a solution to (2.21). Applying the estimates in §2.5, we can solve by iteration (2.14) for $-\frac{3}{4} < s < \frac{3}{2}$, $s \neq \frac{1}{2}$, with a compatibility condition for $\frac{1}{2} < s < \frac{3}{2}$.

Let $\dfrac{x_-^{\lambda-1}}{\Gamma(\lambda)} = \dfrac{(-x)_+^{\lambda-1}}{\Gamma(\lambda)}$. For $\lambda \in \mathbb{C}$, let $\mathcal{J}_-^\lambda$ be convolution with the distribution $\dfrac{x_-^{\lambda-1}}{\Gamma(\lambda)}$ and define

$$\mathcal{L}_+^\lambda(h)(x,t) = e^{i\pi\lambda}\mathcal{J}_-^\lambda[\mathcal{L}^0(\mathcal{I}_{-\frac{\lambda}{3}}h)(-,t)](x)$$

We prove, using the Mellin transform of the right side of the Airy function, that

$$\lim_{x\to 0+} \mathcal{L}_+^\lambda(h)(x,t) = \tfrac{1}{3}e^{i\pi\lambda}h(t)$$

and, actually, it is continuous at $x = 0$ for $\lambda > -2$. Note that

$$(\partial_t + \partial_x^3)\mathcal{L}_+^\lambda(h)(x,t) = e^{i\pi\lambda}\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-\frac{\lambda}{3}-\frac{2}{3}}h(t)$$

This enables us to solve (2.1) for $-\frac{3}{4} < s < \frac{3}{2}$, $s \neq \frac{1}{2}$, with a compatibility condition for $\frac{1}{2} < s < \frac{3}{2}$. Our results are therefore

Theorem 1. *Suppose* $-\frac{3}{4} < s < \frac{1}{2}$. *Then we have local well-posedness of* (2.1) *for* $(\phi, f) \in H^s(\mathbb{R}_x^+) \times H^{\frac{s+1}{3}}(\mathbb{R}_t^+)$ *and local well-posedness of* (2.14) *for* $(\phi, f, g) \in H^s(\mathbb{R}_x^-) \times H^{\frac{s+1}{3}}(\mathbb{R}_t^+) \times H^{s/3}(\mathbb{R}_t^+)$. *Suppose* $\frac{1}{2} < s < \frac{3}{2}$. *Then we have local well-posedness of* (2.1) *for* $(\phi, f) \in H^s(\mathbb{R}_x^+) \times H^{\frac{s+1}{3}}(\mathbb{R}_t^+)$, *provided* $\phi(0) = f(0)$ *and local well-posedness of* (2.14) *for* $(\phi, f, g) \in H^s(\mathbb{R}_x^-) \times H^{\frac{s+1}{3}}(\mathbb{R}_t^+) \times H^{s/3}(\mathbb{R}_t^+)$, *provided* $\phi(0) = f(0)$.

The uniqueness component of "local well-posedness" is meant as uniqueness of the corresponding integral equation formulation with the auxiliary condition that $u \in X_{s,b} \cap D_\alpha$. Since there are many ways to rewrite the problems (2.1) and (2.14) as integral equations, this is a serious issue. [BSZ04] introduce the notion of a mild solution, one that can be approximated by smoother solutions, a prove uniqueness of mild solutions for the problems (2.1) and (2.14) themselves. The solutions we construct are mild solutions.

Finally, we consider the finite-length interval problem:

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & \text{in } (0,1)\times(0,T) \\ u(0,t) = g_3(t) & \text{on } (0,T) \\ u(1,t) = g_1(t) & \text{on } (0,T) \\ \partial_x u(1,t) = g_2(t) & \text{on } (0,T) \\ u(x,0) = \phi & \text{on } (0,1) \end{cases} \tag{2.24}$$

with $\phi(x) \in H^s((0,1))$, $g_1(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$, $g_2(t) \in H^{\frac{s}{3}}(\mathbb{R}^+)$, $g_3(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$. This is accomplished by making use of both operators $\mathcal{L}_-^\lambda$ and $\mathcal{L}_+^\lambda$. The equation relating the desired boundary functions to the needed "input" functions for the forcing operators is a Fredholm equation. We obtain local well-posedness for this system for $-\frac{3}{4} < s < \frac{3}{2}$, $s \neq \frac{1}{2}$, with the compatibility conditions $g_3(0) = \phi(0)$ and $g_1(0) = \phi(1)$ for $\frac{1}{2} < s < \frac{3}{2}$.

Theorem 2. (2.24) *is locally wellposeded for* $-\frac{3}{4} < s < \frac{3}{2}$, $s \neq \frac{1}{2}$, *for* $(\phi, g_3, g_1, g_2) \in H^s((0,1)) \times H^{\frac{s+1}{3}}(\mathbb{R}^+) \times H^{\frac{s+1}{3}}(\mathbb{R}^+) \times H^{\frac{s}{3}}(\mathbb{R}^+)$, *with the compatibility conditions* $g_3(0) = \phi(0)$ *and* $g_1(0) = \phi(1)$ *for* $\frac{1}{2} < s < \frac{3}{2}$.

Surveys of the literature are given in [BSZ02] [BSZ03] and [CK02]. We briefly mention some of the more recent contributions, besides [CK02]: The nonlinear problem on the line segment for $s \geq 0$ is treated in [BSZ03] and on the right half-line for $s > \frac{3}{4}$ in [BSZ02], via a Laplace transform technique. Inverse scattering techniques have been applied to the nonlinear problem on the right half-line [AKS97] [Fok02] [Hab02], and the linear problem on the line segment [FP01b]. Global results for the right half-line are obtained by [Fam01a] [Fam01b] [Fam03], in the latter paper for $s \geq 0$. The left half-line problem was considered by [MS02].

2.2 Needed lemmas from other sources

Lemma 6 ([CK02] Prop. 2.4). *If* $\frac{1}{2} < \alpha < \frac{3}{2}$, *then*

$$H_0^\alpha(\mathbb{R}^+) = \{f \in H^\alpha(\mathbb{R}^+) \mid \mathrm{Tr}(f) = 0\}$$

Lemma 7 ([JK95] Lemma 3.5). *For* $0 \leq \alpha < \frac{1}{2}$,

$$\|\chi_{(0,+\infty)} g\|_{H^\alpha(\mathbb{R})} \leq c\|g\|_{H^\alpha(\mathbb{R})}$$

Corollary 1. *For* $-\frac{1}{2} < \alpha \leq 0$,

$$\|\chi_{(0,+\infty)} g\|_{H^\alpha(\mathbb{R})} \leq c\|g\|_{H^\alpha(\mathbb{R})}$$

Proof. This follows from Lemma 7 by duality. □

Lemma 8 ([JK95] Lemma 3.7). *If* $\frac{1}{2} < \alpha < \frac{3}{2}$, *then*

$$\int_0^{+\infty} \frac{|g(x) - g(0)|^2}{x^{2\alpha}}\, dx \le c\|g\|_{H^\alpha(\mathbb{R})}^2$$

Lemma 9 ([JK95] Lemma 3.8). *If* $\frac{1}{2} < \alpha < \frac{3}{2}$, *then*

$$\|\chi_{(0,+\infty)}g\|_{H^\alpha(\mathbb{R})} \le c\left[\|g\|_{H^\alpha(\mathbb{R})} + \left(\int_0^{+\infty} \frac{|g(x)|^2}{x^{2\alpha}}\, dx\right)^{1/2}\right]$$

Lemma 10. *For* $0 \le \alpha < \frac{1}{2}$, $\theta \in C_0^\infty(\mathbb{R})$,

$$\|\theta h\|_{H^\alpha} \le c\|h\|_{\dot{H}^\alpha}$$

where $c = c(\theta)$. *For* $-\frac{1}{2} < \alpha \le 0$,

$$\|\theta h\|_{\dot{H}^\alpha} \le c\|h\|_{H^\alpha}$$

where $c = c(\theta)$.

2.3 The Riemann-Liouville fractional integral

For Re $\lambda > 0$ define, for $h \in C_0^\infty(\mathbb{R})$,

$$\mathcal{J}_\lambda h(t) = \frac{1}{\Gamma(\lambda)} \int_{-\infty}^t (t-s)^{\lambda-1} h(s)\, ds \tag{2.25}$$

By integration by parts, for Re $\lambda > 0$,

$$\mathcal{J}_\lambda h(t) = \frac{1}{\Gamma(\lambda+k)} \int_{-\infty}^t (t-s)^{\lambda+k-1} h(s)\, ds \tag{2.26}$$

Since (2.26) makes sense for Re $\lambda > -k$, we can extend J_λ to all $\lambda \in \mathbb{C}$ by selecting $k > -$Re λ, taking (2.26) as the definition, and checking that it is independent of the choice of k (see [Fri98]). When $h \in C_0^\infty(\mathbb{R}^+)$, we denote (2.25) by $\mathcal{I}_\lambda h(t)$. Thus, for $h \in C_0^\infty(\mathbb{R}^+)$ and Re $\lambda > 0$, we write

$$\mathcal{I}_\lambda h(t) = \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} h(s)\, ds$$

Remark 1. *If $h \in C_0^\infty(\mathbb{R}^+)$, then $\mathcal{I}_\lambda h(t) \in C_0^\infty(\mathbb{R}^+)$.*

Proof. By definition,

$$\mathcal{I}_\lambda h(t) = \frac{1}{\Gamma(\lambda+k)} \int_0^t (t-s)^{\lambda+k-1} h^{(k)}(s)\, ds \tag{2.27}$$

for all k such that Re $\lambda + k > 0$. Pick $n \in \mathbb{N}$ such that Re $\lambda + n > 0$. Then, for $k > n$, (2.27) shows that the first $k-n-1$ derivatives of $\mathcal{I}_\lambda h(t)$ at $t = 0$ are 0. Since k can be taken arbitarily large, $\mathcal{I}_\lambda h \in C_0^\infty(\mathbb{R}^+)$. □

Lemma 11. *Let $\alpha \in \mathbb{C}$. If $\mu_1 \in C_0^\infty(\mathbb{R})$ and $\mu_2 \in C^\infty(\mathbb{R})$ such that $\mu_2 = 1$ on a neighborhood of $(-\infty, b]$, where $b = \sup\{\, t \,|\, t \in \text{supp } \mu_1 \,\}$, then*

$$\mu_1 \mathcal{J}_\alpha \mu_2 h = \mu_1 \mathcal{J}_\alpha h \tag{2.28}$$

If $\mu_2 \in C_0^\infty(\mathbb{R})$ and $\mu_1 \in C^\infty(\mathbb{R})$ such that $\mu_1 = 1$ on a neighborhood of $[a, +\infty)$, where $a = \inf\{\, t \,|\, t \in \text{supp } \mu_2 \,\}$, then

$$\mu_1 \mathcal{J}_\alpha \mu_2 h = \mathcal{J}_\alpha \mu_2 h \tag{2.29}$$

Proof. (2.28) is clear from the integral definition if Re $\alpha > 0$. If Re $\alpha < 0$, let $k \in \mathbb{N}$ be such that $-k <$ Re $\alpha \le -k+1$ so that $\mathcal{J}_\alpha = \partial_t^k \mathcal{J}_{\alpha+k}$. Let U be an open set such that

$$\text{supp } \mu_1 \subset (-\infty, b] \subset U \subset \{\, t \,|\, \mu_2(t) = 1 \,\}$$

Then $\forall\ t \in U$, $\mathcal{J}_{\alpha+k} h = \mathcal{J}_{\alpha+k} \mu_2 h$, which implies that $\forall\ t \in (-\infty, b]$, $\partial_t^k \mathcal{J}_{\alpha+k} h = \partial_t^k \mathcal{J}_{\alpha+k} \mu_2 h$, which implies that $\forall\ t \in \mathbb{R}$, $\mu_1 \partial_t^k \mathcal{J}_{\alpha+k} h = \mu_1 \partial_t^k \mathcal{J}_{\alpha+k} \mu_2 h$. (2.29) is clear by the integral definition if Re $\alpha > 0$. If Re $\alpha < 0$, let $k \in \mathbb{N}$ be such that $-k <$

Re $\alpha \le -k+1$ so that $\mathcal{J}_\alpha = \mathcal{J}_{\alpha+k}\partial_t^k$. Since supp $\partial_t^j \mu_2 \subset [a,+\infty) \subset \{\, t \mid \mu_1(t) = 1 \,\}$, we have

$$\mu_1 \mathcal{J}_{\alpha+k}(\partial_t^j \mu_2)(\partial_t^{k-j} h) = \mathcal{J}_{\alpha+k}(\partial_t^j \mu_2)(\partial_t^{k-j} h)$$

and thus $\mu_1 \mathcal{J}_{\alpha+k}\partial_t^k \mu_2 h = \mathcal{J}_{\alpha+k}\partial_t^k \mu_2 h$. □

Lemma 12. *For $\gamma \in \mathbb{R}$, $s \in \mathbb{R}$,*

$$\|\mathcal{J}_{i\gamma} h\|_{H^s(\mathbb{R})} \le \cosh(\tfrac{1}{2}\pi\gamma)\|h\|_{H^s(\mathbb{R})} \tag{2.30}$$

Proof. We have the formula (see [Fri98], p. 110, exercise 8.7),

$$\left(\frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\right)^{\widehat{}}(\xi) = e^{-\frac{1}{2}\pi\lambda i}\xi_+^{-\lambda} + e^{+\frac{1}{2}\pi\lambda i}\xi_-^{-\lambda} \tag{2.31}$$

for $\lambda \notin \mathbb{N}$. Putting $\lambda = i\gamma$, we obtain

$$\left(\frac{x_+^{i\gamma-1}}{\Gamma(i\gamma)}\right)^{\widehat{}}(\xi) = e^{\frac{1}{2}\pi\gamma}\xi_+^{-i\gamma} + e^{-\frac{1}{2}\pi\gamma}\xi_-^{-i\gamma}$$

This gives the pointwise bound

$$\left|\left(\frac{x_+^{i\gamma-1}}{\Gamma(i\gamma)}\right)^{\widehat{}}(\xi)\right| \le 2\cosh(\tfrac{1}{2}\pi\gamma)$$

from which (2.30) is immediate. □

Lemma 13. *If $0 \le \text{Re }\alpha < +\infty$ and $s \in \mathbb{R}$, then*

$$\|\mathcal{I}_{-\alpha} h\|_{H_0^s(\mathbb{R}^+)} \le c e^{\frac{1}{2}\text{Im }\alpha}\|h\|_{H_0^{s+\alpha}(\mathbb{R}^+)} \tag{2.32}$$

$$\|\mathcal{J}_{-\alpha} h\|_{H^s(\mathbb{R})} \le c e^{\frac{1}{2}\text{Im }\alpha}\|h\|_{H^{s+\alpha}(\mathbb{R})} \tag{2.33}$$

Proof. (2.33) is immediate from (2.31). (2.32) then follows from (2.33) by Remark 1 and a density argument. □

Lemma 14. *If* $0 \le \operatorname{Re} \alpha < +\infty$*,* $s \in \mathbb{R}$*,* $\mu, \mu_2 \in C_0^\infty(\mathbb{R})$

$$\|\mu \mathcal{I}_\alpha h\|_{H_0^s(\mathbb{R}^+)} \le c e^{\frac{1}{2}\operatorname{Im} \alpha} \|h\|_{H_0^{s-\alpha}(\mathbb{R}^+)} \qquad c = c(\mu) \tag{2.34}$$

$$\|\mu \mathcal{J}_\alpha \mu_2 h\|_{H^s(\mathbb{R})} \le c e^{\frac{1}{2}\operatorname{Im} \alpha} \|h\|_{H^{s-\alpha}(\mathbb{R})} \qquad c = c(\mu, \mu_2) \tag{2.35}$$

where $c = c(\mu, \mu_2)$*.*

Proof. We first explain how (2.34) follows from (2.35). Given μ, let $b = \sup\{\, t \,|\, t \in \operatorname{supp} \mu \,\}$. Take $\mu_2 \in C_0^\infty(\mathbb{R})$, $\mu_2 = 1$ on $[0, b]$. Then, when restricting to $h \in C_0^\infty(\mathbb{R}^+)$, we have $\mu \mathcal{I}_\alpha h = \mu \mathcal{J}_\alpha \mu_2 h$. By Remark 1 and a density argument, we obtain (2.34). Now we prove (2.35). We first need the special case $s = 0$.
Claim. If $k \in \mathbb{Z}_{\ge 0}$, then $\|\mu \mathcal{J}_k \mu_2 h\|_{L^2(\mathbb{R})} \le c \|h\|_{H^{-k}(\mathbb{R})}$, where $c = c(\mu, \mu_2)$.
Proof of claim. If $k \in \mathbb{N}$, then ($g \in C_0^\infty(\mathbb{R})$, $\|g\|_{L^2} \le 1$)

$$\begin{aligned}
\|\mu \mathcal{J}_k \mu_2 h\|_{L^2} &= \frac{1}{\Gamma(k)} \sup_g \int_t \mu(t) \int_{s=-\infty}^{t} (t-s)^{k-1} \mu_2(s) h(s)\, ds\, g(t)\, dt \\
&= \frac{1}{\Gamma(k)} \sup_g \int_s h(s)\, \mu_2(s) \int_{t=s}^{+\infty} \mu(t)(t-s)^{k-1} g(t)\, dt\, ds \\
&\le \frac{1}{\Gamma(k)} \|h\|_{H^{-k}} \left\| \mu_2(s) \int_{t=s}^{+\infty} \mu(t)(t-s)^{k-1} g(t)\, dt \right\|_{H^k(ds)} \\
&\le c \|h\|_{H^{-k}} \|g\|_{L^2}
\end{aligned}$$

The case $k = 0$ is trivial. *End proof of claim.*
To prove (2.35), we first take $\alpha = k \in \mathbb{Z}_{\ge 0}$, $s = m \in \mathbb{Z}$, $h \in C_0^\infty(\mathbb{R})$.
Case 1. $m \ge 0$.

$$\begin{aligned}
\|\mu \mathcal{J}_k \mu_2 h\|_{H^m} &\le \|\mu \mathcal{J}_k \mu_2 h\|_{L^2} + \sum_{j=0}^{m} \|\mu^{(j)} \mathcal{J}_{k-m+j} \mu_2 h\|_{L^2} \\
&\le c(\|h\|_{H^{-k}} + \sum_{j=0}^{m} \|h\|_{H^{m-k-j}}) \le c\|h\|_{H^{m-k}}
\end{aligned}$$

by appealing to the claim or Lemma 13.

Case 2. $m < 0$. Let $\mu_3 = 1$ on supp μ, $\mu_3 \in C_0^\infty(\mathbb{R}^+)$.

$$\mu \mathcal{J}_k \mu_2 h = \mu \left(\frac{d}{dt}\right)^{-m} \mathcal{J}_{k-m} \mu_2 h = \mu \left(\frac{d}{dt}\right)^{-m} \mu_3 \mathcal{J}_{k-m} \mu_2 h$$

and therefore

$$\|\mu \mathcal{J}_k \mu_2 h\|_{H^m} \leq \|\mu_3 \mathcal{J}_{k-m} \mu_2 h\|_{L^2}$$

and we conclude by applying the claim.

Next, we extend to $\alpha = k + i\gamma$ for $k, \gamma \in \mathbb{R}$, as follows. Let $\mu_3 = 1$ on a neighborhood of $(-\infty, b]$, where $b = \sup\{\, t \mid t \in \text{supp}\, \mu \,\}$, and let $\mu_4 = 1$ on a neighborhood of $[a, +\infty)$, where $a = \inf\{\, t \mid t \in \text{supp}\, \mu_2 \,\}$, so that $\mu_3 \mu_4 \in C_0^\infty(\mathbb{R})$. By Lemma 11,

$$\mu \mathcal{J}_{k+i\gamma} \mu_2 h = \mu \mathcal{J}_{i\gamma} \mu_3 \mu_4 \mathcal{J}_k \mu_2 h$$

By Lemma 12,

$$\|\mu \mathcal{J}_{k+i\gamma} \mu_2 h\|_{H^m} \leq c \cosh(\tfrac{1}{2}\pi\gamma) \|\mu_3 \mu_4 \mathcal{J}_k \mu_2 h\|_{H^m}$$

which is bounded as above. We can now apply interpolation to complete the proof. □

2.4 The Duhamel forcing operator class

In this section, we rigorously define the classes of operators $\mathcal{L}_-^\lambda$ and $\mathcal{L}_+^\lambda$ for $-2 \leq \operatorname{Re} \lambda \leq 1$ and deduce their properties. For $h \in C_0^\infty(\mathbb{R}^+)$, let

$$\mathcal{U}h(x,t) = \int_0^t S(t-t')\delta_0(x) h(t')\, dt' \tag{2.36}$$

$$= \int_0^t A\left(\frac{x}{(t-t')^{1/3}}\right) \frac{h(t')}{(t-t')^{1/3}}\, dt' \tag{2.37}$$

Fact 1. *Let $h \in C_0^\infty(\mathbb{R}^+)$. Then for fixed $t \geq 0$, $\mathcal{U}h(x,t)$ and $\partial_x \mathcal{U}h(x,t)$ are contin-*

uous in x for all $x \in \mathbb{R}$. Also, $\mathcal{U}h(x,t)$ and $\partial_x \mathcal{U}h(x,t)$ satisfy the decay bounds

$$\begin{aligned} |\mathcal{U}h(x,t)| &\leq c_k \langle t \rangle^{k+1} \|h\|_{H^k} \langle x \rangle^{-k} \quad \forall\, k \geq 0 \\ |\partial_x \mathcal{U}h(x,t)| &\leq c_k \langle t \rangle^{k+1} \|h\|_{H^k} \langle x \rangle^{-k} \quad \forall\, k \geq 0 \end{aligned} \tag{2.38}$$

For fixed t, $\partial_x^2 \mathcal{U}h(x,t)$ is continuous in x for $x \neq 0$ and has a step discontinuity of size $h(t)$ at $x = 0$. Also, $\partial_x^2 \mathcal{U}h(x,t)$ satisfies the decay bounds

$$|\partial_x^2 \mathcal{U}h(x,t)| \leq c_k \langle t \rangle^{k+2} \|h\|_{H^{k+2}} \langle x \rangle^{-k} \qquad \forall\, k \geq 0 \tag{2.39}$$

Proof. To establish (2.38), it suffices to show that $\langle \xi \rangle |\partial_\xi^k \widehat{\mathcal{U}h}(\xi,t)| \in L^1_\xi$, $\forall\, k \geq 0$. Let

$$\phi(\xi,t) = \int_0^t e^{i(t-t')\xi} h(t')\, dt'$$

We have

$$\partial_\xi^k \phi(\xi,t) = i^k \int_0^t (t-t')^k e^{i(t-t')\xi} h(t')\, dt' \tag{2.40}$$

By integration by parts,

$$\begin{aligned} \partial_\xi^k \phi(\xi,t) = {} & \frac{i(-1)^{k+1} k!}{\xi^{k+1}} \int_0^t e^{i(t-t')\xi} \partial_{t'} h(t')\, dt' + \frac{i(-1)^k k!}{\xi^{k+1}} h(t) \\ & + \frac{i(-1)^{k+1}}{\xi^{k+1}} \int_0^t e^{i(t-t')\xi} \partial_{t'} \sum_{\substack{\alpha+\beta=k \\ \alpha \leq k-1}} c_{\alpha,\beta} \partial_{t'}^\alpha (t-t')^k \partial_{t'}^\beta h(t')\, dt' \end{aligned} \tag{2.41}$$

By (2.40) and (2.41), $|\partial_\xi^k \phi(\xi,t)| \leq c_k \langle t \rangle^{k+1} \|h\|_{H^k} \langle \xi \rangle^{-k-1}$. Since $\widehat{\mathcal{U}h}(\xi,t) = \phi(\xi^3,t)$, we have $|\partial_\xi^k \widehat{\mathcal{U}h}(\xi,t)| \leq c_k \langle t \rangle^{k+1} \|h\|_{H^k} \langle \xi \rangle^{-k-3}$. By integration by parts in (2.36),

$$\partial_x^3 \mathcal{U}h(x,t) = \delta_0(x) h(t) - \mathcal{U}(\partial_t h)(x,t) \tag{2.42}$$

To see that $\partial_x^2 \mathcal{U}h(x,t) \to 0$ as $x \to \pm\infty$, we first note that for $x < -1$,

$$\partial_x^2 \mathcal{U}h(x,t) = \partial_x^2 \mathcal{U}h(-1,t) - \int_x^{-1} \partial_y^3 \mathcal{U}h(y,t)\, dy$$

By (2.42) and (2.38), we can send $x \to -\infty$ and obtain that $\partial_x^2 \mathcal{U}h(x,t) \to c$, for some constant c, as $x \to -\infty$. Since

$$\partial_x \mathcal{U}h(0,t) = \int_{-\infty}^{0} \partial_x^2 \mathcal{U}h(y,t)\, dy$$

we must have $c = 0$. We can similarly show that $\partial_x^2 \mathcal{U}h(x,t) \to 0$ as $x \to +\infty$. For $x < 0$, use

$$\partial_x^2 \mathcal{U}h(x,t) = \int_{-\infty}^{x} \partial_y^3 \mathcal{U}h(y,t)\, dy$$

and for $x > 0$, use

$$\partial_x^2 \mathcal{U}h(x,t) = -\int_{x}^{+\infty} \partial_y^3 \mathcal{U}h(y,t)\, dy$$

together with (2.38) and (2.42) to obtain the bound (2.39). □

Define, for Re $\lambda > 0$,

$$\mathcal{U}_-^{\lambda} h(x,t) = \left[\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} * \mathcal{U}h(-,t) \right](x) \tag{2.43}$$

and, with $\dfrac{x_-^{\lambda-1}}{\Gamma(\lambda)} = \dfrac{(-x)_+^{\lambda-1}}{\Gamma(\lambda)}$, define

$$\mathcal{U}_+^{\lambda} h(x,t) = e^{i\pi\lambda} \left[\frac{x_-^{\lambda-1}}{\Gamma(\lambda)} * \mathcal{U}h(-,t) \right](x) \tag{2.44}$$

By integration by parts in (2.43), the decay bounds provided by Fact 1, and (2.42),

$$\begin{aligned} \mathcal{U}_-^{\lambda} h(x,t) &= \left[\frac{x_+^{(\lambda+3)-1}}{\Gamma(\lambda+3)} * \partial_x^3 \mathcal{U}h(-,t) \right](x) \\ &= \frac{x_+^{(\lambda+3)-1}}{\Gamma(\lambda+3)} h(t) - \int_{-\infty}^{x} \frac{(x-y)^{(\lambda+3)-1}}{\Gamma(\lambda+3)} \mathcal{U}(\partial_t h)(y,t)\, dy \end{aligned} \tag{2.45}$$

For Re $\lambda > -3$, we may thus take (2.45) as the definition for $\mathcal{U}_-^{\lambda}(h)$. By integration by parts in (2.44), the decay bounds provided by Fact 1, and (2.42),

$$\begin{aligned}\mathcal{U}_+^\lambda h(x,t) &= -e^{i\pi\lambda}\left[\frac{x_-^{(\lambda+3)-1}}{\Gamma(\lambda+3)} * \partial_x^3 \mathcal{U}h(-,t)\right](x) \\ &= -e^{i\pi\lambda}\frac{x_-^{(\lambda+3)-1}}{\Gamma(\lambda+3)}h(t) + e^{i\pi\lambda}\int_{-\infty}^x \frac{(-x+y)^{(\lambda+3)-1}}{\Gamma(\lambda+3)}\mathcal{U}(\partial_t h)(y,t)\,dy \quad (2.46)\end{aligned}$$

For Re $\lambda > -3$, we may thus take (2.46) as the definition for $\mathcal{U}_-^\lambda(h)$.

Fact 2. *Let $h \in C_0^\infty(\mathbb{R}^+)$, and fix $t \geq 0$. We have*

$$\mathcal{U}_\pm^{-2}h = \partial_x^2\mathcal{U}h, \qquad \mathcal{U}_\pm^{-1}h = \partial_x\mathcal{U}h, \qquad \mathcal{U}_\pm^0 h = \mathcal{U}h$$

Also, $\mathcal{U}_\pm^{-2}(h)(x,t)$ has a step discontinuity of size $h(t)$ at $x = 0$, otherwise for $x \neq 0$, $\mathcal{U}_\pm^{-2}(h)(x,t)$ is continuous in x. For $\lambda > -2$, $\mathcal{U}_\pm^\lambda h(x,t)$ is continuous in x for all $x \in \mathbb{R}$. For $-2 \leq \lambda \leq 1$, $\mathcal{U}_-^\lambda(h)(x,t)$ satisfies the decay bounds

$$\begin{aligned}|\mathcal{U}_-^\lambda(h)(x,t)| &\leq c_{k,\lambda,h,t}\langle x\rangle^{-k} && \forall\, x \leq 0, \quad \forall\, k \geq 0 \\ |\mathcal{U}_-^\lambda(h)(x,t)| &\leq c_{\lambda,h,t}\langle x\rangle^{\lambda-1} && \forall\, x \geq 0\end{aligned}$$

For $-2 \leq \lambda \leq 1$, $\mathcal{U}_+^\lambda(h)(x,t)$ satisfies the decay bounds

$$\begin{aligned}|\mathcal{U}_+^\lambda(h)(x,t)| &\leq c_{k,\lambda,h,t}\langle x\rangle^{-k} && \forall\, x \geq 0, \quad \forall\, k \geq 0 \\ |\mathcal{U}_+^\lambda(h)(x,t)| &\leq c_{\lambda,h,t}\langle x\rangle^{\lambda-1} && \forall\, x \leq 0\end{aligned}$$

Proof. We only prove the bounds for $\mathcal{U}_-(h)$, since the corresponding results for $\mathcal{U}_+(h)$ are obtained similarly. Assume $x \geq 2$. Let $\psi(y) = 1$ for $y \leq \frac{1}{4}$ and $\psi(y) = 0$ for

$y \geq \frac{3}{4}$. Then

$$\begin{aligned}
\mathcal{U}_-^\lambda h(x,t) &= \frac{x_+^{(\lambda+3)-1}}{\Gamma(\lambda+3)} * \partial_x^3 \mathcal{U}h(-,t) \\
&= \int_{-\infty}^x \frac{(x-y)^{\lambda+2}}{\Gamma(\lambda+3)} \psi\left(\frac{y}{x}\right) \partial_y^3 \mathcal{U}h(y,t)\,dy \\
&\quad + \int_{-\infty}^x \frac{(x-y)^{\lambda+2}}{\Gamma(\lambda+3)} \left[1-\psi\left(\frac{y}{x}\right)\right] \partial_y^3 \mathcal{U}(h)(y,t)\,dy \\
&= \mathrm{I} + \mathrm{II}
\end{aligned}$$

In I, $y \leq \frac{3}{4}x$, integrate by parts,

$$\begin{aligned}
\mathrm{I} &= -\int_{-\infty}^x \partial_y^3 \left[\frac{(x-y)^{\lambda+2}}{\Gamma(\lambda+3)} \psi\left(\frac{y}{x}\right)\right] \mathcal{U}h(y,t)\,dy \\
&= \int_{-\infty}^x \frac{(x-y)^{\lambda-1}}{\Gamma(\lambda)} \psi\left(\frac{y}{x}\right) \mathcal{U}h(y,t)\,dy \\
&\quad + \sum_{j=1}^3 \int_{-\infty}^x \frac{(-1)^j (x-y)^{\lambda+j-1}}{\Gamma(\lambda+j)} \frac{1}{x^j} \psi^{(j)}\left(\frac{y}{x}\right) \mathcal{U}h(y,t)\,dy
\end{aligned}$$

In the first of these terms, since $y \leq \frac{3}{4}x$, $(x-y)^{\lambda-1} \leq (\frac{1}{4})^{\lambda-1} x^{\lambda-1}$. In the second term, $\frac{1}{4}x \leq y \leq \frac{3}{4}x$, and thus we can use the decay of $\mathcal{U}h(y)$. In II, $y \geq \frac{1}{4}x$, apply (2.42),

$$\begin{aligned}
\mathrm{II} &= \int_{-\infty}^x \frac{(x-y)^{\lambda+2}}{\Gamma(\lambda+3)} \left[1-\psi\left(\frac{y}{x}\right)\right] (\delta_0(y)h(t) - \mathcal{U}(\partial_t h)(y,t))\,dy \\
&= -\int_{-\infty}^x \frac{(x-y)^{\lambda+2}}{\Gamma(\lambda+3)} \left[1-\psi\left(\frac{y}{x}\right)\right] \mathcal{U}(\partial_t h)(y,t)\,dy
\end{aligned}$$

Since $y \geq \frac{1}{4}x$, we have by Fact 1, $|\mathcal{U}(\partial_t h)(y,t)| \leq c_k \langle t\rangle^{2k+1} \|h\|_{H^{2k+1}} \langle x\rangle^{-k} \langle y\rangle^{-k}$, which establishes the bound. $\square$

Fact 3. *For fixed t, the function $f_-(t,\cdot) : \mathbb{C} \to \mathbb{C}$ given by*

$$f_-(t,\lambda) = \begin{cases} \lim\limits_{x\to 0^-} \mathcal{U}_-^\lambda h(x,t) & -3 < \text{Re }\lambda \le -2 \\ \mathcal{U}_-^\lambda h(0,t) & -2 < \text{Re }\lambda \end{cases}$$

is analytic in λ for Re $\lambda > -3$*, and moreover,*

$$f_-(t,\lambda) = \tfrac{2}{3}\sin(\tfrac{\pi}{3}\lambda + \tfrac{\pi}{6})\mathcal{I}_{\frac{\lambda}{3}+\frac{2}{3}} h(t) \tag{2.47}$$

For fixed t, the function $f_+(t,\cdot) : \mathbb{C} \to \mathbb{C}$ given by

$$f_+(t,\lambda) = \begin{cases} \lim\limits_{x\to 0^+} \mathcal{U}_+^\lambda h(x,t) & -3 < \text{Re }\lambda \le -2 \\ \mathcal{U}_+^\lambda h(0,t) & -2 < \text{Re }\lambda \end{cases}$$

is analytic in λ for Re $\lambda > -3$*, and moreover,*

$$f_+(t,\lambda) = \tfrac{1}{3}e^{i\pi\lambda}\mathcal{I}_{\frac{\lambda}{3}+\frac{2}{3}} h(t) \tag{2.48}$$

In order to prove this, we need to compute the Mellin transform of each side of the Airy function.

Lemma 15 (Mellin transform of left side of Airy function). *If* $0 <$ Re $\lambda < \frac{1}{4}$*, then*

$$\int_0^{+\infty} x^{\lambda-1} A(-x)\, dx = \tfrac{1}{3\pi}\Gamma(\lambda)\Gamma(-\tfrac{1}{3}\lambda + \tfrac{1}{3})\cos(\tfrac{2\pi}{3}\lambda - \tfrac{\pi}{6}) \tag{2.49}$$

Proof. Owing to the decay of the Airy function $A(-x) \le c\langle x\rangle^{-1/4}$ for $x \ge 0$, the given expression is defined as an absolutely convergent integral. In the calculation, we assume that λ is real and $0 < \lambda < \frac{1}{4}$, and by analyticity, this suffices to establish (2.49). Let

$$A_{1,\epsilon}(x) = \tfrac{1}{2\pi}\int_{\xi=0}^{+\infty} e^{ix\xi}e^{i\xi^3}e^{-\epsilon\xi}\, d\xi \qquad \forall\ \epsilon > 0$$

The van der Corput lemma (as stated, for example, on p. 334 of [Ste93]) yields the bound $|A_{1,\epsilon}(-x)| \le c\langle x\rangle^{-1/4}$ uniformly in $\epsilon > 0$. By dominated convergence,

$$\int_0^{+\infty} x^{\lambda-1} A_{1,0}(-x)\,dx = \lim_{\delta\to 0^+}\lim_{\epsilon\to 0^+}\int_0^{+\infty} x^{\lambda-1}e^{-\delta x}A_{1,\epsilon}(-x)\,dx \tag{2.50}$$

We shall compute

$$\int_0^{+\infty} x^{\lambda-1}e^{-\delta x}A_{1,\epsilon}(-x)\,dx \tag{2.51}$$

We have, by Fubini,

$$(2.51) = \tfrac{1}{2\pi}\int_{\xi=0}^{+\infty} e^{i\xi^3}e^{-\epsilon\xi}\int_{x=0}^{+\infty} x^{\lambda-1}e^{-\delta x}e^{-ix\xi}\,dx\,d\xi \tag{2.52}$$

By changing variable $x \to x\xi^{-1}$ and shifting contour to

$$\gamma:\ re^{i\theta}, \quad \theta = -\tfrac{\pi}{2}, \quad 0 \le r < +\infty$$

we obtain

$$\int_{x=0}^{+\infty} x^{\lambda-1}e^{-\delta x}e^{-ix\xi}\,dx = \xi^{-\lambda}e^{-i\lambda\frac{\pi}{2}}\int_{r=0}^{+\infty} r^{\lambda-1}e^{\frac{\delta}{\xi}ir}e^{-r}\,dr = \xi^{-\lambda}e^{-i\lambda\frac{\pi}{2}}\Gamma(\lambda,\delta\xi^{-1})$$

where we have defined

$$\Gamma(\lambda,z) = \int_{r=0}^{+\infty} r^{\lambda-1}e^{irz}e^{-r}\,dr$$

This is an absolutely convergent integral for $\xi \in \mathbb{C}$ provided Im $z > -1$ (when $z \neq \mathbb{R}$ we will just require $|z| < 1$).

$$(2.52) = \tfrac{1}{2\pi}e^{-i\lambda\frac{\pi}{2}}\int_{\xi=0}^{+\infty} e^{i\xi^3}e^{-\epsilon\xi}\xi^{-\lambda}\Gamma(\lambda,\delta\xi^{-1})\,d\xi \tag{2.53}$$

Shift contour to:

$$\begin{aligned}\gamma:\ & re^{i\theta}, \quad \theta = 0, \quad 0 \le r \le 2\delta^3 \\ & re^{i\theta}, \quad r = 2\delta^3, \quad 0 \le \theta \le \tfrac{\pi}{2} \\ & re^{i\theta}, \quad \theta = \tfrac{\pi}{2}, \quad 2\delta^3 \le r < +\infty\end{aligned}$$

$$\begin{aligned}(2.53) = {} & e^{-i\lambda\frac{\pi}{2}}\tfrac{1}{6\pi}\int_0^{8\delta^3} e^{i\eta}e^{-\epsilon\eta^{1/3}}\Gamma(\lambda,\delta\eta^{-1/3})\eta^{-\frac{2+\lambda}{3}}\,d\eta \\ & + e^{-i\lambda\frac{2\pi}{3}}e^{i\frac{\pi}{6}}\tfrac{1}{6\pi}\int_{r=8\delta^3}^{+\infty} e^{-r}e^{-\frac{\sqrt{3}}{2}\epsilon r^{1/3}}e^{-\frac{1}{2}i\epsilon r^{1/3}}\Gamma(\lambda,\delta r^{-1/3}e^{-i\frac{\pi}{6}})r^{-\frac{2+\lambda}{3}}\,dr \\ & + \tfrac{1}{6\pi}ie^{-i\lambda\frac{\pi}{2}}2^{1-\lambda}\delta^{1-\lambda}\int_{\rho=0}^{\frac{\pi}{2}} e^{-8\delta^3\sin\rho}e^{-2\epsilon\delta\cos\frac{\rho}{3}}e^{i8\delta^3\cos\rho}e^{-2i\epsilon\delta\sin\frac{\rho}{3}} \\ & \qquad \times \Gamma(\lambda,\tfrac{1}{2}e^{-i\rho/3})e^{i(\frac{1}{3}-\frac{\lambda}{3})\rho}\,d\rho\end{aligned}$$

Send $\epsilon \to 0^+$, by dominated convergence

$$\begin{aligned}\lim_{\epsilon\to 0^+}(2.51) = {} & e^{-i\lambda\frac{\pi}{2}}\tfrac{1}{6\pi}\int_0^{8\delta^3} e^{i\eta}\Gamma(\lambda,\delta\eta^{-1/3}),\eta^{-\frac{\lambda+2}{3}}\,d\eta \\ & + \tfrac{1}{6\pi}e^{i\frac{\pi}{6}}e^{-i\lambda\frac{2\pi}{3}}\int_{8\delta^3}^{+\infty} e^{-r}\Gamma(\lambda,\delta r^{-1/3}e^{-i\frac{\pi}{6}})r^{-\frac{2+\lambda}{3}}\,dr \\ & + i\tfrac{1}{6\pi}e^{-i\lambda\frac{\pi}{2}}2^{1-\lambda}\delta^{1-\lambda}\int_{\theta=0}^{\pi/2} e^{-8\delta^3\sin\theta}e^{i8\delta^3\cos\theta} \\ & \qquad \times \Gamma(\lambda,\tfrac{1}{2}e^{-i\theta/3})e^{i(\frac{1}{3}-\frac{\lambda}{3})\theta}\,d\theta\end{aligned}$$

Send $\delta \to 0^+$, and by dominated convergence, the first and third term $\to 0$ and

$$\lim_{\delta\to 0^+}\lim_{\epsilon\to 0^+}(2.51) = \tfrac{1}{6\pi}e^{-i\lambda\frac{2\pi}{3}}e^{i\frac{\pi}{6}}\Gamma(\lambda)\Gamma(\tfrac{1}{3}-\tfrac{\lambda}{3})$$

and hence, by (2.50),

$$\int_0^{+\infty} x^{\lambda-1}e^{-\delta x}A_{1,0}(-x)\,dx = \tfrac{1}{6\pi}e^{-i\lambda\frac{2\pi}{3}}e^{i\frac{\pi}{6}}\Gamma(\lambda)\Gamma(\tfrac{1}{3}-\tfrac{\lambda}{3}) \tag{2.54}$$

Similarly, if we define

$$A_{2,\epsilon}(x) = \tfrac{1}{2\pi} \int_{\xi=-\infty}^{0} e^{ix\xi} e^{i\xi^3} e^{\epsilon\xi}\, d\xi$$

we have, by Fubini,

$$\int_0^{+\infty} x^{\lambda-1} A_{2,0}(-x)\, dx = \lim_{\delta\to 0^+} \lim_{\epsilon\to 0^+} \int_0^{+\infty} x^{\lambda-1} e^{-\delta x} A_{2,\epsilon}(-x)\, dx$$

By a similar computation,

$$\lim_{\delta\to 0^+} \lim_{\epsilon\to 0^+} \int_0^{+\infty} x^{\lambda-1} e^{-\delta x} A_{2,\epsilon}(-x)\, dx = \tfrac{1}{6\pi} e^{i\lambda\frac{2\pi}{3}} e^{i\frac{\pi}{6}} \Gamma(\lambda)\Gamma(\tfrac{1}{3} - \tfrac{\lambda}{3})$$

and hence

$$\int_0^{+\infty} x^{\lambda-1} A_{2,0}(-x)\, dx = \tfrac{1}{6\pi} e^{i\lambda\frac{2\pi}{3}} e^{-i\frac{\pi}{6}} \Gamma(\lambda)\Gamma(\tfrac{1}{3} - \tfrac{\lambda}{3}) \tag{2.55}$$

Adding (2.54) and (2.55), we obtain (2.49). □

Lemma 16 (Mellin transform of right side of Airy function). *If* Re $\lambda > 0$, *then*

$$\int_0^{+\infty} x^{\lambda-1} A(x)\, dx = \tfrac{1}{3\pi} \Gamma(\lambda)\Gamma(\tfrac{1}{3} - \tfrac{1}{3}\lambda) \cos(\tfrac{\pi}{3}\lambda + \tfrac{\pi}{6}) \tag{2.56}$$

Note that although $\Gamma(\frac{1}{3} - \frac{1}{3}\lambda)$ *has poles at* $\lambda = 1, 4, 7, \cdots$, $\cos(\frac{\pi}{3}\lambda + \frac{\pi}{6})$ *vanishes at these positions.*

Proof. In order to make the calculation that follows rigorous, we need to insert appropriate decay factors as we did in the proof of Lemma 15. Since (2.56) is analytic in λ, it suffices to establish (2.56) for λ real, $0 < \lambda < 1$. Let

$$A_1(x) = \tfrac{1}{2\pi} \int_{\xi=0}^{+\infty} e^{ix\xi} e^{i\xi^3}\, d\xi \tag{2.57}$$

$$A_2(x) = \tfrac{1}{2\pi} \int_{\xi=-\infty}^{0} e^{ix\xi} e^{i\xi^3}\, d\xi \tag{2.58}$$

By integration by parts in (2.57) and (2.58), $|A_1(x)| \le c_k \langle x\rangle^{-k}$, $\forall\, k$, and $|A_2(x)| \le$

$c_k\langle x\rangle^{-k}$, $\forall\, k$. By Fubini,

$$\int_{x=0}^{+\infty} x^{\lambda-1} A_1(x)\, dx = \tfrac{1}{2\pi}\int_{\xi=0}^{+\infty} e^{i\xi^3}\left[\int_{x=0}^{+\infty} x^{\lambda-1} e^{ix\xi}\, dx\right] d\xi \tag{2.59}$$

Changing contour from

$$\gamma_1 : \quad re^{i\theta}, \quad \theta = 0, \quad 0 \leq r < +\infty$$

to

$$\gamma_2 : \quad re^{i\theta}, \quad \theta = \tfrac{\pi}{2}, \quad 0 \leq r < +\infty$$

in the integral

$$\int_\gamma z^{\lambda-1} e^{iz\xi}\, dz$$

we obtain, for $\xi > 0$, that

$$\int_{x=0}^{+\infty} x^{\lambda-1} e^{ix\xi}\, dx = e^{i\frac{\pi}{2}\lambda}\xi^{-\lambda}\Gamma(\lambda) \tag{2.60}$$

Also, by Fubini,

$$\int_{x=0}^{+\infty} x^{\lambda-1} A_2(x)\, dx = \tfrac{1}{2\pi}\int_{\xi=-\infty}^{0} e^{i\xi^3}\left[\int_{x=0}^{+\infty} x^{\lambda-1} e^{ix\xi}\, dx\right] d\xi \tag{2.61}$$

Changing contour from

$$\gamma_1 : \quad re^{i\theta}, \quad \theta = 0, \quad 0 \leq r < +\infty$$

to

$$\gamma_2 : \quad re^{i\theta}, \quad \theta = -\tfrac{\pi}{2}, \quad 0 \leq r < -\infty$$

in the integral

$$\int_\gamma z^{\lambda-1} e^{iz\xi}\, dz$$

we obtain, for $\xi < 0$, that

$$\int_{x=0}^{+\infty} x^{\lambda-1} e^{ix\xi}\, dx = e^{-i\frac{\pi}{2}\lambda}(-\xi)^{-\lambda}\Gamma(\lambda) \tag{2.62}$$

By (2.59) and (2.60),

$$\begin{aligned}\int_{x=0}^{+\infty} x^{\lambda-1} A_1(x)\, dx &= \tfrac{1}{2\pi}\Gamma(\lambda)e^{i\frac{\pi}{2}\lambda}\int_{\xi=0}^{+\infty} e^{i\xi^3}\xi^{-\lambda}\, d\xi \\ &= \tfrac{1}{6\pi}\Gamma(\lambda)e^{i\frac{\pi}{2}\lambda}\int_{\eta=0}^{+\infty} e^{i\eta}\eta^{-\frac{\lambda}{3}-\frac{2}{3}}\, d\eta \end{aligned} \tag{2.63}$$

Change contour from

$$\gamma_1: \quad re^{i\theta}, \quad \theta = 0, \quad 0 \le r < +\infty$$

to

$$\gamma_2: \quad re^{i\theta}, \quad \theta = \tfrac{\pi}{2}, \quad 0 \le r < +\infty$$

in the integral

$$\int_\gamma e^{iz} z^{-\frac{\lambda}{3}-\frac{2}{3}}\, dz$$

we get

$$\int_{\eta=0}^{+\infty} e^{i\eta}\eta^{-\frac{\lambda}{3}-\frac{2}{3}}\, d\eta = e^{i\frac{\pi}{2}(-\frac{\lambda}{3}+\frac{1}{3})}\Gamma(\tfrac{1}{3} - \tfrac{1}{3}\lambda) \tag{2.64}$$

and thus, by (2.63) and (2.64),

$$\int_{x=0}^{+\infty} x^{\lambda-1} A_1(x)\, dx = \tfrac{1}{6\pi} e^{i\frac{\pi}{3}\lambda+\frac{\pi}{6}}\Gamma(\lambda)\Gamma(\tfrac{1}{3} - \tfrac{1}{3}\lambda) \tag{2.65}$$

A similar calculation with (2.61) shows

$$\int_{x=0}^{+\infty} x^{\lambda-1} A_2(x)\, dx = \tfrac{1}{6\pi} e^{-i\frac{\pi}{3}\lambda-\frac{\pi}{6}}\Gamma(\lambda)\Gamma(\tfrac{1}{3} - \tfrac{1}{3}\lambda) \tag{2.66}$$

Adding (2.65) and (2.66) gives (2.56). □

Proof of Fact 3. From (2.45),

$$f_-(t,\lambda) = \int_{-\infty}^{0} \frac{(-y)^{\lambda+2}}{\Gamma(\lambda+3)} \mathcal{U}(\partial_t h)(y,t)\, dy \tag{2.67}$$

and from (2.46),

$$f_+(t,\lambda) = e^{i\pi\lambda} \int_{0}^{+\infty} \frac{y^{\lambda+2}}{\Gamma(\lambda+3)} \mathcal{U}(\partial_t h)(y,t)\, dy \tag{2.68}$$

The right-hand side is convergent for Re $\lambda > -3$ by Fact 1. By complex differentiation under the integral sign, (2.67) demonstrates that $f_-(t,\lambda)$ is analytic in λ for Re $\lambda > -3$. We shall only compute (2.47) for $0 < \lambda < \frac{1}{4}$, λ real. By analyticity, the result will extend to the full range Re $\lambda > -3$. For the computation in the range $0 < \lambda < \frac{1}{4}$, we use the representation (2.43) in place of (2.67) to give

$$\mathcal{U}_-^{\lambda} h(0,t) = \int_{y=-\infty}^{0} \frac{(-y)^{\lambda-1}}{\Gamma(\lambda)} \mathcal{U}h(y,t)\, dy$$

By definition,

$$\mathcal{U}h(y,t) = \int_0^t A\left(\frac{y}{(t-t')^{1/3}}\right) \frac{h(t')}{(t-t')^{1/3}}\, dt'$$

By the decay for $A(-y)$, $y \geq 0$, we can apply Fubini and (2.49) to obtain

$$\begin{aligned} f_t(\lambda) &= \tfrac{1}{3\pi}\Gamma\left(-\tfrac{1}{3}\lambda + \tfrac{1}{3}\right) \cos\left(\tfrac{2\pi}{3}\lambda - \tfrac{\pi}{6}\right) \int_{t'=0}^{t} (t-t')^{(\frac{1}{3}\lambda+\frac{2}{3})-1} h(t')\, dt' \\ &= \tfrac{1}{3\pi}\Gamma\left(-\tfrac{1}{3}\lambda + \tfrac{1}{3}\right) \Gamma\left(\tfrac{1}{3}\lambda + \tfrac{2}{3}\right) \cos\left(\tfrac{2\pi}{3}\lambda - \tfrac{\pi}{6}\right) \mathcal{I}_{\frac{1}{3}\lambda+\frac{2}{3}}(h)(t) \end{aligned} \tag{2.69}$$

Using the identities $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$, $\cos x = \sin(\frac{\pi}{2} - x)$, and $\sin 2x = 2\cos x \sin x$,

$$f_t(\lambda) = \frac{\cos\left(\frac{2\pi}{3}\lambda - \frac{\pi}{6}\right)}{3\sin\left(-\frac{\pi}{3}\lambda + \frac{\pi}{3}\right)} \mathcal{I}_{\frac{1}{3}\lambda+\frac{2}{3}}(h)(t) = \tfrac{2}{3}\sin\left(\tfrac{\pi}{3}\lambda + \tfrac{\pi}{6}\right) \mathcal{I}_{\frac{1}{3}\lambda+\frac{2}{3}}(h)(t) \tag{2.70}$$

By complex differentiation under the integral sign, (2.68) demonstrates that $f_+(t,\lambda)$ is analytic in λ for Re $\lambda > -3$. We shall only compute (2.48) for $0 < \lambda$, λ real. By analyticity, the result will extend to the full range Re $\lambda > -3$. For the computation in the range $0 < \lambda$, we use the representation (2.44) in place of (2.68) to give

$$\mathcal{U}_-^\lambda h(0,t) = e^{i\pi\lambda}\int_{y=0}^{+\infty} \frac{y^{\lambda-1}}{\Gamma(\lambda)}\mathcal{U}h(y,t)\,dy$$

By the decay for $A(y)$, $y \geq 0$, we can apply Fubini

$$f_+(t,\lambda) = \tfrac{1}{3\pi}\Gamma(\tfrac{1}{3}-\tfrac{1}{3}\lambda)\cos(\tfrac{\pi}{3}\lambda+\tfrac{\pi}{6})e^{i\pi\lambda}\int_0^t (t-t')^{(\frac{1}{3}\lambda+\frac{2}{3})-1}h(t')\,dt'$$

Using the same identities as above,

$$f_+(t,\lambda) = \tfrac{1}{3}e^{i\pi\lambda}\mathcal{I}_{\frac{\lambda}{3}+\frac{2}{3}}(h)(t)$$

□

Fact 4. *In the sense of distributions*

$$(\partial_t + \partial_x^3)\mathcal{U}_-^\lambda h(x,t) = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}h(t)$$

and

$$(\partial_t + \partial_x^3)\mathcal{U}_+^\lambda h(x,t) = e^{i\pi\lambda}\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}h(t)$$

Proof. This is straightforward from the definition and differentiation under the integral sign. □

Let

$$\mathcal{L}_\pm^\lambda h(x,t) = \mathcal{U}_\pm^\lambda(\mathcal{I}_{-\frac{\lambda}{3}-\frac{2}{3}}h)$$

$\mathcal{L}_\pm h$ inherits versions of all of the above properties for $\mathcal{U}_\pm$, which we summarize in the following lemma.

Lemma 17 (Properties of $\mathcal{L}_\pm^\lambda$ for $-2 \leq$ Re $\lambda \leq 1$).

$$\mathcal{L}_-^{-2}h = \mathcal{L}_+^{-2}h = \partial_x^2\mathcal{U}h, \quad \mathcal{L}_-^{-1}h = \mathcal{L}_+^{-1}h = \partial_x\mathcal{U}\mathcal{I}_{-1/3}h, \quad \mathcal{L}_-^{0}h = \mathcal{L}_+^{0}h = \mathcal{U}\mathcal{I}_{-2/3}h$$

$\mathcal{L}_\pm^{-2}h(x,t)$ has a step discontinuity in x of size $h(t)$ at $x=0$, and for $x\neq 0$, $\mathcal{L}_\pm^{-2}h(x,t)$ is continuous in x. For $-2<\operatorname{Re}\lambda\leq 1$, $\mathcal{L}_-^\lambda h(x,t)$ and $\mathcal{L}_+^\lambda h(x,t)$ are continuous in x for all $x\in\mathbb{R}$. For $-2\leq\operatorname{Re}\lambda\leq 1$,

$$\begin{aligned} |\mathcal{L}_-^\lambda(h)(x,t)| &\leq c_{k,\lambda,h,t}\langle x\rangle^{-k} && \forall\, x\leq 0, \quad \forall\, k\geq 0 \\ |\mathcal{L}_-^\lambda(h)(x,t)| &\leq c_{\lambda,h,t}\langle x\rangle^{\lambda-1} && \forall\, x\geq 0 \end{aligned}$$

and

$$\begin{aligned} |\mathcal{L}_+^\lambda(h)(x,t)| &\leq c_{k,\lambda,h,t}\langle x\rangle^{-k} && \forall\, x\geq 0, \quad \forall\, k\geq 0 \\ |\mathcal{L}_+^\lambda(h)(x,t)| &\leq c_{\lambda,h,t}\langle x\rangle^{\lambda-1} && \forall\, x\leq 0 \end{aligned}$$

For $-2\leq\operatorname{Re}\lambda\leq 1$,

$$\lim_{x\to 0^-}\mathcal{L}_-^\lambda h(x,t) = \tfrac{2}{3}\sin(\tfrac{\pi}{3}\lambda+\tfrac{\pi}{6})h(t) \tag{2.71}$$

$$\lim_{x\to 0^+}\mathcal{L}_+^\lambda h(x,t) = \tfrac{1}{3}e^{i\pi\lambda}h(t) \tag{2.72}$$

(for $-2<\operatorname{Re}\lambda\leq 1$, $\lim_{x\to 0^-}\mathcal{L}_-^\lambda h(x,t)$ can be replaced by $\mathcal{L}_-^\lambda h(0,t)$, and $\lim_{x\to 0^+}\mathcal{L}_+^\lambda h(x,t)$ can be replaced by $\mathcal{L}_+^\lambda h(0,t)$). Moreover, in the sense of distributions,

$$(\partial_t+\partial_x^3)\mathcal{L}_-^\lambda h(x,t) = \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-\frac{\lambda}{3}-\frac{2}{3}}h(t)$$

and

$$(\partial_t+\partial_x^3)\mathcal{L}_+^\lambda h(x,t) = e^{i\pi\lambda}\frac{x_-^{\lambda-1}}{\Gamma(\lambda)}\mathcal{I}_{-\frac{\lambda}{3}-\frac{2}{3}}h(t)$$

Of course, $\mathcal{L}_-^\lambda h(x,0)=0$ and $\mathcal{L}_+^\lambda h(x,0)=0$.

2.5 Outline of the needed estimates

We introduce the following Bourgain spaces, for $b < \frac{1}{2}$ and $\alpha > \frac{1}{2}$,

$$\begin{aligned} \|u\|_{X_{s,b}}^2 &= \iint_{\xi,\tau} (1+|\xi|)^{2s}(1+|\tau-\xi^3|)^{2b}|\hat{u}(\xi,\tau)|^2\,d\xi\,d\tau \\ \|u\|_{Y_{s,b}}^2 &= \iint_{\xi,\tau} (1+|\tau|)^{2s/3}(1+|\tau-\xi^3|)^{2b}|\hat{u}(\xi,\tau)|^2\,d\xi\,d\tau \\ \|u\|_{D_\alpha}^2 &= \iint_{|\xi|\le 1} (1+|\tau|)^{2\alpha}|\hat{u}(\xi,\tau)|^2\,d\xi\,d\tau \end{aligned} \tag{2.73}$$

The D_α norm is a low frequency correction for the $X_{s,b}$ norm that is needed in order for the bilinear estimates to hold.

We need the following estimates, and require $b < \frac{1}{2}$.

Space Traces.

$$\begin{aligned} \sup_{t\in\mathbb{R}} \|S(t)\phi(x)\|_{H^s(\mathbb{R}_x)} &\le c\|\phi\|_{H^s(\mathbb{R}_x)} \\ \sup_{t\in\mathbb{R}} \|\mathcal{L}_\pm^\lambda(h)(x,t)\|_{H^s(\mathbb{R}_x)} &\le c\|h\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)} \qquad s-\tfrac{5}{2} < \lambda < s+\tfrac{1}{2} \\ \sup_{t\in\mathbb{R}} \left\| \int_0^t S(t-t')w(x,t')\,dt' \right\|_{H^s(\mathbb{R}_x)} &\le c\|w\|_{X_{s,-b}} \end{aligned}$$

Time Traces.

$$\begin{aligned} \sup_{x\in\mathbb{R}} \|\theta(t)S(t)\phi(x)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} &\le c\|\phi\|_{H^s(\mathbb{R}_x)} \\ \sup_{x\in\mathbb{R}} \|\theta(t)\mathcal{L}_\pm^\lambda(h)(x,t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} &\le c\|h\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)} \qquad -2 < \lambda < 1 \\ \sup_{x\in\mathbb{R}} \left\| \theta(t)\int_0^t S(t-t')w(x,t')\,dt' \right\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} & \\ \le \begin{cases} c\|w\|_{X_{s,-b}} & \text{if } -1 \le s \le \frac{1}{2} \\ c(\|w\|_{Y_{s,-b}} + \|w\|_{X_{s,-b}}) & \text{for any } s \end{cases} \end{aligned}$$

Derivative Time Traces.

$$\sup_{x\in\mathbb{R}} \|\theta(t)\partial_x S(t)\phi(x)\|_{H^{\frac{s}{3}}(\mathbb{R}_t)} \leq c\|\phi\|_{H^s(\mathbb{R}_x)}$$

$$\sup_{x\in\mathbb{R}} \|\theta(t)\partial_x \mathcal{L}_{\pm}^{\lambda}(h)(x,t)\|_{H^{\frac{s}{3}}(\mathbb{R}_t)} \leq c\|h\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)} \qquad -1 < \lambda < 2$$

$$\sup_{x\in\mathbb{R}} \left\|\theta(t)\partial_x \int_0^t S(t-t')w(x,t')\,dt'\right\|_{H^{\frac{s}{3}}(\mathbb{R}_t)}$$

$$\leq \begin{cases} c\|w\|_{X_{s,-b}} & \text{if } 0 \leq s \leq \frac{3}{2} \\ c(\|w\|_{Y_{s,-b}} + \|w\|_{X_{s,-b}}) & \text{for any } s \end{cases}$$

$X_{s,b}$ **Bourgain Space Estimates**.

$$\|\theta(t)S(t)\phi(x)\|_{X_{s,b}\cap D_\alpha} \leq c\|\phi\|_{H^s(\mathbb{R}_x)}$$

$$\|\theta(t)\mathcal{L}_{\pm}^{\lambda}(h)(x,t)\|_{X_{s,b}\cap D_\alpha} \leq c\|h\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)} \qquad s-1 \leq \lambda < s+\tfrac{1}{2},\ \lambda < \tfrac{1}{2},\ \alpha \leq \tfrac{s-\lambda+2}{3}$$

$$\left\|\theta(t)\int_0^t S(t-t')w(x,t')\,dt'\right\|_{X_{s,b}\cap D_\alpha} \leq c\|w\|_{X_{s,-b}} \qquad \alpha \leq 1-b$$

Bilinear Estimates. For $s > -\frac{3}{4}$, $\exists\, b = b(s) < \frac{1}{2}$ such that $\forall\, \alpha > \frac{1}{2}$,

$$\|\partial_x(uv)\|_{X_{s,-b}} \leq c\|u\|_{X_{s,b}\cap D_\alpha}\|v\|_{X_{s,b}\cap D_\alpha}$$

For $-\frac{3}{4} < s < 3$, $\exists\, b = b(s) < \frac{1}{2}$ such that $\forall\ \alpha > \frac{1}{2}$,

$$\|\partial_x(uv)\|_{Y_{s,-b}} \leq c\|u\|_{X_{s,b}\cap D_\alpha}\|v\|_{X_{s,b}\cap D_\alpha}$$

2.6 $X_{s,b}$ estimates

Lemma 18 ([CK02] Lemma 5.1). *If $s \in \mathbb{R}$, $0 < b < 1$, and $\theta(t) \in C_0^\infty(\mathbb{R})$, then*

$$\|\theta(t)u(x,t)\|_{X_{s,b}} \leq c\|u\|_{X_{s,b}} \tag{2.74}$$

$$\|\theta(t)u(x,t)\|_{D_\alpha} \leq c\|u\|_{D_\alpha} \tag{2.75}$$

where $c = c(\theta)$.

Proof. See [CK02]. □

Lemma 19 ($X_{s,b}$ estimate for the group). *If $s \in \mathbb{R}$, $b < \frac{1}{2}$, $\alpha > \frac{1}{2}$, and $\theta(t) \in C_0^\infty(\mathbb{R})$, then*

$$\|\theta(t)S(t)\phi\|_{X_{s,b}\cap D_\alpha} \leq c\left(\int_\tau (1+|\tau|)^{2\alpha}|\hat{\theta}(\tau)|^2\,d\tau\right)^{1/2}\|\phi\|_{H^s(\mathbb{R})}$$

where c is independent of $\theta(t)$.

Proof. Since $[\theta(t)S(t)\phi]\hat{}\,(\xi,\tau) = \hat{\theta}(\tau-\xi^3)\hat{\phi}(\xi)$, we have

$$\begin{aligned}
\|\theta(t)S(t)\phi\|_{X_{s,b}}^2 &\leq \int_\xi \left[\int_\tau (1+|\tau-\xi^3|)^{2b}|\hat{\theta}(\tau-\xi^3)|^2\,d\tau\right](1+|\xi|)^{2s}|\hat{\phi}(\xi)|^2\,d\xi \\
&\leq \int_\tau (1+|\tau|)^{2b}|\hat{\theta}(\tau)|^2\,d\tau \int_\xi (1+|\xi|)^{2s}|\hat{\phi}(\xi)|^2\,d\xi
\end{aligned}$$

Also,

$$\begin{aligned}
\|\theta(t)S(t)\phi\|_{D_\alpha}^2 &\leq \int_{|\xi|\leq 1}\left[\int_\tau (1+|\tau|)^{2\alpha}|\hat{\theta}(\tau-\xi^3)|^2\,d\tau\right]|\hat{\phi}(\xi)|^2\,d\xi \\
&\leq c\int_{|\xi|\leq 1}\left[\int_\tau (1+|\tau-\xi^3|)^{2\alpha}|\hat{\theta}(\tau-\xi^3)|^2\,d\tau\right]|\hat{\phi}(\xi)|^2\,d\xi \\
&\leq c\int_\tau (1+|\tau|)^{2\alpha}|\hat{\theta}(\tau)|^2\,d\tau \int_{|\xi|\leq 1}|\hat{\phi}(\xi)|^2\,d\xi
\end{aligned}$$

□

Lemma 20 ($X_{s,b}$ estimate for the Duhamel forcing operator class). *Let $s \in \mathbb{R}$, $s-1 \le \lambda < s+\frac{1}{2}$, $\lambda < \frac{1}{2}$, $\alpha \le \frac{s-\lambda+2}{3}$, and $0 \le b < \frac{1}{2}$. Then*

$$\|\theta(t)\mathcal{L}_{\pm}^{\lambda}(h)(x,t)\|_{X_{s,b}\cap D_\alpha} \le c\|h\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)} \tag{2.76}$$

Remark 2. *Note that by the assumption $\lambda < s+\frac{1}{2}$, we have $\frac{s-\lambda+2}{3} > \frac{1}{2}$, and thus we may take $\frac{1}{2} < \alpha \le \frac{s-\lambda+2}{3}$, which is needed in order to validate the bilinear estimates.*

Proof. By [Fri98] p. 110,

$$r_\lambda(\xi) := \left(\frac{x_+^{\lambda-1}}{\Gamma(\lambda)}\right)^{\widehat{}}(\xi) = e^{-\frac{1}{2}i\pi\lambda}\xi_+^{-\lambda} + e^{\frac{1}{2}i\pi\lambda}\xi_-^{-\lambda} \qquad \text{if } \lambda \ne 1, 2, \dots$$

Since $\lambda < \frac{1}{2}$, we may interpret the right-hand side as a sum of locally square-integrable functions. By Lemma 13, it suffices to prove

$$\left\|\theta(t)\mathcal{J}^\lambda\left[\int_0^t S(t-t')\delta_0(-)h(t')\,dt'\right](x)\right\|_{X_{s,b}} \le c\|h\|_{H_0^{\frac{s-\lambda-1}{3}}(\mathbb{R}^+)} \tag{2.77}$$

By a density argument, it suffices to prove (2.77) for $h \in C_0^\infty(\mathbb{R}^+)$. Taking $\widehat{}$ to mean the Fourier transform in the x-variable only,

$$\left(\theta(t)\mathcal{J}^\lambda \int_0^t S(t-t')\delta_0(-)h(t')\,dt'\right)^{\widehat{}}(\xi) = \theta(t)r_\lambda(\xi)\int_0^t e^{i(t-t')\xi^3}h(t')\,dt' \tag{2.78}$$

Let $\psi(\tau) \in C^\infty(\mathbb{R})$ that is 1 for $|\tau| \le \frac{1}{2}$ and supp $\psi \subset [-1,1]$. Using that $2\chi_{(0,t)}(t') =$

sgn $t' +$ sgn $(t-t')$, we obtain the decomposition

$$\left(\theta(t)\mathcal{J}^{\lambda}\int_0^t S(t-t')\delta_0(-)h(t')\,dt'\right)^{\widehat{}}(\xi)$$

$$= \theta(t)r_\lambda(\xi)\int_\tau \frac{e^{it\tau}-e^{it\xi^3}}{\tau-\xi^3}\psi(\tau-\xi^3)\hat{h}(\tau)\,d\tau$$

$$+\,\theta(t)r_\lambda(\xi)\int_\tau e^{it\tau}\frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3}\hat{h}(\tau)\,d\tau$$

$$-\,\theta(t)r_\lambda(\xi)\int_\tau e^{it\xi^3}\frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3}\hat{h}(\tau)\,d\tau$$

$$= \hat{u}_1(\xi,t)+\hat{u}_{2,1}(\xi,t)-\hat{u}_{2,2}(\xi,t)$$

We now address each of the three terms separately.

Term u_1. We shall prove that for all $s\in\mathbb{R}$, and all $\lambda<\frac{1}{2}$, we have

$$\|u_1\|_{X_{s,b}\cap D_\alpha}\le c\|h\|_{H^{\frac{s-\lambda-1}{3}}} \tag{2.79}$$

Using the power series expansion for $e^{it(\tau-\xi^3)}$,

$$\hat{u}_1(\xi,t)=\sum_{k=1}^{\infty}\frac{i^k t^k\theta(t)}{k!}r_\lambda(\xi)e^{it\xi^3}\int_\tau(\tau-\xi^3)^{k-1}\psi(\tau-\xi^3)\hat{h}(\tau)\,d\tau$$

Set $\hat{\phi}_k(\xi)=r_\lambda(\xi)\int_\tau(\tau-\xi^3)^{k-1}\psi(\tau-\xi^3)\,\hat{h}(\tau)\,d\tau$ and $\theta_k(t)=i^k t^k\theta(t)$. Then

$$u_1(x,t)=\sum_{k=1}^{+\infty}\frac{1}{k!}\theta_k(t)S(t)\phi_k(x)$$

By Lemma 19,

$$\|u_1\|_{X_{s,b}\cap D_\alpha}\le\sum_{k=1}^{+\infty}\frac{1}{k!}\|\theta_k\|_{H^1}\|\phi_k\|_{H^s} \tag{2.80}$$

Now

$$\begin{aligned}\|\phi_k\|_{H^s}^2 &\leq \int_\xi |\xi|^{-2\lambda} \left[\int_{|\tau-\xi^3|\leq 1} |\hat{h}(\tau)| \, d\tau \right]^2 (1+|\xi|)^{2s} \, d\xi \\ &\leq \int_\eta |\eta|^{-\frac{2\lambda}{3}} |\eta|^{-\frac{2}{3}} (1+|\eta|)^{\frac{2s}{3}} \int_{|\tau-\eta|\leq 1} |\hat{h}(\tau)|^2 \, d\tau \, d\eta \\ &= \int_\tau |\hat{h}(\tau)|^2 \left[\int_{|\tau-\eta|\leq 1} |\eta|^{-\frac{2\lambda}{3}} |\eta|^{-\frac{2}{3}} (1+|\eta|)^{\frac{2s}{3}} \, d\eta \right] d\tau \\ &\leq \int_\tau (1+|\tau|)^{\frac{2(s-\lambda-1)}{3}} |\hat{h}(\tau)|^2 \, d\tau \end{aligned} \tag{2.81}$$

Combining (2.80) and (2.81) gives (2.79).

<u>Term $u_{2,2}$.</u> We prove that for all $s \in \mathbb{R}$ and all $\lambda < \frac{1}{2}$, we have

$$\|u_{2,2}\|_{X_{s,b} \cap D_\alpha} \leq c \|h\|_{H^{\frac{s-\lambda-1}{3}}} \tag{2.82}$$

By definition,

$$\hat{u}_{2,2}(\xi, t) = \theta(t) e^{it\xi^3} \left[r_\lambda(\xi) \int_\tau \hat{h}(\tau) \frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3} \, d\tau \right]$$

so

$$u_{2,2}(x,t) = \theta(t) S(t) \phi(x)$$

where

$$\hat{\phi}(\xi) = r_\lambda(\xi) \int_\tau \hat{h}(\tau) \frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3} \, d\tau$$

By Lemma 19, it suffices to show

$$\|\phi\|_{H^s} \leq c \|h\|_{H^{\frac{s-\lambda-1}{3}}} \tag{2.83}$$

We claim that

$$\int_\tau \hat{h}(\tau) \frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3} \, d\tau = \int_\tau \hat{h}(\tau) \beta(\tau-\xi^3) \, d\tau \tag{2.84}$$

where $\beta \in \mathcal{S}(\mathbb{R})$, and to prove this we will use the fact that supp $h \subset [0, +\infty)$. Let

$$\hat{g}(\tau) = \frac{1 - \psi(-\tau)}{\tau}$$

Then

$$g(t) = \frac{i}{2}\text{sgn } t - \frac{i}{4\pi} \int_s \text{sgn}(t-s)\hat{\psi}(s)\, ds$$

Let $\alpha \in C^\infty(\mathbb{R})$ be such that $\alpha(t) = 1$ for $t > 0$ and $\alpha(t) = -1$ for $t < -1$, and set

$$f(t) = \frac{i}{2}\alpha(t) - \frac{i}{4\pi} \int_s \text{sgn}(t-s)\hat{\psi}(s)\, ds$$

To show that $f \in \mathcal{S}(\mathbb{R})$, note that by the definition and since $\hat{\psi} \in \mathcal{S}$, we have $f \in C^\infty(\mathbb{R})$. If $t > 0$, then since $\frac{1}{2\pi}\int \hat{\psi}(\tau)\, d\tau = \psi(0) = 1$, we have

$$\begin{aligned} f(t) &= \frac{i}{2} - \frac{i}{4\pi} \int_s \text{sgn}(t-s)\hat{\psi}(s)\, ds \\ &= \frac{i}{2\pi} \int_{s>t} \hat{\psi}(s)\, ds \end{aligned}$$

If $t < -1$, then likewise we have

$$\begin{aligned} f(t) &= -\frac{i}{2} - \frac{i}{4\pi} \int_s \text{sgn}(t-s)\hat{\psi}(s)\, ds \\ &= \frac{i}{2\pi} \int_{s<t} \hat{\psi}(s)\, ds \end{aligned}$$

which provide the decay at ∞ estimates for f and all of its derivatives, establishing that $f \in \mathcal{S}(\mathbb{R})$. Since $f(t) = g(t)$ for $t > 0$ and $h \in C_0^\infty(\mathbb{R}^+)$ we have

$$\begin{aligned}
&\int_\tau \hat{h}(\tau) \frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3}\, d\tau \\
&\quad = -(\hat{h} * \hat{g})(\xi^3) \\
&\quad = -2\pi \widehat{hg}(\xi^3) \\
&\quad = -2\pi \widehat{hf}(\xi^3) \\
&\quad = \int_\tau \hat{h}(\tau)\beta(\tau-\xi^3)\, d\tau
\end{aligned}$$

where $\beta(\tau) = -\hat{f}(-\tau)$, and $\beta \in \mathcal{S}(\mathbb{R})$ since $f \in \mathcal{S}(\mathbb{R})$, thus establishing (2.84). Now we return to the proof of the estimate (2.83).

$$\begin{aligned}
\|\phi\|_{H^s}^2 &\le \int_\xi |\xi|^{-2\lambda} \left[\int_\tau |\hat{h}(\tau)||\beta(\tau-\xi^3)|\, d\tau \right]^2 \langle \xi \rangle^{2s}\, d\xi \\
&\le \int_\tau \int_\xi |\xi|^{-2\lambda} |\hat{h}(\tau)|^2 |\beta(\tau-\xi^3)\langle \tau-\xi^3\rangle|^2 \langle \xi \rangle^{2s}\, d\tau\, d\xi \\
&\le \int_\tau |\hat{h}(\tau)|^2 \left[\int_\eta \langle \eta \rangle^{\frac{2s}{3}} |\eta|^{-\frac{2\lambda}{3}} |\eta|^{-\frac{2}{3}} |\beta(\tau-\eta)|^2 \langle \tau-\eta\rangle^2\, d\eta \right] d\tau
\end{aligned}$$

and since $\lambda < \frac{1}{2}$,

$$\le \int_\tau \langle \tau \rangle^{\frac{2(s-\lambda-1)}{3}} |\hat{h}(\tau)|^2\, d\tau$$

Term $u_{2,1}$. We shall prove that for $s \in \mathbb{R}$, $s-1 \le \lambda < s+\frac{1}{2}$, $\lambda < \frac{1}{2}$, we have

$$\|u_{2,1}\|_{X_{s,b}} \le c\|h\|_{H^{\frac{s-\lambda-1}{3}}} \tag{2.85}$$

If, in addition, $\alpha \le \frac{s-\lambda+2}{3}$, then

$$\|u_{2,1}\|_{D_\alpha} \le c\|h\|_{H^{\frac{s-\lambda-1}{3}}} \tag{2.86}$$

By Lemma 18 (2.74),

$$\|u_{2,1}\|^2_{X_{s,b}} = \int_\xi \int_\tau |\xi|^{-2\lambda}\langle \tau - \xi^3\rangle^{2b-2}\langle \xi\rangle^{2s}|\hat{h}(\tau)|^2 \, d\tau \, d\xi$$
$$= \int_\tau |\hat{h}(\tau)|^2 G(\tau)\, d\tau \tag{2.87}$$

where

$$G(\tau) = \int_\eta |\eta|^{-\frac{2\lambda}{3}}|\eta|^{-\frac{2}{3}}\langle \tau - \eta\rangle^{2b-2}\langle \eta\rangle^{\frac{2s}{3}}\, d\eta$$

We now show that

$$G(\tau) \le c\langle \tau\rangle^{\frac{2(s-\lambda-1)}{3}} \tag{2.88}$$

Case 1. For $|\tau| \le 1$, we have $G(\tau) \le 1$. To prove this,

$$G(\tau) \le \int_\eta |\eta|^{-\frac{2\lambda}{3}-\frac{2}{3}}\langle \eta\rangle^{2b-2+\frac{2s}{3}}\, d\eta$$

Since $\lambda < \frac{1}{2}$, we have $-\frac{2\lambda}{3} - \frac{2}{3} > -1$, and since $s - \lambda - 1 \le 0$ and $b < \frac{1}{2}$, we have $\frac{2(s-\lambda-1)}{3} + 2b - 2 < -1$, so the integral is finite.

Case 2. If $|\tau| \ge 1$ and $|\eta| \sim |\tau|$ or $|\eta| >> |\tau|$, then $|\eta| \ge \frac{1}{2}$ and

$$G(\tau) \le \int_\eta \langle \eta\rangle^{\frac{2(s-\lambda-1)}{3}}\langle \tau - \eta\rangle^{2b-2}\, d\eta$$

Since $s-\lambda-1 \le 0$, we have $\langle \eta\rangle^{\frac{2(s-\lambda-1)}{3}} \le \langle \tau\rangle^{\frac{2(s-\lambda-1)}{3}}$. Apply this bound, and carry out the η integral (using that $b < \frac{1}{2}$), to get $|G(\tau)| \le \langle \tau\rangle^{\frac{2(s-\lambda-1)}{3}}$.

Case 3. Suppose $|\eta| << |\tau|$. Since $\lambda < \frac{1}{2}$ (so that $-\frac{2}{3}(\lambda + 1) > -1$), and since $-1 < \frac{2}{3}(s - \lambda - 1) \le 0$, and since $b \le \frac{1}{2}$, we have

$$G(\tau) \le c\langle \tau\rangle^{2b-2}\left(\int_{|\eta|\le\frac{1}{2}|\tau|} |\eta|^{\frac{2(-\lambda-1)}{3}}\langle \eta\rangle^{\frac{2s}{3}}\, d\eta\right)$$
$$\le c\langle \tau\rangle^{2b-2}\langle \tau\rangle^{\frac{2(s-\lambda-1)}{3}+1} \le c\langle \tau\rangle^{\frac{2(s-\lambda-1)}{3}}$$

This concludes the proof of (2.88), and thus by (2.87), we have (2.85).

Now we proceed to (2.86). By Lemma 18 (2.75),

$$\|u_{2,1}\|_{D_\alpha}^2 \leq \int_{|\xi|\leq 1}\int_\tau |r_\lambda(\xi)|^2 \langle\tau\rangle^{2\alpha}\langle\tau-\xi^3\rangle^{-2}|\hat{h}(\tau)|^2\,d\tau\,d\xi \tag{2.89}$$

For $|\xi|\leq 1$, we have $\langle\tau-\xi^3\rangle \sim \langle\tau\rangle$, and since $\lambda<\frac{1}{2}$, we can carry out the ξ integral in (2.89) to obtain

$$(2.89) \leq c\int_\tau \langle\tau\rangle^{2\alpha-2}|\hat{h}(\tau)|^2\,d\tau \tag{2.90}$$

Since $\alpha-1\leq\frac{s-\lambda-1}{3}$, we obtain (2.86) from (2.89). □

Lemma 21 ($X_{s,b}$ estimate for the Duhamel inhomogeneous term). *If $s\in\mathbb{R}$, $0\leq b<\frac{1}{2}$, and $\alpha\leq 1-b$, then*

$$\left\|\theta(t)\int_0^t S(t-t')w(x,t')\,dt'\right\|_{X_{s,b}\cap D_\alpha} \leq c\|w\|_{X_{s,-b}}$$

Proof. Let $\psi\in C_0^\infty(\mathbb{R})$ that is 1 for $|\tau|\leq\frac{1}{2}$ and supp $\psi\subset[-1,1]$. Using that $2\chi_{(0,t)}(t')=\text{sgn}\ t'+\text{sgn}\ (t-t')$, ($\widehat{\ }$ denotes the Fourier transform in the x-variable only)

$$\begin{aligned}
&\left[\theta(t)\int_0^t S(t-t')w(x,t')\,dt'\right]^{\widehat{}}(\xi)\\
&\quad=\theta(t)\int_0^t e^{i(t-t')\xi^3}\hat{w}(\xi,t')\,dt'\\
&\quad=\ \theta(t)e^{it\xi^3}\int_\tau \frac{e^{it(\tau-\xi^3)}-1}{\tau-\xi^3}\psi(\tau-\xi^3)\hat{w}(\xi,\tau)\,d\tau\\
&\qquad+\theta(t)\int_\tau e^{it\tau}\frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3}\hat{w}(\xi,\tau)\,d\tau\\
&\qquad-\theta(t)e^{it\xi^3}\int_\tau \frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3}\hat{w}(\xi,\tau)\,d\tau\\
&\quad=\hat{u}_1(\xi,t)+\hat{u}_{2,1}(\xi,t)-\hat{u}_{2,2}(\xi,t)
\end{aligned}$$

We now treat each of these three terms separately.

Term $u_{2,2}$. Let

$$\hat{\phi}(\xi) = \int_\tau \frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3}\hat{w}(\xi,\tau)\,d\tau$$

Then $u_{2,2} = \theta(t)S(t)\phi(x)$. By Lemma 19, $\|u_{2,2}\|_{X_{s,b}\cap D_\alpha} \le c\|\phi\|_{H^s}$, and therefore it suffices to show $\|\phi\|_{H^s} \le c\|h\|_{H^{\frac{s-\lambda-1}{3}}}$.

$$\begin{aligned}
\|\phi\|_{H^s}^2 &\le \int \langle\xi\rangle^{2s}\left[\int_\tau \langle\tau-\xi^3\rangle^{-1}|\hat{w}(\xi,\tau)|\,d\tau\right]^2 d\xi \\
&\le \int_\xi\int_\tau \langle\xi\rangle^{2s}\langle\tau-\xi^3\rangle^{-2b}|\hat{w}(\xi,\tau)|^2\,d\xi\,d\tau \\
&\le c\|w\|_{X_{s,-b}}^2
\end{aligned}$$

Term u_1. Let $\hat{\phi}_k(\xi) = \int_\tau (\tau-\xi^3)^{k-1}\psi(\tau-\xi^3)\hat{w}(\xi,\tau)\,d\tau$, and $\theta_k(t) = i^k t^k\theta(t)$. Then

$$u_1(x,t) = \sum_{k=1}^{+\infty}\frac{1}{k!}\theta_k(t)S(t)\phi_k(x)$$

By Lemma 19, it suffices to prove $\|\phi_k\|_{H^s} \le \|w\|_{X_{s,-b}}$. This follows from

$$\int_\xi \langle\xi\rangle^{2s}\left[\int_{|\tau-\xi^3|\le 1}|\hat{w}(\xi,\tau)|\,d\tau\right]^2 d\xi \le \int_\xi\int_\tau \langle\xi\rangle^{2s}\langle\tau-\xi^3\rangle^{-2b}|\hat{w}(\xi,\tau)|^2\,d\tau\,d\xi$$

Term $u_{2,1}$. By Lemma 18 (2.74), and since $b<\frac{1}{2}$,

$$\|u_{2,1}\|_{X_{s,b}}^2 \le \int_\xi\int_\tau \langle\tau-\xi^3\rangle^{-2+2b}|\hat{w}(\xi,\tau)|^2\langle\xi\rangle^{2s}\,d\tau\,d\xi \le c\|w\|_{X_{s,-b}}^2$$

Also, by Lemma 18 (2.75), provided $\alpha \le 1-b$,

$$\|u_{2,1}\|_{D_\alpha}^2 \le \int_{|\xi|\le 1}\int_\tau \langle\tau-\xi^3\rangle^{-2+2\alpha}|\hat{w}(\xi,\tau)|^2\,d\tau\,d\xi \le \|w\|_{X_{s,-b}}^2$$

□

2.7 Space traces estimates

Lemma 22 (Space traces estimate for the group). *If $s \in \mathbb{R}$, then*

$$S(t)\phi(x) \in C(\,t \in (-\infty,+\infty);\; H^s(\mathbb{R}_x)\,)$$
$$\sup_{t\in\mathbb{R}} \|S(t)\phi(x)\|_{H^s(\mathbb{R}_x)} \leq c\|\phi\|_{H^s(\mathbb{R}_x)}$$

Proof. This is immediate from the definition of $S(t)\phi(x)$. □

Lemma 23 (Space traces estimate for the Duhamel forcing operator class). *Let $s \in \mathbb{R}$, $s - \frac{5}{2} < \lambda < s + \frac{1}{2}$, $\lambda < 1$. If supp $h \subset [0,1]$,*

$$\mathcal{L}^{\lambda}_{\pm}(h)(x,t) \in C(\,t \in (-\infty,+\infty);\; H^s(\mathbb{R}_x)\,)$$
$$\sup_t \|\mathcal{L}^{\lambda}_{-}(h)(x,t)\|_{H^s(\mathbb{R}_x)} \leq c\|h\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)} \tag{2.91}$$

Proof. We shall begin by proving

$$\sup_t \left\| \mathcal{J}^{\lambda}\left[\int_0^t S(t-t')\delta_0(-)h(t')\,dt'\right](x)\right\|_{\dot{H}^s(\mathbb{R}_x)} \leq c\|h\|_{H_0^{\frac{s-\lambda-1}{3}}(\mathbb{R}_t^+)} \tag{2.92}$$

By a density argument, it suffices to assume that $h \in C_0^{\infty}(\mathbb{R}^+)$, with supp $h \subset [0,1]$. Let $\phi \in \dot{H}^{-s}$ such that $\|\phi\|_{\dot{H}^{-s}} \leq 1$.

$$\begin{aligned}
&\int_x \mathcal{J}^{\lambda}\left[\int_0^t S(t-t')\delta_0(-)h(t')\,dt'\right](x)\,\overline{\phi(x)}\,dx \\
&\quad= \int_{t'} \overline{S(t'-t)[\mathcal{J}^{\lambda}]^*\phi\Big|_{x=0}}\,\chi_{(-\infty,t)}(t')h(t')\,dt' \\
&\quad\leq \left\| S(t-t')[\mathcal{J}^{\lambda}]^*\phi\Big|_{x=0}\right\|_{\dot{H}_{t'}^{\frac{1-s+\lambda}{3}}} \left\|\chi_{(-\infty,t)}(t')h(t')\right\|_{\dot{H}_{t'}^{\frac{s-\lambda-1}{3}}}
\end{aligned} \tag{2.93}$$

Since $-\frac{1}{2} < \frac{s-\lambda-1}{3} < \frac{1}{2}$, by Lemma 10, we have

$$\left\|\chi_{(-\infty,t)}(t')h(t')\right\|_{\dot{H}_{t'}^{\frac{s-\lambda-1}{3}}} \leq c\left\|\chi_{(-\infty,t)}(t')h(t')\right\|_{H_{t'}^{\frac{s-\lambda-1}{3}}} \tag{2.94}$$

Applying Lemma 7/ Corollary 1 to the right-hand side of (2.94),

$$\left\|\chi_{(-\infty,t)}(t')h(t')\right\|_{\dot H_{t'}^{\frac{s-\lambda-1}{3}}} \leq c\|h\|_{H_{t'}^{\frac{s-\lambda-1}{3}}} \tag{2.95}$$

By Lemma 25,

$$\left\| S(t-t')[\mathcal{J}^\lambda]^*\phi \Big|_{x=0} \right\|_{\dot H_{t'}^{\frac{1-s+\lambda}{3}}} \leq c\|[\mathcal{J}^\lambda]^*\phi\|_{\dot H_x^{-s+\lambda}}$$

Since $\left([\mathcal{J}^\lambda]^*\phi\right)\widehat{\ }(\xi) = \overline{r_\lambda(\xi)}\hat\phi(\xi)$, we have, for $\lambda < 1$

$$\|[\mathcal{J}^\lambda]^*\phi\|_{\dot H_x^{-s+\lambda}} \leq c\|\phi\|_{\dot H_x^{-s}} \tag{2.96}$$

Combining (2.93), (2.95), and (2.96) establishes (2.92). As a consequence, we have, for $h \in C_0^\infty(\mathbb{R}^+)$,

$$\left\| \mathcal{J}^\lambda \left[\int_0^t S(t-t')\delta_0(-)h(t')\, dt' \right](x) \right\|_{H^s} \leq c\|h\|_{H^{\frac{s-\lambda-1}{3}}} \tag{2.97}$$

Indeed, if $s \leq 0$, it is immediate that (2.92) implies (2.97). If $s \geq 0$, then appeal to (2.92) for s and also for $s = 0$. (2.91) now follows from (2.97) by Lemma 13. ☐

Lemma 24 (Space traces estimate for the Duhamel inhomogeneous term). *Let $s \in \mathbb{R}$ and $0 \leq b < \frac{1}{2}$. If $u \in X_{s,-b}$,*

$$\theta(t)\int_0^t S(t-t')u(-,t')\, dt' \in C(\, t \in (-\infty,+\infty);\ H^s(\mathbb{R}_x)\,)$$
$$\sup_t \left\| \theta(t)\int_0^t S(t-t')u(-,t')\, dt' \right\|_{H^s(\mathbb{R}_x)} \leq c\|u\|_{X_{s,-b}}$$

Proof. By writing $2\chi_{(0,t)}(t') = \operatorname{sgn} t' + \operatorname{sgn}(t-t')$, we have the identity

$$\theta(t)\int_0^t S(t-t')u(-,t')\, dt' = \theta(t)\int_\xi e^{ix\xi}\int_\tau \frac{e^{it\tau}-e^{it\xi^3}}{\tau-\xi^3}\hat u(\xi,\tau)\, d\tau\, d\xi$$

Thus,

$$\begin{aligned}
&\left\| \theta(t) \int_0^t S(t-t')u(-,t')\,dt' \right\|^2_{H^s(\mathbb{R}_x)} \\
&\quad = \int_\xi (1+|\xi|)^{2s} \left| \theta(t) \int_{|\tau-\xi^3|\le 1} \frac{e^{it\tau}-e^{it\xi^3}}{\tau-\xi^3} \hat{u}(\xi,\tau)\,d\tau \right|^2 d\xi \\
&\qquad + \int_\xi (1+|\xi|)^{2s} \left| \theta(t) \int_{|\tau-\xi^3|\ge 1} \frac{e^{it\tau}-e^{it\xi^3}}{\tau-\xi^3} \hat{u}(\xi,\tau)\,d\tau \right|^2 d\xi \\
&\quad = \mathrm{I} + \mathrm{II}
\end{aligned}$$

Since $\left| e^{it\xi^3}\theta(t) \dfrac{e^{it(\tau-\xi^3)}-1}{\tau-\xi^3} \right| \le 2|t\theta(t)| \le c$ for $|\tau-\xi^3| \le 1$, we have

$$\mathrm{I} \le c \int (1+|\xi|)^{2s} \left[\int_{|\tau-\xi^3|\le 1} |\hat{u}(\xi,\tau)|\,d\tau \right]^2 d\xi \le c\|u\|^2_{X_{s,-b}}$$

by Cauchy-Schwarz. For $|\tau-\xi^3| \ge 1$, we have $\left| \theta(t) \dfrac{e^{it\tau}-e^{it\xi^3}}{\tau-\xi^3} \right| \le (1+|\tau-\xi^3|)^{-1}$, and therefore, by Cauchy-Schwarz, we have that

$$\mathrm{II} \le c\|u\|^2_{X_{s,-b}}$$

□

2.8 Time traces estimates

Lemma 25 (Time traces estimate for the group). *Let $s \in \mathbb{R}$. If $\theta(t) \in C_0^\infty(\mathbb{R})$, then*

$$\begin{aligned}
&\theta(t)S(t)\phi(x) \in C(\,x \in (-\infty,+\infty);\ H^{\frac{s+1}{3}}(\mathbb{R}_t)\,) \\
&\sup_{x\in\mathbb{R}} \|\theta(t)S(t)\phi(x)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \le c\|\phi\|_{H^s(\mathbb{R}_x)}
\end{aligned} \tag{2.98}$$

where $c = c(\theta)$.

Proof. The identity

$$\|S(t)\phi(x)\|_{\dot{H}^{\frac{s+1}{3}}(\mathbb{R}_t)} = \|\phi\|_{\dot{H}^s} \tag{2.99}$$

is standard (see references in [CK02]). From this we shall deduce (2.98).

Case 1. $s \geq 0$. In this case, $\frac{s+1}{3} \geq \frac{1}{3}$.

$$\|\theta(t)S(t)\phi(x)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \leq \|\theta(t)S(t)\phi(x)\|_{H^{\frac{1}{3}}(\mathbb{R}_t)} + \|\theta(t)S(t)\phi(x)\|_{\dot{H}^{\frac{s+1}{3}}(\mathbb{R}_t)} \tag{2.100}$$

By Lemma 10,

$$\|\theta(t)S(t)\phi(x)\|_{H^{\frac{1}{3}}(\mathbb{R}_t)} \leq c\|\theta(t)S(t)\phi(x)\|_{\dot{H}^{\frac{1}{3}}(\mathbb{R}_t)} \tag{2.101}$$

and thus we may apply (2.99) to obtain

$$\|\theta(t)S(t)\phi(x)\|_{H^{\frac{1}{3}}(\mathbb{R}_t)} \leq c\|\phi\|_{L^2} \tag{2.102}$$

The second piece of (2.100) is bounded by (2.99) giving, for $s \geq 0$,

$$\|\theta(t)S(t)\phi(x)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \leq c\|\phi\|_{L^2} + c\|\phi\|_{\dot{H}^s} \leq c\|\phi\|_{H^s}$$

Case 2. $s \leq 0$. Divide ϕ into low and high frequency pieces as $\phi = \phi_L + \phi_H$.

$$\|\theta(t)S(t)\phi_L(x)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \leq \|\theta(t)S(t)\phi_L(x)\|_{H^{\frac{1}{3}}(\mathbb{R}_t)} \leq c\|\phi_L\|_{L^2}$$

by (2.102).

$$\|\theta(t)S(t)\phi_H(x)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \leq \|\theta(t)S(t)\phi_H(x)\|_{\dot{H}^{\frac{s+1}{3}}(\mathbb{R}_t)} \tag{2.103}$$

by Lemma 10 if $-1 \leq s \leq 0$ (so that $0 \leq \frac{s+1}{3} \leq \frac{1}{3}$) and directly from the definitions of the norms if $s \leq -1$ (so that $\frac{s+1}{3} \leq 0$). We may then apply (2.99) to bound the right-hand side of (2.103). □

Lemma 26 (Time traces for the Duhamel forcing operator class). *Let $s \in \mathbb{R}$ and $-2 < \lambda < 1$. If supp $h \subset [0,1]$, then*

$$\begin{aligned} &\theta(t)\mathcal{L}_{\pm}^{\lambda}(h)(x,t) \in C(\, x \in (-\infty,+\infty);\ H^{\frac{s+1}{3}}(\mathbb{R}_t)\,) \\ &\sup_x \left\|\theta(t)\mathcal{L}_{\pm}^{\lambda}(h)(x,t)\right\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \le c\|h\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)} \end{aligned} \tag{2.104}$$

Proof. By Lemma 13, it suffices to prove

$$\sup_x \left\|\theta(t)\mathcal{J}^{\lambda}\left[\int_0^t S(t-t')\delta_0(-)h(t')\,dt'\right](x)\right\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \le c\|h\|_{H_0^{\frac{s-\lambda-1}{3}}(\mathbb{R}_t)} \tag{2.105}$$

To prove (2.105), we begin by proving that for $h \in C_0^{\infty}(\mathbb{R})$ (not necessarily supported in $[0,1]$), we have

$$\sup_x \left\|\mathcal{J}^{\lambda}\left[\int_{-\infty}^t S(t-t')\delta_0(-)h(t')\,dt'\right](x)\right\|_{\dot{H}^{\frac{s+1}{3}}(\mathbb{R}_t)} \le c\|h\|_{\dot{H}^{\frac{s-\lambda-1}{3}}(\mathbb{R}_t)} \tag{2.106}$$

Proving (2.106) is equivalent to proving

$$\sup_x \left\|D_t^{\frac{s+1}{3}}\mathcal{J}^{\lambda}\left[\int_{-\infty}^t S(t-t')\delta_0(-)D_{t'}^{\frac{-s+\lambda+1}{3}}h(t')\,dt'\right](x)\right\|_{L_t^2} \le c\|h\|_{L_t^2} \tag{2.107}$$

By the change of variable $t'' = t - t'$, we have

$$\begin{aligned} D_t^{\frac{s+1}{3}} &\int_{-\infty}^t S(t-t')\delta_0(-)D_{t'}^{\frac{-s+\lambda+1}{3}}h(t')\,dt' \\ &= D_t^{\frac{s+1}{3}}\int_0^{+\infty} S(t')\delta_0(-)D_{t'}^{\frac{-s+\lambda+1}{3}}h(t-t')\,dt' \\ &= \int_0^{+\infty} S(t')\delta_0(-)D_{t'}^{\frac{\lambda+2}{3}}h(t-t')\,dt' \\ &= \int_{-\infty}^t S(t-t')\delta_0(-)D_{t'}^{\frac{\lambda+2}{3}}h(t')\,dt' \end{aligned}$$

and therefore

$$
\begin{aligned}
&D_t^{\frac{s+1}{3}} \mathcal{J}^\lambda \left[\int_{-\infty}^t S(t-t')\delta_0(-) D_{t'}^{\frac{-s+\lambda+1}{3}} h(t')\, dt' \right](x) \\
&\quad = \mathcal{J}^\lambda \left[\int_{-\infty}^t S(t-t')\delta_0(-) D_{t'}^{\frac{\lambda+2}{3}} h(t')\, dt' \right]
\end{aligned}
$$

Using that $2\chi_{(0,t)}(t') = \chi_{(-\infty,+\infty)}(t') - 2\chi_{(-\infty,0)}(t') + \text{sgn}\,(t-t')$, we obtain the identity

$$
\begin{aligned}
&2\mathcal{J}^\lambda \left[\int_{-\infty}^t S(t-t')\delta_0(-) D_{t'}^{\frac{\lambda+2}{3}} h(t')\, dt' \right] \\
&= \mathcal{J}^\lambda \left[\int_{-\infty}^{+\infty} S(t-t')\delta_0(-) D_{t'}^{\frac{\lambda+2}{3}} h(t')\, dt' \right] \\
&\quad - 2\mathcal{J}^\lambda \left[\int_{-\infty}^{0} S(t-t')\delta_0(-) D_{t'}^{\frac{\lambda+2}{3}} h(t')\, dt' \right] \\
&\quad + \int_\tau e^{it\tau} \left[\lim_{\epsilon\to 0^+} \int_{|\tau-\xi^3|>\epsilon} e^{ix\xi} \frac{|\tau|^{\frac{\lambda+2}{3}} r_\lambda(\xi)}{\tau-\xi^3}\, d\xi \right] \hat{h}(\tau)\, d\tau \\
&= \mathrm{I} - 2\mathrm{II} + \mathrm{III}
\end{aligned}
$$

We shall begin by treating Term III.

$$
r_\lambda(\xi) = \left(\frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \right)^{\widehat{}}(\xi) = e^{-\frac{1}{2}\pi i\lambda} \xi_+^{-\lambda} + e^{\frac{1}{2}i\pi\lambda}\xi_-^{-\lambda} \qquad \lambda \neq 1, 2, \dots
$$

We keep $\lambda < 1$, so $r_\lambda(\xi)$ is locally integrable function. We want to show that

$$
\lim_{\epsilon\to 0} \int_{|\tau-\xi^3|>\epsilon} e^{ix\xi} \frac{|\tau|^{\frac{\lambda+2}{3}}}{\tau-\xi^3} r_\lambda(\xi)\, d\xi \tag{2.108}
$$

is bounded (independent of τ). Changing variable to $\tau^{1/3}\xi$, and using that

$$
r_\lambda(\tau^{1/3}\xi) = \tau_+^{-\lambda/3}(c_1\xi_+^{-\lambda} + c_2\xi_-^{-\lambda}) + \tau_-^{-\lambda/3}(c_1\xi_-^{-\lambda} + c_2\xi_+^{-\lambda})
$$

we get

$$(2.108) = \chi_{\tau>0} \int_\xi e^{i\tau^{1/3}x\xi} \frac{c_1\xi_+^{-\lambda} + c_2\xi_-^{-\lambda}}{1-\xi^3}\,d\xi + \chi_{\tau<0} \int_\xi e^{i\tau^{1/3}x\xi} \frac{c_1\xi_-^{-\lambda} + c_2\xi_+^{-\lambda}}{1-\xi^3}$$

The treatment of both integrals is similar, so we will only consider the first of the two. Let $\psi(\xi) = 1$ near $\xi = 1$, and 0 outside $[\frac{1}{2}, \frac{3}{2}]$. Then this term breaks into

$$c_1 \int_\xi e^{ix\tau^{1/3}\xi} \frac{\psi(\xi)\xi_+^{-\lambda}}{1-\xi^3}\,d\xi + \int_\xi e^{ix\tau^{1/3}\xi} \frac{(1-\psi(\xi))(c_1\xi_+^{-\lambda} + c_2\xi_-^{-\lambda})}{1-\xi^3}\,d\xi = \mathrm{I}_a + \mathrm{I}_b$$

Term I_b is an L^1 function (provided $\lambda > -2$), so this term is okay. Term I_a is

$$c_1 \int_\xi e^{ix\tau^{1/3}\xi} \frac{\psi(\xi)\xi_+^{-\lambda}}{1+\xi+\xi^2} \frac{1}{1-\xi}\,d\xi$$

This becomes convolution of a Schwartz class function with a phase shifted sgn x function, which is bounded. This completes the treatment of Term III. Now we turn to Term I.

$$\mathrm{I} = \mathcal{J}_x^\lambda D_t^{\frac{\lambda+1}{3}} \int_{t'=\infty}^{+\infty} S(t-t')\delta_0(x) D_{t'}^{1/3} h(t')\,dt'$$

Let $g \in L_x^1 L_t^2$. Then

$$\begin{aligned}
&\int_t \int_x \mathcal{J}_x^\lambda D_t^{\frac{\lambda+1}{3}} \int_{t'=\infty}^{+\infty} S(t-t')\delta_0(x) D_{t'}^{1/3} h(t')\,dt'\, \overline{g(x,t)}\,dx\,dt \\
&\quad = \int_x \int_{t'} S(-t')\delta_0(x) D_{t'}^{1/3} h(t')\,dt'\, \overline{\int_t S(-t) D_t^{\frac{\lambda+1}{3}} [\mathcal{J}_x^\lambda]^* g(x,t)\,dt}\,dx \\
&\quad \le \left\| \int_{t'} S(-t')\delta_0(x) D_{t'}^{1/3} h(t')\,dt' \right\|_{L_x^2} \left\| \int_t S(-t) D_t^{\frac{\lambda+1}{3}} [\mathcal{J}_x^\lambda]^* g(x,t)\,dt \right\|_{L_x^2} \qquad (2.109)
\end{aligned}$$

To address the second term in (2.109), let $\phi \in L^2_x$.

$$\int_x \int_t S(-t) D_t^{\frac{\lambda+1}{3}} [\mathcal{J}_x^\lambda]^* g(x,t)\, dt\, \overline{\phi(x)}\, dx = \int_x \int_t g(x,t)\, \overline{\mathcal{J}_x^\lambda D_t^{\frac{\lambda+1}{3}} S(t)\phi(x)}\, dx\, dt$$
$$\leq \|g\|_{L^1_x L^2_t} \left\| \mathcal{J}_x^\lambda D_t^{\frac{\lambda+1}{3}} S(t)\phi(x) \right\|_{L^\infty_x L^2_t} \tag{2.110}$$

For $\phi \in L^2_x$, by the change of variable $\eta = \xi^3$,

$$\mathcal{J}_x^\lambda D_t^{\frac{\lambda+1}{3}} S(t)\phi(x) = \int_\eta e^{it\eta} e^{ix\eta^{1/3}} |\eta|^{\frac{\lambda-1}{3}} r_\lambda(\eta^{1/3}) \hat{\phi}(\eta^{1/3})\, d\eta$$

Since $|r_\lambda(\eta^{1/3})| \leq c|\eta|^{-\frac{\lambda}{3}}$,

$$\|\mathcal{J}_x^\lambda D_t^{\frac{\lambda+1}{3}} S(t)\phi(x)\|_{L^\infty_x L^2_t} \leq c \left(\int |\eta|^{-2/3} |\hat{\phi}(\eta^{1/3})|^2\, d\eta \right)^{1/2} = c\|\phi\|_{L^2_x} \tag{2.111}$$

(2.110) and (2.111) together give

$$\left\| \int_t S(-t) D_t^{\frac{\lambda+1}{3}} [\mathcal{J}_x^\lambda]^* g(x,t)\, dt \right\|_{L^2_x} \leq c\|g\|_{L^1_x L^2_t} \tag{2.112}$$

To address the first term in (2.109), let $\phi(x) \in L^2_x$.

$$\int_x \int_{t'} S(-t')\delta_0(x) D_{t'}^{1/3} h(t')\, dt'\, \overline{\phi(x)}\, dx = \int_{t'} h(t')\, \overline{D_{t'}^{1/3} S(t')\phi\big|_{x=0}}\, dt'$$
$$\leq c\|h\|_{L^2_{t'}} \sup_x \|D_{t'}^{1/3} S(t')\phi(x)\|_{L^2_{t'}}$$

and hence by Lemma 25, we have

$$\left\| \int_{t'} S(-t')\delta_0(x) D_{t'}^{1/3} h(t')\, dt' \right\|_{L^2_x} \leq c\|h\|_{L^2_t} \tag{2.113}$$

(2.113) and (2.112) inserted in (2.109) give the bound for term I. Term II is treated similarly. This completes the proof of (2.106). Now we deduce (2.105) from (2.106).

Suppose supp $h \subset [-1,2]$. Let α be such that $\lambda - \frac{1}{2} < \alpha < \frac{1}{2}$ (note that $\lambda < 1$ implies $\lambda - \frac{1}{2} < \frac{1}{2}$). Then $\frac{\alpha+1}{3} < \frac{1}{2}$ and $\frac{\alpha-\lambda-1}{3} > -\frac{1}{2}$, and hence by Lemma 10 and (2.106),

$$\sup_x \left\| \theta(t)\mathcal{J}^\lambda \left[\int_{-\infty}^t S(t-t')\delta_0(-)h(t')\,dt' \right](x) \right\|_{H^{\frac{\alpha+1}{3}}} \leq c\|h\|_{H^{\frac{\alpha-\lambda-1}{3}}} \tag{2.114}$$

Case 1. $s < \frac{1}{2}$. Take α satisfying the above conditions and $\alpha \geq s$. Write h as a sum of low frequency and high frequency pieces $h = h_L + h_H$. Let $\theta_2(t) = 1$ on $[0,1]$ with supp $\theta_2 \subset [-1,2]$. Then by Lemma 10 and (2.106),

$$\begin{aligned} &\sup_x \left\| \theta(t)\mathcal{J}^\lambda \left[\int_{-\infty}^t S(t-t')\delta_0(-)\theta_2(t')h_H(t')\,dt' \right](x) \right\|_{H^{\frac{s+1}{3}}} \\ &\quad \leq c\|h_H\|_{\dot{H}^{\frac{s-\lambda-1}{3}}} \sim c\|h_H\|_{H^{\frac{s-\lambda-1}{3}}} \end{aligned} \tag{2.115}$$

Also, since $s \leq \alpha$,

$$\begin{aligned} &\sup_x \left\| \theta(t)\mathcal{J}^\lambda \left[\int_{-\infty}^t S(t-t')\delta_0(-)\theta_2(t')h_L(t')\,dt' \right](x) \right\|_{H^{\frac{s+1}{3}}} \\ &\quad \leq c \sup_x \left\| \theta(t)\mathcal{J}^\lambda \left[\int_{-\infty}^t S(t-t')\delta_0(-)\theta_2(t')h_L(t')\,dt' \right](x) \right\|_{H^{\frac{\alpha+1}{3}}} \\ &\quad \leq c\|h_L\|_{H^{\frac{\alpha-\lambda-1}{3}}} \sim c\|h_L\|_{H^{\frac{s-\lambda-1}{3}}} \end{aligned} \tag{2.116}$$

by (2.114).

Case 2. $s \geq \frac{1}{2}$. Since $\lambda < 1$ and $s \geq \frac{1}{2}$, we have $\frac{s-\lambda-1}{3} > -\frac{1}{2}$, and therefore by Lemma 10,

$$\|h\|_{\dot{H}^{\frac{s-\lambda-1}{3}}} \leq c\|h\|_{H^{\frac{s-\lambda-1}{3}}} \tag{2.117}$$

Combining (2.106) and (2.117), we obtain

$$\sup_x \left\| \theta(t)\mathcal{J}^\lambda \left[\int_{-\infty}^t S(t-t')\delta_0(-)h(t')\,dt' \right](x) \right\|_{\dot{H}_t^{\frac{s+1}{3}}} \leq c\|h\|_{H^{\frac{s-\lambda-1}{3}}} \tag{2.118}$$

Pick an $\alpha < \frac{1}{2}$ satisfying the conditions cited above, and then combine (2.118) with

(2.114) to obtain (2.105) in this case. □

Lemma 27 (Time Traces for the Duhamel inhomogeneous term). *Let* $b < \frac{1}{2}$. *Then*

$$\theta(t) \int_0^t S(t-t')w(x,t')\,dt' \in C(\, x \in (-\infty, +\infty);\; H^{\frac{s+1}{3}}(\mathbb{R}_t)\,)$$

$$\sup_{-\infty<x<+\infty} \left\| \theta(t) \int_0^t S(t-t')w(x,t')\,dt' \right\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \leq \begin{cases} c(\|w\|_{Y_{s,-b}} + \|w\|_{X_{s,-b}}) & \text{if } s \in \mathbb{R} \\ c\|w\|_{X_{s,-b}} & \text{if } -1 \leq s \leq \frac{1}{2} \end{cases}$$

Proof. Let $\psi \in C_0^\infty(\mathbb{R})$ that is 1 for $|\tau| \leq \frac{1}{2}$ and supp $\psi \subset [-1,1]$. Using that $2\chi_{(0,t)}(t') = \text{sgn}\, t' + \text{sgn}\, (t-t')$, ($\widehat{\ }$ denotes the Fourier transform in the x-variable only)

$$\begin{aligned}
&\left[\theta(t) \int_0^t S(t-t')w(x,t')\,dt'\right]^{\widehat{}}(\xi) \\
&\quad = \theta(t) \int_0^t e^{i(t-t')\xi^3} \hat{w}(\xi,t')\,dt' \\
&\quad = \theta(t) e^{it\xi^3} \int_\tau \frac{e^{it(\tau-\xi^3)}-1}{\tau-\xi^3} \psi(\tau-\xi^3)\hat{w}(\xi,\tau)\,d\tau \\
&\qquad + \theta(t) \int_\tau e^{it\tau} \frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3} \hat{w}(\xi,\tau)\,d\tau \\
&\qquad - \theta(t) e^{it\xi^3} \int_\tau \frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3} \hat{w}(\xi,\tau)\,d\tau \\
&\quad = \hat{u}_1(\xi,t) + \hat{u}_{2,1}(\xi,t) - \hat{u}_{2,2}(\xi,t)
\end{aligned}$$

The treatment of the terms u_1 and $u_{2,2}$ is similar to that in Lemma 21, and each is bounded by $\|w\|_{X_{s,-b}}$. We now address the term $u_{2,1}$.

$$u_{2,1}(x,t) = \theta(t) \int_\tau e^{it\tau} \int_\xi e^{ix\xi} \frac{1-\psi(\tau-\xi^3)}{\tau-\xi^3} \hat{w}(\xi,\tau)\,d\xi\,d\tau$$

and therefore,

$$
\begin{aligned}
\|u_{2,1}\|^2_{H^{\frac{s+1}{3}}} &\leq \int_\tau \langle\tau\rangle^{\frac{2(s+1)}{3}} \left[\int_\xi \langle\tau-\xi^3\rangle^{-1}|\hat{w}(\xi,\tau)|\,d\xi\right]^2 d\tau \\
&= \int_\tau \langle\tau\rangle^{\frac{2(s+1)}{3}} \left[\int_\eta \langle\tau-\eta\rangle^{-1}|\hat{w}(\eta^{1/3},\tau)|\,\eta^{-2/3}\,d\eta\right]^2 d\tau \qquad (2.119)
\end{aligned}
$$

Apply Cauchy-Schwarz to (2.119) to obtain

$$
\|u_{2,1}\|_{H_t^{\frac{s+1}{3}}} \leq \int_\tau \langle\tau\rangle^{\frac{2(s+1)}{3}} G(\tau) \int_\eta \langle\tau-\eta\rangle^{-2b}|\eta|^{-2/3}|\hat{w}(\eta^{1/3},\tau)|^2\,d\eta\,d\tau \qquad (2.120)
$$

where

$$
G(\tau) = \int_\eta \langle\tau-\eta\rangle^{-2+2b}|\eta|^{-2/3}\,d\eta
$$

We need to show that

$$
G(\tau) \leq c\langle\tau\rangle^{-2/3} \qquad (2.121)
$$

Case 1. If $|\tau| \leq 1$, then

$$
G(\tau) \leq \int_\eta \langle\eta\rangle^{-2+2b}|\eta|^{-2/3}\,d\eta < +\infty
$$

Case 2. If $|\eta| << |\tau|$, then

$$
G(\tau) \leq \langle\tau\rangle^{-2+2b} \int_{|\eta|\leq\frac{1}{2}|\tau|} |\eta|^{-2/3}\,d\eta \leq \langle\tau\rangle^{-2+2b}|\tau|^{1/3}
$$

Case 3. $|\eta| \sim |\tau| \geq 1$, or $|\eta| >> |\tau| \geq 1$.

$$
G(\tau) \leq |\tau|^{-2/3} \int_\eta \langle\tau-\eta\rangle^{-2+2b}\,d\eta
$$

Substituting (2.121) into (2.120), we obtain

$$
\|u_{2,1}(x,t)\|_{H_t^{\frac{s+1}{3}}} \leq c\|w\|_{Y_{s,-b}}
$$

for all $s \in \mathbb{R}$. To bound in terms of the $X_{s,-b}$ norm for $-1 \leq s \leq \frac{1}{2}$, apply Cauchy-Schwarz to (2.119) to otabin

$$\|u_{2,1}(x,t)\|^2_{H_t^{\frac{s+1}{3}}} \leq c \int_\tau \langle\tau\rangle^{\frac{2(s+1)}{3}} G(\tau) \int_\eta \langle\tau-\eta\rangle^{-2b} |\eta|^{-2/3} \langle\eta\rangle^{\frac{2s}{3}} |\hat{w}(\eta^{1/3},\tau)|^2 \, d\eta \, d\tau \tag{2.122}$$

where

$$G(\tau) = \int_\eta \langle\tau-\eta\rangle^{-2+2b} \eta^{-2/3} \langle\eta\rangle^{-2s/3} \, d\eta$$

and we now show that for $-1 \leq s \leq \frac{1}{2}$, we have

$$G(\tau) \leq \langle\tau\rangle^{-\frac{2(s+1)}{3}} \tag{2.123}$$

Case 1. If $|\tau| \leq 1$, then $|G(\tau)| \leq c$. To show this,

$$G(\tau) \leq \int_\eta \langle\eta\rangle^{-2+2b-\frac{2s}{3}} |\eta|^{-\frac{2}{3}} \, d\eta < \infty$$

since $s \geq -1$ and $b < \frac{1}{2}$, we have $-2 + 2b - \frac{2s}{3} < -\frac{1}{3}$.

Case 2. $|\eta| << |\tau|$. Then

$$\begin{aligned} G(\tau) &\leq c\langle\tau\rangle^{-2+2b} \int_{|\eta| \leq \frac{1}{2}|\tau|} \langle\eta\rangle^{-\frac{2s}{3}} |\eta|^{-\frac{2}{3}} \, d\eta \\ &\leq c\langle\tau\rangle^{-2+2b} \begin{cases} \langle\tau\rangle^{-\frac{2s}{3}+\frac{1}{3}} & \text{if } s < \frac{1}{2} \\ 1 + \ln\langle\tau\rangle & \text{if } s = \frac{1}{2} \end{cases} \\ &\leq c\langle\tau\rangle^{\frac{-2(s+1)}{3}} \qquad\qquad \text{if } s \leq \frac{1}{2} \end{aligned}$$

Case 3. $|\eta| \sim |\tau| \geq 1$ or $|\eta| >> |\tau| \geq 1$.

$$G(\tau) \leq \int_\eta \langle\tau-\eta\rangle^{-2+2b} |\eta|^{-2/3} \langle\eta\rangle^{-\frac{2s}{3}} \, d\eta$$

$$\leq \langle\tau\rangle^{-\frac{2}{3}-\frac{2s}{3}} \int_\eta \langle\tau-\eta\rangle^{-2+2b} \, d\eta$$

since $s \geq -1$.

This completes the proof of (2.123). Substituting (2.123) into (2.122), we obtain

$$\|u_{2,1}(x,t)\|_{H_t^{\frac{s+1}{3}}} \leq c\|w\|_{X_{s,-b}}$$

for $-1 \leq s \leq \frac{1}{2}$. □

2.9 Derivative time traces estimates

The following lemmas are corollaries of the lemmas in the previous section.

Lemma 28 (Derivative time traces estimate for the group). *Let $s \in \mathbb{R}$. If $\theta(t) \in C_0^\infty(\mathbb{R})$, then*

$$\theta(t)\partial_x S(t)\phi(x) \in C(\, x \in (\infty, +\infty);\ H^{\frac{s}{3}}(\mathbb{R}_t)\,)$$
$$\sup_{x\in\mathbb{R}} \|\theta(t)\partial_x S(t)\phi(x)\|_{H^{\frac{s}{3}}(\mathbb{R}_t)} \leq c\|\phi\|_{H^s(\mathbb{R}_x)}$$

where $c = c(\theta)$.

Lemma 29 (Derivative time traces for the Duhamel forcing operator class). *Let $s \in \mathbb{R}$ and $-1 < \lambda < 2$. If supp $h \subset [0,1]$, then*

$$\theta(t)\partial_x \mathcal{L}_\pm^\lambda(h)(x,t) \in C(\, x \in (\infty, +\infty);\ H^{\frac{s}{3}}(\mathbb{R}_t)\,)$$
$$\sup_x \left\|\theta(t)\partial_x \mathcal{L}_\pm^\lambda(h)(x,t)\right\|_{H^{\frac{s}{3}}(\mathbb{R}_t)} \leq c\|h\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)}$$

Lemma 30 (Derivative time traces for the Duhamel inhomogeneous term).

Let $b < \frac{1}{2}$. Then

$$\theta(t)\partial_x \int_0^t S(t-t')w(x,t')\,dt' \in C(\, x \in (\infty, +\infty);\; H^{\frac{s}{3}}(\mathbb{R}_t)\,)$$

$$\sup_{-\infty<x<+\infty} \left\| \theta(t)\partial_x \int_0^t S(t-t')w(x,t')\,dt' \right\|_{H^{\frac{s}{3}}(\mathbb{R}_t)}$$

$$\leq \begin{cases} c(\|w\|_{Y_{s,-b}} + \|w\|_{X_{s,-b}}) & \text{if } s \in \mathbb{R} \\ c\|w\|_{X_{s,-b}} & \text{if } 0 \leq s \leq \frac{3}{2} \end{cases}$$

2.10 Bilinear estimates

We shall deduce the needed bilinear estimates, appearing below as Prop. 11 and Prop. 12, by the calculus techniques of [KPV96]. We begin with some elementary integral estimates.

Lemma 31. *If $\frac{1}{4} < b < \frac{1}{2}$, then*

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+|x-\alpha|)^{2b}(1+|x-\beta|)^{2b}} \leq \frac{c}{(1+|\alpha-\beta|)^{4b-1}} \tag{2.124}$$

Proof. By translation, it suffices to prove the inequality for $\beta = 0$, i.e.

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+|x-\alpha|)^{2b}(1+|x|)^{2b}} \leq \frac{c}{(1+|\alpha|)^{4b-1}} \tag{2.125}$$

Case 1. $|\alpha| \leq 1$.

$$\int_{-2}^{+2} \frac{dx}{(1+|x-\alpha|)^{2b}(1+|x|)^{2b}} \leq \int_{-2}^{+2} dx = 4$$

$$\int_{|x|\geq 2} \frac{dx}{(1+|x-\alpha|)^{2b}(1+|x|)^{2b}} \leq c \int \frac{dx}{(1+|x|)^{4b}} \leq c$$

Case 2. $|\alpha| \geq 1$.

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+|x-\alpha|)^{2b}(1+|x|)^{2b}} \leq \int \frac{dx}{|x-\alpha|^{2b}|x|^{2b}} \leq \int \frac{|\alpha| dy}{|\alpha y-\alpha|^{2b}|\alpha y|^{2b}}$$
$$\leq \frac{1}{|\alpha|^{4b-1}} \int \frac{dy}{|y-1|^{2b}|y|^{2b}} \leq \frac{c}{|\alpha|^{4b-1}} \leq \frac{c}{(1+|\alpha|)^{4b-1}}$$

□

Lemma 32. *If* $b < \frac{1}{2}$, *then*

$$\int_{|x|\leq\beta} \frac{dx}{(1+|x|)^{4b-1}|\alpha-x|^{1/2}} \leq \frac{c(1+\beta)^{2-4b}}{(1+|\alpha|)^{\frac{1}{2}}} \tag{2.126}$$

Proof. This is [KPV96] Lemma 2.3 (2.11) with $2b - \frac{1}{2} = 1 - l$. The expression that we wish to bound is

$$\int_{|x|\leq\beta} \frac{dx}{(1+|x|)^{4b-1}|\alpha-x|^{1/2}} \tag{2.127}$$

Case 1. $|\alpha| \leq 1$, $|u| \leq 2$.

$$(2.127) \leq \int_{|u|\leq 2} \frac{du}{|u-\alpha|^{1/2}} = c\left[|u-\alpha|^{1/2}\right]_0^2 \leq c$$

Case 2. $|\alpha| \leq 1$, $|u| \geq 2$. Here, $\frac{1}{2}|u| \leq |u-\alpha| \leq \frac{3}{2}|u|$, and therefore

$$(2.127) \leq \int_{|u|\leq\beta} \frac{1}{(1+|u|)^{4b-\frac{1}{2}}} \leq c\left[(1+|u|)^{\frac{3}{2}-4b}\right]_0^{|\beta|}$$
$$\leq \frac{(1+|\beta|)^{\frac{3}{2}-4b}}{(1+|\alpha|)^{1/2}} \leq \frac{(1+|\beta|)^{2-4b}}{(1+|\alpha|)^{1/2}}$$

Case 3. $|\alpha| \geq 1$, $|u| << |\alpha|$.

$$(2.127) \leq \frac{1}{|\alpha|^{1/2}} \int_{|u|\leq\beta} \frac{du}{(1+|u|)^{4b-1}} \leq \frac{(1+|\beta|)^{2-4b}}{(1+|\alpha|)^{1/2}}$$

Case 4. $|\alpha| \geq 1$, $\frac{1}{2}|\alpha| \leq |u| \leq 2|\alpha|$. We may then assume that $|\beta| \geq \frac{1}{2}|\alpha|$.

$$\begin{aligned}
(2.127) &\leq \frac{1}{(1+|\alpha|)^{4b-1}} \int_{|u|\leq\beta} \frac{du}{|u-\alpha|^{1/2}} \leq \frac{1}{(1+|\alpha|)^{4b-1}} \left[|u-\alpha|^{1/2}\right]_0^{2|\alpha|} \\
&\leq \frac{|\alpha|^{1/2}}{(1+|\alpha|)^{4b-1}} \leq \frac{1}{(1+|\alpha|)^{4b-\frac{3}{2}}} \leq \frac{(1+|\beta|)^{2-4b}}{(1+|\alpha|)^{1/2}}
\end{aligned}$$

Case 5. $|\alpha| \geq 1$, $2|\alpha| \leq |u|$. We may then assume that $|\beta| \geq 2|\alpha|$.

$$\begin{aligned}
(2.127) &\leq \int_{|u|\leq\beta} \frac{du}{(1+|u|)^{4b-1}|u|^{1/2}} \leq c + \int_{|u|\leq\beta} \frac{du}{(1+|u|)^{4b-\frac{1}{2}}} \\
&\leq c + \left[(1+|u|)^{\frac{3}{2}-4b}\right]_0^{|\beta|} \leq c + (1+|\beta|)^{\frac{3}{2}-4b} \\
&\leq (1+|\beta|)^{\frac{3}{2}-4b} \leq \frac{(1+|\beta|)^{2-4b}}{(1+|\alpha|)^{1/2}}
\end{aligned}$$

□

2.10.1 $X_{s,b}$ bilinear estimate

Lemma 33. *If $-\frac{3}{4} < s < -\frac{1}{2}$, then $\exists\, b = b(s) > \frac{1}{2}$ such that $\forall\, \alpha > \frac{1}{2}$, we have*

$$\|\partial_x(uv)\|_{X_{s,-b}} \leq c\|u\|_{X_{s,b}\cap D_\alpha}\|v\|_{X_{s,b}\cap D_\alpha} \tag{2.128}$$

Proof. The proof is modelled on the proof for $b > \frac{1}{2}$ given by [KPV96]. Essentially, we only need to replace one of the calculus estimates ([KPV96] Lemma 2.3 (2.8)) in that paper with a suitable version for $b < \frac{1}{2}$ (Lemma 31). Let $\rho = -s$. It suffices to prove

$$\iint_* \frac{|\xi| d(\xi,\tau)}{\langle\tau-\xi^3\rangle^b\langle\xi\rangle^\rho} \frac{\langle\xi_1\rangle^\rho \hat{g}_1(\xi,\tau_1)}{\beta(\xi_1,\tau_1)} \frac{\langle\xi_2\rangle^\rho \hat{g}_2(\xi_2,\tau_2)}{\beta(\xi_2,\tau_2)} \leq c\|d\|_{L^2}\|g_1\|_{L^2}\|g_2\|_{L^2} \tag{2.129}$$

for $\hat{d} \geq 0$, $\hat{g}_1 \geq 0$, $\hat{g}_2 \geq 0$, where $*$ indicates integration over ξ, ξ_1, ξ_2, subject to the constraint $\xi = \xi_1 + \xi_2$, and over τ, τ_1, τ_2, subject to the constraint $\tau = \tau_1 + \tau_2$, and where $\beta_j(\xi_j,\tau_j) = \langle\tau_j - \xi_j^3\rangle^b + \chi_{|\xi_j|\leq 1}\langle\tau_j\rangle^\alpha$. By symmetry, it suffices to consider the case $|\tau_2 - \xi_2^3| \le |\tau_1 - \xi_1^3|$. We address (2.129) in pieces by the Cauchy-Schwarz method of [KPV96]. We shall assume that $|\xi_1| \ge 1$ and $|\xi_2| \ge 1$, since otherwise, the bound (2.129) reduces to the case $\rho = 0$, which has already been established in [CK02].

Case 1. If $|\tau_2 - \xi_2^3| \le |\tau_1 - \xi_1^3| \le |\tau - \xi^3|$, then we shall show

$$\frac{|\xi|}{\langle \tau - \xi^3 \rangle^b \langle \xi \rangle^\rho} \left(\iint_{\tau_1, \xi_1} \frac{\langle \xi_1 \rangle^{2\rho} \langle \xi_2 \rangle^{2\rho}}{\langle \tau_1 - \xi_1^3 \rangle^{2b} \langle \tau_2 - \xi_2^3 \rangle^{2b}} \, d\xi_1 \, d\tau_1 \right)^{1/2} \le c \tag{2.130}$$

To prove this, we note that

$$\tau - \xi^3 + 3\xi\xi_1\xi_2 = (\tau_2 - \xi_2^3) + (\tau_1 - \xi_1^3) \tag{2.131}$$

By lemma 31 with $\alpha = \xi_1^3$ and $\beta = \xi_1^3 + \tau - \xi^3 + 3\xi\xi_1\xi_2$, we get

$$(2.130) \le \frac{|\xi|}{\langle \tau - \xi^3 \rangle^b \langle \xi \rangle^\rho} \left(\int_{\xi_1} \frac{\langle \xi_1 \rangle^{2\rho} \langle \xi_2 \rangle^{2\rho}}{\langle \tau - \xi^3 + 3\xi\xi_1\xi_2 \rangle^{4b-1}} \, d\xi_1 \right)^{1/2} \tag{2.132}$$

By (2.131), $|\xi\xi_1\xi_2| \le |\tau - \xi^3|$. Substituting $|\xi_1\xi_2| \le |\tau - \xi^3||\xi|^{-1}$ into (2.132) gives

$$(2.132) \le \frac{|\xi|^{1-\rho} \langle \tau - \xi^3 \rangle^{\rho - b}}{\langle \xi \rangle^\rho} \left(\int_{\xi_1} \frac{d\xi_1}{\langle \tau - \xi^3 + 3\xi\xi_1\xi_2 \rangle^{4b-1}} \right)^{1/2} \tag{2.133}$$

Let $u = \tau - \xi^3 + 3\xi\xi_1\xi_2$, so that, by (2.131), we have $|u| \le 2|\tau - \xi^3|$. The corresponding differential is

$$d\xi_1 = \frac{c\,du}{|\xi|^{1/2} |u - (\tau - \frac{1}{4}\xi^3)|^{1/2}}$$

Substituting into (2.133), we obtain

$$(2.133) \le \frac{|\xi|^{\frac{3}{4} - \rho} \langle \tau - \xi^3 \rangle^{\rho - b}}{\langle \xi \rangle^\rho} \left(\int_{|u| \le 2|\tau - \xi^3|} \frac{du}{\langle u \rangle^{4b-1} |u - (\tau - \frac{1}{4}\xi^3)|^{1/2}} \right)^{1/2} \tag{2.134}$$

By Lemma 32,

$$(2.134) \le \frac{\langle \tau - \xi^3 \rangle^{\rho+1-3b}}{\langle \xi \rangle^{2\rho - \frac{3}{4}} \langle \tau - \frac{1}{4}\xi^3 \rangle^{1/4}}$$

This expression is bounded, provided $b \ge \frac{1}{9}\rho + \frac{5}{12}$.

Case 2. $|\tau_2 - \xi_2^3| \le |\tau_1 - \xi_1^3|$, $|\tau - \xi^3| \le |\tau_1 - \xi_1^3|$. In this case, we shall prove the bound

$$\frac{1}{\langle \tau_1 - \xi_1^3 \rangle^b} \left(\iint_{\xi,\tau} \frac{|\xi|^{2-2\rho} |\xi \xi_1 \xi_2|^{2\rho}}{\langle \xi \rangle^{2\rho} \langle \tau - \xi^3 \rangle^{2b} \langle \tau_2 - \xi_2^3 \rangle^{2b}} \, d\xi \, d\tau \right)^{1/2} \le c \tag{2.135}$$

Since

$$(\tau_1 - \xi_1^3) + (\tau_2 - \xi_2^3) - (\tau - \xi^3) = 3\xi\xi_1\xi_2 \tag{2.136}$$

we have, by Lemma 31 with $\alpha = \xi^3$, $\beta = \xi^3 + (\tau_1 - \xi_1^3) - 3\xi\xi_1\xi_2$,

$$(2.135) \le \frac{1}{\langle \tau_1 - \xi_1^3 \rangle^b} \left(\int_\xi \frac{\langle \xi \rangle^{2-4\rho} |\xi\xi_1\xi_2|^{2\rho}}{\langle \tau_1 - \xi_1^3 - 3\xi\xi_1\xi_2 \rangle^{4b-1}} \, d\xi \right)^{1/2} \tag{2.137}$$

We address (2.137) in cases. Cases 2A and 2B differ only in the bound used for $\langle \xi \rangle^{2-4\rho}$, while Case 2C is treated somewhat differently.

Case 2A. $|\xi_1| \sim |\xi|$ or $|\xi_1| << |\xi|$. Here, we use $\langle \xi \rangle^{2-4\rho} \le \langle \xi_1 \rangle^{2-4\rho}$.

Case 2B. $|\xi| << |\xi_1|$ and $[|\tau_1| >> \frac{1}{4}|\xi_1|^3$ or $|\tau_1| << \frac{1}{4}|\xi_1|^3]$. Here, we use $\langle \xi \rangle^{2-4\rho} \le 1$.

Cases 2A and 2B. In the setting of Case 2A, let $g(\xi_1) = \langle \xi_1 \rangle^{1-2\rho}$, and in the setting of Case 2B, let $g(\xi_1) = 1$. Since by (2.136), $|\xi\xi_1\xi_2| \le |\tau_1 - \xi_1^3|$,

$$(2.137) \le g(\xi_1) \langle \tau_1 - \xi_1^3 \rangle^{\rho - b} \left(\int_\xi \frac{d\xi}{\langle \tau_1 - \xi_1^3 - 3\xi\xi_1\xi_2 \rangle^{4b-1}} \right)^{1/2} \tag{2.138}$$

Set $u = \tau_1 - \xi_1^3 - 3\xi\xi_1\xi_2$. Then

$$du = 3\xi_1(\xi_1 - 2\xi) d\xi = c|\xi_1|^{1/2} |u - (\tau_1 - \tfrac{1}{4}\xi_1^3)|^{1/2} d\xi$$

which, upon substituting in (2.138), gives

$$(2.138) \leq \frac{g(\xi_1)\langle \tau_1 - \xi_1^3\rangle^{\rho-b}}{|\xi|^{1/4}} \left(\int_{|u|\leq 2|\tau_1-\xi_1^3|} \frac{du}{\langle u\rangle^{4b-1}|u-(\tau_1-\frac{1}{4}\xi_1^3)|^{1/2}} \right)^{1/2} \quad (2.139)$$

By Lemma 32,

$$(2.139) \leq \frac{g(\xi_1)\langle \tau_1 - \xi_1^3\rangle^{\rho+1-3b}}{|\xi_1|^{1/4}\langle \tau_1 - \frac{1}{4}\xi_1^3\rangle^{1/4}} \quad (2.140)$$

In Case 2A, $g(\xi_1) = \langle \xi_1\rangle^{1-2\rho}$, and (2.140) becomes

$$\frac{\langle \tau_1 - \xi_1^3\rangle^{\rho+1-3b}}{\langle \xi_1\rangle^{2\rho-\frac{3}{4}}\langle \tau_1 - \frac{1}{4}\xi_1^3\rangle^{1/4}}$$

which is bounded provided $b > \frac{1}{9}\rho + \frac{5}{12}$. In Case 2B, $g(\xi_1) = 1$, and (2.140) becomes

$$\frac{\langle \tau_1 - \xi_1^3\rangle^{\rho+1-3b}}{\langle \xi_1\rangle^{1/4}\langle \tau_1 - \frac{1}{4}\xi_1^3\rangle^{1/4}}$$

which is bounded (under the restrictions of Case 2B) provided $b \geq \frac{1}{3}\rho + \frac{1}{4}$.
Case 2C. $|\xi| << |\xi_1|$ and $|\tau_1| \sim \frac{1}{4}|\xi_1|^3$. Here, we return to (2.137) and use that $|\tau_1| \sim \frac{1}{4}|\xi_1|^3$ and $3|\xi\xi_1\xi_2| \leq \frac{1}{4}|\xi_1|^3$ implies $\langle \tau_1 - \xi_1^3 - 3\xi\xi_1\xi_2\rangle \sim \langle \xi_1\rangle^3$. Substituting into (2.137),

$$(2.137) \leq \langle \xi_1\rangle^{3\rho-15b+3} \left(\int_{|\xi|\leq|\xi_1|} \langle \xi\rangle^{2-4\rho}\, d\xi \right)^{1/2} \leq \langle \xi_1\rangle^{\rho-15b+\frac{9}{2}}$$

which is bounded provided $b \geq \frac{1}{15}\rho + \frac{3}{10}$. □

We shall now extend this result by interpolation. From Lemma 33, we have (2.128) for $s = -\frac{5}{8}$ and some $b < \frac{1}{2}$. As a consequence,

$$\begin{aligned}
&\|\partial_x(uv)\|_{X_{\frac{3}{8},-b}} \\
&\quad \leq \|\partial_x(uv)\|_{X_{-\frac{5}{8},-b}} + \|\partial_x[(\partial_x u)v]\|_{X_{-\frac{5}{8},-b}} + \|\partial_x[u(\partial_x v)]\|_{X_{-\frac{5}{8},-b}} \\
&\quad \leq (\|u\|_{X_{-\frac{5}{8},b}\cap D_\alpha} + \|\partial_x u\|_{X_{-\frac{5}{8},b}\cap D_\alpha})(\|v\|_{X_{-\frac{5}{8},b}\cap D_\alpha} + \|\partial_x v\|_{X_{-\frac{5}{8},b}\cap D_\alpha}) \\
&\quad \leq \|u\|_{X_{\frac{3}{8},b}\cap D_\alpha}\|v\|_{X_{\frac{3}{8},b}\cap D_\alpha}
\end{aligned}$$

thus establishing (2.128) for $s = \frac{3}{8}$. Now we can interpolate between the cases $s = -\frac{5}{8}$ and $s = \frac{3}{8}$ to obtain (2.128) for $-\frac{3}{4} < s \leq \frac{3}{8}$. Similarly, we can extend (2.128) to all $s > -\frac{3}{4}$. We thus obtain:

Proposition 11. *If $s > -\frac{3}{4}$, then $\exists\, b = b(s) < \frac{1}{2}$ such that for any $\alpha > \frac{1}{2}$,*

$$\|\partial_x(uv)\|_{X_{s,-b}} \leq \|u\|_{X_{s,b}\cap D_\alpha}\|v\|_{X_{s,b}\cap D_\alpha} \tag{2.141}$$

2.10.2 $Y_{s,b}$ bilinear estimate

Lemma 34. *If $-\frac{3}{4} < s < -\frac{1}{2}$, then $\exists\, b = b(s) < \frac{1}{2}$ such that $\forall\, \alpha > \frac{1}{2}$,*

$$\|\partial_x(uv)\|_{Y_{s,-b}} \leq c\|u\|_{X_{s,b}\cap D_\alpha}\|v\|_{X_{s,b}\cap D_\alpha} \tag{2.142}$$

Proof. Let $\rho = -s$. Note that by the $X_{s,b}$ bilinear estimate Prop. 11, it suffices to prove the lemma under the assumption $|\tau| \leq \frac{1}{8}|\xi|^3$. Constant multiples are routinely omitted from the calculation.

Step 1. If $|\xi_1| \geq 1$, $|\xi_2| \geq 1$, $|\tau_2 - \xi_2^3| \leq |\tau_1 - \xi_1^3|$, $|\tau_1 - \xi_1^3| \leq 1000|\tau - \xi^3|$, and $|\tau| \leq \frac{1}{8}|\xi|^3$, then the expression

$$\frac{|\xi|}{\langle\tau\rangle^{\frac{\rho}{3}}\langle\xi\rangle^{3b}}\left(\int_{\xi_1}\int_{\tau_1}\frac{|\xi_1|^{2\rho}|\xi_2|^{2\rho}}{\langle\tau_1 - \xi_1^3\rangle^{2b}\langle\tau_2 - \xi_2^3\rangle^{2b}}\,d\tau_1\,d\xi_1\right)^{1/2} \tag{2.143}$$

is bounded. *Proof:* Applying Lemma 31, using $\tau_2 - \xi_2^3 = (\tau - \xi^3) - (\tau_1 - \xi_1^3) + 3\xi\xi_1\xi_2$,

$$(2.143) \leq \frac{|\xi|}{\langle\tau\rangle^{\frac{\rho}{3}}\langle\xi\rangle^{3b}} \left(\int_{\xi_1} \frac{|\xi_1|^{2\rho}|\xi_2|^{2\rho}}{\langle \tau - \xi^3 + 3\xi\xi_1\xi_2\rangle^{4b-1}} \, d\xi_1 \right)^{1/2} \tag{2.144}$$

Using that $|\xi_1||\xi_2| \leq \dfrac{|\tau - \xi^3|}{|\xi|}$,

$$(2.144) \leq \frac{|\xi|^{1-\rho}|\tau - \xi^3|^{\rho}}{\langle\xi\rangle^{3b}\langle\tau\rangle^{\rho/3}} \left(\int_{\xi_1} \frac{1}{\langle \tau - \xi^3 + 3\xi\xi_1\xi_2\rangle^{4b-1}} \, d\xi_1 \right)^{1/2} \tag{2.145}$$

Set

$$u = \tau - \xi^3 + 3\xi\xi_1(\xi - \xi_1) \tag{2.146}$$

so that $3\xi(\xi_1 - \frac{1}{2}\xi)^2 = u - (\tau - \frac{1}{4}\xi^3)$, and thus

$$\frac{3}{\sqrt{2}}|\xi||2\xi_1 - \xi| = |\xi|^{1/2}|u - (\tau - \tfrac{1}{4}\xi^3)|^{1/2}$$

Also, $du = 3\xi(\xi - 2\xi_1)\, d\xi_1$. It follows from the hypotheses of this step that the range of integration is a subset of $|u| \leq |\tau - \xi^3|$. With this substitution,

$$(2.145) \leq \frac{|\xi|^{1-\rho}|\tau - \xi^3|^{\rho}}{\langle\xi\rangle^{3b}\langle\tau\rangle^{\rho/3}} \left(\int_{|u| \leq |\tau - \xi^3|} \frac{du}{\langle u\rangle^{4b-1}|\xi|^{1/2}|u - (\tau - \frac{1}{4}\xi^3)|^{1/2}} \right)^{1/2} \tag{2.147}$$

By Lemma 32,

$$(2.147) \leq \frac{|\xi|^{\frac{3}{4}-\rho}|\tau - \xi^3|^{\rho}\langle\tau - \xi^3\rangle^{1-2b}}{\langle\xi\rangle^{3b}\langle\tau\rangle^{\rho/3}\langle\tau - \frac{1}{4}\xi^3\rangle^{1/4}} \tag{2.148}$$

If $|\tau| \leq \frac{1}{8}|\xi|^3$, then (2.148) reduces to

$$\frac{|\xi|^{\frac{3}{4}-\rho}\langle\xi\rangle^{3\rho}\langle\xi\rangle^{3(1-2b)}}{\langle\xi\rangle^{3b}\langle\xi\rangle^{3/4}}$$

and the exponent $2\rho - 9b + 3 \leq 0$ provided $b \geq \frac{2}{9}\rho + \frac{1}{3}$.

Step 2. If $|\xi_1| \geq 1$, $|\xi_2| \geq 1$, $|\tau_2 - \xi_2^3| \leq |\tau_1 - \xi_1^3|$, $|\tau - \xi^3| \leq \frac{1}{1000}|\tau_1 - \xi_1^3|$, and $|\tau| \leq \frac{1}{8}|\xi|^3$, then

$$\frac{|\xi_1|^\rho}{\langle \tau_1 - \xi_1^3 \rangle^b} \left(\int_\xi \int_\tau \frac{|\xi|^2 |\xi_2|^{2\rho}}{\langle \tau \rangle^{2\rho/3} \langle \xi \rangle^{6b} \langle \tau_2 - \xi_2^3 \rangle^{2b}} \, d\xi \, d\tau \right)^{1/2} \tag{2.149}$$

is bounded. *Proof:* Since $|\tau| \leq |\xi|^3$, we have $\dfrac{1}{\langle \xi \rangle^{6b-2\rho}} \leq \dfrac{1}{\langle \tau \rangle^{2b - \frac{2\rho}{3}}}$, and thus

$$(2.149) \leq \frac{|\xi_1|^\rho}{\langle \tau_1 - \xi_1^3 \rangle^b} \left(\int_\xi \int_\tau \frac{|\xi|^2 |\xi_2|^{2\rho}}{\langle \xi \rangle^{2\rho} \langle \tau \rangle^{2b} \langle \tau_2 - \xi_2^3 \rangle^{2b}} \, d\xi \, d\tau \right)^{1/2} \tag{2.150}$$

Carrying out the τ integral and applying Lemma 31,

$$(2.150) \leq \frac{|\xi_1|^\rho}{\langle \tau_1 - \xi_1^3 \rangle^b} \left(\int_\xi \frac{|\xi|^2 |\xi_2|^{2\rho}}{\langle \xi \rangle^{2\rho} \langle \tau_1 - \xi_1^3 - 3\xi\xi_1\xi_2 + \xi^3 \rangle^{4b-1}} \, d\xi \right)^{1/2} \tag{2.151}$$

Case 1. $3|\xi\xi_1\xi_2| \leq \frac{1}{2}|\tau_1 - \xi_1^3|$.

Since $|\tau - \xi^3| << |\tau_1 - \xi_1^3|$ and $|\tau| \leq \frac{1}{8}|\xi|^3$, we have $|\xi|^3 << |\tau_1 - \xi_1^3|$, giving

$$\langle \tau_1 - \xi_1^3 - 3\xi\xi_1\xi_2 + \xi^3 \rangle \sim \langle \tau_1 - \xi_1^3 \rangle$$

and thus

$$(2.151) \leq \frac{|\xi_1|^\rho}{\langle \tau_1 - \xi_1^3 \rangle^{3b - \frac{1}{2}}} \left(\int_\xi \frac{|\xi|^2 |\xi_2|^{2\rho}}{\langle \xi \rangle^{2\rho}} \, d\xi \right)^{1/2} \tag{2.152}$$

Using that $|\xi\xi_1\xi_2| \leq |\tau_1 - \xi_1^3|$,

$$(2.152) \leq \frac{|\tau_1 - \xi_1^3|^\rho}{\langle \tau_1 - \xi_1^3 \rangle^{3b - \frac{1}{2}}} \left(\int_\xi \frac{|\xi|^{2-2\rho}}{\langle \xi \rangle^{2\rho}} \, d\xi \right)^{1/2} \tag{2.153}$$

Carrying out the ξ integral over the region $|\xi| \leq |\tau_1 - \xi_1^3|^{1/3}$ gives

$$\int_\xi \frac{|\xi|^{2-2\rho}}{\langle \xi \rangle^{2\rho}} \, d\xi \leq \langle \tau_1 - \xi_1^3 \rangle^{1 - \frac{4}{3}\rho}$$

and thus

$$(2.153) \leq \langle \tau_1 - \xi_1^3 \rangle^{1+\frac{1}{3}\rho - 3b} \tag{2.154}$$

which is bounded provided $b \geq \frac{5}{12}$.

Case 2. $3|\xi\xi_1\xi_2| \geq \frac{1}{2}|\tau_1 - \xi_1^3|$.

In this case, $|\xi| \leq \frac{1}{10}|\xi_1|$. Indeed, if $|\xi_1| \leq 10|\xi|$, then $3|\xi\xi_1\xi_2| \leq 330|\xi|^3 \leq \frac{1}{3}|\tau_1 - \xi_1^3|$. Let $u = \tau_1 - \xi_1^3 - 3\xi_1(\xi - \xi_1)\xi + \xi^3$, $du = 3\xi_1(-2\xi + \xi_1) + 3\xi^2$. Now $3|\xi|^2 \leq \frac{3}{100}|\xi_1|^2$ and $3|\xi_1(-2\xi + \xi_1)| \geq \frac{12}{5}|\xi_1|^2$, and thus $3|\xi|^2 << 3|\xi_1(2\xi - \xi_1)|$.

$$(2.151) \leq \frac{|\xi_1|^\rho}{\langle \tau_1 - \xi_1^3 \rangle^b} \left(\int_\xi \frac{|\xi|^2 |\xi_2|^{2\rho} |3\xi_1(\xi_1 - 2\xi) + 3\xi^2|}{\langle \xi \rangle^{2\rho} \langle \tau_1 - \xi_1^3 - 3\xi\xi_1\xi_2 + \xi^3 \rangle^{4b-1} |\xi_1(\xi_1 - 2\xi)|} \, d\xi \right)^{1/2} \tag{2.155}$$

Using $|\xi_2| \sim |\xi_1|$, and $|\xi_1(\xi_1 - 2\xi)| \sim |\xi_1|^2$,

$$(2.155) \leq \frac{|\xi_1|^{2\rho - 1}}{\langle \tau_1 - \xi_1^3 \rangle^b} \left(\int_{|u| \leq |\tau_1 - \xi_1^3|} \frac{|\xi|^{2-2\rho}}{\langle u \rangle^{4b-1}} \, du \right)^{1/2} \tag{2.156}$$

Using that $|\xi| \leq \dfrac{|\tau_1 - \xi_1^3|}{|\xi_1|^2}$,

$$(2.156) \leq \frac{|\xi_1|^{2\rho-1} |\tau_1 - \xi_1^3|^{1-\rho}}{|\xi_1|^{2(1-\rho)} \langle \tau_1 - \xi_1^3 \rangle^b} \left(\int_{|u| \leq |\tau_1 - \xi_1^3|} \frac{du}{\langle u \rangle^{4b-1}} \right)^{1/2} \tag{2.157}$$

Carrying out the u integral,

$$(2.157) \leq \frac{|\xi_1|^{4\rho - 3}}{\langle \tau_1 - \xi_1^3 \rangle^{\rho + 3b - 2}} \tag{2.158}$$

which is bounded provided $b \geq \frac{2}{3} - \frac{1}{3}\rho$.

□

Lemma 35. *If $\frac{3}{2} < s < 3$, then $\exists\, b = b(s) < \frac{1}{2}$ such that $\forall\, \alpha > \frac{1}{2}$,*

$$\|\partial_x(uv)\|_{Y_{s,-b}} \leq c\|u\|_{X_{s,b} \cap D_\alpha} \|v\|_{X_{s,b} \cap D_\alpha} \tag{2.159}$$

Proof. It suffices to show

$$\iint_* \frac{|\xi|\langle\tau\rangle^{s/3}\hat{d}(\xi,\tau)}{\langle\tau-\xi^3\rangle^b}\frac{\hat{g}_1(\xi_1,\tau_1)}{\langle\tau_1-\xi_1^3\rangle^b\langle\xi_1\rangle^s}\frac{\hat{g}_2(\xi_2,\tau_2)}{\langle\tau_2-\xi_2^3\rangle^b\langle\xi_2\rangle^s} \leq c\|d\|_{L^2}\|g_1\|_{L^2}\|g_2\|_{L^2}$$

for $\hat{d} \geq 0$, $\hat{g}_1 \geq 0$, $\hat{g}_2 \geq 0$, where $*$ indicates integration over ξ, ξ_1, ξ_2, subject to the constraint $\xi = \xi_1 + \xi_2$, and over τ, τ_1, τ_2, subject to the constraint $\tau = \tau_1 + \tau_2$, under the assumption $|\tau| >> |\xi|^3$, since, for $s > 0$ in the region $|\tau| \leq 2|\xi|^3$, $\|\partial_x(uv)\|_{Y_{s,-b}} \leq c\|\partial_x(uv)\|_{X_{s,-b}}$. We shall show

$$\frac{|\xi|\langle\tau\rangle^{s/3}}{\langle\tau-\xi^3\rangle^b}\left(\int_{\xi_1}\int_{\tau_1}\frac{d\xi_1\, d\tau_1}{\langle\tau_1-\xi_1^3\rangle^{2b}\langle\xi_1\rangle^{2s}\langle\tau_2-\xi_2^3\rangle^{2b}\langle\xi_2\rangle^{2s}}\right)^{1/2} \leq c \tag{2.160}$$

Since $\tau - \xi^3 + 3\xi\xi_1\xi_2 = (\tau_2 - \xi_2^3) + (\tau_1 - \xi_1^3)$, by Lemma 31, we have

$$(2.160) \leq |\xi|\langle\tau\rangle^{\frac{s}{3}-b}\left(\int_{\xi_1}\frac{1}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}}\frac{1}{\langle\tau-\xi^3+3\xi\xi_1\xi_2\rangle^{4b-1}}\, d\xi_1\right)^{1/2} \tag{2.161}$$

Case 1. $|\xi_1| << |\xi_2|$ or $|\xi_2| << |\xi_1|$. In this case, $3|\xi\xi_1\xi_2| << |\xi|^3$, which combined with $|\xi|^3 << |\tau|$, implies $\langle\tau - \xi^3 + 3\xi\xi_1\xi_2\rangle \sim \langle\tau\rangle$. Thus

$$(2.161) \leq \langle\tau\rangle^{\frac{s}{3}-3b+\frac{1}{2}}\left(\int_{\xi_1}\frac{|\xi|^2\, d\xi_1}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}}\right)^{1/2} \leq \langle\tau\rangle^{\frac{s}{3}-3b+\frac{1}{2}}\left(\int_{\xi_1}\frac{d\xi_1}{\langle\xi_1\rangle^{2s-2}\langle\xi_2\rangle^{2s-2}}\right)^{1/2} \tag{2.162}$$

Provided $s > \frac{3}{2}$ and $b > \frac{1}{9}s + \frac{1}{6}$, (2.162) is bounded.

Case 2. $|\xi_1| \sim |\xi_2|$.

Case 2A. $3|\xi\xi_1\xi_2| \sim |\tau|$ or $3|\xi\xi_1\xi_2| >> |\tau|$. Then we ignore $\langle\tau - \xi^3 + 3\xi\xi_1\xi_2\rangle^{4b-1}$ in (2.161) and bound as:

$$(2.161) \leq \left(\int_{\xi_1}\frac{|\xi|^2\langle\tau\rangle^{\frac{2s}{3}-2b}}{\langle\xi_1\rangle^{2s}\langle\xi_2\rangle^{2s}}\, d\xi_1\right)^{1/2} \tag{2.163}$$

Using that $\langle\tau\rangle \leq c\langle\xi\rangle\langle\xi_1\rangle\langle\xi_2\rangle$, $\langle\xi\rangle \leq \langle\xi_1\rangle + \langle\xi_2\rangle$, and $\langle\xi_1\rangle \sim \langle\xi_2\rangle$,

$$(2.163) \leq \left(\int \frac{1}{\langle\xi_1\rangle^{2s+6b-2}}\, d\xi_1\right)^{1/2} \tag{2.164}$$

Thus, we need $2s + 6b - 2 > 1$, which is automatically satisfied if $s > \frac{3}{2}$ and $b > 0$.
Case 2B. $3|\xi\xi_1\xi_2| << |\tau|$. Here, we just follow the method of Case 1. □

Interpolating between the results of Lemma 34 and Lemma 35, we obtain

Proposition 12. *If* $-\frac{3}{4} < s < 3$, *then* $\exists\, b = b(s) < \frac{1}{2}$ *such that* $\forall\, \alpha > \frac{1}{2}$,

$$\|\partial_x(uv)\|_{Y_{s,-b}} \leq c\|u\|_{X_{s,b}\cap D_\alpha}\|v\|_{X_{s,b}\cap D_\alpha} \tag{2.165}$$

2.11 Well-posedness theorems for the left half-line

We shall consider two regions.

Region I. $-\frac{3}{2} < s < \frac{1}{2} \quad \Longrightarrow \quad \begin{cases} -\frac{1}{6} < \frac{s+1}{3} < \frac{1}{2} \\ -\frac{1}{2} < \frac{s}{3} < \frac{1}{6} \end{cases}$

No compatibility conditions are needed. (The nonlinear result will only hold for $s > -\frac{3}{4}$.)

Region II. $\frac{1}{2} < s < \frac{3}{2} \quad \Longrightarrow \quad \begin{cases} \frac{1}{2} < \frac{s+1}{3} < \frac{5}{6} \\ \frac{1}{6} < \frac{s}{3} < \frac{1}{2} \end{cases}$

We need $f(0) = \phi(0)$, but nothing connecting g and $\partial_x\phi$.

If we were to enter the next admissible region, $\frac{3}{2} < s < \frac{7}{2}$, we would need the additional compatibility condition $g(0) = \partial_x\phi(0)$.

Let

$$\begin{aligned} Z_s = \{\, w \mid\ & w \in C(\, x \in (-\infty, +\infty);\ H^{\frac{s+1}{3}}(\mathbb{R}_t)\,),\ \sup_{x\in\mathbb{R}} \|w(x,-)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} < +\infty, \\ & \partial_x w \in C(\, x \in (-\infty, +\infty); H^{s/3}(\mathbb{R}_t)\,),\ \sup_{x\in\mathbb{R}} \|\partial_x w(x,-)\|_{H^{s/3}(\mathbb{R}_t)} < +\infty \\ & w \in C(\, t \in (-\infty, +\infty);\ H^s(\mathbb{R}_x)\,),\ \sup_{t\in\mathbb{R}} \|w(x,t)\|_{H^s(\mathbb{R}_x)} < +\infty \\ & w \in X_{s,b} \cap D_\alpha \,\} \end{aligned}$$

with norm

$$\begin{aligned} \|w\|_{Z_s} = \ & \sup_{x\in\mathbb{R}} \|w(x,-)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} + \sup_{x\in\mathbb{R}} \|\partial_x w(x,-)\|_{H_t^{s/3}} \\ & + \sup_{t\in\mathbb{R}} \|w(x,t)\|_{H^s(\mathbb{R}_x)} + \|w\|_{X_{s,b}\cap D_\alpha} \end{aligned}$$

Let

$$\begin{aligned} V_s = \{\, (\phi, f, g)\ \mid \phi(x) \in H^s(\mathbb{R}_x^-),\ f(t) \in H^{\frac{s+1}{3}}(\mathbb{R}_t^+),\ g(t) \in H^{\frac{s}{3}}(\mathbb{R}_t^+) \\ \text{and } \phi, f, g \text{ satisfy the compatibility conditions for the} \\ \text{region in which } s \text{ lies}\} \end{aligned}$$

with norm

$$\|(\phi, f, g)\|_{V_s} = \|\phi\|_{H^s(\mathbb{R}_x^-)} + \|f\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t^+)} + \|g\|_{H^{s/3}(\mathbb{R}_t^+)}$$

Theorem 3 (Homogeneous solution operator for left half-line). *Let s be in either region I or II as described above, and let $b < \frac{1}{2}$. Fix $T > 0$. Then there exists $\alpha = \alpha(s) > \frac{1}{2}$ and a bounded linear operator*

$$\mathrm{HS}_T : V_s \longrightarrow Z_{s,b,\alpha}$$

such that, with $(\phi, f, g) \in V_s$ *and* $u = HS_T(\phi, f, g)$*, we have*

$$\begin{cases} \partial_t u + \partial_x^3 u = 0 & \text{in } (-\infty, 0) \times (0,T) \quad (2.166) \\ u(x,0) = \phi(x) & \text{on } (-\infty, 0) \quad (2.167) \\ u(0,t) = f(t) & \text{on } (0,T) \quad (2.168) \\ \partial_x u(0,t) = g(t) & \text{on } (0,T) \quad (2.169) \end{cases}$$

where (2.166) *holds in the sense of distributions,* (2.167) *holds in the sense of* $C(\, t \in (-\infty, +\infty);\, H^s(\mathbb{R}_x)\,)$, (2.168) *holds in the sense of* $C(\, x \in (-\infty, +\infty);\, H^{\frac{s+1}{3}}(\mathbb{R}_t)\,)$*, and* (2.169) *holds in the sense of* $C(\, x \in (-\infty, +\infty);\, H^{\frac{s}{3}}(\mathbb{R}_t)\,)$

Proof. Collectively, the estimates require that

$$\begin{aligned} &-1 < \lambda < \tfrac{1}{2} \\ &s-1 \leq \lambda < s + \tfrac{1}{2} \\ &\tfrac{1}{2} < \alpha \leq \tfrac{s-\lambda+2}{3} \end{aligned} \tag{2.170}$$

Obtain extensions $\tilde{f} \in H^{\frac{s+1}{3}}(\mathbb{R}_t)$ of $f \in H^{\frac{s+1}{3}}(\mathbb{R}_t^+)$, $\tilde{g} \in H^{\frac{s}{3}}(\mathbb{R}_t)$ of $g \in H^{\frac{s}{3}}(\mathbb{R}_t^+)$, and $\tilde{\phi} \in H^s(\mathbb{R}_x)$ of $\phi \in H^s(\mathbb{R}_x^-)$ of comparable size. Let $\Psi_1(t) \in C^\infty(\mathbb{R})$ be such that $\Psi_1(t) = 1$ on $[-T, T]$, and supp $\Psi_1(t) \subset [-2T, 2T]$, and let $\Psi_2(t) = \Psi_1(t/2)$, $\Psi_3(t) = \Psi_1(t/4)$. By Lemma 25,

$$\| \Psi_1(t) S(t) \tilde{\phi} \big|_{x=0} \|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} \leq c \|\tilde{\phi}\|_{H^s(\mathbb{R}_x)}$$

and by Lemma 28,

$$\| \Psi_1(t) \partial_x S(t) \tilde{\phi} \big|_{x=0} \|_{H^{\frac{s}{3}}(\mathbb{R}_t)} \leq c \|\tilde{\phi}\|_{H^s(\mathbb{R}_x)}$$

Let

$$f_1(t) = \chi_{(0,+\infty)}(t) \Psi_1(t) \left[\tilde{f}(t) - S(t)\tilde{\phi}\big|_{x=0} \right]$$

If s is in Region I ($-\frac{3}{2} < s < \frac{1}{2}$), then $-\frac{1}{6} < \frac{s+1}{3} < \frac{1}{2}$, and therefore by Lemma 7, Corollary 1,

$$\|f_1\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t)} \le \|\tilde{f}\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} + \|\tilde{\phi}\|_{H^s(\mathbb{R}_x)} \tag{2.171}$$

If s is in Region II ($\frac{1}{2} < s < \frac{3}{2}$), then $\frac{1}{2} < \frac{s+1}{3} < \frac{5}{6}$ and by the compatibility condition $f(0) = \phi(0)$, Lemmas 9 and 8, we have the same bound (2.171). Let

$$g_1(t) = \chi_{(0,+\infty)}(t)\Psi_1(t)\left[\tilde{g}(t) - \partial_x S(t)\tilde{\phi}\big|_{x=0}\right]$$

If s is in either Region I or II, then $-\frac{1}{2} < \frac{s}{3} < \frac{1}{2}$, and therefore by Lemma 7, Corollary 1,

$$\|g_1\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t)} \le \|\tilde{g}\|_{H^{\frac{s}{3}}(\mathbb{R}_t)} + \|\tilde{\phi}\|_{H^s(\mathbb{R}_x)} \tag{2.172}$$

Pick $\lambda_1 \neq \lambda_2$ such that the conditions (2.170) are satisfied. Let $h_1,\ h_2 \in H_0^{\frac{s+1}{3}}(\mathbb{R}_t)$ solve the system

$$\begin{aligned} f_1(t) &= \tfrac{2}{3}\sin(\tfrac{\pi}{3}\lambda_1 + \tfrac{\pi}{6})h_1(t) + \tfrac{2}{3}\sin(\tfrac{\pi}{3}\lambda_2 + \tfrac{\pi}{6})h_2(t) \\ \Psi_3(t)\mathcal{I}_{1/3}g_1(t) &= \tfrac{2}{3}\sin(\tfrac{\pi}{3}\lambda_1 - \tfrac{\pi}{6})h_1(t) + \tfrac{2}{3}\sin(\tfrac{\pi}{3}\lambda_2 - \tfrac{\pi}{6})h_2(t) \end{aligned}$$

(Note that the determinant of the 2×2 coefficient matrix is nonzero.) Then

$$\|h_1\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)} + \|h_2\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)} \le c\|f_1\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)} + c\|g_1\|_{H_0^{\frac{s}{3}}(\mathbb{R}_t^+)}$$

Also, supp $h_1 \subset [0, 4T]$, supp $h_2 \subset [0, 4T]$. Now set

$$u(x,t) = \Psi_2(t)S(t)\tilde{\phi}(x,t) + \Psi_2(t)\mathcal{L}_-^{\lambda_1}(h_1)(x,t) + \Psi_2(t)\mathcal{L}_-^{\lambda_2}(h_2)(x,t)$$

Then, by (2.71) and Lemma 26,

$$\begin{aligned} \lim_{x\to 0^-} \|f(t) - u_1(x,t)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} &= 0 \\ \lim_{x\to 0^-} \|g(t) - u_1(x,t)\|_{H^{\frac{s}{3}}(\mathbb{R}_t)} &= 0 \end{aligned}$$

Thus, u is a solution to the problem (2.166)-(2.169), and satisfies all the needed estimates by Lemmas 19, 20, 22, 23, 25, 26, 28, 29. ☐

Theorem 4 (Inhomogeneous solution operator for left half-line). *Let s be in Region I or II, as described above, and let $b < \frac{1}{2}$. Fix $T > 0$. There exists $\alpha = \alpha(s) > \frac{1}{2}$ and a bounded linear operator*

$$\mathrm{IHS}_T : \begin{cases} X_{s,-b} \cap Y_{s,-b} \longrightarrow Z_{s,b,\alpha} & \text{for any } s \\ X_{s,-b} \longrightarrow Z_{s,b,\alpha} & 0 \le s \le \frac{1}{2} \end{cases}$$

such that, with $w \in X_{s,-b}$ for $0 \le s \le \frac{1}{2}$ or $w \in X_{s,-b} \cap Y_{s,-b}$ for any s, and $u = \mathrm{IHS}_T(w)$, we have

$$\begin{cases} \partial_t u + \partial_x^3 u = w & \text{in } (-\infty, 0) \times (0,T) & (2.173) \\ u(x,0) = 0 & \text{on } (-\infty, 0) & (2.174) \\ u(0,t) = 0 & \text{on } (0,T) & (2.175) \\ \partial_x u(0,t) = 0 & \text{on } (0,T) & (2.176) \end{cases}$$

where (2.173) *holds in the sense of distributions,* (2.174) *holds in the sense of* $C(\, t \in (-\infty, +\infty);\ H^s(\mathbb{R}_x)\,)$, (2.175) *holds in the sense of* $C(\, x \in (-\infty, +\infty);\ H^{\frac{s+1}{3}}(\mathbb{R}_t)\,)$, *and* (2.176) *holds in the sense of* $C(\, x \in (-\infty, +\infty);\ H^{\frac{s}{3}}(\mathbb{R}_t)\,)$.

Proof. We shall require $\alpha \le 1 - b$ in addition to the requirements stated at the beginning of the proof of Theorem 3. Let

$$u_1(x,t) = \Psi_1(t) \int_0^t S(t-t') w(x,t')\, dt'$$

Then u_1 satisfies the needed estimates by Lemmas 21, 24, 27, 30. Let $f(t) = u_1(0,t)$ and $g(t) = \partial_x u_1(0,t)$. By Lemma 27 and 30,

$$\|f\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t^+)} + \|g\|_{H^{\frac{s}{3}}(\mathbb{R}_t^+)} \le c \begin{cases} \|w\|_{X_{s,-b}} & 0 \le s \le \frac{1}{2} \\ \|w\|_{X_{s,-b}} + \|w\|_{Y_{s,-b}} & \text{for any } s \end{cases}$$

Set $u_2 = \mathrm{HS}_T(f, g, 0)$, and then set $u = u_1 - u_2$. $\square$

Theorem 5 (KdV on the left half-line). *Let s lie in either Region I or II, and suppose we are given initial data $\phi(x) \in H^s(\mathbb{R}^-)$, boundary data $f(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$, and derivative boundary data $g(t) \in H^{\frac{s}{3}}(\mathbb{R}^+)$, satisfying compatibility conditions needed for the region in which s lies. Then*

$$\exists\, T = T(\|\phi\|_{H^s}, \|f\|_{H^{\frac{s+1}{3}}}, \|g\|_{H^{\frac{s}{3}}}) > 0$$

and $u(x,t)$ such that

$$u \in (X_{s,b} \cap D_\alpha) \cap C(\, t \in (-\infty, +\infty);\, H^s(\mathbb{R}_x)\,) \cap C(\, x \in (-\infty, +\infty);\, H^{\frac{s+1}{3}}(\mathbb{R}_t)\,)$$

$$\partial_x u \in C(\, x \in (-\infty, +\infty);\; H^{\frac{s}{3}}(\mathbb{R}_t)\,)$$

satisfying

$$\begin{aligned} \|u\|_{X_{s,b}\cap D_\alpha} &+ \sup_x \|u(x,-)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} + \sup_x \|\partial_x u(x,-)\|_{H^{\frac{s}{3}}(\mathbb{R}_t)} + \sup_t \|u(-,t)\|_{H^s(\mathbb{R}_x)} \\ &\leq c(\|\phi\|_{H^s(\mathbb{R}^-)} + \|f\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)} + \|g\|_{H^{\frac{s}{3}}(\mathbb{R}^+)}) \end{aligned}$$

and

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & \text{in } (-\infty, 0) \times (0, T) \quad (2.177) \\ u(x,0) = \phi & \text{on } (-\infty, 0) \quad (2.178) \\ u(0,t) = f & \text{on } (0, T) \quad (2.179) \\ \partial_x u(0,t) = g & \text{on } (0, T) \quad (2.180) \end{cases}$$

where (2.177) *holds in the sense of distributions,* (2.178) *holds in the sense of $C(\, t \in (-\infty, +\infty);\, H^s(\mathbb{R}_x)\,)$,* (2.179) *holds in the sense of $C(\, x \in (-\infty, +\infty);\, H^{\frac{s+1}{3}}(\mathbb{R}_t)\,)$, and* (2.180) *holds in the sense of $C(\, x \in (-\infty, +\infty);\, H^{\frac{s}{3}}(\mathbb{R}_t)\,)$*

Proof. By a scaling argument, it suffices to prove the result for $T = 1$ provided the initial-boundary data triple is sufficiently small in norm. This follows from Theorems 3 and 4 by a standard contraction argument, with $w = \partial_x(u^2)$. For $0 \leq s \leq \frac{1}{2}$, we appeal to the bilinear estimate Prop. 11, while for $-\frac{3}{4} < s < 0$ and $\frac{1}{2} < s < \frac{3}{2}$, we appeal to the bilinear estimates Prop. 11 and Prop. 12. $\square$

2.12 Well-posedness theorems for the right half-line

The results for the right half-line are proved in an analogous fashion, so we shall only state the results.

Region I. $-\frac{3}{2} < s < \frac{1}{2} \quad \Longrightarrow \quad \begin{cases} -\frac{1}{6} < \frac{s+1}{3} < \frac{1}{2} \\ -\frac{1}{2} < \frac{s}{3} < \frac{1}{6} \end{cases}$

No compatibility conditions are needed. (The nonlinear result will only hold for $s > -\frac{3}{4}$.)

Region II. $\frac{1}{2} < s < \frac{3}{2} \quad \Longrightarrow \quad \begin{cases} \frac{1}{2} < \frac{s+1}{3} < \frac{5}{6} \\ \frac{1}{6} < \frac{s}{3} < \frac{1}{2} \end{cases}$

We need $f(0) = \phi(0)$.

Let

$$\begin{aligned} Z_s = \{\, w \mid\ & w \in C(\, x \in (-\infty, +\infty);\ H^{\frac{s+1}{3}}(\mathbb{R}_t)\,),\ \sup_{x\in\mathbb{R}} \|w(x,-)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} < +\infty, \\ & w \in C(\, t \in (-\infty, +\infty);\ H^s(\mathbb{R}_x)\,),\ \sup_{t\in\mathbb{R}} \|w(x,t)\|_{H^s(\mathbb{R}_x)} < +\infty \\ & w \in X_{s,b} \cap D_\alpha \,\} \end{aligned}$$

with norm

$$\|w\|_{Z_s} = \sup_{x\in\mathbb{R}} \|w(x,-)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} + \sup_{t\in\mathbb{R}} \|w(x,t)\|_{H^s(\mathbb{R}_x)} + \|w\|_{X_{s,b}\cap D_\alpha}$$

Let

$$\begin{aligned} V_s = \{\, (\phi, f)\ \mid \phi(x) \in H^s(\mathbb{R}_x^+),\ f(t) \in H^{\frac{s+1}{3}}(\mathbb{R}_t^+),\ & \text{and } \phi, f \text{ satisfy the} \\ & \text{compatibility conditions for the region in which } s \text{ lies}\} \end{aligned}$$

with norm

$$\|(\phi, f)\|_{V_s} = \|\phi\|_{H^s(\mathbb{R}_x^+)} + \|f\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t^+)}$$

Theorem 6 (Homogeneous solution operator for right half-line). *Let s be in either region I or II as described above, and let $b < \frac{1}{2}$. Fix $T > 0$. Then there exists $\alpha = \alpha(s) > \frac{1}{2}$ and a bounded linear operator*

$$\mathrm{HS}_T : V_s \longrightarrow Z_{s,b,\alpha}$$

such that, with $(\phi, f) \in V_s$ and $u = HS_T(\phi, f)$, we have

$$\begin{cases} \partial_t u + \partial_x^3 u = 0 & \textit{in } (0,+\infty)\times(0,T) \quad (2.181) \\ u(x,0) = \phi(x) & \textit{on } (0,+\infty) \quad (2.182) \\ u(0,t) = f(t) & \textit{on } (0,T) \quad (2.183) \end{cases}$$

where (2.181) *holds in the sense of distributions,* (2.182) *holds in the sense of $C(\, t \in (-\infty,+\infty);\, H^s(\mathbb{R}_x)\,)$, and* (2.183) *holds in the sense of*
$C(\, x \in (-\infty,+\infty);\, H^{\frac{s+1}{3}}(\mathbb{R}_t)\,)$.

Theorem 7 (Inhomogeneous solution operator for right half-line). *Let s be in Region I or II, as described above, and let $b < \frac{1}{2}$. Fix $T > 0$. There exists $\alpha = \alpha(s) > \frac{1}{2}$ and a bounded linear operator*

$$\mathrm{IHS}_T : \begin{cases} X_{s,-b} \cap Y_{s,-b} \longrightarrow Z_{s,b,\alpha} & \textit{for any } s \\ X_{s,-b} \longrightarrow Z_{s,b,\alpha} & -1 \le s \le \frac{1}{2} \end{cases}$$

such that, with $w \in X_{s,-b}$ for $-1 \le s \le \frac{1}{2}$ or $w \in X_{s,-b} \cap Y_{s,-b}$ for any s, and $u = \mathrm{IHS}_T(w)$, we have

$$\begin{cases} \partial_t u + \partial_x^3 u = w & \textit{in } (0,+\infty)\times(0,T) \quad (2.184) \\ u(x,0) = 0 & \textit{on } (0,+\infty) \quad (2.185) \\ u(0,t) = 0 & \textit{on } (0,T) \quad (2.186) \end{cases}$$

where (2.184) *holds in the sense of distributions,* (2.185) *holds in the sense of $C(\, t \in (-\infty,+\infty);\, H^s(\mathbb{R}_x)\,)$, and* (2.186) *holds in the sense of*
$C(\, x \in (-\infty,+\infty);\, H^{\frac{s+1}{3}}(\mathbb{R}_t)\,)$.

Theorem 8 (KdV on the right half-line). *Let s lie in either Region I or II, and suppose we are given initial data $\phi(x) \in H^s(\mathbb{R}^+)$ and boundary data $f(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$, satisfying compatibility conditions needed for the region in which s lies. Then*

$$\exists\, T = T(\|\phi\|_{H^s}, \|f\|_{H^{\frac{s+1}{3}}}) > 0$$

and $u(x,t)$ such that

$$u \in (X_{s,b} \cap D_\alpha) \cap C(\, t \in (-\infty,+\infty);\, H^s(\mathbb{R}_x)\,) \cap C(\, x \in (-\infty,+\infty);\, H^{\frac{s+1}{3}}(\mathbb{R}_t)\,)$$

satisfying

$$\begin{aligned} &\|u\|_{X_{s,b}\cap D_\alpha} + \sup_x \|u(x,-)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} + \sup_t \|u(-,t)\|_{H^s(\mathbb{R}_x)} \\ &\qquad \le c(\|\phi\|_{H^s(\mathbb{R}^+)} + \|f\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)}) \end{aligned}$$

and

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & \text{in } (0,+\infty)\times(0,T) \quad (2.187) \\ u(x,0) = \phi & \text{on } (0,+\infty) \quad (2.188) \\ u(0,t) = f & \text{on } (0,T) \quad (2.189) \end{cases}$$

where (2.187) *holds in the sense of distributions,* (2.188) *holds in the sense of* $C(\, t \in (-\infty,+\infty);\, H^s(\mathbb{R}_x)\,)$*, and* (2.189) *holds in the sense of* $C(\, x \in (-\infty,+\infty);\, H^{\frac{s+1}{3}}(\mathbb{R}_t)\,)$.

2.13 Initial-boundary value problem on the line segment

In this section, we will solve the initial-boundary value problem on the line segment $0 \le x \le 1$:

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & \text{in } (0,1)\times(0,T) \\ u(0,t) = g_3(t) & \text{on } (0,T) \\ u(1,t) = g_1(t) & \text{on } (0,T) \\ \partial_x u(1,t) = g_2(t) & \text{on } (0,T) \\ u(x,0) = \phi & \text{on } (0,1) \end{cases}$$

with $\phi(x) \in H^s((0,1))$, $g_1(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$, $g_2(t) \in H^{\frac{s}{3}}(\mathbb{R}^+)$, $g_3(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$. We will consider two regions:
Region I. $-\frac{3}{4} < s < \frac{1}{2}$. Here, we do not need any compatibility conditions.
Region II. $\frac{1}{2} < s < \frac{3}{2}$. Here, we need $g_3(0) = \phi(0)$ and $g_1(0) = \phi(1)$.

Theorem 9 (KdV on the line segment $[0,1]$). *Let s lie in either Region I or II. Given initial data $\phi(x) \in H^s((0,1))$, and boundary data $g_1(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$, $g_2(t) \in H^{\frac{s}{3}}(\mathbb{R}^+)$, $g_3(t) \in H^{\frac{s+1}{3}}(\mathbb{R}^+)$ satisfying compatibility conditions needed for the region in which s lies. Then*

$$\exists\, T = T(\|\phi\|_{H^s}, \|g_1\|_{H^{\frac{s+1}{3}}}, \|g_2\|_{H^{\frac{s}{3}}}, \|g_3\|_{H^{\frac{s+1}{3}}}) > 0$$

and $u(x,t)$ such that

$$u \in (X_{s,b} \cap D_\alpha) \cap C(\, t \in (-\infty,+\infty);\ H^s(\mathbb{R}_x)\,) \cap C(\, x \in (-\infty,+\infty);\ H^{\frac{s+1}{3}}(\mathbb{R}_t)\,)$$

$$\partial_x u \in C(\, x \in (-\infty,+\infty);\ H^{\frac{s}{3}}(\mathbb{R}_t)\,)$$

satisfying

$$\|u\|_{X_{s,b}\cap D_\alpha} + \sup_x \|u(x,-)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)}$$
$$+ \sup_x \|\partial_x u(x,-)\|_{H^{\frac{s}{3}}(\mathbb{R}_t)} + \sup_t \|u(-,t)\|_{H^s(\mathbb{R}_x)}$$
$$\leq c(\|\phi\|_{H^s(0,1)} + \|g_1\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)} \|g_2\|_{H^{\frac{s}{3}}(\mathbb{R}^+)} + \|g_3\|_{H^{\frac{s+1}{3}}(\mathbb{R}^+)})$$

and

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = 0 & \text{in } (0,1)\times(0,T) \quad (2.190) \\ u(0,t) = g_3(t) & \text{on } (0,T) \quad (2.191) \\ u(1,t) = g_1(t) & \text{on } (0,T) \quad (2.192) \\ \partial_x u(1,t) = g_2(t) & \text{on } (0,T) \quad (2.193) \\ u(x,0) = \phi(x) & \text{on } (0,1) \quad (2.194) \end{cases}$$

where (2.190) *holds in the sense of distributions,* (2.194) *holds in the sense of*

$$C(\, t \in (-\infty,+\infty);\; H^s(\mathbb{R}_x)\,)$$

(2.191)*,* (2.192) *hold in the sense of*

$$C(\, x \in (-\infty,+\infty);\; H^{\frac{s+1}{3}}(\mathbb{R}_t)\,)$$

and (2.193) *holds in the sense of*

$$C(\, x \in (-\infty,+\infty);\; H^{\frac{s}{3}}(\mathbb{R}_t)\,)$$

Let $\mathcal{C}$ be the space of smooth functions defined on $[0,1]$ that vanish, together with all derivatives, at 0, but with no restriction for 1. For $\sigma \geq 0$, let $\mathcal{H}^\sigma$ be the closure of $\mathcal{C} \cap H^\sigma((0,1))$ in $H^\sigma((0,1))$. Thus, $\mathcal{H}^\sigma$ will be the Sobolev space on $(0,1)$ with a vanishing condition at 0.

Claim 6. *Let* $\theta \in C^\infty(\mathbb{R})$*,* $\theta(t) = 1$ *for* $-1 \leq y \leq 1$*,* $\operatorname{supp} \theta \subset [-2,2]$*, and* $\lambda \geq -2$*. Then for* k *arbitrarily large, the operator* $\theta(t)\mathcal{L}_+^\lambda h(1,t) : H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+) \to H_0^k(\mathbb{R}_t^+)$ *is* *bounded.*

Proof. It suffices to prove the result for $\theta(t)\mathcal{U}_+^\lambda h(1,t)$. By (2.46),

$$\begin{aligned}\theta(t)\mathcal{U}_+^\lambda h(1,t) &= \theta(t)e^{i\pi\lambda}\int_{y=1}^{+\infty}\frac{(-1+y)^{\lambda+2}}{\Gamma(\lambda+3)}\mathcal{U}(\partial_t h)(y,t)\,dy \\ &= \theta(t)e^{i\pi\lambda}\int_{y=1}^{+\infty}\frac{(-1+y)^{\lambda+2}}{\Gamma(\lambda+3)} \\ &\qquad\qquad \left[\int_0^t A\left(\frac{y}{(t-t')^{1/3}}\right)\theta(2(t-t'))\frac{\partial_{t'}h(t')}{(t-t')^{1/3}}\,dt'\right]dy \\ &= \theta(t)\int_{t'} H(t-t')h(t')\,dt'\end{aligned}$$

where

$$H(t) = \begin{cases} e^{i\pi\lambda}\dfrac{\theta(2t)}{t^{1/3}}\displaystyle\int_{y=1}^{+\infty}\frac{(y-1)^{\lambda+2}}{\Gamma(\lambda+3)}A\left(\frac{y}{t^{1/3}}\right)dy & \text{if } t>0 \\ 0 & \text{if } t\le 0\end{cases}$$

Using the rapid decay of the Airy function and all of its derivatives on the right, we obtain $H(t)\in C_0^\infty(\mathbb{R})$. ∎

We need a special case of the Rellich theorem.

Theorem 10 (Rellich). *For $k>\frac{s+1}{3}$, the inclusion $i: H^k((0,1))\to H^{\frac{s+1}{3}}((0,1))$ is compact.*

See, for example, [Tay96], p. 286, Prop. 4.4. Let $i_1:\mathcal{H}^k\to H^k((0,1))$ be the inclusion. Then, by the Rellich theorem, $i\circ i_1:\mathcal{H}^k\to H^{\frac{s+1}{3}}((0,1))$ is compact. But since the image $i\circ i_1(\mathcal{H}^k)\subset\mathcal{H}^{\frac{s+1}{3}}$ and $\mathcal{H}^{\frac{s+1}{3}}$ is a closed subset of $H^{\frac{s+1}{3}}((0,1))$, we have that $i\circ i_1:\mathcal{H}^k\to\mathcal{H}^{\frac{s+1}{3}}$ is compact. By precomposing with extension and post-composing with a projection, the map from Claim 6, $\theta(t)\mathcal{L}_+^\lambda h(1,t): H_0^{\frac{s+1}{3}}(\mathbb{R}_t^+)\to H_0^k(\mathbb{R}_t^+)$ becomes $\theta(t)\mathcal{L}_+^\lambda h(1,t):\mathcal{H}^{\frac{s+1}{3}}\to\mathcal{H}^k$, and thus by the above, the map $\theta(t)\mathcal{L}_+^\lambda h(1,t):\mathcal{H}^{\frac{s+1}{3}}\to\mathcal{H}^{\frac{s+1}{3}}$ is compact.

Proof of Theorem 9. By the contraction method of §2.11, it suffices to solve

$$\begin{cases} \partial_t u + \partial_x^3 u = 0 & \text{in } (0,1)\times(0,T) \\ u(0,t) = f_3(t) & \text{on } (0,T) \\ u(1,t) = f_1(t) & \text{on } (0,T) \\ \partial_x u(1,t) = f_2(t) & \text{on } (0,T) \\ u(x,0) = 0 & \text{on } (0,1) \end{cases}$$

with

$$\|u\|_{X_{s,b}\cap D_\alpha} + \sup_x \|u(x,-)\|_{H^{\frac{s+1}{3}}(\mathbb{R}_t)} + \sup_x \|\partial_x u(x,-)\|_{H^{\frac{s}{3}}(\mathbb{R}_t)} + \sup_t \|u(-,t)\|_{H^s(\mathbb{R}_x)}$$
$$\leq c(\|f_1\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)} + \|f_2\|_{H_0^{\frac{s}{3}}(\mathbb{R}^+)} + \|f_3\|_{H_0^{\frac{s+1}{3}}(\mathbb{R}^+)})$$

for given f_1, f_2, and f_3.

We shall position two forcing operators $\mathcal{L}_1$ and $\mathcal{L}_2$ at the right endpoint $x = 1$ and one forcing operator $\mathcal{L}_3$ at the left endpoint $x = 0$.

$$\mathcal{L}_1(h_1)(x,t) = \mathcal{L}_-^{\lambda_1}(h_1)(x-1,t)$$
$$\mathcal{L}_2(h_2)(x,t) = \mathcal{L}_-^{\lambda_2}(h_2)(x-1,t)$$
$$\mathcal{L}_3(h_3)(x,t) = \mathcal{L}_+^{\lambda_3}(h_3)(x,t)$$

Let $\mathcal{L}(h_1,h_2,h_3) = \mathcal{L}_1 h_1 + \mathcal{L}_2 h_3 + \mathcal{L}_3 h_3$, so that, in the sense of distributions,

$$\begin{aligned}(\partial_t + \partial_x^3)&\mathcal{L}(h_1,h_2,h_3)(x,t) \\ &= \frac{(x-1)_+^{\lambda_1 - 1}}{\Gamma(\lambda_1)}\mathcal{I}_{-\frac{\lambda_1}{3}-\frac{2}{3}}(h_1)(t) + \frac{(x-1)_+^{\lambda_2 - 1}}{\Gamma(\lambda_2)}\mathcal{I}_{-\frac{\lambda_1}{3}-\frac{2}{3}}(h_2)(t) \\ &\quad + e^{i\pi\lambda_3}\frac{x_-^{\lambda_3-1}}{\Gamma(\lambda_3)}\mathcal{I}_{-\frac{\lambda_3}{3}-\frac{2}{3}}(h_3)(t)\end{aligned}$$

The key feature is, of course, that $(\partial_t + \partial_x^3)\mathcal{L}(h_1,h_2,h_3) = 0$ for $0 < x < 1$. We need to appropriately select $h_1(t) \in \mathcal{H}^{\frac{s+1}{3}}$, $h_2(t) \in \mathcal{H}^{\frac{s+1}{3}}$, $h_3(t) \in \mathcal{H}^{\frac{s+1}{3}}$ so that

$\mathcal{L}(h_1,h_2,h_3)(x,t)$ meets the specified boundary conditions:

$$\begin{aligned} \mathcal{L}(h_1,h_2,h_3)(1,t) &= f_1(t) \in \mathcal{H}^{\frac{s+1}{3}} \\ \partial_x\mathcal{L}(h_1,h_2,h_3)(1,t) &= f_2(t) \in \mathcal{H}^{\frac{s}{3}} \\ \mathcal{L}(h_1,h_2,h_3)(0,t) &= f_3(t) \in \mathcal{H}^{\frac{s+1}{3}} \end{aligned} \tag{2.195}$$

The set of conditions (2.195) translates into the matrix equality

$$\begin{bmatrix} f_1(t) \\ \mathcal{I}_{1/3}f_2(t) \\ f_3(t) \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\sin(\frac{\pi}{3}\lambda_1+\frac{\pi}{6}) & \frac{2}{3}\sin(\frac{\pi}{3}\lambda_2+\frac{\pi}{6}) & \mathcal{L}_3\big|_{x=1} \\ \frac{2}{3}\sin(\frac{\pi}{3}\lambda_1-\frac{\pi}{6}) & \frac{2}{3}\sin(\frac{\pi}{3}\lambda_2-\frac{\pi}{6}) & \mathcal{I}_{1/3}(\partial_x\mathcal{L}_3)\big|_{x=1} \\ \mathcal{L}_1\big|_{x=0} & \mathcal{L}_2\big|_{x=0} & \frac{1}{3}e^{i\pi\lambda_3} \end{bmatrix} \begin{bmatrix} h_1(t) \\ h_2(t) \\ h_3(t) \end{bmatrix} \tag{2.196}$$

or $f=(E+K)h$, with $f=(f_1,\mathcal{I}_{1/3}f_2,f_3)^T$, $h=(h_1,h_2,h_3)^T$, and $E,K:[\mathcal{H}^{\frac{s+1}{3}}]^3 \to [\mathcal{H}^{\frac{s+1}{3}}]^3$

$$E = \begin{bmatrix} \frac{2}{3}\sin(\frac{\pi}{3}\lambda_1+\frac{\pi}{6}) & \frac{2}{3}\sin(\frac{\pi}{3}\lambda_2+\frac{\pi}{6}) & 0 \\ \frac{2}{3}\sin(\frac{\pi}{3}\lambda_1-\frac{\pi}{6}) & \frac{2}{3}\sin(\frac{\pi}{3}\lambda_2-\frac{\pi}{6}) & 0 \\ \mathcal{L}_1\big|_{x=0} & \mathcal{L}_2\big|_{x=0} & \frac{1}{3}e^{i\pi\lambda_3} \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 0 & \mathcal{L}_3\big|_{x=1} \\ 0 & 0 & \mathcal{I}^t_{1/3}\partial_x\mathcal{L}_3\big|_{x=1} \\ 0 & 0 & 0 \end{bmatrix}$$

Note that E is invertible; in fact

$$E^{-1} = \begin{bmatrix} \frac{\sqrt{3}\sin(\frac{\pi}{3}\lambda_2-\frac{\pi}{6})}{\sin(\frac{\pi}{3}\lambda_2-\frac{\pi}{3}\lambda_1)} & \frac{-\sqrt{3}\sin(\frac{\pi}{3}\lambda_2+\frac{\pi}{6})}{\sin(\frac{\pi}{3}\lambda_2-\frac{\pi}{3}\lambda_1)} & 0 \\ \frac{-\sqrt{3}\sin(\frac{\pi}{3}\lambda_1-\frac{\pi}{6})}{\sin(\frac{\pi}{3}\lambda_2-\frac{\pi}{3}\lambda_1)} & \frac{\sqrt{3}\sin(\frac{\pi}{3}\lambda_1+\frac{\pi}{6})}{\sin(\frac{\pi}{3}\lambda_2-\frac{\pi}{3}\lambda_1)} & 0 \\ A_1 & A_2 & 3e^{-i\pi\lambda_3} \end{bmatrix}$$

where

$$A_1 = \frac{3\sqrt{3}e^{-i\pi\lambda_3}\sin(\frac{\pi}{3}\lambda_1-\frac{\pi}{6})}{\sin(\frac{\pi}{3}\lambda_2-\frac{\pi}{3}\lambda_1)}\mathcal{L}_2\big|_{x=0} - \frac{3\sqrt{3}e^{-i\pi\lambda_3}\sin(\frac{\pi}{3}\lambda_2-\frac{\pi}{6})}{\sin(\frac{\pi}{3}\lambda_2-\frac{\pi}{3}\lambda_1)}\mathcal{L}_1\big|_{x=0}$$

and

$$A_2 = \frac{-3\sqrt{3}e^{-i\pi\lambda_3}\sin(\frac{\pi}{3}\lambda_1 + \frac{\pi}{6})}{\sin(\frac{\pi}{3}\lambda_2 - \frac{\pi}{3}\lambda_1)}\mathcal{L}_2\big|_{x=0} + \frac{3\sqrt{3}e^{-i\pi\lambda_3}\sin(\frac{\pi}{3}\lambda_2 + \frac{\pi}{6})}{\sin(\frac{\pi}{3}\lambda_2 - \frac{\pi}{3}\lambda_1)}\mathcal{L}_1\big|_{x=0}$$

Also, note that K is compact. We have $f = (E + K)h \Longleftrightarrow E^{-1}f = (I + E^{-1}K)h$, which is a Fredholm equation, since $E^{-1}K$ is compact. By the Fredholm alternative, $\mathrm{ran}(I + E^{-1}K)$ is closed, and $\dim \ker(I + E^{-1}K) = \dim \ker(I + E^{-1}K)^* < \infty$.

Claim 7. $\ker\,(I + E^{-1}K) = \{0\}$.

Proof. Suppose $(I + E^{-1}K)h = 0$. Since $E^{-1}K$ has the form

$$E^{-1}K = \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}$$

we have $h_1 = 0$ and $h_2 = 0$. The equivalent equation $(E + K)h = 0$ gives $h_3(t) = -3e^{-i\pi\lambda_3}\mathcal{L}_1 h_1(0,t) - 3e^{-i\pi\lambda_3}\mathcal{L}_2 h_2(0,t)$ (see (2.196)) and therefore $h_3 = 0$. □

Thus, $(I + E^{-1}K)$ is invertible, and therefore, given f, we may set $h = (E + K)^{-1}f$, and $\mathcal{L}(h_1, h_2, h_3)(x,t)$ will meet the boundary conditions (2.195). □

CHAPTER 3
THE INITIAL-BOUNDARY VALUE PROBLEM FOR 1D NLS ON THE HALF-LINE

3.1 Introduction

[CK02] introduce a new versatile method for treating nonlinear initial-boundary value problems. Their method involves the introduction of a Duhamel forcing operator

$$\int_0^t S(t-t')\delta_0(x)h(t')\,dt' \tag{3.1}$$

where $S(t)$ denotes the linear solution group, to take care of setting the boundary values at $x=0$. [CK02] treat the generalized KdV equation on the right half-line. In the present chapter, I adapt their method to treat NLS on the half-line. The initial-boundary value problem for NLS on the right half-line is: Given $f \in H^{\frac{2s+1}{4}}(\mathbb{R}_t^+)$, $\phi \in H^s(\mathbb{R}^+)$, with the necessary compatibility conditions relating f and ϕ at 0, find u solving

$$\begin{cases} i\partial_t u + \partial_x^2 u + \lambda u|u|^{\alpha-1} = 0 & \text{for } (x,t) \in (0,+\infty) \times (0,T) \\ u(0,t) = f(t) & \text{for } t \in (0,T) \\ u(x,0) = \phi(x) & \text{for } x \in (0,+\infty) \end{cases} \tag{3.2}$$

The left half-line problem is actually the same problem since $u(x,t)$ solves the left-hand problem for $\phi(x)$ and $f(t)$ iff $u(-x,t)$ solves the right-hand problem for $\phi(-x)$ and $f(t)$.

The present chapter is a synthesis of the techniques in [CK02] with the standard proof of local well-posedness for NLS on the line $\mathbb{R}$, using the Strichartz estimates (see [Caz03]). We take the *space traces* and *mixed norm* estimates for the group $e^{it\partial_x^2}$ and the Duhamel inhomogeneous solution operator

$$\int_0^t e^{i(t-t')\partial_x^2} h(x,t')\,dt'$$

used in this standard proof and add to them local smoothing or *time traces* estimates. We also need to introduce a Duhamel forcing operator (3.1), examine its continuity and decay properties for $h \in C_0^\infty(\mathbb{R})$, and prove *space traces* estimates, *time traces* estimates, and *mixed norm* estimates for it. We then present a solution to the problem (3.2) by the contraction method for $s = 0$, $1 < \alpha < 5$ and $s = 1$, $1 < \alpha < +\infty$ with the compatibility condition $\phi(0) = f(0)$. The L^2-critical case $s = 0$ and $\alpha = 5$ is also treated using the method of [CW89].

Theorem 11. *There is local well-posedness of* (3.2) *for* $(\phi, f) \in L^2(\mathbb{R}_x^+) \times H^{1/4}(\mathbb{R}_t^+)$ *and* $1 < \alpha \leq 5$ *(for* $\alpha = 5$*, the time of existence depends on the initial and boundary data itself, not just on the corresponding norms). There is local well-posedness of* (3.2) *for* $(\phi, f) \in H^1(\mathbb{R}_x^+) \times H^{3/4}(\mathbb{R}_t^+)$ *and* $1 < \alpha < +\infty$*, with the compatibility condition* $\phi(0) = f(0)$*.*

The uniqueness component of "local well-posedness" is in reference to the integral equation formulation of (3.2). Since there are many ways to rewrite (3.2) as an integral equation, this is a serious limitation. [BSZ04] resolve this issue by introducing the notion of a mild solution, i.e. one that can be approximated by smoother solutions, and prove uniqueness of mild solutions. The solutions we construct are mild solutions.

The primary new feature of the results obtained here, in comparison with earlier work on the problem and related problems, is the limited regularity required on the boundary data $f(t)$.

[Fok02] in the cubic (integrable) $\alpha = 3$ case with Schwartz initial data $\phi(x)$ and sufficiently smooth boundary data $f(t)$, obtain a solution by reformulating as a 2×2 matrix Riemann-Hilbert problem. In this setting, [BdMFS03] obtain an explicit representation for $\partial_x u(0,t)$. Inverse scattering techniques are also applied to the problem in [Vu01].

[SB01] consider a bounded or unbounded domain Ω in $\mathbb{R}^n$, with H^1 initial data, smooth compactly supported boundary data, and nonlinearity giving a positive contribution to the energy, and obtain the existence of a global solution (uniqueness is only addressed in limited circumstances). This solution is obtained as a limit of solutions to approximate problems with the help of *a priori* identities. Earlier, [CB91] and [Bu00] had obtained solutions to the $1D$ problem for $\alpha > 3$, $\lambda > 0$ on the right half line for initial data in $H^2(\mathbb{R}^+)$ and boundary data in C^2, using semigroup techniques and *a priori* estimates. The multi-dimensional problem in a domain $\Omega \subset \mathbb{R}^n$, with smooth boundary and boundary condition $u\big|_{\partial\Omega} = 0$, had been considered previously: [BG80] treated Ω bounded, $n = 2$; [Tsu83] treated Ω unbounded, $n \geq 3$; [Tsu91] treated Ω an exterior domain for $n \geq 2$; [Tsu89] treated Ω bounded, $n = 2$; [Wan00] generalized [BG80] to initial data in H^s, $1 < s \leq 2$. Some global considerations via semigroups and Brezis-Gallouet and Brezis-Wainger-type inequalities are discussed in [Den00a] [Den00d] [Den00c] [Den00b], while in [LC96], there are some global results on the exterior of the unit ball for radial solutions. Sufficient conditions for blow-up of solutions with $u\big|_{\partial\Omega} = 0$ are given in [BM96].

Variants of the problem have also been addressed, and we now list some recent contributions. [Wed03b] [Wed03a] has considered the Schrödinger operator with potential $-\partial_x^2 + V$ in $1D$ on the half-line with C^2 boundary data by considering it as a Sturm-Liouville problem and proving estimates on the Jost solution. Questions concerning the spectrum of $-\Delta^2 + V$ on the half-space or half-line are considered in [Pea02], [Bel98], [Eas98]. A quantization of the cubic nonlinear Schrödinger equation is addressed in [GLM98] [GLM99] by means of a "boundary exchange algebra". The nonlocal problem with Δ replaced by a pseudodifferential operator is considered in [KNS99]. Time dependent boundaries are considered via the inverse scattering Riemann-Hilbert method in [Pel00] and [FP01a]. Nonsmooth boundaries are considered in [Khu98]. Nonlinearities containing derivatives ("derivative NLS") are considered in [Meš98]. The line segment problem $0 < x < L$ for the linear equation has been considered by a Green's function approach [Bre97] and by inverse scattering [FP01b], while [GS03] treat the line segment problem for nonlinear problem by rewriting it as a nonlinear dynamical system for suitable sets of algebro-geometric spectral data.

3.2 Needed lemmas from other sources

Lemma 36 (in the proof of [CK02] Lemma 2.8). *Let* $\theta \in C_0^\infty(\mathbb{R})$*, and* $0 \leq \alpha < \frac{1}{2}$*. Then*

$$\|\theta h\|_{H^\alpha} \leq c\|h\|_{\dot{H}^\alpha} \tag{3.3}$$

$$\|\theta h\|_{\dot{H}^{-\alpha}} \leq c\|h\|_{H^{-\alpha}} \tag{3.4}$$

where $c = c(\theta, \alpha)$*.*

Proof. We shall prove (3.3). $\theta h = \theta \mathcal{I}_\alpha D^\alpha h$, where $\mathcal{I}_\alpha$ is the standard fractional integral operator. Taking $q \geq 2$ such that $\frac{1}{q} = \frac{1}{2} - \alpha$ and p such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, we obtain by Hölder and the theorem on fractional integration,

$$\|\theta h\|_2 \leq \|\theta\|_p \|\mathcal{I}_\alpha D^\alpha h\|_q \leq c\|D^\alpha h\|_2$$

By the a version of the Leibniz rule for fractional derivatives, (see the appendix to [KPV93b]),

$$\|D^\alpha(\theta h)\|_2 \leq c\|\theta\|_\infty \|D^\alpha h\|_2 + \|(D^\alpha \theta)h\|_2$$

Also, with the same p and q as above,

$$\|(D^\alpha \theta)h\|_2 \leq \|D^\alpha \theta\|_p \|\mathcal{I}_\alpha D^\alpha h\|_q \leq c\|\theta\|_{H^1} \|D^\alpha h\|_2$$

by Hölder, Sobolev imbedding, and the theorem on fractional integration. (3.4) follows from (3.3) by duality. □

Lemma 37 ([CK02] Prop. 2.4). *If* $\frac{1}{2} < \alpha < \frac{3}{2}$*, then*

$$H_0^\alpha(\mathbb{R}^+) = \{f \in H^\alpha(\mathbb{R}^+) \mid \operatorname{Tr}(f) = 0\}$$

Lemma 38 ([JK95] Lemma 3.7). *If* $\frac{1}{2} < \alpha < \frac{3}{2}$*, then*

$$\int_0^{+\infty} \frac{|g(x) - g(0)|^2}{x^{2\alpha}}\, dx \leq c\|g\|_{H^\alpha(\mathbb{R})}^2$$

Lemma 39 ([JK95] Lemma 3.8). *If* $\frac{1}{2} < \alpha < \frac{3}{2}$, *then*

$$\|\chi_{(0,+\infty)} g\|_{H^\alpha(\mathbb{R})} \leq c \left[\|g\|_{H^\alpha(\mathbb{R})} + \left(\int_0^{+\infty} \frac{|g(x)|^2}{x^{2\alpha}} \, dx \right)^{1/2} \right]$$

Lemma 40 (in the proof of [JK95] Lemma 3.5). *If* $0 \leq \alpha < \frac{1}{2}$, *then*

$$\|\chi_{(0,+\infty)} f\|_{H^\alpha} \leq c\|f\|_{H^\alpha}$$

where $c = c(\alpha)$.

Lemma 41 (Lemma 13). *If* $0 \leq \alpha < +\infty$ *and* $s \in \mathbb{R}$, *then*

$$\|\mathcal{I}_{-\alpha} h\|_{H_0^s(\mathbb{R}^+)} \leq c\|h\|_{H_0^{s+\alpha}(\mathbb{R}^+)}$$

Lemma 42 (Lemma 14). *If* $0 \leq \alpha < +\infty$, $s \in \mathbb{R}$, $\mu \in C_0^\infty(\mathbb{R})$

$$\|\mu \mathcal{I}_\alpha h\|_{H_0^s(\mathbb{R}^+)} \leq c\|h\|_{H_0^{s-\alpha}(\mathbb{R}^+)}$$

where $c = c(\mu)$.

3.3 Estimates for the linear solution operator

Let $B(x) = ce^{ix^2/4}$, so that $\hat{B}(\xi) = e^{-i\xi^2}$. Define

$$e^{it\partial_x^2}\phi(x) = \int_{-\infty}^{+\infty} e^{ix\xi} e^{-it\xi^2} \widehat{\phi}(\xi) \, d\xi \tag{3.5}$$

$$= \int_{-\infty}^{+\infty} B\left(\frac{x-x'}{t^{1/2}}\right) \frac{1}{t^{1/2}} \phi(x') \, dx' \tag{3.6}$$

Then the solution to the homogeneous initial value problem on $\mathbb{R}$ is $w(x,t) = e^{it\partial_x^2}\phi(x)$, i.e.

$$\begin{cases} i\partial_t w + \partial_x^2 w = 0 & \text{in } \mathbb{R}_x \times (-\infty, +\infty)_t \\ w\big|_{t=0} = \phi & \text{on } \mathbb{R}_x \end{cases}$$

Lemma 43 (Decay estimate). *If* $1 \le p \le 2$*, then for a given* t*,*

$$\|w(-,t)\|_{L_x^{p'}} \le \frac{c}{|t|^{\frac{1}{p}-\frac{1}{2}}} \|\phi\|_{L_x^p}$$

where c *is independent of* t*.*

Proof. The case $p = 2$ follows from Plancherel and the representation (3.5), while the case $p = 1$ follows from the representation (3.6). The result for $1 < p < 2$ then follows by interpolation. □

Lemma 44 (Continuity). *If* $\phi \in C_0^\infty(\mathbb{R})$*, then for fixed* t *and* $\forall\ k \in \mathbb{N}$*,* $\partial_x^k w(x,t)$ *is continuous; and for fixed* x *and* $\forall\ k \in \mathbb{N}$*,* $\partial_t^k w(x,t)$ *is continuous. Also,* $|\partial_x^k \partial_t^l w(x,t)| \le c(1+|x|)^{-N}(1+|t|)^N$*,* $c = c(N,k,l,\phi)$*.*

Proof. We use that $\partial_x^k e^{it\partial_x^2} = e^{it\partial_x^2}\partial_x^k$ and $\partial_t^k e^{it\partial_x^2} = i^k e^{it\partial_x^2}\partial_x^{2k}$, and apply dominated convergence to the representation (3.5), since for $\phi \in C_0^\infty(\mathbb{R})$, we have $\hat{\phi} \in \mathcal{S}(\mathbb{R})$. The second assertion follows by integration by parts. □

Lemma 45 (Space traces estimate for the group).

$$\sup_t \|w(-,t)\|_{\dot{H}^s(\mathbb{R}_x)} \le c\|\phi\|_{\dot{H}^s(\mathbb{R}_x)} \qquad -\infty < s < +\infty$$

$$\sup_t \|w(-,t)\|_{H^s(\mathbb{R}_x)} \le c\|\phi\|_{H^s(\mathbb{R}_x)} \qquad -\infty < s < +\infty$$

$$w(-,t) \in C(\ t \in (-\infty,+\infty); H^s(\mathbb{R}_x)\ ) \qquad -\infty < s < +\infty$$

Proof. The bounds follow straight from the Fourier multiplier representation (3.5) of $e^{it\partial_x^2}$. The continuity follows from dominated convergence together with Lemma 44. □

Lemma 46 (Time traces estimate for the group). *Let* $\Psi(t) \in C_0^\infty(\mathbb{R})$ *be* 1 *on*

$[-T, T]$, $T > 0$. *Then*

$$\sup_x \|w(x,-)\|_{\dot{H}^{\frac{2s+1}{4}}(\mathbb{R}_t)} \le c\|\phi\|_{\dot{H}^s(\mathbb{R}_x)} \qquad -\infty < s < +\infty \qquad (3.7)$$

$$\sup_x \|\Psi(-)w(x,-)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} \le c\|\phi\|_{H^s(\mathbb{R}_x)} \qquad -\infty < s < +\infty \qquad (3.8)$$

$$w(-,t) \in C(\, x \in (-\infty,+\infty); H^{\frac{2s+1}{4}}(\mathbb{R}_t)\,) \qquad -\infty < s < +\infty$$

Proof. First we prove the result for the homogeneous Sobolev norms, i.e. (3.7).

$$\begin{aligned} w(x,t) &= \int_\xi e^{ix\xi} e^{-it\xi^2} \hat{\phi}(\xi)\, d\xi \\ &= \int_{\xi=-\infty}^{0} e^{ix\xi} e^{-it\xi^2} \hat{\phi}(\xi)\, d\xi + \int_{\xi=0}^{+\infty} e^{ix\xi} e^{-it\xi^2} \hat{\phi}(\xi)\, d\xi \\ &= \int_{\eta=-\infty}^{0} e^{it\eta} \left[e^{-ix(-\eta)^{1/2}} \hat{\phi}(-(-\eta)^{1/2}) + e^{ix(-\eta)^{1/2}} \hat{\phi}((-\eta)^{1/2}) \right] \tfrac{1}{2}(-\eta)^{-1/2}\, d\eta \end{aligned}$$

Hence

$$\begin{aligned} \|w(x,-)\|^2_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} \le& \int_{\eta=-\infty}^{0} |\eta|^{s-\frac{1}{2}} |\hat{\phi}((-\eta)^{1/2})|^2\, d\eta \\ &+ \int_{\eta=-\infty}^{0} |\eta|^{s-\frac{1}{2}} |\hat{\phi}(-(-\eta)^{1/2})|^2\, d\eta \\ =& \int_\xi |\xi|^{2s} |\hat{\phi}(\xi)|^2\, d\xi \end{aligned}$$

establishing (3.7). Now we turn to the result for the inhomogeneous Sobolev norms, namely (3.8).

Case 1. $0 \le s < +\infty$. In this case, $\frac{1}{4} \le \frac{2s+1}{4}$ and

$$\|\Psi(t)w(x,t)\|_{L^2(\mathbb{R}_t)} \le \|\Psi(t)w(x,t)\|_{H^{1/4}(\mathbb{R}_t)} \le \|w(x,t)\|_{\dot{H}^{1/4}(\mathbb{R}_t)} \le \|\phi\|_{L^2(\mathbb{R}_x)}$$

where the second inequality is Lemma 36 and the third inequality is (3.7). Combining this with (3.7), we get (3.8) for this s-range.

Case 2. $-\frac{1}{2} \le s \le 0$. In this case, $0 \le \frac{2s+1}{4} \le \frac{1}{4}$, and we separate ϕ into low and high frequencies as $\phi = \phi_L + \phi_H$. Observe

$$\begin{aligned}\|\Psi(t)e^{it\partial_x^2}\phi_H(x)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} &\le c\|e^{it\partial_x^2}\phi_H(x)\|_{\dot{H}^{\frac{2s+1}{4}}(\mathbb{R}_t)} \\ &\le c\|\phi_H\|_{\dot{H}^s(\mathbb{R}_x)} \le c\|\phi\|_{H^s(\mathbb{R}_x)}\end{aligned}$$

where the first inequality is Lemma 36, the second is (3.7), and the third holds because ϕ_H contains only high frequencies. Also

$$\begin{aligned}\|\Psi(t)e^{it\partial_x^2}\phi_L(x)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} &\le \|\Psi(t)e^{it\partial_x^2}\phi_L(x)\|_{H^{\frac{1}{4}}(\mathbb{R}_t)} \le c\|e^{it\partial_x^2}\phi_L(x)\|_{\dot{H}^{\frac{1}{4}}(\mathbb{R}_t)} \\ &\le c\|\phi_L\|_{L^2(\mathbb{R}_x)} \le c\|\phi\|_{H^s(\mathbb{R}_x)}\end{aligned}$$

where the second inequality is Lemma 36, the third is (3.7), and the fourth holds because ϕ_L contains only low frequencies.

Case 3. $-\infty < s \le -\frac{1}{2}$. Then $\frac{2s+1}{4} \le 0$. Again, we separate ϕ into low and high frequencies as $\phi = \phi_L + \phi_H$. Then

$$\begin{aligned}\|\Psi(t)e^{it\partial_x^2}\phi_L(x)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} &\le \|\Psi(t)e^{it\partial_x^2}\phi_L(x)\|_{H^{1/4}(\mathbb{R}_t)} \le \|e^{it\partial_x^2}\phi_L(x)\|_{\dot{H}^{1/4}(\mathbb{R}_t)} \\ &\le \|\phi_L\|_{L^2(\mathbb{R}_x)} \le \|\phi\|_{H^s(\mathbb{R}_t)}\end{aligned}$$

where the second inequality follows from Lemma 36 and the third is (3.7). Also,

$$\|\Psi(t)e^{it\partial_x^2}\phi_H(x)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} \le \|e^{it\partial_x^2}\phi_H(x)\|_{\dot{H}^{\frac{2s+1}{4}}(\mathbb{R}_t)} \le \|\phi_H\|_{\dot{H}^s(\mathbb{R}_x)} \le \|\phi\|_{H^s(\mathbb{R}_t)}$$

The continuity follows from the bound (3.8), Lemma 44, and dominated convergence. □

Lemma 47 (Mixed-norm estimate for the group). *Suppose* $\frac{2}{q'} = \frac{1}{2} - \frac{1}{p'}$, $2 <$

$p' \le +\infty$, $4 \le q' < +\infty$. *Then*

$$\|w\|_{L_t^{q'} L_x^{p'}} \le c\|\phi\|_{L_x^2} \tag{3.9}$$

$$\|\partial_x w\|_{L_t^{q'} L_x^{p'}} \le c\|\phi\|_{\dot{H}_x^1} \tag{3.10}$$

Proof. In the steps below, the supremum is taken over $g \in L_t^q L_x^p$ such that $\|g\|_{L_t^q L_x^p} \le 1$.

$$\begin{aligned}
\|w\|_{L_t^{q'} L_x^{p'}} &= \sup_g \int_t \int_x e^{it\partial_x^2} \phi(x) \overline{g(x,t)}\, dx\, dt \\
&= \sup_g \int_x \phi(x) \overline{\int_t e^{-it\partial_x^2} g(x,t)\, dt}\, dx \\
&\le \sup_g \|\phi\|_{L_x^2} \left\| \int_t e^{-it\partial_x^2} g(x,t)\, dt \right\|_{L_x^2} \\
&= c\|\phi\|_{L_x^2}
\end{aligned}$$

where the last step uses Lemma 58. This proves (3.9). Because ∂_x commutes with $e^{it\partial_x^2}$, (3.10) follows from (3.9). □

3.4 Estimates for the Duhamel forcing operator

For $h \in C_0^\infty(\mathbb{R}^+)$, define the Duhamel forcing operator

$$\begin{aligned}
w(x,t) &= \int_0^t e^{i(t-t')\partial_x^2} \delta_0(x) h(t') dt' \\
&= \int_0^t B\left(\frac{x}{(t-t')^{1/2}} \right) \frac{h(t')}{(t-t')^{1/2}}\, dt' \qquad (3.11)
\end{aligned}$$

Lemma 48 (Continuity). *For $h \in C_0^\infty(\mathbb{R}^+)$, define $w(x,t)$ as above. For fixed t, $w(x,t)$ and $\partial_t w(x,t)$ are continuous in x. For fixed x, $w(x,t)$ is continuous in t, and* *for fixed $x \neq 0$, $\partial_x w(x,t)$ is continuous in t. We have the pointwise bound*

$$|\theta(t)w(x,t)| \leq c(1+|x|)^{-N} \tag{3.12}$$

$$|\theta(t)\partial_t w(x,t)| \leq c(1+|x|)^{-N} \tag{3.13}$$

$$|\theta(t)\partial_x w(x,t)| \leq c(1+|x|)^{-N} \tag{3.14}$$

where $c = c(\theta, h)$.

Proof. It is clear from the definition (3.11) and dominated convergence that, for fixed t, $w(x,t)$ is continuous in x, and for fixed x, $w(x,t)$ is continuous in t. The bound (3.12) is deduced as follows. Let

$$\phi(\xi,t) = \theta(t)\int_0^t e^{-i(t-t')\xi} h(t')\,dt'$$

By integration by parts and the fundamental theorem of calculus,

$$|\partial_\xi^k \theta(t)\phi(\xi,t)| \leq c(1+|\xi|)^{-k-1}$$

where $c = c(\theta, h)$, and thus

$$|\partial_\xi^k \theta(t)\phi(\xi^2,t)| \leq c(1+|\xi|)^{-k-2}$$

We have

$$\theta(t)w(x,t) = \int_\xi e^{ix\xi}\phi(\xi^2,t)\,d\xi \tag{3.15}$$

and by integration by parts, we obtain (3.12). By integration by parts

$$\partial_t \int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' = \int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)\partial_{t'}h(t')\,dt'$$

from which we obtain that $\partial_t w(x,t)$ is continuous in x for fixed t, and the bound (3.13). Now we establish (3.14). Integration by parts gives

$$\partial_x^2\left[\int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt'\right] = i\delta_0(x)h(t) - i\int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)\partial_{t'}h(t')\,dt'$$

and thus

$$\partial_x \int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt'$$
$$= \tfrac{1}{2}i(\text{sgn }x)h(t) - i\int_{x'=0}^{x}\int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)\partial_{t'}h(t')\,dt'\,dx' + c(t)$$

Since the left-hand side is odd, $c(t) = 0$. This gives that $\partial_x w(x,t)$ is continuous in t for fixed $x \neq 0$ and the bound

$$\left|\partial_x \int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt'\right| \le c \tag{3.16}$$

From (3.15) and integration by parts, we obtain that

$$\left|\partial_x \int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt'\right| \le c|x|^{-N} \tag{3.17}$$

for each N, $c = c(N)$. Combining (3.16) and (3.17), we obtain (3.14) □

Lemma 49. *$i\partial_t w(x,t) + \partial_x^2 w(x,t) = \delta_0(x)h(t)$ in $\mathcal{S}'(\mathbb{R}^2)$, $w(x,0) = 0$ and $w(0,t) = B(0)\Gamma(\frac{1}{2})\mathcal{I}_{1/2}(h)$.*

Proof. This is clear from the definition of $w(x,t)$. □

Below are two identities that follow from writing $2\chi_{(0,t)}(t') = 1 - 2\chi_{(-\infty,0)}(t') + \text{sgn}(t-t')$ and applying dominated convergence.

Lemma 50. *For*

$$h \in \mathcal{D}_\otimes = \left\{ h \;\middle|\; h(x,t) = \sum_{i=1}^{N} h_i^1(x)h_i^2(t),\ h_i^1(x),\ h_i^2(t) \in C_0^\infty(\mathbb{R}) \right\}$$

we have

$$2\int_0^t e^{i(t-t')\partial_x^2}h(x,t')\,dt' = \int_{-\infty}^{+\infty} e^{i(t-t')\partial_x^2}h(x,t')\,dt' - 2\int_{-\infty}^{0} e^{i(t-t')\partial_x^2}h(x,t')\,dt'$$
$$- \frac{i}{\pi}\int_\tau e^{it\tau}\left[\lim_{\epsilon\to 0^+}\int_{|\tau+\xi^2|>\epsilon} e^{ix\xi}\frac{\hat{h}(\xi,\tau)}{\tau+\xi^2}\,d\xi\right]d\tau$$

Lemma 51. *For $h \in C_0^\infty(\mathbb{R})$,*

$$
\begin{aligned}
&2\int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' \\
&\quad = \int_{-\infty}^{+\infty} e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' - 2\int_{-\infty}^{0} e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' \\
&\qquad - \begin{cases} \dfrac{i}{\pi}\displaystyle\int_\xi e^{ix\xi}\left[\lim_{\epsilon\to 0^+}\int_{|\tau+\xi^2|>\epsilon}\frac{e^{it\tau}\hat{h}(\tau)}{\tau+\xi^2}\,d\tau\right]d\xi \\ \dfrac{i}{\pi}\displaystyle\int_\tau e^{it\tau}\hat{h}(\tau)\left[\lim_{\epsilon\to 0^+}\int_{|\tau+\xi^2|>\epsilon}\frac{e^{ix\xi}}{\tau+\xi^2}\,d\xi\right]d\tau \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&2\partial_x\int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' \\
&\quad = \partial_x\int_{-\infty}^{+\infty} e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' - 2\partial_x\int_{-\infty}^{0} e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' \\
&\qquad + \frac{1}{\pi}\int_\tau e^{it\tau}\hat{h}(\tau)\left[\lim_{\epsilon\to 0^+}\int_{|\tau+\xi^2|>\epsilon}\frac{\xi\, e^{ix\xi}}{\tau+\xi^2}\,d\xi\right]d\tau
\end{aligned}
$$

Lemma 52.

$$\int_\xi e^{ix\xi}\frac{d\xi}{\xi^2+1} = \pi e^{-|x|} \tag{3.18}$$

$$\lim_{\epsilon\to 0^+}\int_{|\xi^2-1|>\epsilon} e^{ix\xi}\frac{d\xi}{\xi^2-1} = -\pi\sin|x| \tag{3.19}$$

As a consequence,

$$\lim_{\epsilon\to 0^+}\int_{|\tau+\xi^2|>\epsilon}\frac{e^{ix\xi}}{\tau+\xi^2}\,d\xi = \begin{cases} \dfrac{\pi e^{-|x||\tau|^{1/2}}}{|\tau|^{1/2}} & \text{if } \tau>0 \\ \dfrac{-\pi\sin(|x||\tau|^{1/2})}{|\tau|^{1/2}} & \text{if } \tau<0 \end{cases} \tag{3.20}$$

and

$$\lim_{\epsilon\to 0^+}\int_{|\tau+\xi^2|>\epsilon}\frac{\xi\, e^{ix\xi}}{\tau+\xi^2}\,d\xi=\begin{cases} i\pi(\operatorname{sgn} x)e^{-|x||\tau|^{1/2}} & \textit{if } \tau>0\\ i\pi(\operatorname{sgn} x)\cos(|x||\tau|^{1/2}) & \textit{if } \tau<0\end{cases} \tag{3.21}$$

Proof. (3.18) is deduced by observing that both sides satisfy $(1-\partial_x^2)(\cdots)=\delta_0(x)$, and therefore, differ by an affine function. But as $x\to\pm\infty$, both sides go to 0. (3.19) is proved using a partial fraction decomposition. □

Lemma 53.

$$\left|\int_{\xi=0}^{+\infty} e^{\pm i\xi^2}e^{-\alpha\xi}\,d\xi\right|\le c \qquad \alpha>0 \tag{3.22}$$

$$\left|\int_{\xi=0}^{+\infty} e^{\pm i\xi^2}e^{i\alpha\xi}\,d\xi\right|\le c \qquad \alpha\in\mathbb{R} \tag{3.23}$$

where c is independent of α. As a consequence,

$$\left|\int_{\xi=0}^{+\infty} e^{i\lambda\xi^2}e^{-\alpha\xi}\,d\xi\right|\le c|\lambda|^{-1/2} \qquad \alpha>0,\ \lambda\in\mathbb{R} \tag{3.24}$$

$$\left|\int_{\xi=0}^{+\infty} e^{i\lambda\xi^2}e^{i\alpha\xi}\,d\xi\right|\le c|\lambda|^{-1/2} \qquad \alpha\in\mathbb{R},\ \lambda\in\mathbb{R} \tag{3.25}$$

Proof. (3.22) and (3.23) are change of contour calculations. To obtain (3.24) from (3.22) and (3.25) from (3.23), use the substitution $|\lambda|^{1/2}\xi'=\xi$. □

Lemma 54 (Space traces estimate for the Duhamel forcing operator). *If supp $h\subset[0,+\infty)$, then*

$$\sup_t \|w(-,t)\|_{\dot H^s(\mathbb{R}_x)}\le c\|h\|_{\dot H^{\frac{2s-1}{4}}(\mathbb{R}_t)} \qquad -\tfrac12<s<\tfrac32 \tag{3.26}$$

If, in adddition, supp $h\subset[0,1]$, then

$$w(-,t)\in C(\,t\in(-\infty,+\infty);\ H^s(\mathbb{R}_x)\,) \qquad 0\le s\le 1 \tag{3.27}$$

$$\sup_t \|w(-,t)\|_{H^s(\mathbb{R}_x)}\le c\|h\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}_t^+)} \qquad -\tfrac12<s<\tfrac32 \tag{3.28}$$

Proof. First we establish (3.26). By Lemma 51, for $h \in C_0^\infty(\mathbb{R}^+)$,

$$\begin{aligned} w(x,t) &= \int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' \\ &= \int_{-\infty}^{+\infty} e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' + \int_\xi e^{ix\xi}\left[\lim_{\epsilon\to 0^+}\int_{|\tau+\xi^2|>\epsilon} e^{it\tau}\frac{\hat{h}(\tau)}{\tau+\xi^2}\,d\tau\right]d\xi \\ &= \mathrm{I}+\mathrm{II} \end{aligned}$$

We estimate term II by applying a weighted inequality for the Hilbert transform.

$$\begin{aligned} \|\mathrm{II}\|^2_{\dot{H}^s(\mathbb{R}_x)} &= \int |\xi|^{2s}\left|\int e^{it\tau}\frac{\hat{h}(\tau)}{\tau+\xi^2}\,d\tau\right|^2 d\xi \\ &= 2\int_0^{+\infty} |\xi|^{2s}\left|\int e^{it\tau}\frac{\hat{h}(\tau)}{\tau+\xi^2}\,d\tau\right|^2 d\xi \end{aligned}$$

Let $\eta = -\xi^2$. Then

$$\begin{aligned} &= \int_{\eta=-\infty}^0 |\eta|^{s-\frac{1}{2}}\left|\int \frac{e^{it\tau}\hat{h}(\tau)}{\eta-\tau}\,d\tau\right|^2 d\eta \\ &\le \int_\eta |\eta|^{s-\frac{1}{2}}\left|\int \frac{e^{it\tau}\hat{h}(\tau)}{\eta-\tau}\,d\tau\right|^2 d\eta \end{aligned}$$

Since $-\frac{1}{2} < s < \frac{3}{2}$, we have $-1 < s-\frac{1}{2} < 1$, so $|\eta|^{s-\frac{1}{2}}$ is an A_2 weight, and therefore

$$\begin{aligned} &\le \int |\tau|^{s-\frac{1}{2}}|\hat{h}(\tau)|^2 d\tau \\ &= \|h\|^2_{\dot{H}^{\frac{2s-1}{4}}(\mathbb{R}_t)} \end{aligned}$$

We estimate the term I by duality. Taking the supremum over $\phi \in L^2(\mathbb{R}_x)$,

$\|\phi\|_{L^2(\mathbb{R}_x)} \le 1$, we have,

$$\begin{aligned}
\|\mathrm{I}\|^2_{\dot H^s(\mathbb{R}_x)} &= \sup_\phi \int_x \left[D_x^s \int_{-\infty}^{+\infty} e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' \right] \overline{\phi(x)}dx \\
&= \sup_\phi \int_x \left[\int_{-\infty}^{+\infty} e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' \right] \overline{D_x^s\phi(x)}dx \\
&= \sup_\phi \int_{t'} \int_x e^{i(t-t')\partial_x^2}\delta_0(x)\overline{D_x^s\phi(x)}\,dx\,h(t')\,dt' \\
&= \sup_\phi \int_{t'} \overline{e^{-i(t-t')\partial_x^2} D_x^s\phi(x)\big|_{x=0}}\,h(t')\,dt' \\
&\le \sup_\phi \left\| e^{-i(t-t')\partial_x^2} D_x^s\phi(x)\big|_{x=0} \right\|_{\dot H^{\frac{1-2s}{4}}(\mathbb{R}_t)} \|h\|_{\dot H^{\frac{2s-1}{4}}(\mathbb{R}_t)} \\
&\le \sup_\phi \|\phi\|_{L^2(\mathbb{R}_x)} \|h\|_{\dot H^{\frac{2s-1}{4}}(\mathbb{R}_t)}
\end{aligned}$$

by Lemma 46. Now we turn to the estimate (3.28), adding the assumption that supp $h \subset [0,1]$.

Case 1. $-\frac{1}{2} < s \le 0$. Then $-\frac{1}{2} < \frac{2s-1}{4} \le -\frac{1}{4}$.

$$\|w(-,t)\|_{H^s(\mathbb{R}_x)} \le \|w(-,t)\|_{\dot H^s(\mathbb{R}_x)} \le c\|h\|_{\dot H^{\frac{2s-1}{4}}(\mathbb{R}_t)} \le c\|h\|_{H^{\frac{2s-1}{4}}(\mathbb{R}_t)}$$

where, in the last inequality, we have applied Lemma 36.

Case 2. $0 \le s \le \frac{1}{2}$. Then $-\frac{1}{4} \le \frac{2s-1}{4} \le 0$. We need the following two bounds:

$$\|w(-,t)\|_{L^2(\mathbb{R}_x)} \le c\|h\|_{\dot H^{-1/4}(\mathbb{R}_t)} \le c\|h\|_{H^{-1/4}(\mathbb{R}_t)} \le c\|h\|_{H^{\frac{2s-1}{4}}(\mathbb{R}_t)} \tag{3.29}$$

where, in the second to last inequality, we have applied Lemma 36; and

$$\|w(-,t)\|_{\dot H^s(\mathbb{R}_x)} \le c\|h\|_{\dot H^{\frac{2s-1}{4}}(\mathbb{R}_t)} \le c\|h\|_{H^{\frac{2s-1}{4}}(\mathbb{R}_t)} \tag{3.30}$$

where, in the last inequality, we have applied Lemma 36.

Case 3. $\frac{1}{2} \le s < \frac{3}{2}$. Then $0 \le \frac{2s-1}{4} < \frac{1}{2}$. (3.29) and (3.30) hold true in this context as well, except that in the last inequality in (3.30), we do not apply Lemma 36 (instead the bound is obvious from the definitions of the norms).

Remark: The next range we can treat is $\frac{3}{2} < s < \frac{7}{2}$ (one gets $s = \frac{3}{2}$ by interpolation). To achieve the bound in this range, we apply ∂_x^2 to the operator:

$$\begin{aligned}\int_0^t \partial_x^2 e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' &= \int_0^t -i\partial_t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' \\ &= i\int_0^t [\partial_{t'} e^{i(t-t')\partial_x^2}\delta_0(x)]h(t')\,dt' \\ &= -i\int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)\,\partial_{t'}h(t')\,dt' + i\delta_0(x)h(t)\end{aligned}$$

Let $\varphi(x) \in C_0^\infty$ be a cutoff with $\varphi(x) = 1$ on $[-1, 1]$. Set

$$\eta(x) = \varphi(x)\begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

Then define an operator

$$Uh(t) = \int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt' - i\eta(x)h(t)$$

so that

$$\partial_x^2 Uh(t) = -i\int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)\,\partial_{t'}h(t')\,dt'$$

By the above result and the fact that $\|h\|_{L^\infty} \leq c\|h\|_{H^{\frac{2s-1}{4}}}$ in this range of s, we have

$$\|Uh\|_{H^s(\mathbb{R}_x)} \leq c\|h\|_{H^{\frac{2s-1}{4}}(\mathbb{R}_t)}$$

and therefore

$$\left\|\int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt'\right\|_{H^s(\mathbb{R}^+)} \leq c\|h\|_{H^{\frac{2s-1}{4}}(\mathbb{R})}$$

Note that, however, in this range ($\frac{3}{2} < s < \frac{7}{2}$) and in higher ranges, we are forced to use the $H^s(\mathbb{R}^+)$ norm instead of the $H^s(\mathbb{R})$ norm, so that we hide the Dirac masses at the origin that one gets upon differentiating in x. *End of Remark.*

To obtain continuity in the range $-\frac{1}{2} < s \leq 1$, approximate $h \in H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)$ by $h_1 \in C_0^\infty(\mathbb{R}^+)$ using the bound (3.28), and for $h_1 \in C_0^\infty(\mathbb{R}^+)$, use dominated convergence and Lemma 48. $\square$

Lemma 55 (Time traces estimate for the Duhamel forcing operator). *If supp $h \subset [0,+\infty)$, then*

$$\sup_x \|w(x,-)\|_{\dot{H}^{\frac{2s+1}{4}}(\mathbb{R}_t)} \leq c\|h\|_{\dot{H}^{\frac{2s-1}{4}}(\mathbb{R}_t)} \qquad -\infty < s < +\infty \tag{3.31}$$

If supp $h \subset [0,1]$ and $\Psi(t) \in C_0^\infty((-2,2))$, then

$$\Psi(-)w(x,-) \in C(\, x \in (-\infty,+\infty);\ H_0^{\frac{2s+1}{4}}(\mathbb{R}_t^+)\,) \qquad 0 \leq s \leq 1 \tag{3.32}$$

$$\sup_x \|\Psi(-)w(x,-)\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}_t)} \leq c\|h\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}_t^+)} \qquad -\infty < s < +\infty \tag{3.33}$$

Proof. To prove (3.31), it suffices to show that for any $h \in \dot{H}_t^{\frac{2s-1}{4}}$ (not neccessarily supported in $[0,+\infty)$),

$$\left\| \int_{-\infty}^t e^{i(t-t')\partial_x^2}\delta_0(x)\, h(t')\, dt' \right\|_{\dot{H}_t^{\frac{2s+1}{4}}} \leq c\|h\|_{\dot{H}_t^{\frac{2s-1}{4}}} \tag{3.34}$$

This is equivalent to the inequality, for $h \in L^2(\mathbb{R}_t)$,

$$\left\| D_t^{\frac{2s+1}{4}} \int_{-\infty}^t e^{i(t-t')\partial_x^2}\delta_0(x) D_{t'}^{\frac{1-2s}{4}} h(t')\, dt' \right\|_{L^2(\mathbb{R}_t)} \leq c\|h\|_{L^2(\mathbb{R}_t)}$$

The change of variable $t'' = t - t'$ gives that

$$\begin{aligned}
&D_t^{\frac{2s+1}{4}} \int_{-\infty}^{t} e^{i(t-t')\partial_x^2}\delta_0(x) D_{t'}^{\frac{1-2s}{4}} h(t')\, dt' \\
&\quad = D_t^{\frac{2s+1}{4}} \int_{0}^{+\infty} e^{it'\partial_x^2}\delta_0(x) D_{t'}^{\frac{1-2s}{4}} h(t-t')\, dt' \\
&\quad = \int_{0}^{+\infty} e^{it'\partial_x^2}\delta_0(x) D_{t'}^{1/2} h(t-t')\, dt' \\
&\quad = \int_{-\infty}^{t} e^{i(t-t')\partial_x^2}\delta_0(x) D_{t'}^{1/2} h(t')\, dt'
\end{aligned}$$

By Lemma 51 and (3.20),

$$\begin{aligned}
&2\int_{-\infty}^{t} e^{i(t-t')\partial_x^2}\delta_0(x) D_{t'}^{1/2} h(t')\, dt' \\
&\quad = \int_{-\infty}^{+\infty} e^{i(t-t')\partial_x^2}\delta_0(x) D_{t'}^{1/2} h(t')\, dt' + i\int_{\tau=-\infty}^{0} e^{it\tau}\sin(|x||\tau|^{1/2})\hat{h}(\tau)\, d\tau \\
&\qquad - i\int_{\tau=0}^{+\infty} e^{it\tau} e^{-|x||\tau|^{1/2}}\hat{h}(\tau)\, d\tau \\
&\quad = \mathrm{I} + \mathrm{II} + \mathrm{III}
\end{aligned}$$

It is clear that

$$\|\mathrm{II}\|_{L_t^2} + \|\mathrm{III}\|_{L_t^2} \le c\|h\|_{L_t^2}$$

We next address term I. In the steps below, the supremum is taken over $\phi \in L^2(\mathbb{R}_x)$ with $\|\phi\|_{L^2(\mathbb{R}_x)} \le 1$.

$$\begin{aligned}
&\left\| \int_{t=-\infty}^{t=+\infty} e^{-it\partial_x^2} D_t^{1/4} g(-,t)\, dt \right\|_{L_x^2} \\
&\quad = \sup \int_x \int_t e^{-it\partial_x^2} D_t^{1/4} g(x,t)\, dt\, \overline{\phi(x)}\, dx \\
&\quad = \sup \int_t \int_x g(x,t) \overline{D_t^{1/4} e^{it\partial_x^2}\phi(x)}\, dx\, dt \\
&\quad \le c\|g\|_{L_x^1 L_t^2} \|D_t^{1/4} e^{it\partial_x^2}\phi\|_{L_x^\infty L_t^2} \\
&\quad \le c\|g\|_{L_x^1 L_t^2}
\end{aligned}$$

by Lemma 46, and hence

$$\left\| \int_{t=-\infty}^{t=+\infty} e^{-it\partial_x^2} D_t^{1/4} g(-,t)\, dt \right\|_{L_x^2} \leq c\|g\|_{L_x^1 L_t^2} \tag{3.35}$$

Also, by (the proof of) Lemma 54,

$$\left\| \int_{t=-\infty}^{+\infty} e^{-it\partial_x^2} \delta_0(x) D_t^{1/4} h(t)\, dt \right\|_{L_x^2} \leq c\|h\|_{L_t^2} \tag{3.36}$$

Now

$$\int_{t'=-\infty}^{+\infty} e^{i(t-t')\partial_x^2} \delta_0(x) D_{t'}^{1/2} h(t')\, dt' = D_t^{1/4} \int_{t'=-\infty}^{+\infty} e^{i(t-t')\partial_x^2} \delta_0(x) D_{t'}^{1/4} h(t')\, dt'$$

For $g \in L_x^1 L_t^2$ such that $\|g\|_{L_x^1 L_t^2} \leq 1$,

$$\begin{aligned}
&\int_x \int_t \left[\int_{t'=-\infty}^{+\infty} e^{i(t-t')\partial_x^2} \delta_0(x) D_{t'}^{1/2} h(t')\, dt' \right] \overline{g(x,t)}\, dx\, dt \\
&\quad = \int_x \int_{t'} e^{-it'\partial_x^2} \delta_0(x) D_{t'}^{1/4} h(t')\, dt' \overline{\int_t e^{-it\partial_x^2} D_t^{1/4} g(x,t)\, dt}\, dx \\
&\quad \leq \left\| \int_{t'} e^{-it'\partial_x^2} \delta_0(x) D_{t'}^{1/4} h(t')\, dt' \right\|_{L_x^2} \left\| \int_t e^{-it\partial_x^2} D_t^{1/4} g(x,t)\, dt \right\|_{L_x^2} \\
&\quad \leq c\|h\|_{L_t^2}
\end{aligned}$$

by (3.35) and (3.36), and therefore

$$\left\| \int_{t'=-\infty}^{+\infty} e^{i(t-t')\partial_x^2} \delta_0(x) D_{t'}^{1/2} h(t')\, dt' \right\|_{L_x^\infty L_t^2} \leq c\|h\|_{L_t^2}$$

concluding the treatment of Term I.

Case 1. $-\frac{1}{2} < s < \frac{1}{2}$. In this case, $0 < \frac{2s+1}{4} < \frac{1}{2}$, $-\frac{1}{2} < \frac{2s-1}{4} < 0$.

$$\|\Psi(t) w(x,t)\|_{H_t^{\frac{2s+1}{4}}} \leq c\|w(x,t)\|_{\dot{H}_t^{\frac{2s+1}{4}}} \leq c\|h\|_{\dot{H}_t^{\frac{2s-1}{4}}} \leq c\|h\|_{H_t^{\frac{2s-1}{4}}}$$

Case 2. $s \geq \frac{1}{2}$. In this case, $\frac{2s+1}{4} \geq \frac{1}{2}$, $\frac{2s-1}{4} \geq 0$,

$$\|\Psi(t)w(x,t)\|_{\dot{H}_t^{\frac{2s+1}{4}}} \leq c\|h\|_{\dot{H}_t^{\frac{2s-1}{4}}} \leq c\|h\|_{H_t^{\frac{2s-1}{4}}}$$

$$\|\Psi(t)w(x,t)\|_{H_t^{1/4}} \leq c\|h\|_{H_t^{-1/4}}$$

Case 3. $s \leq -\frac{1}{2}$. In this case, $\frac{2s+1}{4} \leq 0$ and $\frac{2s-1}{4} \leq -\frac{1}{2}$. Split h into high frequences and low fequencies as $h = h_L + h_H$. We have

$$\int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\,dt'$$
$$= \int_{-\infty}^t e^{i(t-t')\partial_x^2}\delta_0(x)h_L(t')\,dt' + \int_{-\infty}^t e^{i(t-t')\partial_x^2}\delta_0(x)h_H(t')\,dt'$$

Let $w_L(x,t)$ be the first term and $w_H(x,t)$ be the second term.

$$\|\Psi(t)w_L(x,t)\|_{H_t^{\frac{2s+1}{4}}} \leq c\|\Psi(t)w_L(x,t)\|_{H_t^{1/4}} \leq c\|w_L(x,t)\|_{\dot{H}_t^{1/4}}$$
$$\leq c\|h_L\|_{\dot{H}_t^{-1/4}} \leq c\|h_L\|_{H_t^{-1/4}}$$

$$\|\Psi(t)w_H(x,t)\|_{H_t^{\frac{2s+1}{4}}} \leq c\|w_H(x,t)\|_{\dot{H}_t^{\frac{2s+1}{4}}} \leq c\|h_H\|_{\dot{H}_t^{\frac{2s-1}{4}}}$$

□

Lemma 56 (Mixed-norm estimate for the Duhamel forcing operator). *Suppose supp* $h \subset [0,1]$. *If* $\frac{2}{q'} = \frac{1}{2} - \frac{1}{p'}$, $2 < p' \leq +\infty$, $4 \leq q' < +\infty$, *then*

$$\|w(x,t)\|_{L_t^{q'}L_x^{p'}} \leq c\|h\|_{\dot{H}^{-1/4}(\mathbb{R}_t)} \leq c\|h\|_{H^{-1/4}(\mathbb{R}_t)} \tag{3.37}$$

$$\|\partial_x w(x,t)\|_{L_t^{q'}L_x^{p'}} \leq c\|h\|_{\dot{H}^{1/4}(\mathbb{R}_t)} \leq c\|h\|_{H^{1/4}(\mathbb{R}_t)} \tag{3.38}$$

Proof. Step 1. [for (3.37)]. In the calculation below, the supremum is taken over

$g \in L_t^q L_x^p$ with $\|g\|_{L_t^q L_x^p} \le 1$.

$$\begin{aligned}
&\left\| \int_{t'=-\infty}^{+\infty} e^{i(t-t')\partial_x^2} \delta_0(x) h(t')\, dt' \right\|_{L_t^{q'} L_x^{p'}} \\
&\quad = \sup_g \int_t \int_x \overline{g(x,t)} \left[\int_{t'} e^{i(t-t')\partial_x^2} \delta_0(x) h(t')\, dt' \right] dx dt \\
&\quad = \sup \int_x \overline{\int_t e^{-it\partial_x^2} g(x,t)\, dt} \int_{t'} e^{-it'\partial_x^2} \delta_0(x) h(t')\, dt' dx \\
&\quad \le \sup_g \left\| \int_t e^{-it\partial_x^2} g(x,t)\, dt \right\|_{L_x^2} \left\| \int_{t'} e^{-it'\partial_x^2} \delta_0(x) h(t')\, dt' \right\|_{L_x^2} \\
&\quad \le \sup_g c \|g\|_{L_t^q L_x^p} \|h\|_{\dot{H}^{-1/4}(\mathbb{R})} \\
&\quad \le c \|h\|_{\dot{H}^{-1/4}(\mathbb{R})}
\end{aligned}$$

where, in the second to last step, we used Lemmas 58 and 54.

Step 2. [for (3.38)]. In the calculation below, the supremum is taken over $g \in L_t^q L_x^p$ with $\|g\|_{L_t^q L_x^p} \le 1$.

$$\begin{aligned}
&\left\| \partial_x \int_{t'=-\infty}^{+\infty} e^{i(t-t')\partial_x^2} \delta_0(x) h(t')\, dt' \right\|_{L_t^{q'} L_x^{p'}} \\
&\quad = \sup_g \int_t \int_x \overline{\partial_x g(x,t)} \left[\int_{t'} e^{i(t-t')\partial_x^2} \delta_0(x) h(t')\, dt' \right] dx dt \\
&\quad = \sup_g \int_x \overline{\partial_x \int_t e^{-it\partial_x^2} g(x,t)\, dt} \int_{t'} e^{-it'\partial_x^2} \delta_0(x) h(t')\, dt' dx \\
&\quad \le \sup_g \left\| \partial_x \int_t e^{-it\partial_x^2} g(x,t)\, dt \right\|_{\dot{H}_x^{-1}} \left\| \int_{t'} e^{-it'\partial_x^2} \delta_0(x) h(t')\, dt' \right\|_{\dot{H}_x^1} \\
&\quad \le \sup_g c \|g\|_{L_t^q L_x^p} \|h\|_{\dot{H}^{1/4}(\mathbb{R})} \\
&\quad \le c \|h\|_{\dot{H}^{1/4}(\mathbb{R})}
\end{aligned}$$

where, in the second to last step, we used Lemmas 58 and 54.

Step 3. For (3.37), by Lemma 51 and (3.20), it remains to show that

$$\left\| \int_{\tau=-\infty}^{0} e^{it\tau} \frac{\sin(|x||\tau|^{1/2})}{|\tau|^{1/4}} \hat{h}(\tau)\, d\tau + \int_{\tau=0}^{+\infty} e^{it\tau} \frac{e^{-|x||\tau|^{1/2}}}{|\tau|^{1/4}} \hat{h}(\tau)\, d\tau \right\|_{L_t^{q'} L_x^{p'}} \leq c\|h\|_{L_t^2} \tag{3.39}$$

(3.39) has been proved in the case $p' = 2$ and $q' = \infty$ in Lemma 54. We will now prove (3.39) in the case $p' = \infty$ and $q' = 4$, and the inequality (3.39) for general p', q' subject to the relation $\frac{2}{q'} = \frac{1}{2} - \frac{1}{p'}$ will follow by interpolation.

Case $\tau > 0$. By duality, it suffices to prove

$$\left\| \iint e^{-it\tau} \tau^{-1/4} e^{-|x|\tau^{1/2}} f(x,t)\, dx\, dt \right\|_{L^2_{\tau>0}} \leq \|f\|_{L_t^{4/3} L_x^1} \tag{3.40}$$

Writing out the $L^2_{\tau>0}$ norm squared, we obtain

$$\begin{aligned} &\left\| \iint e^{it\tau} \tau^{-1/4} e^{-|x|\tau^{1/2}} f(x,t)\, dx\, dt \right\|^2_{L^2_{\tau>0}} \\ &\quad = \int_{x,t,s,y} K(x,t,s,y) f(x,t) \overline{f(y,s)}\, dxdtdyds \end{aligned}$$

where

$$K(x,t,y,s) = \int_{\tau=0}^{\infty} e^{i(t-s)\tau} \tau^{-1/2} e^{-(|x|+|y|)\tau^{1/2}}\, d\tau$$

Setting $\tau = \xi^2$ and applying (3.24) gives

$$|K(x,t,y,s)| \leq \frac{c}{|t-s|^{1/2}} \tag{3.41}$$

Then (3.40) follows by the theorem on fractional integration.

Case $\tau < 0$. We need to prove

$$\left\| \int_{\tau=-\infty}^{0} e^{\pm ix(-\tau)^{1/2}} e^{it\tau} |\tau|^{-1/4} \hat{h}(\tau)\, d\tau \right\|_{L_t^4 L_x^\infty} \leq c\|h\|_{L^2}$$

By duality, it suffices to prove

$$\left\| \iint_{x,t} e^{\mp ix(-\tau)^{1/2}} |\tau|^{-1/4} e^{-it\tau} f(x,t)\, dxdt \right\|_{L^2_{\tau<0}} \le c\|f\|_{L_t^{4/3} L_x^1} \tag{3.42}$$

Writing out the $L^2_{\tau<0}$ norm squared, we obtain

$$\begin{aligned} &\left\| \iint_{x,t} e^{\mp ix(-\tau)^{1/2}} |\tau|^{-1/4} e^{-it\tau} f(x,t)\, dxdt \right\|^2_{L^2_{\tau<0}} \\ &= \int_{x,t,y,s} K(x,t,y,s) f(x,t)\overline{f(y,s)} dydsdxdt \end{aligned}$$

where the kernel K is defined by

$$\begin{aligned} K(x,t,y,s) &= \int_0^{+\infty} e^{\pm i(y-x)(-\tau)^{1/2}} e^{i(s-t)\tau} (-\tau)^{-1/2}\, d\tau \\ &= \int_{\xi=0}^{+\infty} e^{\pm i(y-x)\xi} e^{i(s-t)\xi^2}\, d\xi \end{aligned}$$

and hence by (3.25),

$$|K(x,t,y,s)| \le \frac{c}{|t-s|^{1/2}}$$

Then (3.42) follows by the theorem on fractional integration.

Step 4. For (3.38), we need to show, by Lemma 51, that

$$\left\| \int_{\tau=-\infty}^0 e^{it\tau} \frac{\cos(|x||\tau|^{1/2})}{|\tau|^{1/4}} \hat{h}(\tau)\, d\tau + \int_{\tau=0}^{+\infty} e^{it\tau} \frac{e^{-|x||\tau|^{1/2}}}{|\tau|^{1/4}} \hat{h}(\tau)\, d\tau \right\|_{L_t^{q'} L_x^{p'}} \le c\|h\|_{L_t^2} \tag{3.43}$$

(3.43) has been proved in the case $p' = 2$ and $q' = \infty$ in Lemma 54. We will now prove (3.43) in the case $p' = \infty$ and $q' = 4$, and the inequality (3.39) for general p', q' subject to the relation $\frac{2}{q'} = \frac{1}{2} - \frac{1}{p'}$ will follow by interpolation.
The proof follows the method of Step 3. □

3.5 Estimates for the Duhamel inhomogeneous solution operator

Let

$$\begin{aligned} w(x,t) &= \int_0^t e^{i(t-t')\partial_x^2} h(x,t')\, dt' \\ &= \int_0^t B\left(\frac{x-x'}{(t-t')^{1/2}}\right) \frac{h(x',t')}{(t-t')^{1/2}}\, dt' \end{aligned}$$

Lemma 57 (Continuity). *Let $h \in C_0^\infty(\mathbb{R}^2)$. Then $\forall\ k, l = 0, 1, 2, \ldots$ and fixed t, $\partial_x^k \partial_t^l w(x,t)$ is a continuous function of x; and for fixed x, $\partial_x^k \partial_t^l w(x,t)$ is a continuous function of t. Also,*

$$|\partial_x^k \partial_t^l w(x,t)| \le c(1+|x|)^{-N}(1+|t|)^N$$

where $c = c(k,l,N,h)$.

Lemma 58 (Space traces estimate for the Duhamel inhomogeneous operator). *Let $\frac{2}{q'} = \frac{1}{2} - \frac{1}{p'}$, $2 < p' \le +\infty$, $4 \le q' < +\infty$. Then*

$$w(x,-) \in C(\ x \in (-\infty,+\infty);\ H^s(\mathbb{R}_x)\ ) \qquad s = 0,1 \tag{3.44}$$

$$\sup_t \|w(x,t)\|_{L_x^2} \le c\|h\|_{L_t^q L_x^p} \tag{3.45}$$

$$\sup_t \|\partial_x w(x,t)\|_{L_x^2} \le c\|\partial_x h\|_{L_t^q L_x^p} \tag{3.46}$$

Proof.

$$\left\| \int_{t'=0}^{t} e^{-it'\partial_x^2} h(x,t')\, dt' \right\|_{L_x^2}^2 = \int_x \int_{t'=0}^{t} e^{-it'\partial_x^2} h(x,t')\, dt' \overline{\int_{t''=0}^{t} e^{-it''\partial_x^2} h(x,t'')\, dt''}\, dx$$

$$= \int_{t'=0}^{t} \int_{t''=0}^{t} \int_x e^{i(t''-t')\partial_x^2} h(x,t')\, \overline{h(x,t'')}\, dx\, dt''\, dt'$$

$$= \int_{t''=0}^{t} \int_x \left[\int_{t'=0}^{t} e^{i(t''-t')\partial_x^2} h(x,t')\, dt' \right] \overline{h(x,t'')}\, dx\, dt''$$

$$\leq \left\| \int_{t'=0}^{t} e^{i(t''-t')\partial_x^2} h(x,t')\, dt' \right\|_{L_{t''}^{q'} L_x^{p'}} \|h\|_{L_{t''}^{q} L_x^{p}}$$

$$\leq \|h\|_{L_t^q L_x^p}^2$$

where in the last step, a variant of Lemma 60 was applied. The result then follows since

$$\left\| \int_{t'=0}^{t} e^{-it'\partial_x^2} h(x,t')\, dt' \right\|_{L_x^2} = \left\| \int_{t'=0}^{t} e^{i(t-t')\partial_x^2} h(x,t')\, dt' \right\|_{L_x^2}$$

The result for $\partial_x w$ follows immediately since ∂_x passes through the integral and commutes with $e^{i(t-t')\partial_x^2}$. The continuity statement (3.44) follows from Lemma 57, dominated convergence, and the bounds (3.45) and (3.46). □

Lemma 59 (Time traces for the inhomogeneous Duhamel operator).

$$\sup_x \|\Psi(-)w(x,-)\|_{H^{1/4}(\mathbb{R}_t)} \leq c\|h\|_{L_t^q L_x^p} \tag{3.47}$$

$$\sup_x \|\Psi(-)w(x,-)\|_{H^{3/4}(\mathbb{R}_t)} \leq c\|h\|_{L_x^\infty H_t^{-1/4}} + c\|h\|_{L_t^q W_x^{p,1}} \tag{3.48}$$

$$\Psi(-)w(x,-) \in C(\, (-\infty,+\infty); H^{\frac{2s+1}{4}}(\mathbb{R}_t)\,) \qquad s = 0,1 \tag{3.49}$$

Proof. We first establish

$$\sup_x \|w(x,-)\|_{\dot{H}^{1/4}(\mathbb{R}_t)} \leq c\|h\|_{L_t^q L_x^p} \tag{3.50}$$

By Lemma 50,

$$
\begin{aligned}
&D_t^{1/4}\int_0^t e^{i(t-t')\partial_x^2}h(x_0,t')\,dt' \\
&\quad = \tfrac{1}{2}D_t^{1/4}\int_{-\infty}^{+\infty} e^{i(t-t')\partial_x^2}h(x_0,t')\,dt' - D_t^{1/4}\int_{-\infty}^{0} e^{i(t-t')\partial_x^2}h(x_0,t')\,dt' \\
&\qquad + \iint e^{i(x_0\xi+t\tau)}\frac{|\tau|^{1/4}\hat{h}(\xi,\tau)}{\xi^2+\tau}\,d\xi d\tau \\
&\quad = \mathrm{I}-\mathrm{II}+\mathrm{III}
\end{aligned}
$$

First, we treat term I and II. (Term II is treated by following the steps below with $h(x,t)$ replaced by $h(x,t)\chi_{(-\infty,0)}(t)$.) In the steps below, the supremum is taken over $g\in L_t^2$ with $\|g\|_{L_t^2}\le 1$.

$$
\begin{aligned}
\|\,\mathrm{I}\,\|_{L_t^2} &= \int_t \left[D_t^{1/4}\int_{t'} e^{i(t-t')\partial_x^2}h(x_0,t')\,dt'\right]\overline{g(t)}\,dt \\
&= \int_t \left[\int_{t'}\int_x e^{i(t-t')\partial_x^2}h(x_0-x,t')\delta_0(x)\,dx\,dt'\right]\overline{D_t^{1/4}g(t)}\,dt \\
&= \int_t \left[\int_{t'}\int_x e^{-it'\partial_x^2}h(x_0-x,t')e^{it\partial_x^2}\delta_0(x)\,dx\,dt'\right]\overline{D_t^{1/4}g(t)}\,dt \\
&= \int_x\int_{t'} e^{-it'\partial_x^2}h(x_0-x,t')\,dt'\,\overline{\int_t e^{-it\partial_x^2}\delta_0(x)D_t^{1/4}g(t)\,dt}\,dx \\
&= \int_x H(x)\overline{G(x)}\,dx
\end{aligned}
$$

Now

$$\|H(x)\|_{L_x^2}\le c\|h\|_{L_t^qL_x^p}$$

by (the proof of) Lemma 58, and

$$\|G(x)\|_{L_x^2}\le c\|g\|_{L_t^2}$$

by (the proof of) Lemma 54. Therefore

$$\|\,\mathrm{I}\,\|_{L_t^2}\le \|H(x)\|_{L_x^2}\|G(x)\|_{L_x^2}\le c\|g\|_{L_t^2}\|h\|_{L_t^qL_x^p}$$

Next we show

$$\| \text{ III } \|_{L_t^2} = \left\| \iint e^{i(x\xi + t\tau)} \frac{|\tau|^{1/4} \hat{h}(\xi,\tau)}{\xi^2 + \tau} \, d\xi d\tau \right\|_{L_t^2} \leq c \|h\|_{L_t^q L_x^p} \tag{3.51}$$

We first prove (3.51) in the case $p = 2$ and $q = 1$.

$$\begin{aligned}
&\left(\int_\tau |\tau|^{1/2} \left| \int_\xi e^{ix\xi} \frac{\hat{h}(\xi,\tau)}{\tau + \xi^2} \, d\xi \right|^2 \, d\tau \right)^{1/2} \\
&\quad = \left(\int_\tau |\tau|^{1/2} \left| \int_t e^{it\tau} \int_\xi e^{ix\xi} \frac{\hat{h}^x(\xi,t)}{\tau + \xi^2} \, d\xi dt \right|^2 \, d\tau \right)^{1/2} \\
&\quad \leq \int_t \left(\int_\tau |\tau|^{1/2} \left| \int_\xi e^{ix\xi} \frac{\hat{h}^x(\xi,t)}{\tau + \xi^2} \, d\xi \right|^2 \, d\tau \right)^{1/2} dt \\
&\quad \leq \int_t F_1(t) + F_2(t) \, dt
\end{aligned}$$

by Minkowskii's integral inequality, where

$$F_1(t) = \left(\int_\tau |\tau|^{1/2} \left| \int_{\xi=-\infty}^0 e^{ix\xi} \frac{\hat{h}^x(\xi,t)}{\tau + \xi^2} \, d\xi \right|^2 \, d\tau \right)^{1/2}$$

and

$$F_2(t) = \left(\int_\tau |\tau|^{1/2} \left| \int_{\xi=0}^{+\infty} e^{ix\xi} \frac{\hat{h}^x(\xi,t)}{\tau + \xi^2} \, d\xi \right|^2 \, d\tau \right)^{1/2}$$

To estimate $F_1(t)$, we set $\eta = -\xi^2$ (so that $\xi = (-\eta)^{1/2}$) and obtain

$$\begin{aligned}
F_1(t) &= \left(\int_\tau |\tau|^{1/2} \left| \int_{\eta=-\infty}^0 \frac{e^{ix(-\eta)^{1/2}} \hat{h}^x((-\eta)^{1/2}, t)(-\eta)^{-1/2}}{\tau - \eta} \, d\eta \right|^2 \, d\tau \right)^{1/2} \\
&\leq \left(\int_{\tau=-\infty}^0 |\tau|^{-1/2} |\hat{h}^x((-\tau)^{1/2}, t)|^2 \, d\tau \right)^{1/2} \\
&= \left(\int_{\xi=0}^\infty |\hat{h}^x(\xi, t)|^2 \, d\xi \right)^{1/2} \leq \|h(-, t)\|_{L_x^2}
\end{aligned}$$

since $|\tau|^{1/2}$ is an A_2 weight, the substitution $\xi = (-\tau)^{1/2}$, and finally Plancherel. To estimate $F_2(t)$, we set $\eta = -\xi^2$ (so that $\xi = -(-\eta)^{1/2}$) and obtain

$$\begin{aligned}
F_2(t) &= \left(\int_\tau |\tau|^{1/2} \left| \int_{\eta=-\infty}^0 \frac{e^{-ix(-\eta)^{1/2}} \hat{h}^x(-(-\eta)^{1/2}, t)(-\eta)^{-1/2}}{\tau - \eta} \, d\eta \right|^2 \, d\tau \right)^{1/2} \\
&\leq \left(\int_{\tau=-\infty}^0 |\tau|^{-1/2} |\hat{h}^x(-(-\tau)^{1/2}, t)|^2 \, d\tau \right)^{1/2} = \left(\int_{\xi=-\infty}^0 |\hat{h}^x(\xi, t)|^2 \, d\xi \right)^{1/2} \\
&\leq \|h(-, t)\|_{L_x^2}
\end{aligned}$$

since $|\tau|^{1/2}$ is an A_2 weight, the substitution $\xi = -(-\tau)^{1/2}$, and finally Plancherel. We next prove (3.51) for $p = 1$ and $q = \frac{4}{3}$.

Case $\tau > 0$. By (3.20),

$$\left\| \int_{\tau=0}^{+\infty} \int_{\xi} e^{ix_0\xi} e^{it\tau} \frac{|\tau|^{1/4}\hat{h}(\xi,\tau)}{\tau+\xi^2} \, d\xi \, d\tau \right\|_{L_t^2}^2$$
$$= \int_{\tau=0}^{+\infty} |\tau|^{-1/2} \left| \int_x e^{-|x|\tau^{1/2}} \hat{h}^t(x_0 - x, \tau) \, dx \right|^2 d\tau$$
$$= \int_{\xi=0}^{+\infty} \left| \int_x e^{-|x|\xi} \hat{h}^t(x_0 - x, \xi^2) \, dx \right|^2 d\xi$$
$$= \int_{\xi=0}^{+\infty} \left| \int_x \int_t e^{-|x|\xi} e^{-it\xi^2} h(x_0 - x, t) \, dx \, dt \right|^2 d\xi$$
$$= \int_{\xi=0}^{+\infty} \int_x \int_t e^{-|x|\xi} e^{-it\xi^2} h(x_0 - x, t) \, dx \, dt \, \overline{\int_y \int_s e^{-|y|\xi} e^{-is\xi^2} h(x_0 - y, s) \, dy \, ds} \, d\xi$$
$$= \int_{x,t,y,s} K(x,t,y,s) h(x_0 - x, t) \overline{h(x_0 - y, s)} \, dx \, dt \, dy \, ds$$

where

$$K(x,t,y,s) = \int_{\xi=0}^{+\infty} e^{-(|x|+|y|)\xi} e^{i(s-t)\xi^2} \, d\xi$$

By (3.24),

$$|K(x,t,y,s)| \leq \frac{C}{|t-s|^{1/2}}$$

and the result follows from the theorem on fractional integration.

Case $\tau < 0$. By (3.20)

$$\begin{aligned}
&\left\| \int_{\tau=-\infty}^{0} \int_\xi e^{ix_0\xi} e^{it\tau} \frac{|\tau|^{1/4}\hat{h}(\xi,\tau)}{\tau+\xi^2}\, d\xi\, d\tau \right\|_{L_t^2}^2 \\
&\leq \int_{\tau=-\infty}^{0} |\tau|^{-1/2} \left| \int_x e^{\pm i|x||\tau|^{1/2}} \hat{h}^t(x_0-x,\tau)\ dx \right|^2 d\tau \\
&= \int_{\xi=0}^{+\infty} \left| \int_x e^{\pm i|x|\xi} \hat{h}^t(x_0-x,\xi^2)\, dx \right|^2 d\xi \\
&= \int_{\xi=0}^{+\infty} \left| \int_x \int_t e^{\pm i|x|\xi} e^{-it\xi^2} h(x_0-x,t)\, dx\, dt \right|^2 d\xi \\
&= \int_{\xi=0}^{+\infty} \int_x \int_t e^{\pm i|x|\xi} e^{-it\xi^2} h(x_0-x,t)\, dx\, dt \overline{\int_y \int_s e^{\pm i|y|\xi} e^{-is\xi^2} h(x_0-y,s)\, dy\, ds}\, d\xi \\
&= \int_{x,t,y,s} K(x,t,y,s) h(x_0-x,t)\, \overline{h(x_0-y,s)}\, dx\, dt\, dy\, ds
\end{aligned}$$

where

$$K(x,t,y,s) = \int_{\xi=0}^{+\infty} e^{\pm i(|x|-|y|)\xi} e^{i(s-t)\xi^2}\, d\xi$$

By (3.25),

$$|K(x,t,y,s)| \leq \frac{c}{|t-s|^{1/2}}$$

and the result follows from the theorem on fractional integration. (3.50) now follows by interpolation between the cases $p=2$, $q=1$, and $p=1$, $q=\frac{4}{3}$. (3.47) follows from (3.50) by Lemma 36. To prove (3.48), we first prove

$$\sup_x \|\partial_x w(x,-)\|_{\dot{H}^{-1/4}(\mathbb{R}_t)} \leq c\|h\|_{L_t^q L_x^p} \tag{3.52}$$

By Lemma 50,

$$\begin{aligned}
&\partial_x D_t^{-1/4} \int_0^t e^{i(t-t')\partial_x^2} h(x_0, t')\, dt' \\
&\quad = \tfrac{1}{2}\partial_x D_t^{-1/4} \int_{-\infty}^{+\infty} e^{i(t-t')\partial_x^2} h(x_0, t')\, dt' - \partial_x D_t^{-1/4} \int_{-\infty}^{0} e^{i(t-t')\partial_x^2} h(x_0, t')\, dt' \\
&\qquad + \iint e^{i(x_0\xi + t\tau)} \frac{\xi\, |\tau|^{-1/4} \hat{h}(\xi,\tau)}{\xi^2 + \tau}\, d\xi d\tau \\
&\quad = \mathrm{I} - \mathrm{II} + \mathrm{III}
\end{aligned}$$

Terms I and II are established by the same method as in the proof of (3.50). To address Term III, consider separately the cases $p = 2$, $q = 1$, and $p = 1$, $q = \frac{4}{3}$. For the case $p = 2$, $q = 1$, use that $|\tau|^{-1/2}$ is an A_2 weight, and follow the previous method. For the case $p = 1$, $q = \frac{4}{3}$, use the formula for

$$\int_\xi e^{ix_0\xi} \frac{|\tau|^{-1/4}\xi}{\tau + \xi^2}\, d\xi$$

provided by (3.21), and follow the previous method. (3.52) follows by interpolation. Note that

$$\partial_t \int_0^t e^{i(t-t')\partial_x^2} h(x, t')\, dt' = h(x,t) + i \int_0^t e^{i(t-t')\partial_x^2} \partial_x^2 h(x,t)\, dt'$$

Apply the $H^{-1/4}$ norm to this equality, and add to it (3.47), to obtain by (3.52) that

$$\left\| \Psi(t) \int_0^t e^{i(t-t')\partial_x^2} h(x,t)\, dt' \right\|_{L_x^\infty H_t^{3/4}} \le \|h\|_{L_x^\infty H_t^{-1/4}} + \|h\|_{L_t^q W_x^{p,1}}$$

The continuity statement (3.49) follows from Lemma 57, dominated convergence, and the bounds (3.47) and (3.48). □

Lemma 60 (Mixed-norm estimate for the inhomogeneous Duhamel opera-

tor). *If* $\frac{2}{q'} = \frac{1}{2} - \frac{1}{p'}$, $2 < p' \le +\infty$, *and* $4 \le q' < +\infty$, *then*

$$\|w\|_{L_t^{q'} L_x^{p'}} \le c\|h\|_{L_t^q L_x^p}$$
$$\|\partial_x w\|_{L_t^{q'} L_x^{p'}} \le c\|\partial_x h\|_{L_t^q L_x^p}$$

Proof.

$$\|w\|_{L_x^{p'}} \le \int_0^t \|e^{i(t-t')\partial_x^2} h(-,t')\|_{L_x^{p'}}\, dt' \le c\int_{-\infty}^{+\infty} \frac{\|h(-,t')\|_{L_x^p}}{|t-t'|^{\frac{1}{p}-\frac{1}{2}}}\, dt'$$

where the second inequality follows from Lemma 43, since $1 \le p \le 2$. Applying the $L_t^{q'}$ norm, the result follows from the theorem on fractional integration (see [Ste70], Chapter V, Theorem 1), provided the relation $\frac{2}{q'} = \frac{1}{2} - \frac{1}{p'}$ holds and $q' \neq \infty$. □

3.6 Construction of the solution operators

Let $s = 0$ or $s = 1$, and let p', q' be an admissible pair, i.e.

$$4 \le q' < +\infty$$
$$2 < p' \le +\infty$$
$$\frac{2}{q'} = \frac{1}{2} - \frac{1}{p'}$$

and p, q their respective dual exponents.

$$Z_s = C(\, x \in (-\infty, +\infty);\; H^{\frac{2s+1}{4}}(\mathbb{R}_t)\,) \cap C(\, t \in (-\infty, +\infty);\; H^s(\mathbb{R}_x)\,) \cap L_t^{q'} W_x^{p',s}$$

with norm

$$\|u\|_{Z_s} = \sup_x \|u(x,t)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} + \sup_t \|u(x,t)\|_{H^s(\mathbb{R}_x)} + \|u\|_{L_t^{q'} W_x^{p',s}}$$

Let

$$V_s = \{\ (\phi, f)\ \mid\ \phi \in H^s(\mathbb{R}_x^+),\ f \in H^{\frac{2s+1}{4}}(\mathbb{R}_t^+) \text{ and } \phi(0) = f(0) \text{ if } s = 1\ \}$$

with norm

$$\|(\phi, f)\|_{V_s} = \|\phi\|_{H^s(\mathbb{R}^+)} + \|f\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)}$$

Note that the compatibility condition for $s = 1$ is built into the definition of V_s.

Theorem 12 (Homogeneous solution operator). *Let $T_0 > 0$. Then there is a bounded linear operator $HS_{T_0} : V_s \longrightarrow Z_s$, such that, with $w = HS_{T_0}(f, \phi)$, we have*

$$\begin{aligned} i\partial_t w + \partial_x^2 w = 0 \quad & \text{in } \mathcal{D}'((0, T_0) \times \mathbb{R}_x^+) \\ w(x, 0) = \phi(x) \quad & \text{in the sense of } C(\, t \in (-\infty, +\infty);\ H^s(\mathbb{R}_x)\,) \\ w(0, t) = f(t) \quad & \text{in the sense of } C(\, x \in (-\infty, +\infty);\ H^{\frac{2s+1}{4}}(\mathbb{R}_t)\,) \end{aligned}$$

Proof. There is an extension $\tilde{f} \in H^{\frac{2s+1}{4}}(\mathbb{R})$ of $f \in H^{\frac{2s+1}{4}}(\mathbb{R}^+)$. That is,

$$\|\tilde{f}\|_{H^{\frac{2s+1}{4}}(\mathbb{R})} \leq c\|f\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)}$$

$$\tilde{f} = f \text{ pointwise a.e. in } (0, +\infty)$$

Let $\tilde{\phi}(x) \in H_x^s$ be an extension of $\phi(x) \in H^s(\mathbb{R}^+)$ such that $\|\tilde{\phi}\|_{H^s(\mathbb{R})} \leq c\|\phi\|_{H^s(\mathbb{R}^+)}$
Let $\Psi_1(t)$, $\Psi_2(t)$, and $\Psi_3(t)$ be $C_0^\infty(\mathbb{R})$ functions such that

$$\Psi_1(t) = \begin{cases} 1 & \text{if } t \in [-T_0, T_0] \\ 0 & \text{if } t \in \left[-\frac{4}{3}T_0, \frac{4}{3}T_0\right]^c \end{cases}$$

$$\Psi_2(t) = \begin{cases} 1 & \text{if } t \in \left[-\frac{4}{3}T_0, \frac{4}{3}T_0\right] \\ 0 & \text{if } t \in \left[-\frac{5}{3}T_0, \frac{5}{3}T_0\right]^c \end{cases}$$

$$\Psi_3(t) = \begin{cases} 1 & \text{if } t \in \left[-\frac{5}{3}T_0, \frac{5}{3}T_0\right] \\ 0 & \text{if } t \in [-2T_0, 2T_0]^c \end{cases}$$

By Lemma 46,

$$\|\Psi_1(t)e^{it\partial_x^2}\tilde{\phi}\,\big|_{x=0}\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} \leq c\|\tilde{\phi}\|_{H^s(\mathbb{R})} \leq c\|\phi\|_{H^s(\mathbb{R}^+)} \tag{3.53}$$

Furthermore,

$$\|\Psi_1(t)\tilde{f}(t)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} \leq c\|\tilde{f}\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} \leq c\|f\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)} \tag{3.54}$$

Let $f_1(t) = \chi_{(0,+\infty)}\left[\Psi_1(t)\tilde{f}(t) - \Psi_1(t)e^{it\partial_x^2}\tilde{\phi}\,\big|_{x=0}\right]$. Then,

$$\begin{aligned} \|f_1\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} &= \|f_1\|_{H^{\frac{2s+1}{4}}(\mathbb{R})} && \text{by definition of } H_0^{\frac{2s+1}{4}}(\mathbb{R}^+) \\ &\leq c\|\Psi_1(t)\tilde{f}(t) - \Psi_1(t)e^{it\partial_x^2}\tilde{\phi}\,\big|_{x=0}\|_{H^{\frac{2s+1}{4}}(\mathbb{R})} && \text{see note below} \\ &\leq c\|(f,\phi)\| && \text{by (3.53) and (3.54)} \end{aligned} \tag{3.55}$$

where the middle step follows by Lemma 40 if $s = 0$, and the compatibility condition $\phi(0) = f(0)$ and Lemmas 39 and 38 if $s = 1$.

Let $C_B = B(0) \neq 0$, and

$$h_1(t) = \frac{1}{C_B\Gamma(\frac{1}{2})}\mathcal{I}_{-1/2}(f_1)(t)$$

Then

$$\begin{aligned} \|h_1\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)} &\leq c\|\mathcal{I}_{-1/2}(f_1)(t)\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)} \\ &\leq c\|f_1\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} && \text{by Lemma 41} \\ &\leq c\|(f,\phi)\| && \text{by (3.55)} \end{aligned}$$

Claim 1. $\Psi_2\mathcal{I}_{1/2}(h_1) = \dfrac{1}{C_B\Gamma(\frac{1}{2})}f_1$

Proof of Claim 1. Assume, for the moment, that $f_1 \in C_0^\infty(\mathbb{R}^+)$ with $\text{supp}(f_1) \subset [-\frac{4}{3}T_0, \frac{4}{3}T_0]$ (although it is actually in $H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)$) and h_1 is defined as above in terms of f_1. Then,

$$\begin{aligned}
\Psi_2\mathcal{I}_{1/2}(h_1) &= \Psi_2\mathcal{I}_{1/2}\left(\frac{1}{C_B\Gamma(\frac{1}{2})}\mathcal{I}_{-1/2}(f_1)\right) \\
&= \frac{\Psi_2}{C_B\Gamma(\frac{1}{2})}\mathcal{I}_{1/2}\mathcal{I}_{-1/2}(f_1) \\
&= \frac{\Psi_2}{C_B\Gamma(\frac{1}{2})}f_1 \\
&= \frac{1}{C_B\Gamma(\frac{1}{2})}f_1 \qquad\qquad \text{since supp} f_1 \subset [-\tfrac{4}{3}T_0, \tfrac{4}{3}T_0]
\end{aligned}$$

Now return to the general case of $f_1 \in H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)$. Let f_1^k be a sequence in $C_0^\infty(\mathbb{R}^+)$ with $\text{supp}(f_1^k) \subset [-\frac{4}{3}T_0, \frac{4}{3}T_0]$ such that $f_1^k \to f_1$ in $H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)$. Let h_1^k be the corresponding functions defined in terms of f_1^k as above. Then,

$$\begin{aligned}
&\left\|\Psi_2\mathcal{I}_{1/2}(h_1) - \frac{1}{C_B\Gamma(\frac{1}{2})}f_1\right\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} \\
&\quad= \left\|\Psi_2\mathcal{I}_{1/2}(h_1 - h_1^k) + \Psi_2\mathcal{I}_{1/2}(h_1^k) - \frac{1}{C_B\Gamma(\frac{1}{2})}(f_1 - f_1^k) - \frac{1}{C_B\Gamma(\frac{1}{2})}f_1^k\right\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} \\
&\quad= \left\|\Psi_2\mathcal{I}_{1/2}(h_1 - h_1^k) - \frac{1}{C_B\Gamma(\frac{1}{2})}(f_1 - f_1^k)\right\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} \\
&\quad\le \left\|\Psi_2\mathcal{I}_{1/2}(h_1 - h_1^k)\right\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} + \frac{1}{C_B\Gamma(\frac{1}{2})}\left\|f_1 - f_1^k\right\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} \\
&\quad\le \left\|(h_1 - h_1^k)\right\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)} + \frac{1}{C_B\Gamma(\frac{1}{2})}\left\|f_1 - f_1^k\right\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)}
\end{aligned}$$

where, in the last step, we applied Lemma 42

$$\leq \frac{1}{C_B\Gamma(\frac{1}{2})}\left\|\mathcal{I}_{-1/2}(f_1 - f_1^k)\right\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)} + \frac{1}{C_B\Gamma(\frac{1}{2})}\left\|f_1 - f_1^k\right\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)}$$
$$\leq \frac{2}{C_B\Gamma(\frac{1}{2})}\left\|f_1 - f_1^k\right\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)}$$

Now let $k \to +\infty$. *End proof of Claim 1.*

Claim 2. $\Psi_2\mathcal{I}_{1/2}(h_1) = \Psi_2\mathcal{I}_{1/2}(\Psi_3 h_1)$ holds in $H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)$. (Recall h_1 in $H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)$).
Proof of Claim 2. First, we'll prove the identity assuming $h_1 \in C_0^\infty(\mathbb{R}^+)$.

$$\begin{aligned}\Psi_2(t)\mathcal{I}_{1/2}(\Psi_3 h_1)(t) &= \frac{\Psi_2(t)}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-1/2}\Psi_3(s)h_1(s)\,ds \\ &= \frac{\Psi_2(t)}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-1/2}h_1(s)\,ds\end{aligned}$$

since supp $\Psi_2 \subset [-\frac{5}{3}, \frac{5}{3}T_0]$ and $\Psi_3(s) = 1$ on this interval

$$= \Psi_2(t)\mathcal{I}_{1/2}(h_1)$$

For the general case, let h_1^k be a sequence in $C_0^\infty(\mathbb{R}^+)$ such that $h_1^k \to h_1$ in $H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)$.

$$\begin{aligned}&\|\Psi_2\mathcal{I}_{1/2}(\Psi_3 h_1) - \Psi_2\mathcal{I}_{1/2}(h_1)\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} \\ &\quad\leq \|\Psi_2\mathcal{I}_{1/2}(\Psi_3(h_1 - h_1^k)) - \Psi_2\mathcal{I}_{1/2}(h_1 - h_1^k) \\ &\qquad\qquad + \Psi_2\mathcal{I}_{1/2}(\Psi_3 h_1^k) - \Psi_2\mathcal{I}_{1/2}(h_1^k)\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} \\ &\quad\leq \|\Psi_2\mathcal{I}_{1/2}(\Psi_3(h_1 - h_1^k))\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} + \|\Psi_2\mathcal{I}_{1/2}(h_1 - h_1^k)\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)}\end{aligned}$$

By Lemma 42, the first piece is

$$\begin{aligned}
&\|\Psi_2\mathcal{I}_{1/2}(\Psi_3(h_1-h_1^k))\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} \\
&\quad\le \|\Psi_3(h_1-h_1^k)\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)} \\
&\quad\le \|h_1-h_1^k\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)}
\end{aligned}$$

The second piece is

$$\|\Psi_2\mathcal{I}_{1/2}(h_1-h_1^k)\|_{H_0^{\frac{2s+1}{4}}(\mathbb{R}^+)} \le \|h_1-h_1^k\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)}$$

Now let $k\to\infty$. *End proof of Claim 2.*

Combining Claim 1 and Claim 2, we get

$$\frac{1}{C_B\Gamma(\frac{1}{2})}f_1 = \Psi_2\mathcal{I}_{1/2}(\Psi_3 h_1)$$

Set $h=\Psi_3 h_1$, and then $h\in H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)$, and

$$\|h\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)} \le c\|h_1\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}^+)} \le c\|(f,\phi)\|$$

Let

$$w(x,t)=\Psi_2(t)\left[\int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')dt' + e^{it\partial_x^2}\tilde{\phi}(x)\right]$$

We now check the boundedness properties.

Step 1. Show $w\in C(x\in(-\infty,+\infty);\, H^{\frac{2s+1}{4}}(\mathbb{R}_t))$ and $\sup_x \|w(x,\cdot)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} \le c\|(f,\phi)\|$.

This follows from Lemma 55, which gives

$$\sup_x \left\| \Psi_2(t) \int_0^t e^{i(t-t')\partial_x^2} \delta_0(x) h(t') dt' \right\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} \leq c \|h\|_{H_0^{\frac{2s-1}{4}}(\mathbb{R}_t^+)}$$

and Lemma 46

$$\sup_x \left\| \Psi_2(t) e^{it\partial_x^2} \tilde{\phi}(x) \right\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} \leq c \|\tilde{\phi}\|_{H^s(\mathbb{R})}$$

Step 2. Show $w \in C(t \in (-\infty, +\infty);\, H^s(\mathbb{R}_x))$ and $\sup_t \|w(\cdot, t)\|_{H^s(\mathbb{R}_x)} \leq c\|(f, \phi)\|$
This follows from Lemma 45

$$\sup_t \|\Psi_2(t) e^{it\partial_x^2} \tilde{\phi}\|_{H^s(\mathbb{R}_x)} \leq \sup_t \|e^{it\partial_x^2} \tilde{\phi}\|_{H^s(\mathbb{R}_x)} \leq c \|\tilde{\phi}\|_{H^s(\mathbb{R}_x)}$$

and Lemma 54

$$\sup_t \left\| \Psi_2(t) \int_0^t e^{i(t-t')\partial_x^2} \delta_0(x) h(t') dt' \right\|_{H^s(\mathbb{R}_x)} \leq c \|h\|_{H^{\frac{2s-1}{4}}(\mathbb{R})}$$

Step 3. Show $w \in L_t^{q'} W_x^{p',s}$ and $\|w\|_{L_t^{q'} W_x^{p',s}} \leq c\|(f, \phi)\|$
Lemma 56 gives

$$\left\| \Psi_2(t) \int_0^t e^{i(t-t')\partial_x^2} \delta_0(x) h(t') dt' \right\|_{L_t^{q'} W_x^{p',s}} \leq c \|h\|_{H^{\frac{2s-1}{4}}(\mathbb{R})}$$

Lemma 47 gives

$$\|\Psi_2(t) e^{it\partial_x^2} \tilde{\phi}(x)\|_{L_t^{q'} W_x^{p',s}} \leq c \|\tilde{\phi}\|_{H^s(\mathbb{R})}$$

This concludes the boundedness estimates. Finally, we must check that it is a solution.

By substitution, $w(x,0) = \tilde{\phi}(x)$. On $[0,T_0]$,

$$\begin{aligned} w(0,t) &= \int_0^t e^{i(t-t')\partial_x^2}\delta_0(x)h(t')\big|_{x=0}dt' + e^{it\partial_x^2}\tilde{\phi}\big|_{x=0} \\ &= C_B\Gamma(\tfrac{1}{2})\mathcal{I}_{1/2}(h)(t) + (\tilde{f} - f_1) = \tilde{f} = f \end{aligned}$$

□

Theorem 13 (Inhomogenous solution operator). *For given $T_0 > 0$, there is a bounded linear operator $\mathrm{IHS}_{T_0} : L_t^q L_x^p \to Z_0$ for $s = 0$, and $\mathrm{IHS}_{T_0} : L_t^q W_x^{p,1} \cap L_x^\infty H_t^{-1/4} \to Z_1$ for $s = 1$, such that, with $w = \mathrm{IHS}_{T_0}(h)$, we have*

$$\begin{aligned} i\partial_t w + \partial_x^2 w = h \qquad & \text{in } \mathcal{D}'(\,(0,T_0)\times\mathbb{R}_x^+\,) \\ w(x,0) = 0 \qquad & \text{in the sense of } C(\,t\in(-\infty,\infty);\, H^s(\mathbb{R}_x)\,) \\ w(0,t) = 0 \qquad & \text{in the sense of } C(\,x\in(-\infty,\infty);\, H^{\frac{2s+1}{4}}(\mathbb{R}_t^+)\,) \end{aligned}$$

Proof. Let $\Psi(t) \in C_0^\infty(\mathbb{R})$ such that $\Psi(t) = 1$ on $[-T_0, T_0]$ and $\operatorname{supp}\Psi \subset [-2T_0, 2T_0]$. Let

$$w_1(x,t) = \Psi(t)\int_0^t e^{i(t-t')\partial_x^2}h(x,t')dt'$$

By Lemma 59,

$$\sup_x \|w_1(x,-)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} \le c \begin{cases} \|h\|_{L_t^q L_x^p} & \text{if } s = 0 \\ \|h\|_{L_t^q W_x^{p,s}} + \|h\|_{L_x^\infty H_t^{-1/4}} & \text{if } s = 1 \end{cases}$$

By Lemma 58,

$$\sup_t \|w_1(-,t)\|_{H^s(\mathbb{R}_x)} \le c\|h\|_{L_t^q W_x^{p,s}}$$

By Lemma 60,

$$\|w_1\|_{L_t^{q'} W_x^{p',s}} \le c\|h\|_{L_t^q W_x^{p,s}}$$

Let $f(t) = w_1(0,t)$. Then

$$\|f\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)} \leq \|f\|_{H^{\frac{2s+1}{4}}(\mathbb{R})} = \|w_1(0,t)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)}$$
$$\leq c \begin{cases} \|h\|_{L_t^q L_x^p} & \text{if } s=0 \\ \|h\|_{L_t^q W_x^{p,s}} + \|h\|_{L_x^\infty H_t^{-1/4}} & \text{if } s=1 \end{cases}$$

Let $T = \max\{2T_0, 1\}$, and set $w_2 = \mathrm{HS}_T(f,0)$ (if $s=1$, the compatibility condition $f(0)=0$ is satisfied). Then, by Theorem 12,

$$\sup_x \|w_2(x,-)\|_{H^{\frac{2s+1}{4}}(\mathbb{R}_t)} + \sup_t \|w_2(-,t)\|_{H^s(\mathbb{R}_x)}$$
$$+ \|w_2\|_{L_t^{q'} W_x^{p',s}} \leq c\|f\|_{H^{\frac{2s+1}{4}}(\mathbb{R}^+)} \leq c \begin{cases} \|h\|_{L_t^q L_x^p} & \text{if } s=0 \\ \|h\|_{L_t^q W_x^{p,s}} + \|h\|_{L_x^\infty H_t^{-1/4}} & \text{if } s=1 \end{cases}$$

Let $w = w_1 - w_2$. Then w satisfies the estimates, and solves the equation. □

3.7 Solution to the nonlinear problem

Theorem 14 (L^2 case). *Let $1 < \alpha < 5$. Let $(f,\phi) \in V_0$. Then there exists a $T_0 = T_0(\|\phi\|_{L_x^2}, \|f\|_{H_t^{1/4}}) > 0$ such that*

$$\Lambda w = HS_{T_0}(f,\phi) + IHS_{T_0}(w|w|^{\alpha-1})) \tag{3.56}$$

has a fixed point in the space

$$Z_0 = C(\, t \in (-\infty,+\infty);\, L^2(\mathbb{R}_x)\,) \cap C(\, x \in (-\infty,+\infty);\, H^{1/4}(\mathbb{R}_t)\,) \cap L_t^r L_x^{\alpha+1} \tag{3.57}$$

where $r = \frac{4(\alpha+1)}{\alpha-1}$.

Proof. We apply the estimates with $p = \frac{\alpha+1}{\alpha}$, $p' = \alpha+1$, $q = \frac{4(\alpha+1)}{3\alpha+5}$, $q' = \frac{4(\alpha+1)}{\alpha-1}$.

Then $\alpha p = p'$ and $\frac{q}{p} < \frac{q'}{p'}$, so by Hölder there exists $\theta > 0$ such that

$$\| w|w|^{\alpha-1} \|_{L_t^q L_x^p} \le cT^\theta \|w\|_{L_t^{q'} L_x^{p'}}$$

We can obtain a similar bound for a difference $\Lambda(w_2) - \Lambda(w_1)$, and thus Λ is a contraction on Z_0. □

In the following case, we follow [CW89].

Theorem 15 (L^2 critical case). *Given $(\phi, f) \in V_0$, there exists $T_0 = T_0(\phi, f) > 0$ and $u \in Z_0$ where*

$$Z_0 = C(\, t \in (-\infty, +\infty);\, L^2(\mathbb{R}_x)\,) \cap C(\, x \in (-\infty, +\infty);\, H^{1/4}(\mathbb{R}_t)\,) \cap L_t^6 L_x^6$$

such that

$$\begin{aligned} i\partial_t u + \partial_x^2 u + u|u|^4 &= 0 && \text{in } \mathbb{R}^+ \times (0, T_0) \\ u(x, 0) &= \phi(x) && \text{on } \mathbb{R}^+ \\ u(0, t) &= f(t) && \text{on } (0, T_0) \end{aligned}$$

Proof. Take T_0 sufficiently small so that

$$\|\mathrm{HS}_{T_0}(\phi, f)\|_{L_t^6 L_x^6} \le \delta$$

where δ will be chosen sufficiently small later. Note that T_0 will depend on ϕ and f, not just on the corresponding norms of these functions. c_1 will also be selected as

$\sim \|\phi\|_{L^2_x} + \|f\|_{H^{1/4}_t}$. Let

$$E_0 = \{\ w\ \mid w \in C(\,t \in (-\infty,+\infty);\ L^2(\mathbb{R}_x)\,) \cap C(\,x \in (-\infty,+\infty);\ H^{1/4}(\mathbb{R}_t)\,) \cap L^6_t L^6_x$$

$$\sup_t \|w(-,t)\|_{L^2_x} \leq c_1$$

$$\sup_x \|w(x,-)\|_{H_t^{\frac{2s+1}{4}}} \leq c_1$$

$$\|w\|_{L^6_t L^6_x} \leq 2\delta\ \}$$

Define the map $\Lambda :\ E_0 \to Z_0$ as

$$\Lambda w = \mathrm{HS}_{T_0}(\phi, f) + \mathrm{IHS}_{T_0}(w|w|^4)$$

We claim that $\Lambda :\ E_0 \to E_0$ and that Λ is a contraction on E_0. Given $w \in E_0$, we have

$$\begin{aligned}
\|\Lambda w\|_{L^6_t L^6_x} &\leq \|\mathrm{HS}_{T_0}(\phi, f)\|_{L^6_t L^6_x} + \|\mathrm{IHS}_{T_0}(w|w|^4)\|_{L^6_t L^6_x} \\
&\leq \delta + c\|w|w|^4\|_{L^{6/5}_t L^{6/5}_x} \\
&\leq \delta + c\|w\|^5_{L^6_t L^6_x} \\
&\leq \delta + c(2\delta)^5 \\
&\leq 2\delta
\end{aligned}$$

provided δ is chosen suitably small. We also have

$$\|\Lambda w\|_{L^\infty_t L^2_x} + \|\Lambda w\|_{L^\infty_x H^{1/4}_t} \leq c(\|\phi\|_{L^2_x} + \|f\|_{H^{1/4}_t}) + c\|w\|_{L^6_t L^6_x} \leq c_1$$

Now we show that Λ is a contraction on E_0. We have

$$\begin{aligned}\|\Lambda w_1 - \Lambda w_2\|_{L_t^6 L_x^6} &\le \|\mathrm{IHS}_{T_0}(w_1|w_1|^4 - w_2|w_2|^4)\|_{L_t^6 L_x^6} \\ &\le \|\mathrm{IHS}_{T_0}[(w_1 - w_2)(|w_1|^4 + |w_2|^4)]\|_{L_t^6 L_x^6} \\ &\le \left(\int_t \int_x |w_1 - w_2|^{6/5}(|w_1|^{24/5} + |w_2|^{24/5})\, dx\, dt\right)^{5/6} \\ &\le c\|w_1 - w_2\|_{L_t^6 L_x^6}\left(\|w_1\|^4_{L_t^6 L_x^6} + \|w_2\|^4_{L_t^6 L_x^6}\right) \\ &\le 2c\delta^4 \|w_1 - w_2\|_{L_t^6 L_x^6}\end{aligned}$$

so we also need that $2\delta^4 \le \frac{1}{2}$. The same bound holds for

$$\|\Lambda w_1 - \Lambda w_2\|_{L_t^\infty L_x^2}$$

and

$$\|\Lambda w_1 - \Lambda w_2\|_{L_x^\infty H_t^{1/4}}$$

showing that Λ is a contraction. The fixed point u to Λ in E_0 will be a solution. □

Theorem 16 (H^1 case). *Let $1 < \alpha < \infty$. Let $(f, \phi) \in V_1$ (in particular, f and ϕ must satisfy the compatibility condition $f(0) = \phi(0)$). Then there exists a $T_0 > 0$ such that*

$$\Lambda w = HS_{T_0}(f, \phi) + IHS_{T_0}(w|w|^{\alpha-1})) \tag{3.58}$$

has a fixed point in the space

$$Z_1 = C(\, t \in (-\infty, +\infty);\, H^1(\mathbb{R}_x)\,) \cap C(\, x \in (-\infty, +\infty);\, H^{3/4}(\mathbb{R}_t)\,) \cap L_t^r W_x^{\alpha+1,1} \tag{3.59}$$

where $r = \frac{4(\alpha+1)}{\alpha-1}$.

Proof. We apply the estimates, again, with $p = \frac{\alpha+1}{\alpha}$, $p' = \alpha + 1$, $q = \frac{4(\alpha+1)}{3\alpha+5}$,

$q' = \frac{4(\alpha+1)}{\alpha-1}$. By Hölder

$$\|\partial_x[w|w|^{\alpha-1}]\|_{L_t^q L_x^p} \le c \left(\int \left(\int |\partial_x w|^{p\alpha}\, dx \right)^{\frac{q}{\alpha p}} \left(\int |w|^{\alpha p}\, dx \right)^{\frac{(\alpha-1)q}{\alpha p}} dt \right)^{1/q}$$

Apply Sobolev imbedding to the second piece, and since $\alpha p = p'$ and $\frac{q}{\alpha p} < \frac{q'}{p'}$, by Hölder, we have

$$\|\partial_x[w|w|^{\alpha-1}]\|_{L_t^q L_x^p} \le cT^\theta \|w\|^{\rho_1}_{L_t^{q'} L_x^{p'}} \|w\|^{\rho_2}_{L_t^\infty H_x^1}$$

The other piece, i.e.

$$\|w|w|^{\alpha-1}\|_{L_t^q L_x^p}$$

can also be treated with the help of Sobolev. Finally, we need to bound

$$\|w|w|^{\alpha-1}\|_{L_x^\infty H_t^{-1/4}} \le cT^\theta \|w\|^\alpha_{L_t^\infty L_x^\infty} \le cT^\theta \|w\|^\alpha_{L_t^\infty H_x^1}$$

We can prove similar bounds for $\Lambda(w_2) - \Lambda(w_1)$, and therefore Λ is a contraction on Z_1. $\square$

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