Homework 9 for MATH 1260 Fall 2019

Due Thursday November 21

Reading: Chapter VI Sections 5–6 (especially Section 6).
Turn in solutions to all problems.

1. Section VI.6 Problems 12, 13, 14.

2. Prove that the following are equivalent when $f \sim \sum c_k e^{ik}$ (and $f\colon \mathbb{R} \to \mathbb{C}$ is 2π-periodic) and $\int_{-\pi}^{\pi} |f|^2 < \infty$:

   (a) $\sum_{k=-\infty}^{\infty} |c_k|^2 = \int_{-\pi}^{\pi} |f(x)|^2 \frac{dx}{2\pi}$ (Parseval's Identity).

   (b)
   $$\lim_{N,M\to\infty} \int_{-\pi}^{\pi} \left| \sum_{k=-N}^{M} c_k e^{ikx} - f(x) \right|^2 dx = 0.$$

   (c) For all $\epsilon > 0$ there are N, M and $(b_k)_{k=-N}^{M}$ such that
   $$\int_{-\pi}^{\pi} \left| \sum_{k=-N}^{M} b_k e^{ikx} - f(x) \right|^2 dx < \epsilon.$$

   We say that f is *Parseval* when it satisfies any (and therefore all) of these conditions.

3. Prove that any continuous 2π-periodic function is Parseval.

4. Suppose that f_n is a Parseval function for each n and $\int_{-\pi}^{\pi} |f_n - f|^2 \to 0$ as $n \to \infty$. Show that f is Parseval.

5. We say that a 2π-periodic $f\colon \mathbb{R} \to \mathbb{C}$ is *strictly piecewise continuous* if there exist $-\pi = a_0 < a_1 < \dots a_m = \pi$ such that for each $i = 0, \dots m-1$ there is a continuous function on $[a_i, a_{i+1}]$ that agrees with f on (a_i, a_{i+1}). Prove that every strictly piecewise continuous function is Parseval.

6. Now that we've properly shown that the function in the Example on page 187 of Gamelin (with the series given in (6.5)) is Parseval, do Problem 8 in Section VI.6.

In the lecture I discussed functions $f\colon \mathbb{R} \to \mathbb{C}$ with $f(x+2\pi) \equiv f(x)$, while Gamelin prefers $f(e^{i\theta})$ with $f\colon S^1 \to \mathbb{C}$. In the problems I've written I've used my convention for f, while the problems from Gamelin use Gamelin's convention.

While it really is called Parseval's Identity, no one (outside of this problem set) says that a given function is Parseval. In fact every square-integrable function f is Parseval.