LECTURE 3

1. Extremal length

Conformal metrics. Let S be a Riemann surface. A *conformal metric* on S is a map

$$\rho : TS \to \mathbb{R}_{\geq 0}$$

satisfying $\rho(\lambda v) = |\lambda|\rho(v)$ for all $v \in TS$ and all $\lambda \in \mathbb{C}$. If $z : U \to \mathbb{C}$ is a chart defined on some open set $U \subset S$, then we can write ρ as

$$\rho_U(z)|dz|$$

for some non-negative function ρ_U defined on $z(U)$. Therefore the metric ρ is a rescaling of the euclidean metric at each point. We assume that the functions ρ_U are measurable and locally integrable. It is standard to abuse notation and just write $\rho \equiv \rho(z)|dz|$, where the ρ on the left is a tensor and the ρ on the right is a function.

Suppose that two charts $z : U \to \mathbb{C}$ and $w : V \to \mathbb{C}$ on S overlap, and let $h := w \circ z^{-1}$ be the transition function. If ρ is written locally as $\rho_U(z)|dz|$ and $\rho_V(w)|dw|$ on $U \cap V$, then we have

$$\rho_V(h(z))|h'(z)| = \rho_U(z),$$

since $dw = dh \circ dz = h'(z)dz$.

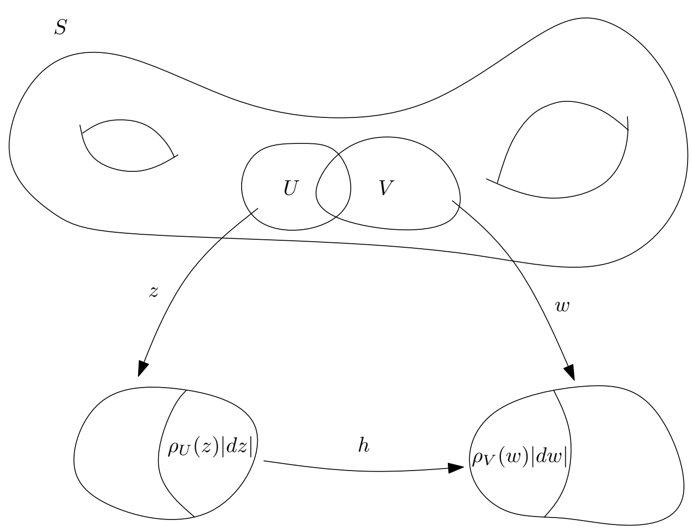

Figure 1. A conformal metric seen in charts.

If γ is a path from an interval I to S which is piecewise smooth (or even locally rectifiable), then we can compute its length in the metric ρ as

$$L_\rho(\gamma) := \int_I \rho(\gamma'(t))\, dt.$$

A conformal metric ρ also defines an area form ρ^2. If the local expression for ρ is $\rho(z)|dz|$, then the local expression for ρ^2 is $\rho(z)^2 dxdy$. These local area forms glue together consistently since under a conformal change of coordinates h two such expression will differ by $|h'|^2$, which is the same as the Jacobian of h.

Extremal length. If Γ is any set of paths on S, we define its *extremal length* $\mathcal{L}(\Gamma)$ as the supremum of the quantity

$$\frac{(\inf_{\gamma\in\Gamma} L_\rho(\gamma))^2}{\int_S \rho^2}$$

taken over all conformal metrics ρ on S for which this quantity is a real number.

The *extremal width (or modulus)* of Γ is defined as

$$\mathcal{W}(\Gamma) := \inf\left\{\int_S \rho^2 \;\middle|\; \rho \text{ is a conformal metric such that } \forall\gamma\in\Gamma, L_\rho(\gamma)\geq 1\right\}.$$

Note that the ratio $(\inf_{\gamma\in\Gamma} L_\rho(\gamma))^2/\int_S \rho^2$ is homogeneous : it remains unchanged under rescaling of ρ. From this observation it follows easily that $\mathcal{W}(\Gamma) = 1/\mathcal{L}(\Gamma)$.

Examples. Here are two crucial examples.

Theorem 1.1. *Let*

$$R_a = \{x+iy\in\mathbb{C} \mid 0\leq x\leq a, 0\leq y\leq 1\} = [0,a]+i[0,1],$$

and let Γ_a be the set of paths $\gamma : [0,1]\to R_a$ that connect the two vertival sides, i.e. which satisfy $\operatorname{Re}\gamma(0) = 0$ and $\operatorname{Re}\gamma(1) = a$. Then

$$\mathcal{L}(\Gamma_a) = a.$$

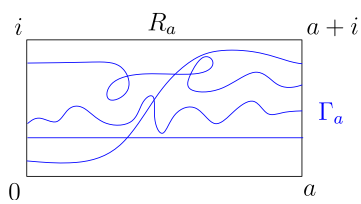

FIGURE 2. The rectangle R_a and the path family Γ_a.

Proof. The euclidean metric $\rho\equiv|dz|$ on R_a has area a and is such that each path $\gamma\in\Gamma$ has ρ-length at least a. Therefore, $\mathcal{L}(\Gamma_a)$ is at least $a^2/a = a$.

Now, given an arbitrary conformal metric ρ, by rescaling we may assume that $\inf_{\gamma\in\Gamma_a} L_\rho(\gamma) = a$. In particular, the horizontal lines have length at least a so that

$$\int_0^a \rho(x+iy)\, dx \geq a$$

for all $y \in [0,1]$, and by integrating over y we get

$$\int_0^1 \int_0^a \rho(x+iy)\,dxdy \geq a.$$

If we square this and use the Cauchy-Schwarz inequality, we obtain

$$a^2 \leq \left(\iint_{R_a} \rho(x+iy)\,dxdy\right)^2 \leq \left(\iint_{R_a} 1\,dxdy\right)\left(\iint_{R_a} \rho(x+iy)^2\,dxdy\right),$$

and thus $a \leq \iint_{R_a} \rho^2$. This shows that

$$\frac{(\inf_{\gamma\in\Gamma_a} L_\rho(\gamma))^2}{\iint_{R_a}\rho^2} \leq \frac{a^2}{a} = a$$

for every conformal metric ρ and hence $\mathcal{L}(\Gamma_a) \leq a$. □

Corollary 1.2. *For $R > 1$, let A_R be the annulus $\{z \in \mathbb{C} \mid 1 < |z| < R\}$ and let Γ_R be the set of paths in the closure $\overline{A_R}$ connecting the two boundary components. Then*

$$\mathcal{L}(\Gamma_R) = \frac{\log R}{2\pi}.$$

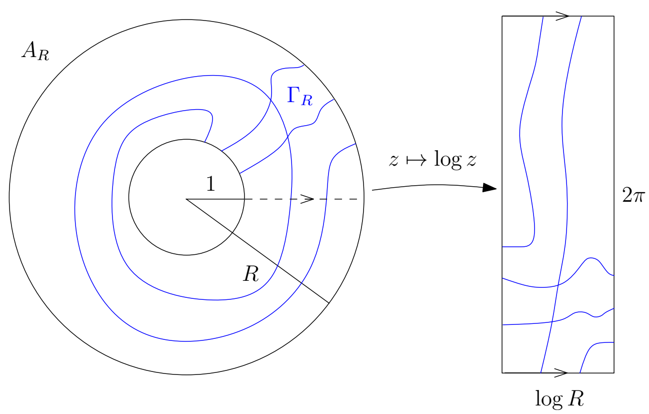

FIGURE 3. Opening up an annulus.

Proof. The logarithm function maps the annulus A_R conformally onto the rectangle

$$(0, \log R) + i[0, 2\pi]$$

with horizontal sides identified. The paths in Γ_R get mapped to paths connecting the two vertical sides, but possibly crossing through a horizontal side and reappearing on the opposite side in between. The calculations from the previous theorem hold nevertheless and we get that the extremal length of Γ_R is the ratio of the base of the above rectangle to its height. □

If A is an annulus in the plane neither of whose boundary components reduces to a point, then A is conformally equivalent to a round annulus A_R for a unique $R > 1$. This is a consequence of the uniformization theorem. We define the *modulus* of A as

$$\operatorname{mod} A := \frac{\log R}{2\pi},$$

which is the same as the extremal length of the set of paths connecting the two boundary components of A.

Suppose that two disjoint annuli A_1 and A_2 are included in A and separate its boundary components. We will use extremal length to prove that

$$\operatorname{mod} A \geq \operatorname{mod} A_1 + \operatorname{mod} A_2.$$

This will follow from a more general principle called *the series law.*

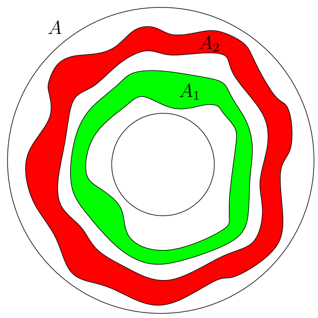

FIGURE 4. Two separating subannuli.

Comparing path families. We say that Γ *extends* H (denoted $\Gamma \succ H$) if every curve $\gamma \in \Gamma$ extends some curve $\eta \in H$.

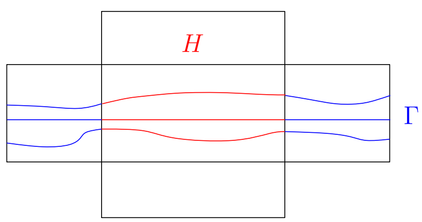

FIGURE 5. The path family Γ extends H.

It is easy to see that if $\Gamma \succ H$, then $\mathcal{L}(\Gamma) \geq \mathcal{L}(H)$. Indeed, if ρ is any conformal metric on S, then for every $\gamma \in \Gamma$ there exists an $\eta \in H$ such that $L_\rho(\gamma) \geq L_\rho(\eta)$, which implies that

$$\inf_{\gamma \in \Gamma} L_\rho(\gamma) \geq \inf_{\eta \in H} L_\rho(\eta),$$

and thus

$$\mathcal{L}(\Gamma) \geq \mathcal{L}(H).$$

Series and parallel laws. If Γ_1 and Γ_2 are two sets of paths on S, then their union

$$\Gamma_1 \cup \Gamma_2$$

is again a set of paths. We can also form the formal sum

$$\Gamma_1 + \Gamma_2 := \{\gamma_1 + \gamma_2 \mid \gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2\},$$

where the path $\gamma_1 + \gamma_2$ can be thought of the path γ_1 followed by the path γ_2.

The *support* $\operatorname{supp}\Gamma$ of a set of paths Γ is defined as the closure of the union of the images of the curves $\gamma \in \Gamma$.

If Γ_1 and Γ_2 have disjoint supports, then the *series law* states that

$$\mathcal{L}(\Gamma_1 + \Gamma_2) = \mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2)$$

and the *parallel law* states that

$$\mathcal{W}(\Gamma_1 \cup \Gamma_2) = \mathcal{W}(\Gamma_1) + \mathcal{W}(\Gamma_2).$$

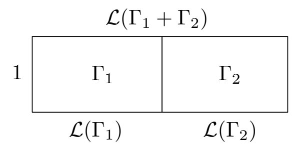

FIGURE 6. The series law.

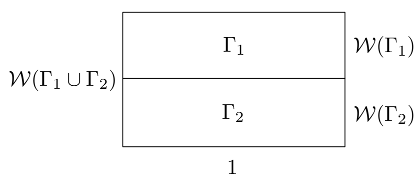

FIGURE 7. The parallel law.

Proof of the series law. For simplicity, let us assume that for $i = 1, 2$, there exists an extremal metric ρ_i for which

$$\mathcal{L}(\Gamma_i) = \frac{\left(\inf_{\gamma \in \Gamma_i} L_{\rho_i}(\gamma)\right)^2}{\int_S \rho_i^2}.$$

It is a simple matter to modify the argument to treat the general case.

We can rescale the metrics so that

$$\mathcal{L}(\Gamma_i) = \inf_{\gamma \in \Gamma_i} L_{\rho_i}(\gamma) = \int_S \rho_i^2$$

for $i = 1, 2$. Moreover, we may assume that the support of ρ_i is included in the support of Γ_i, for setting ρ_i to be zero outside this set decreases the area of ρ_i^2 while keeping the ρ_i-lengths of paths in Γ_i unchanged. In particular, we may assume that ρ_1 and ρ_2 have disjoint supports.

Consider the metric $\rho := \rho_1 + \rho_2$. We have

$$\inf_{\gamma \in \Gamma_1 + \Gamma_2} L_\rho(\gamma) = \inf_{\gamma_1 \in \Gamma_1} L_{\rho_1}(\gamma_1) + \inf_{\gamma_2 \in \Gamma_2} L_{\rho_2}(\gamma_2) = \mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2)$$

and

$$\int_S \rho^2 = \int_S \rho_1^2 + \int_S \rho_2^2 = \mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2),$$

so that $\mathcal{L}(\Gamma_1 + \Gamma_2) \geq \mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2)$.

For the reverse inequality, if ρ is any conformal metric, then define

$$\rho_i := \rho|_{\operatorname{supp}\Gamma_i}$$

and let
$$\ell_i := \inf_{\gamma\in\Gamma_i} L_{\rho_i}(\gamma)$$
and
$$A_i := \int_S \rho_i^2$$
for $i = 1, 2$.

We thus have
$$\inf_{\gamma\in\Gamma_1+\Gamma_2} L_\rho(\gamma) = \ell_1 + \ell_2$$
and
$$\int_S \rho^2 \geq A_1 + A_2.$$
It follows from the Cauchy-Schwarz inequality (or the weighted arithmetic mean – quadratic mean inequality) that
$$\frac{(\inf_{\gamma\in\Gamma_1+\Gamma_2} L_\rho(\gamma))^2}{\int_S \rho^2} \leq \frac{(\ell_1+\ell_2)^2}{A_1+A_2} \leq \frac{\ell_1^2}{A_1} + \frac{\ell_2^2}{A_2}.$$
The details are left as an exercise. Taking supremums we obtain that
$$\mathcal{L}(\Gamma_1 + \Gamma_2) \leq \mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2).$$
□

The proof of the parallel law is similar and left as an exercise.

Let A be an annulus with two disjoint separating subannuli A_1 and A_2, and let $\Gamma_A, \Gamma_{A_1}, \Gamma_{A_2}$ be the sets of paths connecting the boundary components of the corresponding annuli. Then we have
$$\Gamma_A \succ \Gamma_{A_1} + \Gamma_{A_2}$$
and thus
$$\mathrm{mod}(A) = \mathcal{L}(\Gamma_A) \geq \mathcal{L}(\Gamma_{A_1} + \Gamma_{A_2}) = \mathcal{L}(\Gamma_{A_1}) + \mathcal{L}(\Gamma_{A_2}) = \mathrm{mod}(A_1) + \mathrm{mod}(A_2).$$

Quasiconformal diffeomorphisms. Let $K \geq 1$. An orientation-preserving diffeomorphism $f : S \to T$ between Riemann surfaces is said to be K-*quasiconformal* if $\mathrm{Dil}(f) \leq K$.

Theorem 1.3. *Suppose that $f : S \to T$ is a K-quasiconformal diffeomorphism and Γ is a path family on S. Then*
$$\mathcal{L}(f(\Gamma)) \leq K \cdot \mathcal{L}(\Gamma).$$

Proof. Given any conformal metric ρ on T, we define a metric $\hat\rho$ on S by
$$\hat\rho(v) = \sup_{|\lambda|=1} \rho(Df(\lambda v)).$$

For every $\gamma \in \Gamma$, we have
$$\hat\rho(\gamma'(t)) \geq \rho(Df(\gamma'(t))) = \rho((f\circ\gamma)'(t)),$$
and thus
$$\inf_{\gamma\in\Gamma} L_{\hat\rho}(\gamma) \geq \inf_{f(\gamma)\in f(\Gamma)} L_\rho(f(\gamma)).$$

On the other hand, since f is K-quasiconformal, we have
$$\sup_{|\lambda|=1} \rho(Df(\lambda v)) \leq K \inf_{|\lambda|=1} \rho(Df(\lambda v)),$$

from which it follows that

$$\hat{\rho}^2 \leq K f^*(\rho^2)$$

and hence

$$\int_S \hat{\rho}^2 \leq K \int_S f^*(\rho^2) = K \int_T \rho^2.$$

This shows that

$$K \frac{(\inf_{\gamma\in\Gamma} L_{\hat{\rho}}(\gamma))^2}{\int_S \hat{\rho}^2} \geq \frac{(\inf_{f(\gamma)\in f(\Gamma)} L_\rho(f(\gamma)))^2}{\int_S \rho^2}$$

and thus

$$K \cdot \mathcal{L}(\Gamma) \geq \mathcal{L}(f(\Gamma)).$$

□

Corollary 1.4 (Grötzsch). *If $f : R_a \to R_b$ is a K-quasiconformal marking-preserving diffeomorphism, then $b \leq Ka$.*

Proof. If $\Gamma_{a,b}$ denotes the set of paths in $R_{a,b}$ connecting the vertical sides, then we have

$$f(\Gamma_a) \subset \Gamma_b,$$

so that $f(\Gamma_a)$ extends Γ_b and hence

$$b = \mathcal{L}(\Gamma_b) \leq \mathcal{L}(f(\Gamma_a)) \leq K \cdot \mathcal{L}(\Gamma_a) = Ka.$$

□

Recall that if the equality $b = Ka$ holds, then f has to be the stretching map $x + iy \mapsto Kx + iy$. The proof of this fact is left as an exercise.

2. Quadratic differentials

In calculating the extremal length of the set of paths Γ_a connecting the vertical sides of the rectangle R_a, we exploited the existence of an extremal metric (the euclidean metric) as well as a foliation by extremal paths (the horizontal lines).

On arbitrary Riemann surfaces, the appropriate kind of metrics for which this method can be applied come from quadratic differentials.

A *quadratic differential* on a Riemann surface S is a map

$$\varphi : TS \to \mathbb{C}$$

satisfying $\varphi(\lambda v) = \lambda^2 \varphi(v)$ for all $v \in TS$ and all $\lambda \in \mathbb{C}$. Note that $|\varphi|^{1/2}$ is then a conformal metric. If $z : U \to \mathbb{C}$ is a chart defined on some open set $U \subset S$, then φ is equal on U to

$$\varphi_U(z)(dz)^2$$

for some function φ_U defined on $z(U)$. We will be mostly interested in holomorphic or meromorphic quadratic differentials, for which the local functions φ_U are holomorphic or meromorphic.

Suppose that two charts $z : U \to \mathbb{C}$ and $w : V \to \mathbb{C}$ on S overlap, and let $h := w \circ z^{-1}$ be the transition function. If φ is represented both as $\varphi_U(z)dz^2$ and $\varphi_V(w)dw^2$ on $U \cap V$, then we have

$$\varphi_V(h(z))(h'(z))^2 = \varphi_U(z),$$

since $dw = h'(z)dz$.

Natural coordinates. Suppose that $\varphi \equiv \varphi(z)dz^2$ is holomorphic and non-zero in a neighborhood U of $p_0 \in S$.

We may assume that U is simply connected, so that there exists a holomorphic square root of $\varphi(z)$ on U. Define $g : z(U) \to \mathbb{C}$ by

$$g(z) = \int_{z_0}^{z} \sqrt{\varphi(\zeta)}\, d\zeta,$$

where $z_0 := z(p_0)$.

Then $g'(z) = \sqrt{\varphi(z)}$ is non-zero on U so that g is an immersion. We may restrict to a smaller neighborhood $W \subset z(U)$ containing z_0 on which g is injective. The composition $\zeta := g \circ z$ is then called a *natural coordinate* about p_0. By construction, we have

$$d\zeta^2 = (g'(z)dz)^2 = (g'(z))^2 dz^2 = \varphi(z)dz^2 = \varphi.$$

This could be used as a definition for a natural coordinate, that is, any chart ζ in which the quadratic differential φ takes the simple form $d\zeta^2$. Suppose that two natural coordinates ζ and ω overlap and let $h := \omega \circ \zeta^{-1}$ be the transition function. Then we have

$$d\zeta^2 = \varphi = d\omega^2 = (h'(\zeta))^2 d\zeta^2,$$

so that $(h'(\zeta))^2 = 1$, thus $h'(\zeta) = \pm 1$, and hence $\omega = h(\zeta) = \pm\zeta + c$ for some constant $c \in \mathbb{C}$. In other words, natural coordinates are unique up to translation and rotation by π. This implies that the foliation of the plane by horizontal lines can be pulled-back to S under natural coordinates. This gives a foliation off the zeros of φ. We can further equip this foliation with the transverse measure coming from integrating $|dy|$ in the plane. This will allow us to adapt Grötzsch's argument to Riemann surfaces.

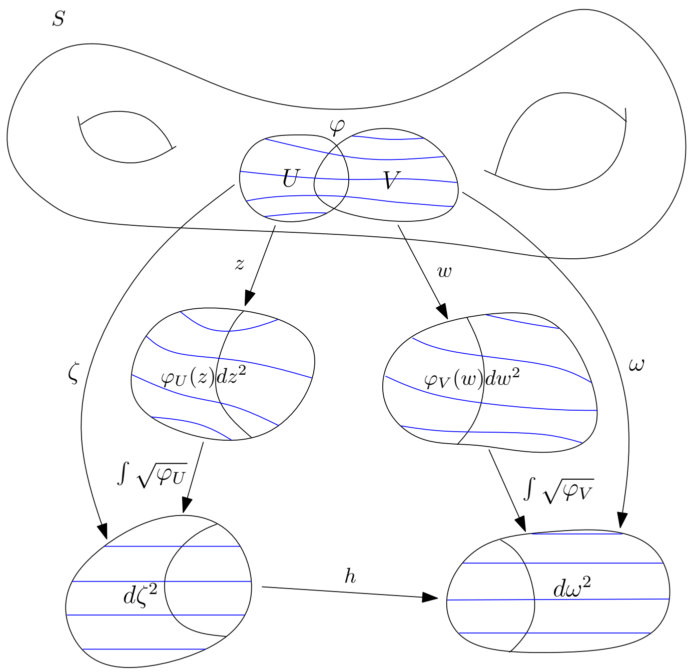

FIGURE 8. Natural coordinates.