A PRIORI BOUNDS FOR SOME INFINITELY RENORMALIZABLE QUADRATICS: I. BOUNDED PRIMITIVE COMBINATORICS

JEREMY KAHN

Abstract. We prove the *a priori* bounds for infinitely renormalizable quadratic polynomials for which we can find an infinite sequence of primitive renormalizations for which the ratios of the periods of successive renormalizations is bounded.

Contents

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1. Introduction

1.1. **Statement of the main result.** In this paper, we will prove *a priori* bounds for infinitely renormalizable quadratic polynomials of bounded primitive type. Let us begin with recalling some basic definitions of the holomorphic renormalization theory. Most of them can be found in any source on the subject, see [L3, L2, McM1].

A quadratic polynomial $f\colon z \mapsto z^2 + c$ is called *primitively renormalizable with period* p if there exist topological disks $V \Supset U \ni 0$ such that $f^p\colon U \to V$ is a quadratic-like map with connected Julia set, and the domains $f^n U$, $n = 0, 1, \dots, p-1$ are pairwise disjoint. This quadratic-like map is called a *renormalization* of f. If there is an infinite sequence of periods $p_0 < p_1 < \dots$ such that f is primitively p_k-renormalizable then f is called *infinitely primitively renormalizable*. If additionally, there exists a B such that $p_{k+1}/p_k \leqslant B$, $k = 0, 1, \dots$, then f is called infinitely primitively renormalizable of *bounded type*. Such a map has *a priori* bounds if there exists an $\varepsilon > 0$ and a sequence of quadratic-like renormalizations $f^{p_k}\colon U_k \to V_k$ such that $\operatorname{mod}(V_k \smallsetminus U_k) \geqslant \varepsilon$.

Main Theorem. *Let f be an infinitely primitively renormalizable quadratic polynomial of bounded type. Then f has a priori bounds.*

In the forthcoming notes [KL2], we will prove *a priori* bounds for a class of infinitely renormalizable maps of unbounded type.

For real quadratics of bounded type *a priori* bounds were proved by Sullivan [S], see also [LS, LY, MS]. They were also proved for a class of complex combinatorics of "high bounded type" [L1].

1.2. **Consequences.** The *a priori* bounds have numerous consequences. Let us list some of them (below $f_c\colon c \mapsto z^2 + c$ stands for an infinitely primitively renormalizable quadratic polynomial of bounded type):

- *The Mandelbrot set is locally connected at c, or equivalently, the polynomial f_c is combinatorially rigid* (see [L1]). The conjecture of local connectivity of the Mandelbrot set (the *MLC Conjecture*) formulated about 20 years ago by Douady and Hubbard (see [DH1]) is a central open problem in holomorphic dynamics. Previously, it was established for all quadratic maps which are not infinitely renormalizable (Yoccoz, see [H]) and for the class of infinitely renormalizable maps of high type mentioned above (see [L1]).
- *The Julia set $J(f_c)$ is locally connected* (see [HJ, J]).

- *The Feigenbaum-Coullet-Tresser Renormalization Conjecture is valid for primitive combinatorics.* This conjecture was established in the work of Sullivan [S], McMullen [McM2] and Lyubich [L2] assuming *a priori* bounds (and thus, unconditionally, for real maps). Now, these results become unconditional for arbitrary primitive combinatorics.
- *Universality and Hairiness of the Mandelbrot set at c.* These properties were conjectured by Milnor [M] and proved in [L2] for maps with *a priori* bounds.
- *The Basic Trichotomy for the measure and Hausdorff dimension of the Julia set $J(f_c)$* which was established in [AL] for maps with *a priori* bounds.

1.3. **Outline of the proof.** We will now give a brief top-down outline of the proof of the Main Theorem.

General strategy: improving of the lengths of the hyperbolic geodesics. Let K_n be the filled Julia set of the renormalization $f^{p_n} : U_n \to V_n$,

$$\mathcal{K}_n = \bigcup_{i=0}^{p_n-1} f^i(K_n),$$

and let γ_n be the peripheral hyperbolic geodesic in $\mathbb{C} \setminus \mathcal{K}_n$ going around K_n. The *a priori* bounds are equivalent to the assertion that the hyperbolic lengths of the γ_n are bounded. Our strategy towards this end is to show that *if the length of some γ_n gets long then it was even longer before*: There exist $M > 0$ and $l > 0$ such that

$$|\gamma_n| > M \Rightarrow |\gamma_{n-l}| > 2M. \tag{1.1}$$

Localization "in principle". The immediate and obvious difficulty in proving (1.1) is that the number of components of $\mathcal{K}_n$ grows exponentially with n, and of course is unbounded as n goes to infinity. While there has been some success in handling an unbounded number of components, there is certainly a vast advantage to working with a bounded number, and this paper is able to make a major step forward with the help of such a localization.

The central idea is that, while n is unbounded, l is bounded, and, because we assume $p_{k+1}/p_k \leqslant B$ (for all k), the number of components of $\mathcal{K}_n$ that are surrounded by γ_{n-l} is at most B^l. Moreover, if we let $f_{n-l} \equiv f^{p_{n-l}} \colon U_{n-l} \to V_{n-l}$ be a renormalization of period p_{n-l} of f, then this f_{n-l} is renormalizable of period p_n/p_{n-l}, and the small Julia sets for this renormalization are exactly the components of $\mathcal{K}_n$ surrounded by γ_{n-l}.

With this in mind, we will first state and prove a theorem about a primitively renormalizable quadratic-like map that will be closely related to (1.1). Before we do so, we will introduce some notation. Suppose that $f: U \to V$ is a quadratic-like map that is primitively p-renormalizable. We let $K \equiv K(f)$ be the filled-Julia set for f. We let $\mathcal{K}_p \equiv \mathcal{K}_p(f)$ be the union of small filled-Julia sets for this renormalization, and K_p be the central small filled-Julia set. We let γ be the closed geodesic in $V \smallsetminus K$ that goes around K, and let γ_p be the hyperbolic geodesic in $V \smallsetminus \mathcal{K}_p$ that goes around K_p.

Theorem 1.2. *There exists $\epsilon > 0$, such that for all p there exists M: Suppose that f is a quadratic-like map that is primitively p-renormalizable. Then, with the notation described just above,*

$$l(\gamma_p) > M \implies l(\gamma) \geqslant \epsilon p\, l(\gamma_p). \tag{1.3}$$

Now suppose f in Theorem 1.2 is the renormalization f_{n-l} described above, and V_{n-l} is taken to be "pretty large", and let $q = p_n/p_{n-k}$. Then the γ_q for the f in Theorem 1.2 is a "good approximation" of the γ_n for the original f, and the γ for the f in Theorem 1.2 is a "good approximation" of the γ_{n-l} for the original f. We can choose q *large but bounded*, and (1.1) would follow from (1.3) if these "good approximations" were actually equalities.

To make them equalities we introduce (in Section 8) the *pseudo-quadratic-like map* and the *canonical renormalization.* A pseudo-quadratic-like map $(i, f): \mathbf{U} \to \mathbf{V}$ is a generalization of a quadratic-like map where the inclusion from $\mathbf{U}$ to $\mathbf{V}$ is replaced by an immersion i that has certain properties that imply the (multivalued) $f \circ i^{-1}$ has a (single-valued) quadratic-like restriction. We can then, given our original quadratic polynomial f, define, for each n, the *canonical renormalization* $f_n: \mathbf{U}_n \to \mathbf{V}_n$ (really (i_n, f_n) but we will suppress the i_n here), which will be a pseudo-quadratic-like map. Then, given n and $l < n$, and letting $q = p_n/p_{n-l}$, we'll see that f_{n-l} has two crucial properties:

1. (1) The geodesic γ around $K(f_{n-l})$ in $\mathbf{V}_{n-l} \smallsetminus K(f_{n-l})$ has length *equal to* γ_{n-l}.
2. (2) f_{n-l} is renormalizable of period q, and the geodesic γ_q (for f_{n-l}) around $K_q(f_{n-l})$ in $\mathbf{V}_{n-l} \smallsetminus \mathcal{K}_q(f_{n-l})$ has length *greater than or equal to* the length of $\gamma_n(f)$.

In Section 9, we then extend Theorem 1.2 to pseudo-quadratic-like maps, and (1.1) and our Main Theorem then follow.

Canonical weighted arc diagrams. Let us now briefly discuss the proof of Theorem 1.2. In Section 2 we develop the theory of the *canonical weighted arc-diagram*, which describes the "thin rectangles" of large modulus which we will then use to estimate $|\gamma_p|$ as the sum of extremal widths of thin rectangles in $V \setminus \mathcal{K}_p$ crossed by γ_p. .

Roughly speaking, the canonical weighted-arc diagram is

$$W_{\text{can}} = \sum W(\alpha)\alpha,$$

where the α are arcs[1] in $V \setminus \mathcal{K}_p$ represented by paths crossing the thin rectangles and the $W(\alpha)$ are their widths. Holomorphic maps between Riemann surfaces, such as covering and inclusions, then induce relations between their canonical WAD's; these relations are described in Section 2. The most subtle of these relations is *domination*, which describes the relation between the canonical WAD's between Riemann surfaces $U \subset V$, and encodes the parallel and series laws for extremal length.

In Section 3 we reduce Theorem 1.2 to a similar statement expressed in terms of the canonical WAD of $V \setminus \mathcal{K}_p$, and then provide an outline of how to use the functorial properties of the canonical WAD (and the Covering Lemma[KL1], which is applied in Section 7) to reduce that statement, in turn, to a purely combinatorial one, which is proven in Sections 5 and 6, using the background in Section 4.

1.4. **Terminology and Notation.** We let:
$\mathbb{N} = \{1, 2, \dots\}$ be the set of natural numbers; $\mathbb{Z}_{\geqslant 0} = \mathbb{N} \cup \{0\}$;
$\mathbb{D} = \{z : |z| < 1\}$ be the unit disk, $\mathbb{T}_r = \{z : |z| = r\}$, $\mathbb{T} = \mathbb{T}_1$;
$\mathbb{A}(r, R) = \{z : r < |z| < R\}$.

A *topological disk* means a simply connected domain in some Riemann surface S. We will say a subset K of $\mathbb{C}$ is an FJ-set ("filled Julia set") if K is compact, connected, and full.

1.5. **Acknowledgment.** This paper was written in collaboration with Mikhail Lyubich following the mathematical work of the author. The primary result for this paper will be combined with the main result of a subsequent paper of Mikhail Lyubich and the author to form one result that is the goal of the series of papers of which this paper is the first.

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[1] An *arc* is a non-trivial homotopy class of properly embedded paths.

2. Canonical weighted arc diagram

2.1. **Arc diagrams.** Let S be a hyperbolic open Riemann surface of finite topology without cusps. It is conformally equivalent to the interior of a compact Riemann surface $\mathbf{S}$ with non-empty boundary.[2] The boundary of $\mathbf{S}$ is called the *ideal boundary* of S. It is canonically attached to S in the sense that any conformal isomorphism $S \to S'$ extends to a homeomorphism $\mathbf{S} \to \mathbf{S}'$.

A *path* in some topological space Z is an embedded interval $\gamma\colon I \to Z$. It is called *open, closed, or semiclosed* depending on the nature of I. An open path $\gamma\colon (0,1) \to S$ is called *proper* if it extends to a closed path $\boldsymbol{\gamma}\colon [0,1] \to \mathbf{S}$ such that $\boldsymbol{\gamma}\{0,1\} \subset \partial\mathbf{S}$. Two proper paths γ_0 and γ_1 in S are called *properly homotopic* if there is a homotopy $\boldsymbol{\gamma}_t$, $t \in [0,1]$, connecting $\boldsymbol{\gamma}_0$ to $\boldsymbol{\gamma}_1$ through a family of proper paths.[3]

An *arc* on S is a class of properly homotopic paths, $\alpha = [\gamma]$. An arc is called *trivial* if it has representing paths $\boldsymbol{\gamma}\colon I \to \mathbf{S}$ in arbitrary small neighborhoods of $\partial\mathbf{S}$. Let $\mathcal{A}(S)$ stand for the set of all non-trivial arcs on S.

Two different arcs are said to be *non-crossing* if they can be represented by non-crossing paths. An *arc diagram* is a family of pairwise non-crossing arcs α_i. Note that any arc diagram consists of at most $3|\chi(S)|$ arcs, where $\chi(S)$ is the Euler characteristic of S.

A *weighted arc diagram (WAD)* on S is an arc diagram $\mathbf{a} = \{\alpha_i\}$ endowed with weights $w_i \in \mathbb{R}_+$. In this case, the arc diagram $\mathbf{a}$ is called the *support* of W.[4] Let $\mathcal{W}(S)$ stand for the set of WAD's on S.

The set $\mathcal{W}(S)$ is *partially ordered*: $X \leqslant Y$ is $X(\alpha) \leqslant Y(\alpha)$ for any $\alpha \in \operatorname{supp} X$. We will also write $X \leqslant Y + c$ if $X(\alpha) \leqslant Y(\alpha) + c$ for any $\alpha \in \operatorname{supp} X$.

The sum of two WAD's, $X + Y$, is well defined whenever any two arcs $\alpha \in \operatorname{supp} X$ and $\beta \in \operatorname{supp} Y$ are either the same or non-crossing. The difference $X - Y$ is always well defined if we let $(X - Y)(\alpha) = 0$ whenever $X(\alpha) \leqslant Y(\alpha)$. Similarly, $X - c$ is well defined for any constant $c \geqslant 0$.

We will make use of two norms on the space of WAD's:

$$\|W\|_\infty = \sup_{\alpha \in \mathcal{A}} W(\alpha); \quad \|W\|_1 = \sum_{\alpha \in \mathcal{A}} W(\alpha).$$

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[2] In what follows, all Riemann surfaces are assumed to be of this type, unless otherwise is explicitly stated.

[3] Note that this homotopy is automatically isotopy.

[4] We can also think of a WAD as a function $\mathcal{A}(S) \to \mathbb{R}_+$ supported on some arc diagram.

If $f\colon U \to V$ is a holomorphic covering between two Riemann surfaces then there is a natural *pull-back* operation $f^*\colon \mathcal{W}(V) \to \mathcal{W}(U)$ acting on the WAD's.

A *proper lamination* $\mathcal{F}$ on S is a Borel set $Z \subset S$ explicitly realized as a union of disjoint proper paths called the *leaves* of $\mathcal{F}$. Any proper lamination[5] can be written $\mathcal{F} = \bigcup_\alpha \mathcal{F}(\alpha)$, where $\mathcal{F}(\alpha)$ comprises the leaves of $\mathcal{F}$ that represent α. The arcs $\alpha \in \mathcal{A}$ for which $\mathcal{F}(\alpha)$ is non-empty assemble an arc diagram. Let us weight each of these arcs with the weight $W_\mathcal{F}(\alpha)$ equal to the *extremal width* $\mathcal{W}(\mathcal{F}(\alpha))$ of the sub-lamination $\mathcal{F}(\alpha)$ (viewed as a path family). In this way we obtain the WAD $W_\mathcal{F} = \sum_\alpha W_\mathcal{F}(\alpha) \cdot \alpha$ corresponding to $\mathcal{F}$.

Note that if $f\colon U \to V$ is a holomorphic covering between two Riemann surfaces and $\mathcal{F}$ is a proper lamination on V then $f^*(\mathcal{F})$ is a proper lamination on U and $W_{f^*(\mathcal{F})} = f^*(W_\mathcal{F})$.

Weighted arc diagrams that are $W_\mathcal{F}$ for some proper lamination $\mathcal{F}$ are called *valid*.

2.2. **Canonical WAD.** Let us consider the universal covering $\pi\colon \mathbb{D} \to \operatorname{int} \mathbf{S}$. Let Γ be the Fuchsian group of deck transformations of π, and let $\Lambda \subset \mathbb{T}$ be its limit set. Since $\mathbf{S}$ has non-empty boundary, Λ is a Cantor set. Moreover, π extends continuously onto $\hat{S} = \bar{\mathbb{D}} \smallsetminus \Lambda$, and the restriction of π to any component I of $\partial \hat{S}$ is a universal covering onto some component I of $\partial \mathbf{S}$.

Let us pick two components, $I \neq J$, of $\partial \hat{S}$. The disk $\mathbb{D}$ with these two intervals as horizontal sides determines a quadrilateral $Q(I,J)$. This quadrilateral can be conformally uniformized, $\phi\colon Q(I,J) \to \mathbf{Q}(a)$, by a standard quadrilateral $\mathbf{Q}(a) = [0,a] \times [0,1]$ in such a way that I and J correspond to the horizontal sides of $\mathbf{Q}(a)$. The *vertical foliation* $\mathcal{F}(I,J)$ on $Q(I,J)$ is the ϕ-pullback of the standard vertical foliation on $\mathbf{Q}(a)$.

Assume now that $a > 2$, and let us cut off from $\mathbf{Q}(a)$ two side squares, $[0,1] \times [0,1]$ and $[a-1,a] \times [0,1]$. We call the left-over rectangle $\mathbf{Q}_{\mathrm{can}}(a)$, and we let $Q_{\mathrm{can}}(I,J) = \phi^{-1}(\mathbf{Q}_{\mathrm{can}}(a))$. The side quadrilaterals that we have cut off from $Q(I,J)$ are called its *buffers*.

Let $\mathcal{F}_{\mathrm{can}}(I,J)$ be the restriction of $\mathcal{F}(I,J)$ to $Q_{\mathrm{can}}(I,J)$. Obviously, for any deck transformation $\gamma \in \Gamma$, we have:

$$\mathcal{F}_{\mathrm{can}}(\gamma(I), \gamma(J)) = \gamma(\mathcal{F}_{\mathrm{can}}(I,J)). \tag{2.1}$$

Lemma 2.2. *The rectangles $Q_{\mathrm{can}}(I,J)$ are pairwise disjoint.*

[5] In what follows, all laminations under consideration are assumed to be proper.

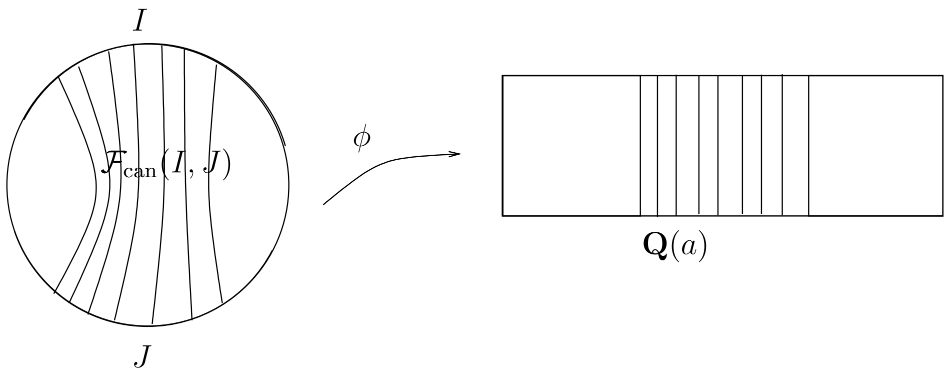

Figure 2.1. The canonical foliation

Proof. Let us consider two rectangles, $Q \equiv Q(I,J)$ and $Q' \equiv Q(I',J')$. Then we can find one interval from each pair, say J and J', such that $J \neq J'$ (so $J \cap J' = \varnothing$), and such that there is a component T of $\mathbb{T} - (J \cup J')$ such that $T \cap (I \cup I') = \varnothing$.

We let B be the buffer of $Q(I,J)$ that has a horizontal side (which is a subset of J) that shares an endpoint with T; we define B' likewise. Then if any vertical leaf γ of $\mathcal{F}_{\rm can}(I,J)$ crossed any vertical leaf γ' of $\mathcal{F}_{\rm can}(I',J')$, then every vertical leaf of B would cross every vertical leaf of B'. This would contradict Lemma 10.6. ☐

This lemma allows us to define the *canonical lamination* $\mathcal{F}_{\rm can}(\hat{S})$ as the union of the laminations $\mathcal{F}_{\rm can}(I,J)$ for all pairs of different components I and J of $\partial\hat{S}$. By (2.1), this lamination is Γ-invariant, and hence it can be pushed forward to $\mathbf{S} \supset S$. In this way we obtain the *canonical lamination on* S:

$$\mathcal{F}_{\rm can}(S) = \mathcal{F}_{\rm can}(\mathbf{S}) = \pi_*(\mathcal{F}_{\rm can}(\hat{S})).$$

The corresponding weighted arc diagram $\alpha \mapsto W_{\rm can}(S,\alpha)$, $\alpha \in \mathcal{A}(S)$, is called the *canonical WAD* on S.[6] By definition, it is valid.

We will now list several basic properties of the canonical WAD. The rest of the theory will be based on these properties in an essentially axiomatic way.

2.3. **Property A: Maximality.** Let $W_{\max}(S)\colon \mathcal{A} \to \mathbb{R}_+$ stand for the functional assigning to an arc $\alpha \in \mathcal{A}$ the extremal width of the family of all proper paths γ in S representing α. (Note that $W_{\max}(S)$ is *not* a WAD as it is not supported on an arc diagram.)

[6]We will use abbreviated notations $W_{\rm can}(S)$ or $W_{\rm can}(\alpha)$ whenever it does not lead to confusion.

Lemma 2.3. *For any valid arc diagram W on S, we have:*

$$W \leqslant W_{\max}(S) \leqslant W_{\mathrm{can}}(S) + 2.$$

Proof. The first inequality is obvious, so let us focus on the second one.

It is trivial for any arc $\alpha \in \mathcal{A}$ with $W_{\max}(\alpha) \leqslant 2$. Let us consider some arc $\alpha \in \mathcal{A}$ with $W_{\max}(\alpha) > 2$. This arc connects two boundary components, σ and ω, of S.

The path family $\mathcal{G}(\alpha)$ representing α lifts to a path family $\hat{\mathcal{G}}(\alpha)$ consisting of all the paths in $\mathbb{D}$ that connect two appropriate arcs on $\mathbb{T}$, I and J, covering σ and ω respectively. Viewing I and J as the horizontal sides of the rectangle $Q(I,J)$ based on $\bar{\mathbb{D}}$, we obtain the desired estimate:

$$\mathcal{W}(\mathcal{G}(\alpha)) \leqslant \mathcal{W}(\hat{\mathcal{G}}(\alpha)) \leqslant \mathcal{W}(Q(I,J)) = W_{\mathrm{can}}(S,\alpha) + 2$$

(where we have made use of Lemma 10.1 for the first estimate). $\square$

2.4. **Property B: Natural behavior under coverings.**

Lemma 2.4. *If $f\colon U \to V$ is a finite-degree covering then $W_{\mathrm{can}}(U) = f^*W_{\mathrm{can}}(V)$.*

Proof. Let $\pi_U\colon \hat{U} \equiv \bar{\mathbb{D}} \smallsetminus \Lambda_U \to U$ and $\pi_V\colon \hat{V} \equiv \bar{\mathbb{D}} \smallsetminus \Lambda_V \to V$ be the universal coverings of U and V, with deck transformations groups Γ_U and Γ_V respectively. Since f has a finite degree, the group Γ_U has a finite index in Γ_V. It follows that $\Lambda_U = \Lambda_V$, so that $\hat{U} = \hat{V}$. Hence $\mathcal{F}_{\mathrm{can}}(\hat{U}) = \mathcal{F}_{\mathrm{can}}(\hat{V}) \equiv \mathcal{F}$. Then

$$\mathcal{F}_{\mathrm{can}}(U) = \mathcal{F}/\Gamma_U = f^*(\mathcal{F}/\Gamma_V) = f^*(\mathcal{F}_{\mathrm{can}}(V)),$$

and the conclusion follows. $\square$

2.5. **Property C: Behavior under partially proper maps.** Let $\mathcal{E}(S)$ stand for the set of ends of the Riemann surface S (that, under our standing assumption, can be identified with the set of boundary components of $\mathbf{S}$). We say that S is *partially marked* if we have chosen a subset $\mathcal{E}_p(S) \subset \mathcal{E}(S)$ of ends that we call “proper”.

A map $e\colon U \to V$ between partially marked Riemann surfaces is called *partially proper* if it is proper on proper ends. An arc $\alpha \in \mathcal{A}(S)$ is called *horizontal* if it connects proper ends of S. Let $\mathcal{A}^{\mathrm{h}}(S)$ stand for the set of horizontal arcs on S, and let $\mathcal{W}^{\mathrm{h}}(S)$ be the set of horizontal WAD’s on S. The *horizontal canonical WAD* $W^{\mathrm{h}}_{\mathrm{can}}(S)$ is the restriction of the canonical WAD to the set of horizontal arcs.

Notice that a partially proper map $e\colon U \to V$ induces a push-forward map on the horizontal arcs, $e_*\colon \mathcal{A}^{\mathrm{h}}(U) \to \mathcal{A}^{\mathrm{h}}(V)$, and hence, a pullback map on the horizontal arc diagrams, $e^*\colon \mathcal{W}^{\mathrm{h}}(V) \to \mathcal{W}^{\mathrm{h}}(U)$, defined by $e^*(Y)(\alpha) = Y(e_*\alpha)$.

Lemma 2.5. *Let U and V be partially marked Riemann surfaces, and let $e\colon U \to V$ be a partially proper holomorphic map. Then*

$$W^{\mathrm{h}}_{\mathrm{can}}(U) \leqslant e^* W^{\mathrm{h}}_{\mathrm{can}}(V).$$

Proof. Let us consider a horizontal arc $\alpha \in \operatorname{supp} W^{\mathrm{h}}_{\mathrm{can}}(U)$. It connects two proper ends, σ and ω, and it lifts to an arc in $\hat{U}$ connecting some components I and J of $\partial \hat{U}$. Let Q be the quadrilateral based on $\hat{U}$ with the horizontal sides I and J. Let $I' = \hat{e}(I)$, $J' = \hat{e}(J)$, and let Q' be the corresponding quadrilateral based on $\hat{V}$. Then $\hat{e}\colon Q \to Q'$ maps the horizontal sides of Q to the corresponding horizontal sides of Q'. By Corollary 10.3, $\mathcal{W}(Q) \leqslant \mathcal{W}(Q')$. But $\mathcal{W}(Q) = W_{\mathrm{can}}(U, \alpha) + 2$, $\mathcal{W}(Q') = W_{\mathrm{can}}(V, e_*(\alpha)) + 2$, and the desired conclusion follows. □

Similarly, an arc $\alpha \in \mathcal{A}(S)$ is called *vertical* if it connects a proper end of S to an improper one. The *vertical canonical WAD* $W^{\mathrm{v}}_{\mathrm{can}}(S)$ is the restriction of the canonical WAD to the set of vertical arcs.

2.6. **Property D: Domination.** Let us now introduce an important relation between WAD's.

An *integer WAD* $\mathbf{a}$ (IWAD) is a WAD with integer coefficients, that is, a formal linear combination $\sum n_i \alpha_i$, where α_i is a arc diagram, and $n_i \in \mathbb{N}$. It will be also convenient to write $\mathbf{a}$ as a formal sum of arcs, $\mathbf{a} = \sum \alpha_j$, where any two different arcs α_j are non-crossing. The order on WAD's induces a natural order on IWAD's.

Let us consider two Riemann surfaces, $U \subset V$. Given a path γ on V, the restriction $\gamma \cap U$ has only finitely many non-trivial components, $(\gamma_i)_{i=1}^n$. They represent a sequence of arcs on U, $I(\gamma) \equiv (\alpha_i \equiv [\gamma_i])_{i=1}^n$, called the *itinerary* of γ.

We say that a sequence of arcs (α_i) on U *arrows* an arc β on V if there exists a path γ representing β such that $I(\gamma) = (\alpha_i)$. We will use notation $(\alpha_i) \longrightarrow \beta$ for the arrow relation.

Remark. Note that the endpoint of γ_i is connected to the beginning of γ_{i+1} by a path that goes through some component K of $V \smallsetminus U$. In this case, the end of U corresponding to this component is *not properly embedded* into V. This remark is useful as it reduces a number of possibilities of how the arc β can be composed by arcs α_j.

We say that a *IWAD* $\mathbf{a} = \sum \alpha_i$ *arrows* β if the arcs α_i can be ordered so that the string of arcs (α_i) arrows β. (In other words, we "abelianize" the arrow relation.) We use the same notation, $\mathbf{a} \longrightarrow \beta$, for this arrow relation.

Let us now consider two WAD's, $X \in \mathcal{W}(U)$ and $Y \in \mathcal{W}(V)$. We say that X *dominates* Y, written

$$X \multimap Y,$$

if we can write

$$X \geqslant \sum_i \sum_j w_{ij}\alpha_{ij}, \quad Y = \sum_i v_i \beta_i$$

where, for each i,

$$(\alpha_{ij})j \longrightarrow \beta_i$$

and

$$\bigoplus_j w_{ij} \geqslant v_i.$$

The basic example comes from laminations on V:

Lemma 2.6. *Given a valid WAD Y on V, there exists a valid WAD X on U such that $X \multimap Y$.*

Proof. Since Y is valid, $Y = W_{\mathcal{F}}$ for some lamination $\mathcal{F}$ on V. For $\beta \in \operatorname{supp} W_{\mathcal{F}}$, let $\mathcal{F}(\beta)$ be the sublamination of $\mathcal{F}$ assembled by the leaves γ representing the arc β.

Let us consider the slice of $\mathcal{F}$ on U, that is, let $\mathcal{H} = \mathcal{F} \cap U$ and $X = W_{\mathcal{H}}$. To any leaf γ of $\mathcal{F}$, let us associate its itinerary $I(\gamma) = (\alpha_j(\gamma))$ on U. Let $\mathcal{I}(\beta)$ stand for the set of all non-trivial itineraries $\mathbf{a} = I(\gamma)$ corresponding to all possible leaves γ of $\mathcal{F}(\beta)$. By definition, $\mathbf{a} \longrightarrow \beta$ for any $\mathbf{a} \in \mathcal{I}(\beta)$. Let $\mathcal{F}(\beta, \mathbf{a})$ stand for the sublamination of $\mathcal{F}(\beta)$ assembled by the leaves γ with itinerary $\mathbf{a}$, i.e., $I(\gamma) = \mathbf{a}$.

For $\mathbf{a} = (\alpha_j) \in \mathcal{I}(\beta)$, let $v(\beta, \mathbf{a}) = W(\mathcal{F}(\beta, \mathbf{a}))$, and let $w_j(\beta, \mathbf{a})$ be the width of the lamination assembled by the segments of $\mathcal{F}(\beta, \mathbf{a}) \cap U$ corresponding to α_j. By the Series Law,

$$\bigoplus_j w_j(\beta, \mathbf{a}) \geqslant v(\beta, \mathbf{a}).$$

Moreover,

$$X = \sum_\beta \sum_{\mathbf{a} \in \mathcal{I}(\beta)} \sum_j w_j(\beta, \mathbf{a})\alpha_j, \quad Y = \sum_\beta \sum_{\mathbf{a} \in \mathcal{I}(\beta)} v(\beta, \mathbf{a})\beta.$$

This would mean that $X \multimap Y$ if we knew that $\mathcal{I}(\beta)$ were finite. The rest of the argument will show that the terms can be grouped so that the decompositions become finite.

Let $T(\beta)$ be the set of all IWAD's $\boldsymbol{\alpha} = \sum \alpha_j$ corresponding to all itineraries $\mathbf{a} = (\alpha_j) \in \mathcal{I}(\gamma)$, and let $\eta\colon \mathcal{I}(\beta) \to T(\beta)$ be the corresponding abelianization projection. Let $(e_i)_{i=1}^n$ be the maximal family of different non-trivial arcs represented by the leaves of $\mathcal{F} \cap U$. Then any

IWAD $\boldsymbol{\alpha} \in T(\beta)$ has a form $\sum n_i e_i$, $n_i \in \mathbb{Z}_+$, and hence represents an element of the free semi-group $(\mathbb{Z}_+)^n$. By Lemma 2.7 below, there is a finite set $B(\beta) \subset T(\beta)$ such that any $\boldsymbol{\alpha} \in T(\beta)$ is greater or equal than some $\mathbf{a}' \in B(\beta)$. Let us show that the labeling set $\mathcal{I}(\beta)$ can be replaced with $B(\beta)$.

Let us select a projection $\pi: \mathcal{I}(\beta) \to B(\beta)$ factored through η and such that $\mathbf{a}' \equiv \pi(\mathbf{a}) \leqslant \eta(\mathbf{a})$. Then the components of $\mathbf{a}' = \sum \alpha'_k$ is a subset of components of $\mathbf{a} = (\alpha_j)$, so that, we can select an injective map $j(k) = j_{\mathbf{a}}(k)$ such that $\alpha'_k = \alpha_{j(k)}$. Let

$$w'_k(\beta, \mathbf{a}') = \sum_{\pi(\mathbf{a})=\mathbf{a}'} w_{j_{\mathbf{a}}(k)}(\beta, \mathbf{a}), \quad v'(\beta, \mathbf{a}') = \sum_{\pi(\mathbf{a})=\mathbf{a}'} v(\beta, \mathbf{a}).$$

Then

$$X \geqslant \sum_{\beta} \sum_{\mathbf{a}' \in B(\beta)} \sum_k w'_k(\beta, \mathbf{a}')\, \alpha'_k, \quad Y = \sum_{\beta} \sum_{\mathbf{a}' \in B(\beta)} v'(\beta, \mathbf{a}')\beta.$$

Moreover, by Lemma 11.8

$$\bigoplus_k w'_k(\beta, \mathbf{a}') \geqslant v'(\beta, \mathbf{a}').$$

Thus, $X \multimap Y$. □

Let us consider a free Abelian semigroup $S = \mathbb{Z}_+^n$ with the standard basis $(e_i)_{i=1}^n$. It is ordered coordinatewise: $\sum x_i e_i \geqslant \sum y_i e_i$ if $x_i \geqslant y_i$ for all i. For any $x \in S$, let $S_x = \{y \in S : \ y \geqslant x\}$ stand for the cone with the vertex at x.

Lemma 2.7. *Let T be an arbitrary subset of the semi-group S. Then there exist a finite subset $B \subset T$ such that $T \subset \bigcup_{x \in B} S_x$.*

Proof. Suppose first that T is non-empty. Let $a = \sum a_i e_i$ be an element of T. Let $\pi_i: \mathbb{Z}_+^n \to \mathbb{Z}$ be the projection onto the i^{th} coordinate. For $i \leqslant n$, and $k < a_i$, $k \in \mathbb{Z}_+$, let $T_k^i = \pi_i^{-1}(\{k\}) \cap T$. Then, by induction on n, there is a finite $B_k^i \subset T_k^i$ such that

$$T_k^i \subset \bigcup_{x \in B_k^i} S_x.$$

Then let

$$B = \{a\} \cup \bigcup_{1 \leqslant i \leqslant n} \bigcup_{k < a_i} B_k^i;$$

it has the desired property. □

We can now prove Property D of canonical WAD's:

Lemma 2.8. *Let $U \subset V$. Then there exists a WAD $B \in W(U)$ with $\|B\|_\infty \leqslant 2$ such that*

$$W_{\mathrm{can}}(U) + B \multimap W_{\mathrm{can}}(V).$$

Proof. Since $W_{\mathrm{can}}(V)$ is valid, Lemma 2.6 gives us a WAD X on U such that $X \multimap W_{\mathrm{can}}(Y)$. By the Maximality Property, $X \leqslant W_{\mathrm{can}}(U) + 2$. Hence there exists a WAD B with $\operatorname{supp} B \subset \operatorname{supp} X$, $\|B\|_\infty \leqslant 2$, and $X \leqslant W_{\mathrm{can}}(U) + B$. The conclusion follows. □

Lemma 2.9. *If*

$$\bigoplus (x_i + b_i) \geqslant y,$$

then

$$\bigoplus x_i \geqslant y - \sum b_i.$$

Proof.

$$\frac{\partial}{\partial x_i} \bigoplus x_k = \frac{(\bigoplus x_k)^2}{x_i^2} \leqslant 1.$$

Therefore

$$\bigoplus (x_i + b_i) \leqslant \bigoplus x_i \ + \ \sum b_i.$$

□

Lemma 2.10. *If $X + B \multimap Y$, then $X \multimap Y - \|B_1\|_1$.*

Proof. Now suppose $X + B \multimap Y$. Formally we can write $X + B \geqslant \sum_i T_i$, $Y = \sum Y_i$, where $T_i = \sum_j w_{ij}\alpha_{ij}$, $Y_i = v_i\beta_i$, and

$$(\alpha_{ij})_j \to \beta_i, \tag{2.11}$$

and

$$\bigoplus_j w_{ij} > v_i.$$

By the general theory of positive vectors in $\mathbb{R}^n$, we can write $T_i = X_i + B_i$, where $X \geqslant \sum X_i$, and $B \geqslant \sum B_i$. So writing $X_i = \sum_j w_{ij}^X \alpha_{ij}$, and $B_i = \sum_j w_{ij}^B \alpha_{ij}$, we obtain

$$\bigoplus_j (w_{ij}^X + w_{ij}^B) \geqslant v_i,$$

so, by Lemma 2.9,

$$\bigoplus_j w_{ij}^X \geqslant v_i - \|B_i\|_1,$$

and therefore (using (2.11))

$$X \geqslant \sum_i X_i \ \multimap \ \sum_i (v_i - \|B_i\|_1)\beta_i \ \geqslant \ Y - \|B\|_1.$$

□

Together with Lemma 2.8, this implies:

Corollary 2.12. *Let $U \subset V$. Then*

$$W_{\mathrm{can}}(U) \multimap W_{\mathrm{can}}(V) - 6|\chi(U)|.$$

2.7. **Property E: Intersection number and hyperbolic length.** Let S be a compact Riemann surface with boundary. We have defined $W_{\mathrm{can}}(S)$, and we have also defined the the weighted arc-diagram M^S in Appendix C. We can relate M^S to W^S_{can} via the following:

Lemma 2.13. *Let Q be a quadrilateral with horizontal sides I_1 and I_2, endowed with the hyperbolic metric. Let Γ_1 and Γ_2 be the hyperbolic geodesics in Π that share the endpoints with I_1 and I_2 respectively. Then*

$$\mathcal{W}(Q) = -\frac{2}{\pi} \log \operatorname{dist}(\Gamma_1, \Gamma_2) + O(1).$$

Proof. We can map Q conformally to the infinite strip

$$\Pi \equiv \{z : 0 < \Im z < \pi\}$$

such that Γ_1 and Γ_2 map to vertical transverse segments with real part 0 and d respectively, where $d \equiv \operatorname{dist}(\Gamma_1, \Gamma_2)$. Then the vertical sides of Q map to $J_1 \equiv [0, d] \times \{0\}$ and $J_2 \equiv [0, d] \times \{\pi\}$. Then $\mathcal{W}(Q)$ is equal to $\mathcal{L}(\Theta)$, where Θ is the family of paths in Π connecting J_1 and J_2. By two applications of the reflection principle (see [A2]), we find that $\mathcal{L}(\Theta) = 4\mathcal{L}(\Theta')$, where Θ' is the family of paths connecting J_1 to the boundary of $\Pi' \equiv \{z : |\Im z| < \pi/2\}$. By the round annulus theorem[McM2]

$$\mathcal{L}(\Theta') = \frac{1}{2\pi} \log \frac{\pi/2}{d} + O(1) = -\frac{1}{2\pi} \log d + O(1)$$

when d is bounded above. The theorem follows. $\square$

We then have the following corollary:

Lemma 2.14. *For any arc $\alpha \in \mathcal{A}(S)$,*

$$|W^S_{can}(\alpha) - \frac{2}{\pi} M^S(\alpha)| < C_0.$$

Proof. Given α, let us consider a lift $\tilde{\alpha}$ to the universal cover. It connects two arcs I_1, I_2 on the circle that cover the boundary curves of S that the endpoints of α lie on. We can lift $h\colon \mathbf{S} \to S$ to $\tilde{h}\colon \boldsymbol{R} \to \mathbb{D}$, where $\boldsymbol{R}$ is the convex hull of the limit set of the deck transformation group. Then letting Γ_1, Γ_2 connect the endpoints of I_1, I_2 respectively, we find that the transverse geodesic arc $\boldsymbol{\alpha}$ lifts to the common perpendicular segment to Γ_1 and Γ_2. So $\operatorname{dist}(\Gamma_1, \Gamma_2) = L(\boldsymbol{\alpha})$, and the Lemma follows from Lemma 2.13. $\square$

From Theorem 12.6.1 and Lemma 2.14 we can immediately conclude:

Corollary 2.15. *Let S be a compact Riemann surface with boundary, and endow* $\operatorname{Int} S$ *with its Poincaré metric. Let γ be a peripheral closed geodesic in* $\operatorname{Int} S$*. Then*

$$L(\gamma) = \pi \langle W_{\text{can}}(S), \gamma\rangle + O(1; \chi(S)).$$

With S as in Corollary 2.15, we let Γ_S be the formal sum of the peripheral closed geodesics in $\operatorname{Int} S$. Then

$$\langle W_{\text{can}}(S), \Gamma_S\rangle = 2\|W_{\text{can}}(S)\|_1,$$

because every arc in S intersects exactly two peripheral geodesics exactly once (or intersects exactly one exactly twice). Therefore, by Corollary 2.15,

$$\|W_{\text{can}}(S)\|_1 = \frac{\pi}{2}|\Gamma| + O(1; p). \tag{2.16}$$

We can use this equation to compare the total weight of $W_{\text{can}}(S)$ between two surfaces:

Theorem 2.17. *Suppose S and S' are the interiors of compact Rieman surfaces, and there is a holomorphic map $i\colon S' \to S$ that is homotopic to a homeomorphism. Then*

$$\|W_{\text{can}}(S)\|_1 \leqslant \|W_{\text{can}}(S')\|_1 + O(1; \chi(S)).$$

Proof. Because i is homotopic to a homeomorphism, it bijectively maps the free homotopy classes of the peripheral geodesics of S' to the same for S. Then, by the Schwarz Lemma, because i is holomorphic, we have $|\Gamma_{S'}| \geqslant |\Gamma_S|$. The Theorem then follows from the application of (2.16) to $\Gamma_{S'}$ and $\Gamma_S|$. □

3. Theorem 1.2 in terms of W_{can}

In this brief section we will prove Theorem 1.2, given a simple theorem about $W_{\text{can}}(V \smallsetminus \mathcal{K}_p)$. We then outline the proof of this latter theorem, which is then proven in Sections 4–7.

3.1. **Quadratic-like maps and their renormalizations.**

For the basic theory of quadratic-like maps, see [DH2].

A *quadratic-like map* is a double branched covering $f : U \to V$, where U and V are topological disc in $\mathbb{C}$, and $U \Subset V \neq \mathbb{C}$. Such a map has a unique critical point, which will be placed at the origin. The *filled Julia set* $K(f)$ is the set of non-escaping points:

$$K(f) = \{z : f^n z \in U,\ n = 0, 1, \dots\}.$$

It is either connected or a Cantor set depending on whether $0 \in K(f)$ or not. (In what follows, we will often skip "filled", since we will never deal with the actual "Julia sets".)

Two quadratic-like maps are called *hybrid equivalent* if they are quasi-conformally conjugate by a map h with $\bar{\partial} h = 0$ a.e. on $K(f)$. The Douady and Hubbard Straightening Theorem asserts that any quadratic-like map with connected Julia set is hybrid equivalent to a quadratic-like restriction of a unique quadratic polynomial.

A quadratic-like map f is called *primitively renormalizable with period* p if there exists a topological disk $U' \ni 0$ such that the domains $f^k(U')$, $k = 0, 1, \dots, p-1$, are contained in U and pairwise disjoint, and $g = f^p : U' \to f^p(U')$ is a quadratic-like map with connected Julia set. In this case, $K_0 = K(g)$, as well as $K_j = f^j(K')$, $j = 0, 1, \dots, p-1$, are called the *little Julia set*, and the invariant set

$$\mathcal{K} = \bigcup_{j=0}^{p-1} K_j$$

is called the *cycle of little Julia sets.*

3.2. **Horizontal and vertical arc diagrams.** In the further applications, the Riemann surface S will be $V \smallsetminus \mathcal{K}$, where V is a topological disk, and $\mathcal{K}$ is the union of p disjoint full continua (in reality, the cycle of little Julia sets of some quadratic-like map).

Under these circumstances, a proper path (and the corresponding arc) in $V \smallsetminus \mathcal{K}$ is called *horizontal* if it connects two little Julia sets, and is called *vertical* if it connects a little Julia set to ∂V. Given an arc diagram W on $V \smallsetminus \mathcal{K}$, let W^{h} and W^{v} stand for its horizontal and vertical parts respectively, and let $W^{\mathrm{v+h}}$ stand for their sum. In particular, we can consider $W^{\mathrm{h}}_{\mathrm{can}}$, $W^{\mathrm{v}}_{\mathrm{can}}$ and $W^{\mathrm{v+h}}_{\mathrm{can}}$.

We can now state a theorem which will be proven in Section 7:

Theorem 3.1. *Let $f\colon \mathbf{U} \to \mathbf{U}$ be primitively renormalizable quadratic-like-map with period p, and let $\mathcal{K}$ be its cycle of little Julia sets. Then there exists $M = M(p)$ such that: If $\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1 > M$ then*

$$\|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1 \geqslant C^{-1} \|W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1,$$

with an absolute constant $C > 0$.

Given Theorem 3.1, we can prove Theorem 1.2:

We are now ready to show that the modulus of a ψ-quadratic-like map is improving under the renormalization.

Let $f\colon \mathbf{U} \to \mathbf{U}$ be a quadratic-like map with filled Julia set K. The simple closed geodesic γ in the annulus $\mathbf{U} \smallsetminus K$ is called the *geodesic associated with* f. Let $|\gamma|$ stand for its hyperbolic length in $\mathbf{U} \smallsetminus K$.

Proof of Theorem 1.2. Let $K_p(j) = f^j(K_p)$ be the j^{th} small filled-Julia set, and let $\gamma_p(j)$ be the peripheral simple closed geodesic in $\mathbf{U} \smallsetminus \mathcal{K}_p$ going around $K_p(j)$. Let Γ be peripheral closed geodesic in $\mathbf{U} \smallsetminus \mathcal{K}_p$ homotopic to $\partial\mathbf{U}$. We let $W_{\text{can}} \equiv W_{\text{can}}(\mathbf{U} \smallsetminus \mathcal{K}')$.

The geodesic Γ intersects each vertical arc once and does not intersect horizontal arcs. By Corollary 2.15,

$$|\Gamma| \geqslant c\|W^{\text{v}}_{\text{can}}\|_1 + O(1). \tag{3.2}$$

Let $W_{\text{can}}|\,j$ be the part of W_{can} supported on the arcs landing at $K_p(j)$. The geodesic $\gamma_p(j)$ does not intersect $W_{\text{can}} - W_{\text{can}}|\,j$ and intersects each arc of $\operatorname{supp} W_{\text{can}}|\,j$ once. By Corollary 2.15,

$$|\gamma_p(j)| = c\|W_{\text{can}}|\,j\|_1 + O(1). \tag{3.3}$$

But by the Schwarz Lemma, the geodesics $\gamma_p(j)$ have comparable lengths (see [McM1, Theorem 9.3]):

$$\frac{1}{2}|\gamma_p| \leqslant |\gamma_p(j)| \leqslant |\gamma_p|.$$

Putting this together with (3.3), we see that all the $\|W_{\text{can}}|\,j\|_1$ are also comparable, provided $|\gamma_p|$ is sufficiently big (bigger than some absolute L_0). Hence

$$\|W_{\text{can}}|0\|_1 \asymp \frac{1}{p}\|W^{\text{v+h}}_{\text{can}}\|_1,$$

and together with (3.3), we obtain:

$$|\gamma_p| \asymp \frac{1}{p}\|W^{\text{v+h}}_{\text{can}}\|_1.$$

Now assume $\|W^{\text{v+h}}_{\text{can}}\|_1 > M(\bar{p})$. Then Theorem 3.1 implies that $\|W_{\text{can}}\|_1$ is comparable with $\|W^{\text{v}}_{\text{can}}\|_1$, and we can conclude

$$|\gamma_p| \leqslant \frac{C}{p}\|W^{\text{v}}_{\text{can}}\|_1,$$

which, combined with (3.2), implies

$$|\gamma_p| \leqslant \frac{C}{p}|\Gamma|.$$

We now need only observe that by the Schwarz Lemma, $|\Gamma| \leqslant |\gamma|$ (since $K \supset \mathcal{K}_p$). $\square$

3.3. **Outline for the proof of Theorem 3.1.** In this informal subsection we will outline the proof of Theorem 1.2 that will be presented in Sections 5, 6, and 7, using the background in Section 4.

Suppose that $f\colon U \to V$ is a primitively p-renormalizable quadratic-like map. Theorem 1.2 relates two hyperbolic lengths in $V \setminus \mathcal{K}_p$, which we will reinterpret in terms of $W_{\text{can}}(V \setminus \mathcal{K}_p)$ in light of eqrefeq:length-wcan. Moreover, letting $K_p(k) = f^k(K_p)$ be the k^{th} small filled-Julia set, and $\gamma_1(k)$ be the geodesic in $V \setminus \mathcal{K}_p$ going around it, we know that $|\gamma_1| \leqslant |\gamma_1(k)| \leqslant 2|\gamma_1|$. It is then not too hard to see that Theorem 1.2 follows from the following:

$$\|W^{\text{v+h}}\|_1 > M \implies \|W^{\text{v}}\| \geqslant \epsilon \|W^{\text{v+h}}\|. \tag{3.4}$$

We will prove (3.4) as follows:

Restrictions of the canonical WAD's and the entropy argument. We consider the nest of the domains $\mathbf{U}^m$, pullbacks of V under the iterates of f, and restrict the canonical WAD to these domains. Let X^m stand for the horizontal parts of these restrictions (with appropriate constants subtracted). We show that $f^* X^m \multimap X^{m+1}$. This allows us to conclude that eventually the *diagrams X^n are aligned with the Hubbard tree* (§4, 5) and that

$$\|X^{10p}\|_1 \leqslant \frac{1}{2}\|X^0\|_1,$$

which implies

$$\|W^{\text{h}}_{\text{can}}(\mathbf{U}^{10p} \setminus \mathcal{K}_p)| \leqslant \frac{1}{2}\|W^{\text{h}}_{\text{can}}(\mathbf{U}^0 \setminus \mathcal{K}_p)\| + O(1;p)$$

(§5). The latter property depends on the positivity of entropy of the Hubbard tree dynamics: this is the main place where we use that the renormalizations are of primitive type.

Push-forward via the Covering Lemma (§7). The horizontal width released under restrictions is turned into the vertical width, which implies

$$\|W_{\text{can}}(\mathbf{U}^{10p} \setminus \mathcal{K}_p)\|_1 \geqslant \frac{1}{2}\|W^{\text{h}}_{\text{can}}(\mathbf{U}^{10p} \setminus \mathcal{K}_p)\|_1 - O(1;p).$$

We can then apply the Covering Lemma[KL1] to push forward $W_{\text{can}}(\mathbf{U}^{10p} \setminus \mathcal{K}_p)$ by f^{10p} and obtain (3.4).

4. Life on Hubbard trees

4.1. **Topological disked trees and aligned graphs.** Let us consider a Riemann surface U with finitely many open Jordan disks $D_j \Subset U$ with disjoint closures, and let $\mathcal{D} = \cup D_j$. An embedded 1-complex

$H_k \subset U \smallsetminus \mathcal{D}$ is called *proper* if all its tips (i.e., valence 1 vertices) belong to $\partial\mathcal{D}$. Assume that we have finitely many disjoint proper graphs H_k such that $H = \mathcal{D} \cup_k H_k$ is simply connected. In this case we say that H is a *(topological) disked tree.*

We say that a path in $H \smallsetminus \mathcal{D}$ is *aligned* with a disked tree H if it connects the boundaries of two disks in $\mathcal{D}$. The arcs α in $U \smallsetminus \mathcal{D}$ represented by aligned paths are also called *aligned with* H. Let $\mathbf{H}$ stand for the family of the aligned paths/arcs. (Since there is a natural one-to-one correspondence between the aligned arcs and the aligned paths, we will not distinguish notationally these families).

Let $\mathbf{G}$ be the abstract graph whose vertices are disks D_i and edges are the paths/arcs of $\mathbf{H}$. We call it the *graph of aligned arcs.*

Lemma 4.1. *The graph* $\mathbf{G}$ *is a tree of complete graphs.*[7]

Proof. Let $\mathcal{D}_k$ stand for the family of the disks D_i that touch the graph H_k. Let $\mathbf{G}_k \subset G$ be the graph whose vertices are disks $D_i \subset \mathcal{D}_k$ and edges are the path of $\mathbf{H}$ contained in $\mathbf{G}_k$. As H_k is path connected, any two disks D_i and D_j of $\mathcal{D}_k$ can be connected by an arc in $\mathbf{G}_k$, so that, the graph $\mathbf{G}_k$ is full.

Since H is a tree, it is easy to check that the graphs $\mathbf{G}_k$ are organized in a tree as well. ☐

4.2. **Superattracting model.** To a primitively p-renormalizable map f, one can associate a certain superattracting quadratic polynomial F of period p as follows (compare [McM1, §B]). First, we can straighten f to a quadratic polynomial, so we can assume that it is a quadratic polynomial in the first place. Then, collapsing the little Julia sets K_j to points c_j, we obtain a topological sphere S. Moreover, the map f descends to a degree two map $f_0\colon S \to S$ with a periodic critical point c_0 (f_0 collapses the sets $-K_j \subset \mathbb{C} \smallsetminus \mathcal{K}$, $j = 1, \ldots, p-1$), to points c_{j+1}. One can check that this map does not have Thurston obstructions (see [DH3]), so it can be realized as a superattracting quadratic polynomial. This polynomial is the desired F.

We call F the *superattracting model* for f. We let $\mathcal{O} = \{F^k(0)\}_{k=0}^{p-1}$ stand for the superattracting cycle of F, and $\mathcal{D}$ stand for its immediate basin of attraction.

Lemma 4.2. *Let* $f\colon (U, \mathcal{K}) \to (V, \mathcal{K})$ *be a primitively renormalizable quadratic-like map, and let* $F\colon (\mathbb{C}, \mathcal{D}) \to (\mathbb{C}, \mathcal{D})$ *be its superattracting model. There is a natural one-to-one correspondence between horizontal/vertical arcs in* $U \smallsetminus \mathcal{K}$ *(resp.,* $V \smallsetminus f^{-1}(\mathcal{K})$*) and horizontal/vertical*

[7]See §11.7 of Appendix B for the definition.

arcs in $\mathbb{C} \smallsetminus \mathcal{D}$ *(resp.* $\mathbb{C} \smallsetminus F^{-1}(\mathcal{D})$*). The correspondence between the horizontal arcs is compatible with the arrow relation.*

This obvious statement allows us to replace f with its superattracting model F, as long as we are dealing with the combinatorics of arc diagrams. The advantage of the model is that it possesses a well-defined Hubbard tree.

4.3. **Disked Hubbard tree.** Let D_k be the component of $\mathcal{D}$ containing c_k. It is known that the $\bar{D}_k$'s are Jordan disks. Given a set $X \subset K(F)$, the *path hull* of X is defined as the smallest path connected closed subset of $K(F)$ containing X satisfying the property that it intersects any component of $\operatorname{int} K(F)$ by the union of internal rays. The *Hubbard tree* $H \equiv H_F$ is defined as the path hull of the basin $\mathcal{D}$. It is a disked tree in the sense of §4.1. Moreover, it is invariant under F; in fact, $F(H) = H$.

Lemma 4.3. *The valence of any disk* D_k *of the Hubbard tree is at most 2.*

4.4. **No periodic horizontal arcs.** Given a proper path γ in $\mathbb{C} \smallsetminus \mathcal{D}$ that begins at some D_i, $i \in \mathbb{Z}/p\mathbb{Z}$, let $F^*\gamma$ stand for the union of the proper lifts of γ that begin at D_{i-1}. (Note that a horizontal path γ has at most one proper lift.)

This lifting operation descends to the level of arcs. An arc $\mathbf{a}$ is called *periodic* if $(F^*)^l(\alpha) \supset \alpha$ for some $l \in \mathbb{Z}_+$. (Of course, the period l must be a multiple of p.)

Lemma 4.4 (see [P], Theorem 5.8)**.** *Periodic horizontal arcs do not exist.*

Proof. Assume that α is such a periodic horizontal arc with period l. Let us endow $\mathbb{C} \smallsetminus \mathcal{O}$ with hyperbolic metric, and let us represent α be a geodesic path $\gamma \subset \mathbb{C} \smallsetminus \mathcal{D}$. It is the shortest representative of α.

Since $(F^*)^l\alpha = \alpha$, the lift $F^*\gamma$ represents the same arc α. so that, it is no shorter than γ. On the other hand, the Schwarz Lemma implies that it is shorter than γ – contradiction. □

4.5. **Expansion property.** Let $F^{-1}\mathbf{H}$ stand for the family of paths in $H \smallsetminus F^{-1}\mathcal{D}$ with endpoints on $F^{-1}\bar{\mathcal{D}}$. Each path of $F^{-1}\mathbf{H}$ is homeomorphically mapped by F onto some path of $\mathbf{H}$.

Let us consider a $|\mathbf{H}| \times |\mathbf{H}|$ matrix $M = M(F)$ defined as follows: $M_{\gamma\delta} = n$ if γ contains n disjoint segments $\gamma_i \in F^{-1}\mathbf{H}$ such that $F\gamma_i = \delta$. One can easily check that the matrix M^k can obtained by applying the same construction to the map F^k. .

Lemma 4.5. *For any $\gamma \in \mathbf{H}$, we have* $\sum_\delta M^p_{\gamma\delta} \geqslant 2$.

Proof. Not that the path $F^p\gamma$ has the same endpoints as γ does. If $\sum_\delta M^p_{\gamma\delta} = 1$ then $F^p\gamma$ would not cross any disks of $\mathcal{D}$. It would follow that $f^p\gamma = \gamma$, which is impossible by Lemma 4.4. □

4.6. **Periodic vertical arcs.** Let $\mathbf{H}^\perp$ be the arc diagram consisting of vertical arcs in $\mathbb{C} \smallsetminus \mathcal{D}$ that do not intersect the Hubbard tree H (up to homotopy).

Lemma 4.6. *If a vertical arc β is periodic then $\beta \in \mathbf{H}^\perp$.*

Proof. Let $(F^*)^l\beta \supset \beta$ for some $l \in \mathbb{Z}_+$. Let $\mathcal{G}$ be the family of paths γ in $\mathbb{C} \smallsetminus \mathcal{D}$ representing β. For a path $\gamma \in \mathcal{G}$, let γ' be its maximal initial segment whose endpoints belong to the Julia set $K(F)$. Let $\mathcal{G}'$ be the family of all paths γ' that can be obtained in this way.

Since $(F^*)^l\beta \supset \beta$, any path $\gamma \in \mathcal{G}$ can be lifted by F^l to a path $\delta \in \mathcal{G}$. Then δ' is a lift of γ', and thus, we obtain the lifting map $L\colon \mathcal{G}' \to \mathcal{G}'$.

Let us endow the punctured plane $S = \mathbb{C} \smallsetminus \mathcal{O}$ with the hyperbolic metric. Let $|\gamma|$ stand for the hyperbolic length of a path γ, and let

$$\mu = \inf_{\gamma' \in \mathcal{G}'} |\gamma'|.$$

This infimum is realized by some hyperbolic geodesic γ'. If $\mu > 0$ then by the Schwarz Lemma, $|L(\gamma')| < |\gamma'|$, contradicting minimality of γ'. Hence $\mu = 0$ and so, $\mathcal{G}$ contains a curve γ that does not intersect the Julia set $K(F)$ (except for the initial point in $\mathcal{D}$). This is the desired vertical representative of β. □

4.7. **Pullbacks of vertical arcs.** Given two families $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of proper arcs (or paths) in a Riemann surface S, the "inner product" $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \equiv \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_S$ is the minimal number of intersection points between path representatives of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

For instance, $\alpha \in \mathbf{H}^\perp$ if and only if $\langle \alpha, \mathbf{H} \rangle = 0$. The *dual* statement is also valid: $\alpha \in \mathbf{H}$ if and only if $\langle \alpha, \mathbf{H}^\perp \rangle = 0$.

Lemma 4.7. *For any vertical arc β, we have:* $\bigcup_{n=0}^{\infty} (F^n)^*(\beta) \supset \mathbf{H}^\perp$.

Proof. Let γ be a vertical path representing β, and let γ_1 be a component of $F^*\gamma$. Since $F\colon \gamma_1 \to \gamma$ is a homeomorphism,

$$\langle \gamma_1, F^{-1}\mathbf{H} \rangle_{\mathbb{C} \smallsetminus F^{-1}\mathcal{D}} = \langle \gamma, \mathbf{H} \rangle_{\mathbb{C} \smallsetminus \mathcal{D}}.$$

Since $H \subset F^{-1}H$,

$$\langle \gamma_1, \mathbf{H} \rangle \leqslant \langle \gamma, \mathbf{H} \rangle. \tag{4.8}$$

Let us now consider a chain of vertical arcs $\beta_n \subset (F^n)^*(\beta)$. By (4.8), the intersection number $\langle \beta_n, \mathbf{H} \rangle$ does not increase with n, and hence eventually stabilizes at some value k. But for a given k, there are only finitely many different vertical arcs β such that $\langle \beta, \mathbf{H} \rangle = k$. Hence $\beta_n = \beta_{n+l}$ for some $l > 0$. By Lemma 4.6, $\beta_n \in \mathbf{H}^\perp$.

But then all further lifts of β_n by the iterates of F belong to $\mathbf{H}^\perp$ as well. Lifting β_n to the critical disk D_0, we obtain two symmetric arcs, σ_1 and σ_2, landing at D_0. By symmetry, these arcs are different. But by Lemma 4.3, for any disk D_k, there exist at most two different arcs of $\mathbf{H}^\perp$ landing on D_k. Thus, σ_1 and σ_2 are the only arcs of $\mathbf{H}^\perp$ landing on D_0. Lifting these arcs further to all the domains D_k, we obtain all the arcs of $\mathbf{H}^\perp$. ☐

4.8. **Trees H^l and the associated objects.** For any $l \in \mathbb{Z}_\geqslant$, we can introduce the following objects:

- $\mathcal{D}^l = F^{-l}(\mathcal{D})$; D^l_k are the components of $\mathcal{D}$;
- $H^l = F^{-l}(H)$; it is a disked tree with disks D^l_k;
- $\mathbf{H}^l$ is the family of the paths/arcs aligned with H^l;
- $\mathbf{G}^l$ is a graph whose vertices are the disks D^l_k and edges are the arcs of $\mathbf{H}^l$; as for $l = 0$ (Lemma 4.1), it is a tree of complete graphs $\mathbf{G}^l_k$.

Notice that F^l maps $\mathbf{G}^l$ onto $\mathbf{G}$, so that each complete graph $\mathbf{G}^l_k$ is mapped onto some complete graph $\mathbf{G}_j$.

By Lemma 11.11 applied to $\mathbf{G}^l$, for any edge $[D_m, D_n]$ of $\mathbf{H}$, there is a unique chain of $\mathbf{G}^l$,

$$D_m = D^l_{k(1)}, D^l_{k(2)}, \ldots, D^l_{k(d)} = D_n, \tag{4.9}$$

connecting D_m to D_n. Here $d = d_{\mathbf{G}^l}(D_m, D_n)$ is the distance between D_m and D_n in $\mathbf{G}^l$. Lemma 4.5 implies:

$$d_{\mathbf{G}^{rp}}(D_m, D_n) \geqslant 2^r. \tag{4.10}$$

In fact, we have:

Lemma 4.11. *At least 2^{r-1} of the disks $D^{rp}_{k(i)}$ belong to $\mathcal{D}^{rp} \smallsetminus \mathcal{D}^{(r-1)p}$.*

Proof. For $r = 1$, the assertion follows from Lemma 4.5 and a remark that all the intermediate disks in the path (4.9) do not belong to $\mathcal{D}$. Since each path of $\mathbf{H}^p$ is mapped homeomorphically under F^p onto a path of $\mathbf{H}$ (transferring chains in $\mathbf{G}^{rp}$ to chains in $\mathbf{G}^{(r-1)p}$), the assertion follows by induction in r. ☐

5. Restrictions of WAD's

5.1. **Restriction of the domains.** Let us consider a quadratic-like map $f\colon U^1 \to U^0$. We let $U^n = f^{-n}U^0$, and observe that $U_{n+1} \subset U_n$, and $f\colon U_{n+1} \to U_n$ is quadratic-like.

Assume f is primitively renormalizable with some period p, and let $\mathcal{K}$ be the cycle of the corresponding little (filled) Julia sets. Let $\tilde{\mathcal{K}} = f^{-1}(\mathcal{K})$. Then we have an embedding $i\colon U^{n+1} \smallsetminus \mathcal{K} \to U^n \smallsetminus \mathcal{K}$, a covering $f\colon U^{n+1} \smallsetminus \tilde{\mathcal{K}} \to U^n \smallsetminus \mathcal{K}$, and an embedding $U^{n+1} \smallsetminus \tilde{\mathcal{K}} \subset U^{n+1} \smallsetminus \mathcal{K}$, that together form a triangle diagram. We will properly mark the Riemann surfaces $U^n \smallsetminus \mathcal{K}$ and $U^n \smallsetminus \tilde{\mathcal{K}}$ by declaring the ends corresponding to the little Julia sets K_i to be proper. Then the maps in the above diagram are partially proper. Moreover, the embedding $U^{n+1} \smallsetminus \tilde{\mathcal{K}} \subset U^{n+1} \smallsetminus \mathcal{K}$ is also proper on ∂U^{n+1}.

5.2. **Domination relations.** Let us select a non-decreasing sequence of numbers $q_n \in \mathbb{R}_+$ such that

$$q_{n+1} \geqslant 3p(q_n + 2). \tag{5.1}$$

Let $X^n = W^{\mathrm{h}}_{\mathrm{can}}(U^n \smallsetminus \mathcal{K})$ and $\hat{X}^n = X^n - q_n$.

Proposition 5.2. *We have:*

(i) $\hat{X}^{n+1} \leqslant i^*\hat{X}^n$;
(ii) $f^*\hat{X}^n \multimap \hat{X}^{n+1}$.

Proof. (i) Since the embedding i is proper on the ends corresponding to the little Julia sets K_j, we have $X^{n+1} \leqslant i^*X^n$ by Property C. This yields the desired inequality since the sequence (q_n) is non-decreasing.

(ii) Let us properly mark the ends of $U^n \smallsetminus \mathcal{K}$ and $U^{n+1} \smallsetminus \tilde{\mathcal{K}}$ corresponding to the sets $\mathcal{K}$ and $\tilde{\mathcal{K}}$ respectively. Since the covering f maps $\mathbf{U}^{n+1}$ to $\mathbf{U}^n$ and maps $\tilde{\mathcal{K}}$ to $\mathcal{K}$, horizontal and vertical arcs of $U^n \smallsetminus \mathcal{K}$ lift respectively to horizontal and vertical arcs of $U^{n+1} \smallsetminus \tilde{\mathcal{K}}$. Hence $f^*X^n = W^{\mathrm{h}}_{\mathrm{can}}(U^{n+1} \smallsetminus \tilde{\mathcal{K}})$.

By Property D,

$$W_{\mathrm{can}}(U^{n+1} \smallsetminus \tilde{\mathcal{K}}) \multimap W^{\mathrm{h}}_{\mathrm{can}}(U^{n+1} \smallsetminus \mathcal{K}) - 6p.$$

But since the embedding $U^{n+1} \smallsetminus \tilde{\mathcal{K}} \subset U^{n+1} \smallsetminus \mathcal{K}$ is proper on ∂U^{n+1}, the itinerary of any horizontal path γ in $U^{n+1} \smallsetminus \mathcal{K}$ consists only of horizontal arcs of $U^{n+1} \smallsetminus \tilde{\mathcal{K}}$. It follows that

$$W^{\mathrm{h}}_{\mathrm{can}}(U^{n+1} \smallsetminus \tilde{\mathcal{K}}) \multimap W^{\mathrm{h}}_{\mathrm{can}}(U^{n+1} \smallsetminus \mathcal{K}) - 6p.$$

Thus, $f^*X^n \multimap X^{n+1} - 6p$. Taking into account Lemma 2.9 and (5.1), we conclude:

$$f^*\hat{X}^n = f^*X^n - q_n \multimap X^{n+1} - 6p - 3pq_n \geqslant \hat{X}^{n+1}. \quad \square$$

In the remainder of Sections 5 and 6, we will reason using only combinatorics and topology, combined with (i) and (ii) above, to conclude our main goal, Lemma 6.6.

5.3. **Topological arrow.** Let us consider two Riemann surfaces, $U \subset V$, and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be multiarcs on U and V respectively. We say that $\boldsymbol{\alpha}$ *topologically arrows* $\boldsymbol{\beta}$, $\boldsymbol{\alpha} \rightsquigarrow \boldsymbol{\beta}$, if for any arc $\beta \in \boldsymbol{\beta}$ there is a sequence (α_k) of arcs with $\alpha_k \in \boldsymbol{\alpha}$ for each k, and $(\alpha_k) \longrightarrow \beta$.

A basic example comes from two WAD's, $X \in W(U)$ and $Y \in W(V)$, such that $X \multimap Y$. Then $\operatorname{supp} X \rightsquigarrow \operatorname{supp} Y$, as follows immediately from the definitions.

5.4. **Invariant horizontal arc diagram.** Given a partially proper embedding $U \subset V$ such that U is a deformation retract of V, we can view horizontal arcs on V as horizontal arcs on U (by retracting them from V to U). When we'd like to emphasize this point of view, we will use notation $\alpha|U$.

Let us say that a horizontal arc diagram $\boldsymbol{\alpha}$ on $U^n \smallsetminus \mathcal{K}$ is *invariant* if

$$f^*\boldsymbol{\alpha} \rightsquigarrow \boldsymbol{\alpha}$$

Proposition 5.3. *There exists $n \leqslant 3p$ such that the horizontal arc diagram $\boldsymbol{\alpha}^n = \operatorname{supp} \hat{X}^n$ is invariant.*

Proof. Since $\hat{X}^{n+1} \leqslant i^*\hat{X}^n$ (by Lemma 5.2 (i)), $\boldsymbol{\alpha}^{n+1} \subset \boldsymbol{\alpha}^n|U^{n+1} \smallsetminus \mathcal{K}$. Since $|\boldsymbol{\alpha}^0| \leqslant 3p$, there exists an $n \leqslant 3p$ such that $\boldsymbol{\alpha}^{n+1} = \boldsymbol{\alpha}^n|U^{n+1} \smallsetminus \mathcal{K}$. Since by Lemma 5.2 (ii), $f^*\boldsymbol{\alpha}^n \rightsquigarrow \boldsymbol{\alpha}^{n+1}$, we are done. $\square$

5.5. **Alignment with the Hubbard tree.** We say that a horizontal arc diagram $\mathbf{a}$ is *aligned with the Hubbard tree* if it is so for the superattracting realization of $\mathbf{a}$.

Lemma 5.4. *Any invariant horizontal arc diagram $\boldsymbol{\alpha}$ is aligned with the Hubbard tree.*

Proof. Let $\boldsymbol{\alpha}$ be an invariant horizontal arc diagram for a superattracting polynomial F. It can be realized by a "disked graph" A whose "vertices" are the disks $\bar{D}_k$ and edges are paths representing the arcs of $\boldsymbol{\alpha}$. Let Δ be the unbounded component of $\mathbb{C} \smallsetminus A$. Then one of the disks $\bar{D}_k$ intersects $\partial\Delta$. A path γ in Δ landing on $\bar{D}_k$ represents a vertical arc β such that $\langle \boldsymbol{\alpha}, \beta \rangle = 0$.

Then $F^*\gamma$ does not intersect $F^{-1}(A)$ (except for the landing points). But by invariance, the arcs of $\boldsymbol{\alpha}$ can be realized as paths in $F^{-1}(A)$. Hence $F^*\gamma$ does not intersect $\boldsymbol{\alpha}$.

Iterating, we see that $\boldsymbol{\alpha}$ does not intersect $(F^n)^*\gamma$ for all $n = 0, 1, \ldots$. By Lemma 4.7, it does not intersect $\mathbf{H}^\perp$ either. So, $\langle \boldsymbol{\alpha}, \mathbf{H}^\perp \rangle = 0$, and we are done. ☐

Putting this together with Proposition 5.3 and Proposition 5.2 (i), we obtain:

Corollary 5.5. *For any $n \geqslant 3p$, the horizontal arc diagram $\boldsymbol{\alpha}^n = \operatorname{supp} \hat{X}^n$ is aligned with the Hubbard tree H.*

6. Entropy argument

6.1. **Domination and electric circuits.** Let us consider two topological disked trees $H \subset H'$, with the families of disks $\{D_j\} \subset \{D'_i\}$. Then the corresponding aligned arc diagrams are related by the topological arrow: $\mathbf{H}' \rightsquigarrow \mathbf{H}$. Let $\mathbf{G}$ and $\mathbf{G}'$ be the corresponding trees of complete graphs.

We say that WAD Y is aligned with H if $\operatorname{supp} Y$ is aligned with H. Such a diagram induces an unplugged electric circuit $\mathcal{C}_Y$ based on $\mathbf{G}$ by letting the conductance of the edge $e \in \mathbf{H}$ be $Y(e)$. The following lemma relates the domination relation between aligned WAD's to the domination relation between the corresponding electric circuits (see §11.6).

Lemma 6.1. *Let WAD's Y and Y' be aligned with trees $H \subset H'$ as above. If $Y' \multimap Y$ then $\mathcal{C}_{Y'} \multimap \mathcal{C}_Y$.*

Proof. By definition of domination, there exist edges $\beta_i \in \mathbf{H}$ concatenated by paths $(\alpha_{ij})_j$ in $\mathbf{H}'$, and positive numbers w_i, v_{ij} such that

$$Y = \sum_i w_i \beta_i, \quad Y' \geqslant \sum_i \sum_j v_{ij} \alpha_{ij}, \tag{6.2}$$

and for any i,

$$w_i \leqslant \bigoplus_j v_{ij}. \tag{6.3}$$

Let $B(e)$ be the family of edges β_i equal to e. For any $\beta_i \in B(e)$, let us consider an auxiliary electric circuit $\mathcal{C}'_i$ whose resistors are $(\alpha_{ij})_j$ with conductances $(v_{ij})_j$ plugged in series with battery ∂e. Then (6.3) translates into an estimate: $w_i \leqslant \mathbf{W}(\mathcal{C}'_i)$, where $\mathbf{W}(\mathcal{C}'_i) = \bigoplus_j v_{ij}$ is the conductance of $\mathcal{C}'_i$.

Each electric circuit $\mathcal{C}'_i$ admits a natural projection into the circuit $\mathcal{C}'_Y$. By Lemma 11.7 and the second inequality of (6.2), $\sum \mathbf{W}(\mathcal{C}_i) \leqslant \mathbf{W}(\mathcal{C}_{Y'}(e))$. Hence

$$Y(e) = \sum_{\beta_i \in B(e)} w_i \leqslant \sum_{\beta_i \in B(e)} \mathbf{W}(\mathcal{C}_i) \leqslant \mathbf{W}(\mathcal{C}_{Y'}(e)),$$

and we are done. □

6.2. **Dynamical setting.** Let us now consider the Hubbard tree H of a superattracting map F with the basin $\mathcal{D} = \cup D_j$, where $j \in \mathbb{Z}/pZ$. If Y is a WAD aligned with H, we let

$$Y|\, D_j = \sum_{\partial\alpha \ni D_j} Y(\alpha)$$

be the local conductances of the associated electric circuit (see §11.5).

Lemma 6.4. *If Y and Z are WAD's aligned with the Hubbard tree H and such that $F^*Y \multimap Z$, then for any two disk D_j we have:*

$$Z|\, D_j \leqslant (\deg F|D_j) \cdot Y|\, D_{j+1}.$$

Proof. We have $F^*Y|\, D_j = \deg(F|D_j) \cdot Y|\, D_{j+1}$. By Lemma 6.1, $\mathcal{C}_{F*Y} \multimap \mathcal{C}_Y$. Hence $F^*Y|\, D_j \geqslant Z|\, D_j$ by Lemma 11.10. Thus,

$$Z|\, D_j \leqslant (\deg F|D_j) \cdot Y|\, D_{j+1}.$$

□

Lemma 6.5. *Let Y and Z be WAD's aligned with the Hubbard trees H. Assume $(F^{rp})^*Y \multimap Z$ for some $r \in \mathbb{Z}_+$ and some WAD Z. Then for any edge α of $\mathbf{H}$, we have:*

$$Z(\alpha) \leqslant \frac{1}{2^{r-2}} \max_j (Y|D_j).$$

Proof. The WAD $(F^{rp})^*Y$ is aligned with the tree $H^{rp} = F^{-rp}H \supset H$ (see §4.8). Let $\mathcal{C}^l$ be the associated electric circuit. Recall that $\mathcal{C}^l(\alpha)$ stands for the restriction of this circuit to the connected component of $\mathbf{G}^l \smallsetminus \mathcal{D}$ attached to $\partial\alpha$. By Lemma 6.1, $\mathcal{C}^l \multimap \mathcal{C}$, so that,

$$Z(\alpha) \leqslant \mathbf{W}(\mathcal{C}^l(\alpha)).$$

Let us consider the path of disks $(D^{rp}_k)_k$ connecting $\partial\alpha$ in the tree of graphs $\mathbf{G}^{rp}$ (see Lemma 11.11). By Lemma 11.12

$$\mathbf{W}(\mathcal{C}^l(\alpha)) \leqslant \bigoplus_k (F^{rp})^*Y|\, D^{rp}_k.$$

If $D_k^{rp} \in \mathcal{D}^{rp} \smallsetminus \mathcal{D}^{(r-1)p}$ then F^{rp} maps D_k^{rp} onto some disk D_j with degree at most 2. Hence

$$(F^{rp})^* Y |\, D_k^{rp} \leqslant 2Y |\, D_j.$$

Since by Lemma 4.11, there are at least 2^{r-1} such disks, we have:

$$\bigoplus_k (F^{rp})^* Y |\, D_k^{rp} \leqslant \frac{1}{2^{r-2}} \max_j (Y |\, D_j).$$

Putting the above estimates together, we obtain the desired. $\square$

6.3. **Loss of the horizontal weight.** Recall now the WAD's $\hat{X}^n = X^n - q_n$ from §5.2. By Lemma 4.2, they can be viewed as WAD's of the superattracting model F. We will keep the same notation for these diagrams.

Lemma 6.6. *Let p be the period of* 0 *under F. Then*

$$\|\hat{X}^{10p}\|_1 \leqslant \frac{1}{4} \|\hat{X}\|_1.$$

Proof. By Corollary 5.5, the WAD's $\hat{X}^n$ are aligned with the Hubbard tree H for $n \geqslant 3p$. By Proposition 5.2 (ii), $(F^{6p})^* \hat{X}^{4p} \multimap \hat{X}^{10p}$. Hence by Lemma 6.5,

$$\hat{X}^{10p}(\alpha) \leqslant \frac{1}{16} \max_j (\hat{X}^{4p} |\, D_j).$$

By Lemma 6.4, $(\hat{X}^{4p} |\, D_j) \leqslant 2(\hat{X}^{3p} |\, D_k)$ for any two disks D_j and D_k. Hence

$$\max(\hat{X}^{4p} |\, D_j) \leqslant \frac{2}{p} \|\hat{X}^{3p}\|_1 \leqslant \frac{2}{p} \|\hat{X}|\|_1,$$

where the last estimate comes from Proposition 5.2(i).

Putting the above estimates together and summing it up over all $\alpha \in \operatorname{supp} X^{10p}$ (taking into account that $\operatorname{supp} X^{10p}$ contains at most at most $3p$ arcs), we obtain:

$$\|\hat{X}^{10p}\|_1 \leqslant \frac{3}{8} \|\hat{X} |\, D_j\|_1.$$

$\square$

Let us now go back to the original map f. The following result shows that after an appropriate restriction of the domain of f, there is a definite loss of the horizontal weight of the associated canonical WAD.

Corollary 6.7. *Let f be a renormalizable ψ-ql map with period p. Then there exists $M = M(p)$ such that*

$$\|W^h_{\mathrm{can}}(\mathbf{U}^{10p} \smallsetminus \mathcal{K})\|_1 \leqslant \frac{1}{2}\|W^h_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1,$$

provided $\|W^h_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1 > M(p)$.

Proof. The last lemma immediately yields

$$\|W^h_{\mathrm{can}}(\mathbf{U}^{10p} \smallsetminus \mathcal{K})\|_1 \leqslant \frac{1}{4}\|W^h_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1 + 3pq_{10p},$$

which implies the desired. □

7. Push-forward Argument

In this section we will derive Theorem 3.1 from the loss of horizontal weight (Corollary 6.7), using some simple observations and the Covering Lemma[KL1].

7.1. **Vertical weight from the loss of horizontal.** We first observe

Lemma 7.1. *For any $n \in \mathbb{Z}_{\geqslant 0}$, we have:*

$$\|W^{\mathrm{v+h}}_{\mathrm{can}}(U \smallsetminus \mathcal{K})\|_1 \leqslant \|W^{\mathrm{v+h}}_{\mathrm{can}}(U^n \smallsetminus \mathcal{K})\|_1 + C(p).$$

Proof. This follows immediately from Theorem 2.17, because $U^n \smallsetminus \mathcal{K} \subset U \smallsetminus \mathcal{K}$, and the inclusion is homotopic to a homeomorphism. □

Combining Corollary 6.7 with Lemma 7.1, we obtain:

Lemma 7.2. *Let f be a renormalizable quadratic-like map with period p. Then there exists $M = M(p)$ such that*

$$\|W^{\mathrm{v}}_{\mathrm{can}}(U^{10p} \smallsetminus \mathcal{K})\|_1 \geqslant \frac{1}{2}\|W^{\mathrm{v+h}}_{\mathrm{can}}(U \smallsetminus \mathcal{K})\|_1,$$

provided $\|W^h_{\mathrm{can}}(U \smallsetminus \mathcal{K})\|_1 > M(p)$.

Our plan in the next two sections is to use the Covering Lemma to "push forward" the $W^{\mathrm{v}}_{\mathrm{can}}(U^{10p} \smallsetminus \mathcal{K})$ to get a lower bound on $W^{\mathrm{v}}_{\mathrm{can}}(U \smallsetminus \mathcal{K})$.

7.2. **Covering Lemma.** Let us state a suitable version of the Covering Lemma[KL1] in the language of the canonical weighted arc diagram.

Theorem 7.3. *Suppose that U and V are simply connected Riemann surfaces, $A = \bigcup_{i=1}^{p} A_i$ is a union of p disjoint compact connected full subsets of U, $B = \bigcup_{i=1}^{n} B_i$ is the same on V, and $f\colon U \to V$ is a holomorphic branched cover of degree D, such that all the critical values of f lie in B, and for each i, A_i is a component of $f^{-1}B_i$, and the degree* *of $f|_{A_i}$ is d (in the sense that A_i and B_i have open neighborhoods A_i' and B_i' such that $f\colon A_i' \to B_i'$ is a degree d branched cover). Suppose further that*

$$\|W^{\mathrm{v}}_{\mathrm{can}}(U \smallsetminus A)\|_1 \geqslant \epsilon \|W^{\mathrm{v+h}}_{\mathrm{can}}(V \smallsetminus B)\|, \tag{7.4}$$

and

$$\|W^{\mathrm{v}}_{\mathrm{can}}(U \smallsetminus A)\|_1 \geqslant M(\epsilon, D, p) \tag{7.5}$$

Then

$$\|W^{\mathrm{v}}_{\mathrm{can}}(U \smallsetminus A)\|_1 \leqslant \frac{12d^2}{\epsilon} \|W^{\mathrm{v}}_{\mathrm{can}} V \smallsetminus A\|_1. \tag{7.6}$$

Before proving this theorem, let us review the notation of [KL1] and related it to $W_{\mathrm{can}}(U \smallsetminus A)$. Accordingly, let U and A be as in the statement of Theorem 7.3. We then define three numbers determined by U and A:

$$X = \mathcal{W}(S, \bigcup_{j=1}^{N} A_j);$$

$$Y = \sum_{j=1}^{N} \mathcal{W}(S, A_j), \tag{7.7}$$

$$Z = \sum_{j=1}^{N} \mathcal{W}(S \smallsetminus \bigcup_{k \neq j} A_k,\ A_j).$$

It is easy to see that $X \leqslant Y \leqslant Z$. We further claim that

$$W^{\mathrm{v}}_{\mathrm{can}}(U \smallsetminus A) \leqslant X \leqslant W^{\mathrm{v}}_{\mathrm{can}}(U \smallsetminus A) + 4p \tag{7.8}$$

and

$$W^{\mathrm{2h+v}}_{\mathrm{can}}(U \smallsetminus A) \leqslant Z \leqslant W^{\mathrm{2h+v}}_{\mathrm{can}}(U \smallsetminus A) + 4p^2 \tag{7.9}$$

To see the lower bound for X in (7.8), we need only observe that every vertical leaf in the canonical *lamination* for $U \smallsetminus A$ contributes to the defining width for X, so X is at least the total width of this vertical lamination. To see the upper bound, we consider the extremal foliation $\mathcal{F}_X$ for the defining path family for X. Then the leaves of $\mathcal{F}_X$ can lie in at most $2p$ homotopy classes α_i, and

$$W(\mathcal{F}_X|_{\alpha_i}) \leqslant W^{U \smallsetminus A}_{\mathrm{can}}(\alpha_i) + 2$$

by maximality. The derivation of (7.9) is similar and left as an exercise for the reader.

We can now reduce Theorem 7.3 to the version of the Quasi-Invariance Law stated in Section 3.1.1 of [KL1].

Proof of Theorem 7.3. We wish to think of the setting of our Theorem as an instance of the setting of the QI Law stated in Section 3 of [KL1], and specialized in Section 3.1.1. So we apply the latter, letting U and V in [KL1] be our U and V, letting $\Lambda'_j = \Lambda_j$ be our A_j, and letting $B'_j = B_j$ be our B_j. Then the Y_U in Section 3.1.1 of [KL1] is our $Y(U, A)$; we will use only that it is at least $X(U, A)$. *Because the critical values of f lie in B*, the $Z_V = Z_V\{B_j, B'_j, CV\}$ in Section 3.1.1 is simply our $Z(V, B)$, and of course X_V in [KL1] is our $X(V, B)$.

The context and direction of the two inequalities in Section 3.1.1 of [KL1] are such that we can replace Y_U with X_U in them. We can then derive, assuming $\|W^{\mathrm{v}}(U \setminus A)\| \geqslant 4p^{\epsilon}$,

$$\begin{aligned} Z_V &\leqslant \|W^{2\mathrm{h}+\mathrm{v}}_{\mathrm{can}}(V \setminus B)\|_1 + 4p^2 && \text{by (7.9)} \\ &\leqslant 2\|W^{\mathrm{h}+\mathrm{v}}_{\mathrm{can}}(V \setminus B)\|_1 + 4p^2 \\ &\leqslant \frac{3}{\epsilon}\|W^{\mathrm{v}}_{\mathrm{can}}(U \setminus A)\|_1 && \text{by (7.4)} \\ &\leqslant \frac{3}{\epsilon} X_U && \text{by (7.8).} \end{aligned}$$

This is the separation assumption (with $\xi := 3/\epsilon$) in Section 3.1.1 of [KL1]; we can then conclude, assuming (7.5) (which implies $X_U \geqslant K(\xi, p, D)$),

$$X_U \leqslant \frac{6d^2}{\epsilon} X_V,$$

which implies

$$\|W^{\mathrm{v}}_{\mathrm{can}}(U \setminus A)\|_1 \leqslant \frac{6d^2}{\epsilon}(\|W^{\mathrm{v}}_{\mathrm{can}}(V \setminus B)\|_1 + 4p).$$

Assuming then that $W^{\mathrm{v}}_{\mathrm{can}}(U \setminus A) \geqslant 48d^2p/\epsilon$, we obtain (7.6). □

7.3. **Vertical part has a definite weight.** We can now prove Theorem 3.1:

Proof of Theorem 3.1. If $\|W^{\mathrm{v}}_{\mathrm{can}}(U \setminus \mathcal{K})\|_1 \geqslant \|W^{\mathrm{h}}_{\mathrm{can}}(U \setminus \mathcal{K})\|_1$, there is nothing to prove. So, we assume that the opposite inequality holds, and then

$$\|W^{\mathrm{h}}_{\mathrm{can}}(U \setminus \mathcal{K})\|_1 > \frac{1}{2}\|W^{\mathrm{v}+\mathrm{h}}_{\mathrm{can}}(U \setminus \mathcal{K})\|_1.$$

Then taking $M = M(p)$ as in Corollary 7.2, we conclude that

$$\|W^{\mathrm{v}}_{\mathrm{can}}(U^{10p} \setminus \mathcal{K})\|_1 \geqslant \frac{1}{2}\|W^{\mathrm{v}+\mathrm{h}}_{\mathrm{can}}(U \setminus \mathcal{K})\|_1, \tag{7.10}$$

provided $\|W_{\mathrm{can}}(U \setminus \mathcal{K})\|_1 > 2M$.

We are now ready to apply Theorem 7.3 to the map $f^{10p}\colon U^{10p} \to U$, where we let $A = B = \mathcal{K}$. We observe that all the hypotheses are satisfied, with $d = 2^{10}$, $D = 2^{10p}$, and $\epsilon = 1/2$. Moreover, by Lemma 7.1 and our assumption, we have

$$\|W^{\mathrm{h+v}}_{\mathrm{can}}(U^{10p} \smallsetminus \mathcal{K})\| \geqslant \|W^{\mathrm{h+v}}_{\mathrm{can}}(U \smallsetminus \mathcal{K})\| - C(p) \geqslant M'(p),$$

so we obtain, as our form of (7.6),

$$\|W^{\mathrm{v}}_{\mathrm{can}}(U \smallsetminus \mathcal{K})\| \geqslant (24 \cdot 2^{20})^{-1} \|W^{\mathrm{v}}_{\mathrm{can}}(U^{10p} \smallsetminus \mathcal{K})\|$$

The Lemma then follows from this and Lemma 7.2. □

8. Pseudo-quadratic-like maps and canonical renormalization

Definition 8.1. A *pseudo-polynomial-like map*[8] (or ψ-polynomial-like map) of degree d is a pair of holomorphic maps $(i, f)\colon \mathbf{U}' \to \mathbf{U}$ and an FJ-set $K \subset \mathbf{U}$ such that

(1) $\mathbf{U}'$ and $\mathbf{U}$ are disks,
(2) i is an immersion,
(3) f is a degree d branched cover,
(4) $i^{-1}(K) = f^{-1}(K)$, and, denoting this set by K',
(5) K' is connected.

We will call K the filled-Julia set of (i, f). Note that K' is connected if and only if all the branch values of f lie in K. Also note that, because f is proper, K' is an FJ-set. We will see that $K \subset \mathbf{U}$ is uniquely determined by $(i, f)\colon \mathbf{U}' \to \mathbf{U}$. We will also see that these conditions imply that $i\colon K' \to K$ is a bijection.

8.1. Finding a polynomial-like restriction.

We can now prove that every ψ-polynomial-like map F has a polynomial-like restriction, with modulus controlled by the modulus for F.

Theorem 8.1. *Let $F = (i, f)\colon \mathbf{U}' \to \mathbf{U}$ be a ψ-polynomial-like map of degree d with filled Julia set K. Then we can find $U \subset \mathbf{U}$ open, with $K \subset U$ such that, letting $U' \equiv f^{-1}U \subset \mathbf{U}'$,*

- *$f\colon U' \to U$ is a degree d holomorphic cover,*
- *$i|_{U'}$ is an embedding,*
- *$i(U') \subset U$, and*
- *$\mathrm{mod}(U \smallsetminus i(U')) \geqslant \mu(d, \mathrm{mod}(\mathbf{U} \smallsetminus K))$, where $\mu(d,\nu) > 0$ for $\nu > 0$.*

[8] Strictly speaking, we are defining a "ψ-polynomial-like map with connected Julia set".

Proof. Let us show that i is an embedding in a neighborhood of K'. To this end let us consider the annuli $\mathbf{U} \setminus K$ and $\mathbf{U}' \setminus K'$, and let us uniformize them by the round annuli:

$$\phi\colon \mathbb{A}(1,r) \to \mathbf{U} \setminus K, \quad \phi'\colon \mathbb{A}(1,r') \to \mathbf{U}' \setminus K', \quad \text{where } r' = r^{1/d}.$$

Let $I = \phi^{-1} \circ i \circ \phi'\colon \mathbb{A}(1,r') \to \mathbb{A}(1,r)$. Since $i(K') \subset K$, the map I is proper near the unit circle $\mathbb{T}$. By the Reflection Principle, it admits an analytic extension to a map $\mathbb{A}(1/r',r') \to \mathbb{A}(1/r,r)$ that restricts to a covering $\mathbb{T} \to \mathbb{T}$ of some degree $k > 0$. (We will use the same notation, I, for this extension).

Let us consider the geodesic $\omega' = \mathbb{T}_{\sqrt{r'}}$ in $\mathbb{A}(1,r')$, and let $\omega = I(\omega')$. Since for given $r, r' > 0$, the family of homotopically nontrivial holomorphic immersions $\mathbb{A}(1,r') \to \mathbb{A}(1,r)$ is compact, the (Euclidean) distance from ω to $\mathbb{T}$ is greater than $\rho = \rho(r',r) > 0$. Let $\Lambda = \mathbb{A}(1/\rho,\rho)$ and let Λ' be the component of $I^{-1}(\Lambda)$ containing $\mathbb{T}$. By the Argument Principle, the map $I\colon \Lambda' \to \Lambda$ is a covering of degree k.

In fact, $k = 1$. Indeed, let γ and γ' be the outer boundaries of Λ and Λ' respectively. Let $\Gamma = \phi(\gamma)$, $\Gamma' = \phi'(\gamma')$, and let Ω and Ω' be the disks bounded by Γ and Γ' respectively. Since $I\colon \gamma' \to \gamma$ is a covering of degree k, so is $i\colon \Gamma' \to \Gamma$. Hence $i\colon \Omega' \to \Omega$ is a branched covering of degree k. If $k > 1$, i would necessarily have a critical point, contradicting the assumption that it is an immersion.

Furthermore, let $F = \phi^{-1} \circ f \circ \phi'\colon \mathbb{A}(1,r') \to \mathbb{A}(1,r)$. Since it is a covering map of degree d, $F(z) = z^d$. Hence $F^{-1}(\mathbb{A}(1,\rho)) = \mathbb{A}(1,\rho')$, where $\rho' = \rho^{1/d}$. Notice that the hyperbolic distance from $\mathbb{T}_\rho$ to $\mathbb{T}$ in $\mathbb{A}(1/r,r)$ is equal to the hyperbolic distance from $\mathbb{T}_{\rho'}$ to $\mathbb{T}$ in $\mathbb{A}(1/r',r')$. Since the map $I\colon \mathbb{A}(1/r',r') \to \mathbb{A}(1/r,r)$ contracts the respective hyperbolic metrics by a uniform factor, we have: $I(\mathbb{T}_{\rho'}) \subset \mathbb{A}(1,\lambda\rho)$, where $\lambda = \lambda(r,d,\rho) < 1$. (Actually $\lambda = \lambda(r,d)$ because $\rho = \rho(r^{1/d},r)$.) Putting all these together, we conclude that

$$W' \equiv I \circ F^{-1}(\mathbb{A}(1,\rho)) \subset \mathbb{A}(1,\lambda\rho) \equiv W.$$

and moreover, the map $F \circ I^{-1}\colon W' \to W$ is a covering of degree d.

Letting

$$U' = K \cup \phi(W'), \quad U = K \cup \phi(W),$$

we see that the map $f \circ i^{-1}\colon U' \to U$ is polynomial-like of degree d and $\mathrm{mod}(U \setminus U') \geqslant \dfrac{1}{2\pi} \log \lambda$, where λ depends only on d and $\mathrm{mod}(\mathbf{U} \setminus K)$. ☐

In the proof we have that $i\colon U' \to i(U)$ is a homemorphism, and hence $i\colon K' \to K$ is a bijection.

The above estimates with poincare metric also show that if $\hat{K} \subset \mathbf{U}$ is compact, and $K \subset \hat{K}$, and $(\hat{K}_i)_{i=0}^{\infty}$ is defined by $\hat{K}_0 = \hat{K}$ and $\hat{K}_{i+1} = i(f^{-1}(\hat{K}))$, then $K = \bigcap_{i=0}^{\infty} \hat{K}_i$. Therefore K is uniquely determined by F.

8.2. **Iterating a ψ-polynomial-like map.** It is not immediately obvious how to iterate a ψ-polynomial-like map. We will briefly review a few simple constructions in the categories of topological spaces and Riemann surfaces, and then use this to construct the iterations.

8.2.1. *A few simple constructions.* Given topological spaces X, Y, Z and continuous maps $f: X \to Z$ and $g: Y \to Z$, we can form the *pullback* $E = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$, along with maps $\pi_X: E \to X$ and $\pi_Y: E \to Y$. With this notation, we observe that π_Y can inherit certain properties from f (and π_X from g).

Lemma 8.2. *We have*

(1) $f \circ \pi_X = g \circ \pi_Y$.
(2) *If f is proper, then π_Y is proper.*
(3) *If g is a local homeomorphism, then π_X is a local homeomorphism.*
(4) *If g is injective (or surjective, or bijective), then π_X is injective (or surjective, or bijective).*

Proof. Statement (1) is immediate.

For Statement (2), suppose $B \subset Y$ is compact. Then

$$\begin{aligned}\pi_Y^{-1}(B) &= \{(x, y) \in X \times B \mid g(x) = h(y)\} \\ &= \{(x, y) \in f^{-1}(g(X)) \times B \mid g(x) = h(y)\}\end{aligned}$$

which is a closed subset of a compact set.

For Statement (3), take $(x, y) \in E$. Then we can find $V \subset Y$ such that $y \in V$ and $g|_V: V \to g(V)$ is a homeomorphism. Let $\hat{V} = \pi_Y^{-1} V$. Then $\pi_X|_{\hat{V}}$ is a homeomorphism, because its inverse is $x \mapsto (x, (g|V)^{-1}(f(x)))$.

Statement (4) is straightforward (and belongs to the category of sets) and is left to the reader. ☐

It will also help to have a few similar observations about composition.

Lemma 8.3. *Suppose that U, V, W are Riemann surfaces and $g: U \to V$ is continuous and $f: V \to W$ are holomorphic. Then*

(1) *if f and $f \circ g$ are holomorphic immersions, then g is a holomorphic immersion, and*
(2) *if $f \circ g$ is proper, then g is proper.*

Proof. For (1), we can locally write $g = (f \circ g)^{-1} \circ f$ for a suitable local branch of $(f \circ g)^{-1}$.

For (2), we suppose $A \subset V$ is compact. Then

$$g^{-1}(A) \subset g^{-1}(f^{-1}(f(A)) = (f \circ g)^{-1}(f(A))$$

is compact. □

We can now prove

Lemma 8.4. *Suppose that U_f, U_i, and U_i are Riemann surfaces, and $f\colon U_f \to U$ is holomorphic and proper, and $i\colon U_i \to U$ is a holomorphic immersion. Then we can find a Riemann surface E and $\hat{i}\colon E \to U_f$ and $\hat{f}\colon E \to U_i$ such that*

(1) $\hat{f} \circ i = \hat{i} \circ f$,
(2) $\hat{i}$ *is holomorphic immersion,*
(3) $\hat{f}$ *is holomorphic and proper, and* $\deg \hat{f} = \deg f$.

Moreover, these objects are unique, in that if E', $\hat{f}'$ and $\hat{i}'$ satisfy Condition 1–3 above, then there is a holomorphic isomorphism $h\colon E' \to E$ such that $\hat{i}' = \hat{i} \circ h$ and $\hat{f}' = \hat{f} \circ h$.

Proof. We can construct E as a topological space as the pullback described above; we relabel π_{U_i} and π_{U_f} as $\hat{f}$ and $\hat{i}$. Then Condition (1) holds, and $\hat{f}$ is proper, and $\hat{i}$ is a local homeomorphism, by Lemma 8.2. We can then pull back the Riemann surface structure on U_f to E by $\hat{i}$ to make $\hat{i}$ holomorphic. Then $\hat{f}$ is holomorphic, because we can write it locally as $\hat{i} \circ f \circ i^{-1}$, for a suitable local branch of i^{-1}.

For a generic $x \in U_i$, the equation $f(y) = i(x)$ has $\deg f$ solutions in y. Therefore $\deg \hat{f} = \deg f$.

Now suppose that E', $\hat{f}'$, and $\hat{i}'$ satisfy Conditions 1–3 above. Then there is a continuous map $h\colon E' \to E$ such that $\hat{i}' = \hat{i} \circ h$ and $\hat{f}' = \hat{f} \circ h$. We can apply Lemma 8.3 to see that h is a holomorphic immersion, and h is proper. Therefore h is a finite-sheeted covering, and $\deg \hat{f}' = (\deg \hat{f})(\deg h)$, so $\deg h = 1$, and h is a holomorphic isomorphism. □

Remark 8.1. While the pullback of $(f, g)\colon (U, V) \to W$ always exists, we have chosen to construct it only in a simple case where we not only have universality, but a strong form of uniqueness.

8.2.2. *Iteration.* We can now construct our iterations of a ψ-polynomial-like map.

Theorem 8.5. *Let $(i, f)\colon (\mathbf{U}', K') \to (\mathbf{U}, K)$ be a degree d ψ-polynomial-like map. Then there exists an essentially unique degree d ψ-polynomial-like-map $(i_1, f_1)\colon (\mathbf{U}'', K'') \to (\mathbf{U}', K')$ such that $i \circ f_1 = f \circ i_1$.*

Proof. By Lemma 8.4. there is a Riemann surface $\mathbf{U}''$ with a holomorphic immersion $i_1: \mathbf{U}'' \to \mathbf{U}'$ and a degree d holomorphic branched cover $f_1: \mathbf{U}'' \to \mathbf{U}'$ such that $i \circ f_1 = f \circ i_1$. (Lemma 8.4 also says that these data are essentially unique.) Moreover, for $z \in \mathbf{U}''$,

$$i_1(z) \in K' \iff (f \circ i_1)(z) \in K \iff (i \circ f_1)(z) \in K \iff f_1(z) \in K'.$$

So $i_1^{-1}(K') = f_1^{-1}(K')$; we denote this set by K''. We observe that $(f_1, i_1): K'' \to K'$ is the topological pullback of $i: K' \to K$ and $f: K' \to K$, and hence i_1 is bijective by Lemma 8.2.

We claim that $\mathbf{U}''$ is a disk. To see this, we observe that $z \in \mathbf{U}''$ is a critical point of f_1 if and only if it is a critical point of $i \circ f_1 = f \circ i_1$, and this occurs exactly when $i_1(z)$ is a critical point of f. Now f has exactly $d-1$ critical points (by Riemann-Hurwitz) and they all lie in K', and $i_1: K'' \to K'$ is a bijection, so f_1 has exactly $d-1$ critical points and hence $\mathbf{U}''$ is a disk (by Riemann-Hurwitz).

The branch values of f_1 map by i to branch values of f, so all these branch values lie in K'. Therefore K'' is connected, and we have verified that $(i_1, f_1): (\mathbf{U}'', K'') \to (\mathbf{U}', K')$ is a degree d ψ-polynomial-like map. $\square$

Let us relabel (i, f) as $(i_0, f_0): (\mathbf{U}^1, K^1) \to (\mathbf{U}^0, K^0)$. We can then inductively apply Theorem 8.5 to construct degree d ψ-polynomial-like maps $(i_n, f_n): (\mathbf{U}^{n+1}, K^{n+1}) \to (\mathbf{U}^n, K^n)$ such that $i_n \circ f_{n+1} = f_n \circ i_{n+1}$ for all n. By a very slight abuse of notation[9], we will often just denote i_n, f_n, and K_n as i, f, and K. This then allows us to refer to the iterations $i^n: \mathbf{U}^{m+n} \to \mathbf{U}^m$ and $f^n: \mathbf{U}^{m+n} \to \mathbf{U}^m$ (where we usually take $m = 0$).

8.3. **Canonical Renormalization.**

We now describe a form of renormalization in the category of ψ-quadratic-like maps that has the crucial property of preserving certain moduli and hyperbolic lengths.

8.3.1. *Localization.* Suppose that S is a Riemann surface, and $A \subset S$ is an FJ-set. The *localization* of S *at* A is a disk X with a holomorphic immersion $\pi: X \to S$ such that $\pi: \pi^{-1}(A) \to A$ is a bijection and $\pi: X \setminus \pi^{-1}(A) \to S \setminus A$ is a covering.

To construct X (and π), we find a neighborhood U of A that is a disk, and take the cover $\pi': X' \to S \setminus A$ corresponding to the fundamental group of the annulus $U \setminus A$. Then there is a natural lift of $U \setminus A$ to X'; we let X be the quotient of the disjoint union of U and X' by identifying points in $U \setminus A$ with their image by this lift. It is clear that X is a Riemann surface, and it is a disk because it is the identification

[9] or clever definition of i, f, and K on the disjoint union $\bigcup_{n \in \mathbb{N}} \mathbf{U}^n$

of a disk with an annulus along a sub-annulus that is a deformation retract. We let $\pi(x) = \pi'(x)$ for $x \in X'$, and $\pi(x) = x$ for $x \in U$. As a general rule of notation, we will let $\hat{A} = \pi^{-1}(A)$.

Now suppose that S' and S are Riemann surfaces, $A' \subset S'$ and $A \subset S$ are FJ-sets, and $g\colon (S', A') \to (S, A)$ is holomorphic map such that A' is a component of A; we let $A'' = g^{-1}(A) \setminus A'$, and $S'' = S' \setminus A''$. We let $\pi'\colon X' \to S \setminus A''$ be localizations of S'' at A', and $\pi\colon X \to S$ and S be the localization of S at A. Then *we leave it to the reader to verify* that there is a unique map ("lift") $\hat{g}\colon X' \to X$ such that $g \circ \pi = \pi \circ \hat{g}$. We will always have $\hat{g}^{-1}(\hat{A}) = \hat{A}'$. If g is an immersion, then $\hat{g}$ is an immersion. The conditions on g, A', A imply that $g\colon A' \to A$ is a branched cover of finite degree $d_{A'}$, in the sense that almost every point of A has $d_{A'}$ preimages in A', and there are simply connected neighborhoods $U' \supset A'$ and $U \supset A$ such that $g\colon U' \to U$ is a degree $d_{A'}$ branched cover. We then observe that if $g\colon S' \to S$ is a branched cover[10] branched only over A, then $\hat{g}\colon X' \to X$ is a branched cover branched only over A, and hence has degree $d_{A'}$. This is indeed the main idea of localization, in that we can localize a map of much higher degree to get a map of a lower degree, equal to a local degree.

8.3.2. *Dynamical Localization.* Now let us suppose that

- $(i, f)\colon (\mathbf{U}', K') \to (\mathbf{U}, K)$ is a ψ-polynomial-like map of degree d,
- $A \subset K$ is an FJ-set (and let $A' = i^{-1}(A)$),
- $B \subset K'$ is connected and disjoint from A (and let $B' = i^{-1}(B)$),
- A' is a component of $f^{-1}(A)$,
- $B' \subset f^{-1}(B)$, and
- all the branch values of f lie in $A \cup B$.

We observe that $i(\mathbf{U}' \setminus f^{-1}(B')) \subset \mathbf{U} \setminus B$, and $f\colon \mathbf{U}' \setminus f^{-1}(B') \to \mathbf{U} \setminus B$ is a branched cover branched only over A. We let $\pi\colon X \to \mathbf{U} \setminus B$ be the localization of $\mathbf{U} \setminus B$ with respect to A, and $\pi'\colon X' \to \mathbf{U}' \setminus f^{-1}(B')$ be the localization of $\mathbf{U}' \setminus f^{-1}(B')$ with respect to A'. Then, as observed above, we can lift i and f to $(\hat{i}, \hat{f})\colon X' \to X$, and $\hat{i}$ is an immersion and $\hat{f}$ is a branched cover, with degree equal to the local degree $d_{A'}$ of f at A'. So we have shown that $(\hat{i}, \hat{f})\colon (X', A') \to (X, A)$ is a ψ-polynomial-like map, of degree $d_{A'}$. We call it the *dynamical renormalization* of f *at A with the removal of B*. Moreover, since $\pi\colon X \setminus \hat{A} \to \mathbf{U} \setminus B \setminus A$ is a covering, the poincare length of the core curve in $X \setminus \hat{A}$ is equal

[10]While we will use this only when g is a finite degree branched cover, it also applies whenever g has path lifting (which of course is non-unique at branch points).

to the poincare length of the closed geodesic in $\mathbf{U} \setminus (A \cup B)$ that goes around A.

8.3.3. *Canonical Renormalization.* Now suppose that $(i, f) \colon (\mathbf{U}^1, K') \to (\mathbf{U}^0, K)$ is a degree 2 ψ-polynomial-like map (which we will call a ψ-quadratic-like map). Suppose that f has a quadratic-like restriction that is primitively renormalizable of period p. Then as before we let $K_p \subset K$ be the central small Julia set of the renormalization, and $\mathcal{K}$ be the union of small Julia sets. We let $K'_p = i^{-1}(K_p)$ and so forth. We let $\mathcal{K}^+ = \mathcal{K} \setminus \mathcal{K}_p$. We observe that $K_p, \mathcal{K}^+$ satisfy all the conditions for A, B in the previous section, when we replace (i, f) with its iteration $(i^p, f^p) \colon \mathbf{U}^p \to \mathbf{U}^0$. So we then let $(\hat{i}, \hat{f}) \colon (X', \hat{K}'_p) \to (X, \hat{K}_p)$ be the dynamical renormalization at K_p with the removal of $\mathcal{K}_p$. We call this the *canonical renormalization* of f (or really (i, f)) of period p.

As we did with polynomial-like maps, if $(i, f) \colon (\mathbf{U}', K') \to (\mathbf{U}, K)$ is a ψ-quadratic-like map, then we let $\gamma(f)$ be the core curve of $\mathbf{U} \setminus K$. Moreover, if f is p-renormalizable, we let γ_p be the closed geodesic around K_p in $\mathbf{U} \setminus \mathcal{K}_p$. We can now prove the a general version of the two "crucial properties" that we stated in Section 1.

Theorem 8.6. *Suppose that $(i, f) \colon \mathbf{U}' \to \mathbf{U}$ is a ψ-quadratic-like map that is primitively renormalizable with periods p and also pq (with $p, q > 1$). Let $(j, g) \colon \mathbf{V}' \to \mathbf{V}$ be the canonical renormalization of f of period p. Then*

(1) $|\gamma(g)| = |\gamma_p(f)|$, *and*
(2) $|\gamma_q(g)| \geqslant |\gamma_{pq}(f)|$.

Proof. Statement (1) has already been observed in the definition of dynamical localization.

To prove (2), we consider the map $\pi \colon \mathbf{V} \to \mathbf{U}$ that maps K_g to $\mathcal{K}_p(0)$ and $\mathbf{V} \setminus K_g$ to $\mathbf{U} \setminus \mathcal{K}_p$. We claim that $\pi(\mathbf{V} \setminus \mathcal{K}_q(g)) \subseteq \mathbf{U} \setminus \mathcal{K}_{pq}$; this follows because $\pi(\mathbf{V} \setminus K_g) \subseteq \mathbf{U} \setminus \mathcal{K}_p \subseteq \mathbf{U} \setminus \mathcal{K}_{pq}$ and $\pi(K_g \setminus \mathcal{K}_q(g)) = \mathcal{K}_p(0) \setminus \mathcal{K}_{pq}$. Statement (2) then follows from the Schwarz Lemma. ☐

9. *A priori* and beau bounds for bounded-primitive renormalization

Let us first prove the generalization of Theorem 3.1 for ψ-quadratic-like maps:

Theorem 9.1. *Let $(i, f) \colon \mathbf{U}' \to \mathbf{U}$ be a primitively renormalizable ψ-quadratic-like-map with period p, and let $\mathcal{K}$ be its cycle of little Julia sets. Then there exists $M = M(p)$ such that: If $\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|_1 > M$ then*

$$\|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|_1 \geqslant C^{-1} \|W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|_1,$$

with an absolute constant $C > 0$.

Proof. We let $\mathbf{U}^0 = \mathbf{U}$, $\mathbf{U}^1 = \mathbf{U}'$, $f_0 = f$, and $i_0 = i$. As described in Section 8, we can form disks $\mathbf{U}^n$ and ψ-quadratic-like maps $(i, f)\colon \mathbf{U}^{n+1} \to \mathbf{U}^n$, such that all these ψ-quadratic-like maps restrict to the same quadratic-like germ. In particular, we the filled-Julia set $K^n \subset U^n$ and also the cycle $\mathcal{K}^n \subset K^n$ of small Julia sets; by a slight abuse of notation, we will just denote these by K and $\mathcal{K}$. We then have (with the slight abuse of notation described at the end of Section 8.1),

(1) a covering $f\colon \mathbf{U}^{n+1} \setminus f^{-1}\mathcal{K} \to \mathbf{U}^n \setminus \mathcal{K}$,
(2) an inclusion $\mathbf{U}^{n+1} \setminus f^{-1}\mathcal{K} \hookrightarrow \mathbf{U}^{n+1} \setminus \mathcal{K}$, and
(3) an immersion $i\colon \mathbf{U}^{n+1} \setminus \mathcal{K} \to \mathbf{U}^n \setminus \mathcal{K}$.

The one significant change from the quadratic-like to the ψ-quadratic-like setting is the third map, which was previously an embedding, and is now an immersion.

Now we need only go through Sections 5, 6, and 7 and see what needs to be changed. The main observation for Sections 5 and 6 is that the results of these sections depend only on properties (i) and (ii) that are stated in Propostion 5.2; they use no other facts about the geometry of f. We therefore need only verify these two properties in our ψ-quadratic-like setting. Property (ii) follows just as before, because we use only the covering $f\colon \mathbf{U}^{n+1} \setminus f^{-1}\mathcal{K} \to \mathbf{U}^n \setminus \mathcal{K}$ and the inclusion $\mathbf{U}^{n+1} \setminus f^{-1}\mathcal{K} \hookrightarrow \mathbf{U}^{n+1} \setminus \mathcal{K}$. For Property (i) we need only observe that the immersion $i\colon \mathbf{U}^{n+1} \setminus \mathcal{K} \to \mathbf{U}^n \setminus \mathcal{K}$ is still proper on the ends corresponding to $\mathcal{K}$, and Lemma 2.5 applies to immersions as well as embeddings.

Finally, we consider the results of Section 7. For Lemma 7.1, we can replace the inclusion $U^n \setminus \mathcal{K} \subset U \setminus \mathcal{K}$ with the immersion $i_n\colon \mathbf{U}^n \setminus \mathcal{K} \to \mathbf{U} \setminus \mathcal{K}$; this is still homotopic to a homeomorphism, so we can apply Theorem 2.17 as before. In a similar vein, examining the proof of Theorem 3.1 in Section 7.3, we find that we need only replace $f^n\colon U^n \to U$ with $f^n\colon \mathbf{U}^n \to \mathbf{U}$, where now this f^n is the iteration described at the end of Section 8.1. □

We can also state and prove the version of Theorem 1.2 for a primitively p-renormalizable ψ-quadratic-like map $\mathbf{f} \equiv (f, i)\colon \mathbf{U}' \to \mathbf{U}$, with K, $\mathcal{K}$ defined as before, and geodesics γ around K and γ_p around the central small Julia set in $\mathbf{U} \setminus \mathcal{K}$:

Theorem 9.2. *There exists $\epsilon > 0$, such that for all p there exists M: Suppose that is a ψ-quadratic-like map that is primitively p-renormalizable. Then, with the notation as described just above,*

$$|\gamma_p| > M \implies |\gamma| \geqslant \epsilon p\, |\gamma_p|. \tag{9.3}$$

Proof. This follows from Theorem 9.1 in exactly the same way that Theorem 1.2 follows from Theorem 3.1. $\square$

We can now prove our "if it's bad now, it was worse earlier" statement for geodesics around the central small Julia sets. We return to our original setting, where f is a quadratic polynomial that is primitively renormalizable of periods $p_1 \mid p_2 \mid \ldots$, with $1 < p_{k+1}/p_k \leqslant B$. We let γ_k be the geodesic around $\mathcal{K}_{p_k}(0)$ in $\mathbb{C} \smallsetminus \mathcal{K}_{p_k}$.

Theorem 9.4. *There exists l such that for all B there exists M: Suppose that f is infinitely B-bounded primitively renormalizable (with notation as above). Then, for all n,*

$$|\gamma_{n+l}| > M \implies |\gamma_n| > 2|\gamma_{n+l}|. \tag{9.5}$$

Proof. Choose l such that $2^l\epsilon > 2$, where ϵ is the ϵ from Theorem 9.2. Let $P = B^l$, and choose M such that M is sufficiently large in Theorem 9.2, for all $p \leqslant P$. We claim that (9.5) holds for these choices (and all n). To see this, suppose that $|\gamma_{n+l}| > M$, and let $\mathbf{g} \equiv (i, g)\colon \mathbf{U}' \to \mathbf{U}$ be the canonical renormalization of f of period p_n. Let $q = p_{n+l}/p_n$; then $\mathbf{g}$ is q-renormalizable, and by Theorem 8.6, we have

$$|\gamma_n| = |\gamma(g)| \text{ and } |\gamma_q(g)| \geqslant |\gamma_{n+l}| > M. \tag{9.6}$$

So we can apply Theorem 9.2 to obtain

$$|\gamma(g)| \geqslant \epsilon q|\gamma_q(g)| \geqslant \epsilon 2^l|\gamma_q(g)| \geqslant 2|\gamma_q(g)|,$$

which combined with (9.6) yields (9.5). $\square$

We can also observe

Lemma 9.7. *For all P there is an $M \equiv M(P)$ such that if f is a primitively renormalizable quadratic polynomial of period $p \leqslant P$, then $|\gamma_p| \leqslant M$.*

Proof. This follows immediately from Theorem 1.2, taking the maximum over all $p \leqslant P$, and noting that $|\gamma(f)| = 0$ for a polynomial. $\square$

Lemma 9.7 can also be proven by a standard compactness argument. The following is probably known, but we include it for completeness:

Theorem 9.8. *Suppose f is primitively p-renomalizable quadratic polynomial, and $|\gamma_p| \leqslant M$. Then f has a (Douady-Hubbard) renormalization of period p and modulus at least $m > 0$, for $m \equiv m(M)$.*

Proof. Let $V_p \subset \mathbb{C}$ be the (open) disk bounded by γ_p; and let V'_p be V_p pulled back along $\mathcal{K}_p$, so V'_p is the unique component of $f^{-p}V_p$ that contains K_p. Then $f^p\colon V'_p \to V_p$ is a degree 2 cover; we claim that $\overline{V'_p} \subset V_p$, and that $\operatorname{mod}(V_p, V'_p) > m(M)$; this would then complete the proof of the Theorem. We let $A_p = V_p \setminus K_p$ and $A'_p = V'_p \setminus K_p$; we need only show the outer boundary $\partial_{\mathrm{out}} A'_p$ of A'_p lies in A_p, with a definite modulus between $\partial_{\mathrm{out}} A'_p$ and $\partial_{\mathrm{out}} A_p$.

Now let $(i,g)\colon U'_p \to U_p$ be the canonical renormalization of f of period p. Then we will also denote by γ_p and A_p their lifts to U_p and likewise for γ'_p and A_p in U'_p; then $g\colon A'_p \to A_p$ and $g\colon \gamma'_p \to \gamma_p$ are degree 2 coverings. We can give $H_p \equiv U_p \setminus K_p$ the poincare metric from doubling along its inner ideal boundary, and likewise $H'_p \equiv U'_p \setminus K'_p$. We then observe that $i\colon H'_p \to H_p$ is a contraction by the Schwarz lemma; it is strictly contracting at each point, because it is not a homeomorphism. The inner boundaries of A'_p and A_p are the same of those of H'_p and H_p, and the outer boundaries for A'_p and A_p (in H'_p and H_p) are the same constant distance from the inner boundaries (because γ_p and γ'_p are core curves for H_p and H'_p). A simple compactness argument shows that i is uniformly contracting on A'_p, depending only on $\mathrm{mod}(A_p) = \pi/(2|\gamma_p|)$. It follows that $\overline{i(A'_p)} \subset A_p$, and $\mathrm{mod}(A_p, i(A'_p)) \geqslant m \equiv m(M) > 0$. We then have the desired inclusion and bound on modulus for our original A'_p and A_p. □

We can now conclude the *a priori* bounds for bounded-primitive type. We return to the same setting as in Theorem 9.4.

Theorem 9.9. *For all B there exists $M, m > 0$: Suppose f is infinitely B-bounded primitively renormalizable, with periods $p_1 \mid p_2 \ldots$. Then, for all n,*

(1) *$|\gamma_n| \leqslant M$, and*
(2) *There is a renormalization of f of period p_n with modulus at least m.*

Proof. Let us first prove Statement (1). We take l as in Theorem 9.4, and given B, we take M to be the greater value of $M(B)$ in Theorem 9.4 and $M(B^l)$ in Lemma 9.7. Now suppose there were an n with $|\gamma_{p_n}| > M$; we take the least such value of n. If $n \leqslant l$ then $p_n \leqslant B^l$, and we contradict Lemma 9.7. If $n > l$, then $|\gamma_{p_{n-l}}| \geqslant 2|\gamma_{p_n}|$ by Theorem 9.4, so n was not minimal. This concludes the proof of (1). Statement (2) then follows from (1) and Theorem 9.8. □

10. Appendix A: Extremal length and width

There is a worth of sources containing background material on extremal length, see, e.g., the book of Ahlfors [A1]. We will briefly summarize the necessary minimum (see also the Appendix of [KL1]).

10.1. **Definitions.** Let $\mathcal{G}$ be a family of curves on a Riemann surface U. Given a (measurable) conformal metric $\mu = \mu(z)|dz|$ on U, let

$$\mu(\mathcal{G}) = \inf_{\gamma\in\mathcal{G}} \mu(\gamma).$$

where $\mu(\gamma)$ stands for the μ-length of γ. The length of $\mathcal{G}$ with respect to μ is defined as

$$\mathcal{L}_\mu(\mathcal{G}) = \frac{\mu(\mathcal{G})^2}{\mathrm{area}_\mu(U)},$$

where area_μ is an area form of μ. Taking the supremum over all conformal metrics μ, we obtain the *extremal length* $\mathcal{L}(\mathcal{G})$ of the family $\mathcal{G}$.

The *extremal width* is the inverse of the extremal length:

$$\mathcal{W}(\mathcal{G}) = \mathcal{L}^{-1}(\mathcal{G}).$$

It can be also defined as follows. Consider all conformal metrics μ such that $\mu(\gamma) \geqslant 1$ for any $\gamma \in \mathcal{G}$. Then $\mathcal{W}(\mathcal{G})$ is the infimum of the areas $\mathrm{area}_\mu(U)$ of all such metrics.

10.2. **Electric circuits laws.** We say that a family $\mathcal{G}$ of curves *overflows* a family $\mathcal{G}'$ if any curve of $\mathcal{G}$ contains some curve of $\mathcal{G}$. Let us say that $\mathcal{G}$ *disjointly overflows* two families, $\mathcal{G}_1$ and $\mathcal{G}_2$, if any curve of $\gamma \in \mathcal{G}$ contains the disjoint union $\gamma_1 \sqcup \gamma_2$ of two curves $\gamma_i \in \mathcal{H}_i$.

The following crucial properties of the extremal length and width show that the former behaves like the resistance in electric circuits, while the latter behaves like conductance.

Series Law. *Let $\mathcal{G}$ be a family of curves that disjointly overflows two other families, $\mathcal{G}_1$ and $\mathcal{G}_2$. Then*

$$\mathcal{L}(\mathcal{G}) \geqslant \mathcal{L}(\mathcal{G}_1) + \mathcal{L}(\mathcal{G}_2),$$

or equivalently,

$$\mathcal{W}(\mathcal{G}) \leqslant \mathcal{W}(\mathcal{G}_1) \bigoplus \mathcal{W}(\mathcal{G}_2).$$

Parallel Law. *For any two families $\mathcal{G}_1$ and $\mathcal{G}_2$ of curves we have:*

$$\mathcal{W}(\mathcal{G}_1 \cup \mathcal{G}_2) \leqslant \mathcal{W}(\mathcal{G}_1) + \mathcal{W}(\mathcal{G}_2).$$

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are contained in two disjoint open sets, then

$$\mathcal{W}(\mathcal{G}_1 \cup \mathcal{G}_2) = \mathcal{W}(\mathcal{G}_1) + \mathcal{W}(\mathcal{G}_2)$$

10.3. **Transformation rules.** Both extremal length and extremal width are conformal invariants. More generally, we have:

Lemma 10.1. *Let $f\colon U \to V$ be a holomorphic map between two Riemann surfaces, and let $\mathcal{G}$ be a family of curves on U. Then*

$$\mathcal{L}(f(\mathcal{G})) \geqslant \mathcal{L}(\mathcal{G}).$$

See Lemma 4.1 of [KL1] for a proof.

Corollary 10.2. *Under the circumstances of the previous lemma, let $\mathcal{H}$ be a family of curves in V satisfying the following lifting property: any curve $\gamma \in \mathcal{H}$ contains an arc that lifts to some curve in $\mathcal{G}$. Then $\mathcal{L}(\mathcal{H}) \geqslant \mathcal{L}(\mathcal{G})$.*

Proof. The lifting property means that the family $\mathcal{H}$ overflows the family $f(\mathcal{G})$. Hence $\mathcal{L}(\mathcal{H}) \geqslant \mathcal{L}(f(\mathcal{G}))$, and the conclusion follows. □

Corollary 10.3. *Let Q and Q' be two quadrilaterals, and let $e\colon Q \to Q'$ be a holomorphic map that maps the horizontal sides of Q to the horizontal sides of Q'. Then $\mathcal{W}(Q) \leqslant \mathcal{W}(Q')$.*

Proof. Let $\mathcal{G}$ (resp. $\mathcal{G}'$) be the family of horizontal curves in Q (resp., in Q'). Since the horizontal sides of Q are mapped to the horizontal sides of Q', these families satisfy the lifting property of the previous Corollary. Hence $\mathcal{L}(\mathcal{G}) \leqslant \mathcal{L}(\mathcal{G}')$, and we are done. □

Lemma 10.4. *Let $f\colon U \to V$ be a branched covering between two compact Riemann surfaces with boundary. Let A be an archipelago in U, $B = f(A)$, and assume that $f\colon A \to B$ is a branched covering of degree d. Then*

$$\mathrm{mod}(V, B) \geqslant d \,\mathrm{mod}(U, A).$$

See Lemma 4.3 of [KL1] for a proof.

Given two compact subsets A and B in a Riemann surface S, let $\mathcal{W}_S(A, B)$ stand for the *extremal width* between them, i.e., the extremal width of the family of curves connecting A to B.

Lemma 10.5. *Let S and S' be two compact Riemann surfaces with boundary. and let $f\colon S \to S'$ be a branched covering of degree D. Let $S' = A' \sqcup B'$, where A' and B' are closed, and let $A = f^{-1}(A')$, $B = f^{-1}(B')$. Then*

$$\mathcal{W}_S(A, B) = D\, \mathcal{W}(A', B').$$

See [A1] for a proof. It makes use of the fact that the extremal width $\mathcal{W}(A, B)$ is achieved on the *harmonic foliation* $\mathcal{F} = \mathcal{F}_S(A, B)$ connecting A and B, i.e., the gradient foliation of the harmonic function ω vanishing on 0 and equal to 1 on B. Hence $\mathcal{W}_S(A, B)$ is equal to the l_1-norm of the associated WAD $W_{\mathcal{F}}$.

10.4. **Non-Intersection Principle.** The following important principle says that two wide quadrilaterals cannot go non-trivially one across the other:

Lemma 10.6. *Let us consider two quadrilaterals, Q_1 and Q_2, endowed with the vertical foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. If $\mathcal{W}(Q_i) \geqslant 1$ then*
- *either there exists a pair of disjoint leaves γ_i of the foliations $\mathcal{F}_i$;*
- *or $\mathcal{W}(Q_1) = \mathcal{W}(Q_2) = 1$, and the rectangles perfectly match in the sense that the vertical sides of one of them coincide with the horizontal sides of the other.*

Proof. Assume that the first option is violated, so that, every leaf of $\mathcal{F}_1$ crosses every leaf of $\mathcal{F}_2$. Let us uniformize our rectangles by standard rectangles, $\phi_i: \mathbf{Q}(a_i) \to Q_i$, where $a_i \geqslant 1$. Let λ_i be the Euclidean metrics on the $\mathbf{Q}(a_i)$, and let $\mu_i = (\phi_i)_*(\lambda_i)$. Since every vertical leave γ of Q_2 crosses every vertical leaf of Q_1, $\mu_1(\gamma) \geqslant a_1$. Hence $a_2 = \mathcal{W}(Q_2) \leqslant 1/a_1$, which implies that $a_1 = a_2 = 1$. Thus, both $\mathbf{Q}(a_i) \equiv \mathbf{Q}$ are the squares.

Moreover, μ_1 must be the extremal metric for $\mathcal{F}_2$. Since the extremal metric is unique (up to scaling), we conclude that $\mu_1 = \mu_2$. Hence $\phi_2^{-1} \circ \phi_1: \mathbf{Q} \to \mathbf{Q}$ is the isometry of the square, and the conclusion follows. ☐

11. Appendix B: Elements of electric engineering

11.1. **Potentials, currents and conductances.** *Electric circuit* $\mathcal{C}$ is

- A connected graph Γ with two marked vertices (*battery*). We let $\mathcal{E} = \mathcal{E}(\Gamma)$ be the set of edges of Γ, $\mathcal{V} = \mathcal{V}(\Gamma)$ be the set of its vertices, and $\mathcal{B} = \{a, b\} \subset \mathcal{V}$ be the battery.
- A *conductance* vector $W = \sum_{e \in \mathcal{E}} W(e)\, e$, where $W(e) > 0$ for all $e \in \mathcal{E}$.

The edges of Γ are interpreted as *resistors* of the circuit with conductances $W(e)$. The inverse numbers $R(e) = W(e)^{-1}$ as their *resistances.*

We write $x \sim y$ for two neighboring vertices of Γ. The vertices of $\mathcal{V} \smallsetminus \mathcal{B}$ will also be called *internal.* If we forget the battery $\mathcal{B}$, we call the circuit "unplugged".

A *potential distribution* on $\mathcal{C}$ is a function $U: \mathcal{V} \to \mathbb{R}$.

Let $\mathcal{E}^*$ stand for the set of all possible *oriented* edges $\mathcal{E}$. An oriented edge e^* can be written as $[x, y]$ where x, y are its endpoints ordered according to the orientation of e. We also write $-e^*$ for the edge e^* with the opposite orientation.

A *current* on $\mathcal{C}$ is an odd function $I: \mathcal{E}^* \to \mathbb{R}$, i.e., $I(-e^*) = -I(e^*)$.

A potential U induces potential differences

$$dU[x, y] = U(y) - U(x)$$

on the oriented edges of Γ (negative *coboundary* of U). It forces current

$$I(e^*) = -W(e)\, dUe^*, \quad e^* \in \mathcal{E}^*,$$

to run through the resistors. The energy $E(e)$ of this current is equal to $I(e)\, dU(e) = W(e)\, (dU(e))^2$ (note that it is independent of the orientation of e), so that, the total energy of this potential distribution is equal to

$$E = E(U) = \sum_{e \in \mathcal{E}} W(e)\, (dU(e))^2.$$

The quantity $\mathbf{U} = U(a) - U(b)$ is called the *battery potential.*

11.2. **Equilibrium.** Let us define the *boundary* of the current I as the following function on the vertices:

$$\partial I(x) = \sum_{y \sim x} I[x, y].$$

We say that the potential distribution is in *equilibrium*, U_{eq}, if the current is *conserved*, i.e., $\partial I \equiv 0$ on $\mathcal{V} \smallsetminus \mathcal{B}$. (In other words, a conserved current is a relative 1-cycle on $(\Gamma, \mathcal{B})$). It is equivalent to saying that the potential U is "W-harmonic" on $\Gamma \smallsetminus \mathcal{B}$.

We say that a potential distribution U is *normalized* if $U(a) = 1$, $U(b) = 0$.

Lemma 11.1. *There exists a unique normalized equilibrium potential distribution on $\mathcal{C}$. This distribution is energy minimizing.*

Proof. The restriction of the energy function E to the affine subspace

$$\mathcal{A} = \{U \in \mathbb{R}^{|\mathcal{E}|} : U(a) = 1,\ U(b) = 0\}$$

is a positive quadratic function. One can easily check that $E(U) \to +\infty$ as $U \to \infty$, $U \in \mathcal{A}$, so that, U has a global minimum U_{eq} on $\mathcal{A}$. Moreover, E is strictly convex, an hence can have at most one critical point. Hence U_{eq} is the only critical point of U on $\mathcal{A}$. Finally the criticality condition gives exactly the conservation law for the corresponding current. □

Remark 1. Since the energy function $E(U)$ depends only on the potential differences, it is invariant under translations $U \mapsto U + t\mathbf{1}$, $t \in \mathbb{R}$, where $\mathbf{1} \equiv 1$. Thus, we can always normalize U so that $U(b) = 0$. Since $E(U)$ is homogeneous of order 2 in U, normalization $\mathbf{U} = \lambda$ would replace the equilibrium distribution U_{eq} by $U_{eq}^{\lambda} = \lambda U_{eq}$. Then the equilibrium current would scale proportionally: $I_{eq}^{\lambda} = \lambda I_{eq}$.

Remark 2. The above lemma is just a solution of the Dirichlet boundary problem by minimizing the Dirichlet integral.

Lemma 11.2. *For the equilibrium current I_{eq}, we have $\partial I_{eq}(a) = -\partial I_{eq}(b)$.*

Proof. We have:

$$0 = \sum_{e^* \in \mathcal{E}^*} I_{eq}(e^*) = \sum_{x \in \mathcal{V}} \partial I_{eq}(x) = \partial I_{eq}(a) + \partial I_{eq}(b),$$

where the first equality is valid since I is odd, the second is a rearrangement of terms, and the last one comes from the conservation law. □

The *total equilibrium current* of the circuit $\mathcal{C}$ with battery potential $\mathbf{U} = \lambda$ is defined as $\mathbf{I} = I^{\lambda}_{eq}(a) = -I^{\lambda}_{eq}(b)$. It depends linearly on the battery potential, so we can define the *total conductance* of $\mathcal{C}$ as

$$\mathbf{W} = \mathbf{W}(\mathcal{C}) = \frac{\mathbf{I}}{\mathbf{U}}.$$

The *total Resistance* of the circuit is the inverse quantity: $\mathbf{R} = 1/\mathbf{W}$.

Let $\mathbf{E} = E(U^{\lambda}_{eq})$ stand for the equilibrium energy of the circuit.

Lemma 11.3. *At the normalized equilibrium, we have:* $\mathbf{E} = \mathbf{W}$.

Proof. Let $\mathcal{A}_0 = \{U \in \mathbb{R}^{|\mathcal{E}|} : U(b) = 0\} = \mathbb{R}^{|\mathcal{E}|-1}$. Then the equilibrium state is the critical point of the quadratic form $E(U) = (QU, U)$ on $\mathcal{A}_0$ subject of the restriction $U(a) = 1$ (here Q is the matrix of $E|\mathcal{A}_0$). By the Lagrange multipliers method, this stationary point satisfies the equation

$$\frac{\partial E}{\partial U} \equiv 2QU = \lambda V,$$

where V is a basic vector in $R^{|\mathcal{E}|-1}$ with $V(a) = 1$, $V(x) = 0$ for $x \in \mathcal{V} \smallsetminus \mathcal{B}$. From here we conclude:

(i) $2E(U) = 2(QU, U) = \lambda(V, U) = \lambda U(a) = \lambda$;

(ii) $\dfrac{\partial E}{\partial U(a)} = \lambda V(a) = \lambda$.

But

$$\frac{\partial E}{\partial U(a)} = \frac{\partial}{\partial U(a)} \sum_{x \sim a} W[a, x](U(a) - U(x))^2 = 2 \sum_{x \sim a} I[a, x] = 2\mathbf{W},$$

and we are done. □

Remark. So, in the normalized situation (when $\mathbf{U} = 1$), we have

$$\mathbf{E} = \mathbf{I} = \mathbf{W}. \tag{11.4}$$

Since $\mathbf{I}$ is proportional to $\mathbf{U}$, $\mathbf{E}$ quadratically depends on $\mathbf{U}$, and $\mathbf{W}$ is independent of $\mathbf{U}$, we obtain by scaling the following physically obvious formulas:

$$\mathbf{E} = \mathbf{U} \cdot \mathbf{I}, \quad \mathbf{W} = \frac{\mathbf{I}}{\mathbf{U}}.$$

11.3. **Series and Parallel Laws.** Given two circuits $\mathcal{C}_1$ and $\mathcal{C}_2$, we can put them in *series*, that is, to match the terminal battery pole of $\mathcal{C}_1$ to the initial battery pole of $\mathcal{C}_2$, and to declare the "free" battery poles a new battery of this "connected sum". More formally, let $\mathcal{B}_i = \{a_i, b_i\}$ be the battery of $\mathcal{C}_i = (\Gamma_i, W_i)$. Then we let $\mathcal{C} = C_1 \sqcup_{a_2=b_1} C_2$ with the battery $\{a_1, b_2\}$ and the conductance vector: $W = W_1 + W_2$. (direct sum).

Lemma 11.5 (Series law)**.** *If $\mathcal{C}$ is a series of two circuits $\mathcal{C}_1$ and $\mathcal{C}_2$ with conductances $\mathbf{W}$, $\mathbf{W}_1$ and $\mathbf{W}_2$ respectively, then*

$$\mathbf{W} = \mathbf{W}_1 \oplus \mathbf{W}_2.$$

Proof. Let us consider the equilibrium state (U, I) of the circuit $\mathcal{C}$. Then the conservation law is valid for both sub-circuits $\mathcal{C}_1$ and $\mathcal{C}_2$, so that they are both in the equilibrium state, with the battery potentials $\mathbf{U}_1 = 1 - U(c)$ and $\mathbf{U}_2 = U(c)$, where $c = b_1 = a_2$. Let $\mathbf{I}_i$ be the corresponding equilibrium currents through the $\mathcal{C}_i$. By the current conservation at vertex c, $\mathbf{I}_1 = \mathbf{I}_2$. In fact, $\mathbf{I}_1 = \partial I(a_1) = \mathbf{I}$ and $\mathbf{I}_2 = -\partial I(b_2) = \mathbf{I}$. Hence

$$\frac{1}{\mathbf{W}_1} = \frac{\mathbf{U}_1}{\mathbf{I}}, \quad \frac{1}{\mathbf{W}_2} = \frac{\mathbf{U}_2}{\mathbf{I}}.$$

Summing these up, we obtain the desired. □

Given two circuits as above, we can also put them in *parallel*, that is, to identify the two pairs of poles as follows: $a_1 \sim a_2$, $b_1 \sim b_2$.

Lemma 11.6 (Parallel Law)**.** *If $\mathcal{C}$ is a parallel of two circuits $\mathcal{C}_1$ and $\mathcal{C}_2$ as above, then*

$$\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2.$$

11.4. **Quotients.** Let us have a finite family of electric circuits $\mathcal{C}_i$ based on graphs Γ_i with batteries $\mathcal{B}_i = \{a_i, b_i\}$ and conductance vectors $\mathcal{W}_i$. Let us identify certain vertices of the disjoint union $\sqcup \Gamma_i$ so that the batteries of different $\mathcal{C}_i$'s get identified ($a_i = a_j$, $b_i = b_j$) and no internal vertices get identified with a battery vertex. If in the quotient graph there are several edges $e_{ij} \in \mathcal{E}_i \equiv \mathcal{E}(\Gamma_i)$ connecting the same pair of vertices, let us identify those edges, too. We obtain a graph Γ. For the edge e of $\mathcal{E} \equiv \mathcal{E}(\Gamma)$ obtained this way, let

$$W(e) = \sum_{e_{ij} \sim e} W(e_{ij}).$$

In this way we obtain a new electric circuit $\mathcal{C} = (\Gamma, \mathcal{B}, \mathcal{W})$ called a quotient of $\sqcup \mathcal{C}_i$.

Let $\mathbf{W} = \mathcal{W}(\mathcal{C})$, $\mathbf{W}_i = \mathcal{W}(\mathcal{C}_i)$.

Lemma 11.7. *If $\mathcal{C}$ is a quotient of $\sqcup \mathcal{C}_i$ then $\mathbf{W} \geqslant \sum \mathbf{W}_i$.*

Proof. Let U be the normalized equilibrium potential for $\mathcal{C}$. By Lemma 11.3, its energy $E(U)$ is equal to $\mathbf{W}$.

The potential U lifts to normalized potentials U_i for the circuits $\mathcal{C}_i$'s. By the same lemma, $\mathbf{W}_i \leqslant E(U_i)$. On the other hand,

$$\sum_i E(U_i) = \sum_i \sum_{e_{ij} \in \mathcal{E}_i} W(e_{ij}) dU_i(e_{ij})^2 = \sum_{e \in \mathcal{E}} dU(e)^2 \sum_{e_{ij} \sim e} W(e_{ij}) = E(U).$$

Putting these pieces together, we obtain the desired inequality. □

Remark 11.1. The above inequality has a clear physical meaning: taking a quotient of a circuit gives more choices for the current to flow, which increases total conductance.

The following arithmetic inequality can be proven by interpreting it in terms of electric circuits:

Lemma 11.8. *Let us consider finite sets of positive numbers w_i and v_{ij}, $1 \leqslant i, j \leqslant n$, such that $w_i \leqslant \oplus_j v_{ij}$. Let $w = \sum w_i$ and $v_j = \sum_i v_{ij}$. Then $w \leqslant \oplus v_j$.*

Proof. Let us consider resistors V_{ij} with conductances v_{ij}. Let $\mathcal{C}_i$ be an electric circuit obtained by plugging the resistors V_{ij} in series. Then

$$\mathbf{W}(\mathcal{C}_i) = \bigoplus_j v_{ij} \geqslant w_i.$$

Let us also consider the quotient $\mathcal{C}$ of $\sqcup \mathcal{C}_i$ obtained by identifying the endpoints of the resistors V_{ij} with the same j. In other words, we plug the V_{ij}'s with the same j in parallel obtaining circuits V_j, and then plug the V_i's in series. Then

$$\mathbf{W}(\mathcal{C}) = \bigoplus \mathbf{W}(V_j) = \oplus v_j.$$

By the previous lemma, $\sum \mathbf{W}(\mathcal{C}_i) \leqslant \mathbf{W}(\mathcal{C})$, which boils down to the desired estimate. □

Remark 11.2. So, the signs of ordinary and harmonic sums can be interchanged following the rule:

$$\sum_i \bigoplus_j v_{ij} \leqslant \bigoplus_j \sum_i v_{ij}.$$

11.5. **Local conductances.** Given a vertex $x \in \mathcal{V}(\Gamma)$, we let

$$W|x = \sum_{y\sim x} W[x,y],$$

and call it the *local conductance at* x.

Lemma 11.9. *We have:* $\mathbf{W} \leqslant \min(W|a, W|b)$.

Proof. By the Maximum Principle for harmonic functions, $0 \leqslant U_{eq}(x) \leqslant 1$ for any $x \in \Gamma$. Hence $|I_{eq}(e^*)| \leqslant W(e)$ for any $e^* \in \mathcal{E}^*$. Together with (11.4), this implies:

$$\mathbf{W} = \mathbf{I} = \sum_{x\sim a} I_{eq}[x,a] \leqslant W|a,$$

and similarly for $W|b$. □

11.6. **Domination.** Let us consider two unplugged electric circuits $\mathcal{C} = (\Gamma, W)$ and $\mathcal{C}' = (\Gamma', W')$ such that:

• Γ' is obtained from Γ by replacing edges $e \in \mathcal{E}(\Gamma)$ with some graphs $\Gamma'(e)$;

• Letting $\mathcal{C}'(e)$ be the restriction of $\mathcal{C}'$ to $\Gamma'(e)$ with battery ∂e, we have: $\mathbf{W}(\mathcal{C}'(e)) \geqslant W(e)$ for any $e \in \mathcal{E}(\Gamma)$.

Under these circumstances we say that $\mathcal{C}'$ *dominates* $\mathcal{C}$, $\mathcal{C}' \multimap \mathcal{C}$.

Lemma 11.10. *If* $\mathcal{C}' \multimap \mathcal{C}$ *then* $W'|x \geqslant W|x$ *for any* $x \in \mathcal{V}(\Gamma)$.

Proof. Indeed, by Lemma 11.9, we have:

$$W'|x = \sum_{\partial e' \ni x} W'(e') \geqslant \sum_{\partial e \ni x} \mathbf{W}(\mathcal{C}'(e)) \geqslant \sum_{\partial e \ni x} W(e) = W|x,$$

where the summation is taken over $e \in \mathcal{E}(\Gamma)$, $e' \in \mathcal{E}(\Gamma')$. □

11.7. **Trees of complete graphs.** In this section we will consider a special class of electric circuits based on trees of complete graphs (TCG). A TCG is an object that can be constructed inductively by the following rules:

• Any complete graph is a TCG;

• If Γ, Γ' are TCG's, $v \in \mathcal{V}(\Gamma)$, $v' \in \mathcal{V}(\Gamma')$, then $\Gamma \underset{v=v'}{\sqcup} \Gamma'$ is a tree of complete graphs as well.

A TCG is called an *interval of complete graphs* if any complete graph involved has a common vertex with at most two other complete graphs, and now three complete graphs have a common vertex.

Given three vertices x, y, z in a graph Γ, we say that a vertex y *separates* x *from* z if x and z belong to different components of $\Gamma \smallsetminus \{y\}$.

We say that a sequence of vertices $(x_0, x_1, \ldots, x_d)$ form a *chain in* Γ if x_i separates x_{i-1} from x_{i+1} for each $i = 1, \ldots, d-1$. Let $SS(x, y)$ stand for the set of vertices separating x from y.

The reader can entertain himself by verifying the following fact:

Lemma 11.11. *Let Γ be a TCG. Then for any two vertices $x, y \in \mathcal{V}(\Gamma)$, the set $SS(x, y) \cup \{x, y\}$ can be uniquely ordered to form a chain $(x = x_0, x_1, \ldots, x_d = y)$. Moreover, for any $i = 0, \ldots, d-1$, the vertices x_i and x_{i+1} belong to the same complete graph $\Gamma_{k(i)}$, and these graphs form an interval of complete graphs.*

We call it "the chain connecting x to y", and we let $d_\Gamma(x, y) = d$.

Lemma 11.12. *Let us consider an electric circuit $\mathcal{C}$ based on a tree of complete graphs, and let $(a = x_0, x_1, \ldots, x_d = b)$ be the chain connecting the poles of the battery. Then*

$$\mathbf{W} \leqslant \bigoplus_{i=1}^{d} W|x_i \leqslant \frac{1}{d_\Gamma(a, b)} \max_{x \in \Gamma} W|x.$$

Proof. The second inequality is trivial, so we only need to prove the first one.

Let G_i be the component of $\Gamma \smallsetminus \{x_i, x_{i+1}\}$ containing the edge (x_i, x_{i+1}). Let $\mathcal{C}_i$ be the restriction of the electric circuit $\mathcal{C}$ to $G_i \cup \{x_i, x_{i+1}\}$ with battery $\{x_i, x_{i+1}\}$. By the Series Law and Lemma 11.9,

$$\mathbf{W} \leqslant \bigoplus_{i=1}^{d} \mathbf{W}(\mathcal{C}_i) \leqslant \bigoplus_{i=1}^{d} W|x_i.$$

□

12. Appendix C: Estimates in hyperbolic geometry

In this appendix we will describe how to measure the geometry of a surface using transverse geodesic arcs. We will then show how to compute the lengths of peripheral closed geodesics of the surface using these measurements.

Suppose T is a compact hyperbolic surface with geodesic boundary. The following lemma appears as the Corollary to section 3.3 of [Ab]:

Lemma 12.1. *There is an $\epsilon_0 > 0$ such that any two distinct closed geodesics on T of length at most ϵ_0 are simple and disjoint.*

Let $\mathbf{S}$ be a compact hyperbolic surface with geodesic boundary. Then a *transverse geodesic arc* for $\mathbf{S}$ is a proper path of minimal length in its proper homotopy class. If $\boldsymbol{\alpha}$ is a path on $\mathbf{S}$, it is a transverse geodesic arc if and only if it is a geodesic arc that meets $\partial\mathbf{S}$ orthogonally, or, equivalently, the double of $\boldsymbol{\alpha} \cup \overline{\boldsymbol{\alpha}}$ in $\mathbf{S} \cup \overline{\mathbf{S}}$ is a closed geodesic.

Let S be a compact Riemann surface with boundary, and endow $\operatorname{Int} S$ with its Poincaré metric. The peripheral geodesics on $\operatorname{Int} S$ bound a compact surface $\mathbf{S}$ with geodesic boundary, called the *convex core* of S. There is a homeomorphism $h\colon \mathbf{S} \to S$ that is isotopic through embeddings to the inclusion $\mathbf{S} \subset S$. We can then form a weighted arc diagram M^S on S as follows: for $\alpha \in \mathcal{A}(S)$, we find the transverse geodesic arc $\boldsymbol{\alpha}$ for $\mathbf{S}$ such that $h(\boldsymbol{\alpha}) \sim \alpha$. Then we let $M^S(\alpha) = -\log L(\boldsymbol{\alpha})$ if $L(\boldsymbol{\alpha}) < \epsilon_0/2$, and $M^S(\alpha) = 0$ otherwise. Then M^S is supported on a set of disjoint arcs, so M^S is a weighted arc-diagram for S. We let $\boldsymbol{M}^S = \bigcup\{\boldsymbol{\alpha} : L(\boldsymbol{\alpha}) < \epsilon_0/2\}$, so $\boldsymbol{M}^S \subset \mathbf{S}$ is the union of the short transverse geodesic arcs of $\mathbf{S}$.

We will call a non-peripheral simple closed geodesic a *dividing* geodesic. The following result appears in [Ab]:

Lemma 12.2. *Let T be a compact hyperbolic Riemann surface with bounded-length geodesic boundary. Then either T is a pair of pants, or there is a bounded-length dividing geodesic on T.*

We say that a hyperbolic surface T is *symmetric* if it admits an isometric orientation-reversing involution, which we will denote by complex conjugation: "$z \mapsto \overline{z}$". Then we let $E_T = \{z \in T : z = \overline{z}\}$, and E_T will be a union of (simple) closed geodesics and transverse geodesic arcs. (E_T depends implicitly on on the choice of involution. Whenever we say "symmetric hyperbolic surface" we will mean "symmetric hyperbolic surface and choice of involution.") Note that $T \smallsetminus E_T$ has two components, call them A_T and $\overline{A_T}$, which are mapped to each other by $z \mapsto \overline{z}$. We prove a symmetric version of Lemma 12.2:

Lemma 12.3. *Every symmetric compact hyperbolic surface T with bounded-length geodesic boundary has a bounded-length symmetric pair of pants decomposition.*

Proof. It suffices to find a single bounded-length symmetric dividing geodesic on the surface, or a symmetric pair of disjoint bounded-length dividing geodesics, because then we can cut the surface T along that geodesic or pair of geodesics, and repeat.

By Theorem 12.2, unless T is a pair of pants, there is a dividing geodesic γ of bounded length on T. If $\gamma \cap E_T = \varnothing$, then $\gamma \cap \overline{\gamma} = \varnothing$, and we are done. Likewise, if $\gamma \subset E_T$, then γ is symmetric, and we are done. Otherwise, let η be a component of $\gamma \cap \mathrm{Cl}\, A_T$. Then $\eta \cup \overline{\eta}$ is a non-trivial non-peripheral simple closed curve so we let τ be the dividing geodesic homotopic to η. Then $L(\tau) \leqslant 2L(\eta) < 2L(\gamma)$, so τ is the desired object. $\square$

We now prove two basic estimates on transverse geodesic arcs on pairs of pants. We denote by $[x, y, z, r, s, t]$ the right-angled hyperbolic hexagon with lengths x, y, z, r, s, t in that order. We will omit lengths that are not specified, so for example $[a, , b, , c,]$ denotes the right-angled hexagon with alternating side lengths a, b, and c. We first estimate the length of one side in a hyperbolic right-angled hexagon, in terms of the lengths of three alternating sides:

Lemma 12.4. *Let $[a, c', b, , c, ,]$ be a hyperbolic right-angled hexagon, and suppose that $a, b, c \leqslant r$. Then $c' = -\log a - \log b + O(1; r)$.*[11]

Proof. We use the formula (from [F]):

$$\cosh c' = \frac{\cosh c + \cosh a \cosh b}{\sinh a \sinh b} = e^{O(1;r)} \frac{1}{ab} \tag{12.5}$$

and recall that $\cosh^{-1} x = \log x + O(1; t)$ whenever $x \geqslant t > 0$. $\square$

We let $\mathcal{P}(a, b, c)$ denote the pair of pants with cuff lengths a, b, and c.

Lemma 12.6. *We have the following two estimates:*

1. *If $P = \mathcal{P}(a, b, c)$ is a pair of pants, and γ is the transverse geodesic arc that connects a and b, then*
$$|\gamma| = -\log a - \log b + O(1),$$
for $a, b, c \leqslant C_0$.
2. *If γ is the transverse geodesic arc that connects a and a in $P(a, b, b)$, then*
$$|\gamma| = -2 \log a + O(1)$$
for $a, b \leqslant C_0$.

Proof. We prove each of the above:

1. We cut P along the three pairwise transversals into two right-angled hexagons. By formula (12.5) these hexagons are equal, and hence each has type $[a/2, \gamma, b/2, , c/2,]$. Apply now Lemma 12.4.

[11]Notation $O(1; r)$ stands for a quantity bounded in terms of r.

(2) We let η be the transversal from the length a cuff to one other. Then we cut along η and γ to obtain the right-angled hexagon $[a/4, \gamma, a/4, \eta, b, \eta]$, and then apply Lemma 12.4.

□

Given a closed geodesic γ and an arc $\alpha \in \mathcal{A}$, let $\langle \gamma, \alpha \rangle$ stand for the intersection number of γ with α, i.e., the minimal number of intersections of γ with the paths representing α. Given a weighted arc diagram $W = \sum W(\alpha)\alpha$, we can define the intersection number $\langle \gamma, W \rangle = \sum W(\alpha) \langle \gamma, \alpha \rangle$ by linearity. We can now prove the main theorem of this appendix:

Theorem 12.6.1. *Let S be a compact Riemann surface with boundary, and endow* $\operatorname{Int} S$ *with its Poincaré metric. Suppose that γ is a peripheral closed geodesic for S. Then*

$$L(\gamma) = 2 \left\langle M^S, \gamma \right\rangle + O(1; \chi(S)).$$

Proof. We let $\mathbf{S}$ be the convex core of S. We find a symmetric bounded-length pair of pants decomposition for $\mathbf{S} \cup \overline{\mathbf{S}}$ extending $\boldsymbol{M}^S \cup \bar{\boldsymbol{M}}^S$. Then we can write $\gamma = \bigcup t_i$, where the segments t_i are interior-disjoint, and each is a transverse arc of one of the pairs of pants. Then $L(t_i) = -\log a_i - \log b_i + O(1)$, where a_i and b_i are the lengths of the cuffs that γ connects (possibly the same cuff). Therefore

$$L(\gamma) = 2 \sum -\log a_i + O(1)$$

where the a_i are the lengths of the cuffs that γ crosses, counted with multiplicity. But

$$2 \sum -\log a_i = 2 \left\langle M^S, \gamma \right\rangle + O(1). \tag{12.7}$$

□

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