

# **The Arithmetic of Dynamical Systems**

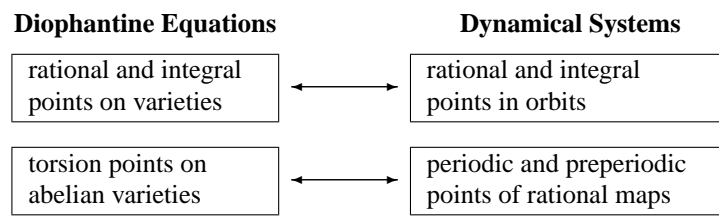
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December 28, 2006



# Preface

This book is designed to provide a path for the reader into an amalgamation of two venerable areas of mathematics, Dynamical Systems and Number Theory. Many of the motivating theorems and conjectures in the new subject of Arithmetic Dynamics may be viewed as the transposition of classical results in the theory of Diophantine equations to the setting of discrete dynamical systems, especially to the iteration theory of maps on the projective line and other algebraic varieties. Although there is no precise dictionary connecting the two areas, the reader will gain a flavor of the correspondence from the following associations:



There are a variety of topics covered in this volume, but inevitably the choice reflects the author's tastes and interests. Many related areas that also fall under the heading of arithmetic or algebraic dynamics have been omitted in order to keep the book to a manageable length. A brief list of some of these omitted topics may be found in the introduction.

## Online Resources

The reader will find additional material, references and errata at

<http://www.math.brown.edu/~jhs/ADSHome.html>

## Acknowledgements

The author has consulted a great many sources in writing this book. Every attempt has been made to give proper attribution for all but the most standard results. Much of the presentation is based on courses taught at Brown University in 2000 and 2004, and the exposition benefits greatly from the comments of the students in those

courses. In addition, the author would like to thank the many students and mathematicians who read drafts and/or offered suggestions and corrections, including Matt Baker, Rob Benedetto, Paul Blanchard, Rex Cheung, Bob Devaney, Graham Everest, Liang-Chung Hsia, Rafe Jones, Daniel Katz, Shu Kawaguchi, Michelle Manes, Patrick Morton, Curt McMullen, Hee Oh, Giovanni Panti, Lucien Szpiro, Tom Tucker, Tom Ward, Xinyi Yuan, Shou-Wu Zhang. An especial thanks is due to Matt Baker, Rob Benedetto and Liang-Chung Hsia for their help in navigating the treacherous shoals of  $p$ -adic dynamics. The author would also like to express his appreciation to John Milnor for a spellbinding survey talk on dynamical systems at Union College in the mid-1980s that provided the initial spark leading eventually to the present volume. Finally, the author thanks his wife Susan for her support and patience during the many hours occupied in writing this book.

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*January 1, 2007*

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# Introduction

A (discrete) dynamical system consists of a set  $S$  and a function  $\phi : S \rightarrow S$  mapping the set  $S$  to itself. This self-mapping permits iteration

$$\phi^n = \underbrace{\phi \circ \phi \circ \dots \circ \phi}_{n \text{ times}} = n^{\text{th}} \text{ iterate of } \phi.$$

(By convention,  $\phi^0$  denotes the identity map on  $S$ .)

For a given point  $\alpha \in S$ , the (forward) orbit of  $\alpha$  is the set

$$\mathcal{O}_\phi(\alpha) = \mathcal{O}(\alpha) = \{\phi^n(\alpha) : n \geq 0\}.$$

The point  $\alpha$  is *periodic* if  $\phi^n(\alpha) = \alpha$  for some  $n \geq 1$ . The smallest such  $n$  is called the *exact period* of  $\alpha$ . The point  $\alpha$  is *preperiodic* if some iterate  $\phi^m(\alpha)$  is periodic. The set of periodic and preperiodic points of  $\phi$  in  $S$  are denoted respectively by

$$\begin{aligned} \text{Per}(\phi, S) &= \{\alpha \in S : \phi^n(\alpha) = \alpha \text{ for some } n \geq 1\}, \\ \text{PrePer}(\phi, S) &= \{\alpha \in S : \phi^{m+n}(\alpha) = \phi^m(\alpha) \text{ for some } n \geq 1, m \geq 0\} \\ &= \{\alpha \in S : \mathcal{O}_\phi(\alpha) \text{ is finite}\}. \end{aligned}$$

We write  $\text{Per}(\phi)$  and  $\text{PrePer}(\phi)$  when the set  $S$  is fixed.

## Principal Goal of Dynamics

Classify the points  $\alpha$  in the set  $S$  according to the behavior of their orbits  $\mathcal{O}_\phi(\alpha)$ .

If  $S$  is simply a set with no additional structure, then typical problems are to describe the sets of periodic and preperiodic points and to describe the possible periods of periodic points. Usually, however, the set  $S$  has some additional structure and one attempts to classify the points in  $S$  according to the interaction of their orbits with that structure. There are many types of additional structures that may be imposed, including algebraic, topological, metric, and analytic.

*Example 0.1. (Finite Sets).* Let  $S$  be a finite set and  $\phi : S \rightarrow S$  a function. Clearly every point of  $S$  is preperiodic, so we ask for a description of the set of periodic points. For example, for each  $n \geq 0$  we ask for the size of the set



$$\text{Per}_n(\phi, S) = \{\alpha \in S : \phi^n(\alpha) = \alpha\}.$$

As a particular example, we consider the case that  $S = \mathbb{F}_p$  is a finite field and look at maps  $\phi : \mathbb{F}_p \rightarrow \mathbb{F}_p$  give by polynomials  $\phi(z) \in \mathbb{F}_p[z]$ . Fermat's little theorem says that

$$\text{Per}(z^p, \mathbb{F}_p) = \mathbb{F}_p \quad \text{and} \quad \text{Per}(z^{p-1}, \mathbb{F}_p) = \{0, 1\},$$

which gives two extremes for the set of periodic points. A much harder question is to fix an integer  $d \geq 2$  and ask for which primes  $p$  is there a polynomial  $\phi$  of degree  $d$  satisfying  $\text{Per}(\phi, \mathbb{F}_p) = \mathbb{F}_p$ ? Similarly, one might fix a polynomial  $\phi(z) \in \mathbb{Z}[z]$  and ask for which primes  $p$  is it true that  $\text{Per}(\phi, \mathbb{F}_p) = \mathbb{F}_p$ ; in particular, are there infinitely many such primes?

In a similar, but more general, vein, one can look at a rational function  $\phi \in \mathbb{F}_p(z)$  inducing a rational map  $\phi : \mathbb{P}^1(\mathbb{F}_p) \rightarrow \mathbb{P}^1(\mathbb{F}_p)$ . Even more generally, one can ask similar questions for a morphism  $\phi : V(\mathbb{F}_p) \rightarrow V(\mathbb{F}_p)$  of any variety  $V/\mathbb{F}_p$ , for example  $V = \mathbb{P}^N$ .

*Example 0.2. (Groups).* Let  $G$  be a group and let  $\phi : G \rightarrow G$  be a homomorphism. Using the group structure, it is often possible to describe the periodic and preperiodic points of  $\phi$  fairly explicitly. The following proposition describes a simple, but important, example. In order to state the proposition, we recall that the *torsion subgroup* of an abelian group  $G$ , denoted  $G_{\text{tors}}$ , is the set of elements of finite order in  $G$ ,

$$G_{\text{tors}} = \{\alpha \in G : \alpha^m = e \text{ for some } m \geq 1\},$$

where  $e$  denotes the identity element of  $G$ .

*Proposition 0.3.* Let  $G$  be an abelian group, let  $d \geq 2$  be an integer and let  $\phi : G \rightarrow G$  be the  $d^{\text{th}}$ -power map  $\phi(\alpha) = \alpha^d$ . Then

$$\text{PrePer}(\phi, G) = G_{\text{tors}}$$

*Proof.* The simple nature of the map  $\phi$  allows us to give an explicit formula for its iterates,

$$\phi^n(\alpha) = \alpha^{d^n}.$$

First take an element  $\alpha \in \text{PrePer}(\phi, G)$ . This means that  $\phi^{m+n}(\alpha) = \phi^m(\alpha)$  for some  $n \geq 1$  and  $m \geq 0$ , so  $\alpha^{d^{m+n}} = \alpha^{d^m}$ . But  $G$  is a group, so we can multiply by  $\alpha^{-d^m}$  to get  $\alpha^{d^{m+n}-d^m} = e$ . The assumptions on  $d$ ,  $m$ , and  $n$  imply that the exponent is positive, so  $\alpha \in G_{\text{tors}}$ .

Next suppose that  $\alpha \in G_{\text{tors}}$ , say  $\alpha^m = e$ , and consider the following sequence of integers modulo  $m$ :

$$d, d^2, d^3, d^4, \dots \text{ modulo } m.$$

Since there are only finitely many residues modulo  $m$ , eventually the sequence has a repeated element, say  $d^i \equiv d^j \pmod{m}$  with  $i > j$ . Then

$$\phi^i(\alpha) = \alpha^{d^i} = \alpha^{d^j} = \phi^j(\alpha), \quad \text{since } \alpha^m = e \text{ and } d^i \equiv d^j \pmod{m},$$

which proves that  $\alpha \in \text{PrePer}(\phi)$ .  $\square$

*Example 0.4. (Topological Spaces).* Let  $S$  be a topological space and let  $\phi : S \rightarrow S$  be a continuous map. For a given  $\alpha \in S$ , one might ask for a description of the accumulation points of  $\mathcal{O}_\phi(\alpha)$ . For example, a point  $\alpha$  is called *recurrent* if it is an accumulation point of  $\mathcal{O}_\phi(\alpha)$ . In other words,  $\alpha$  is recurrent if there is a sequence of integers  $n_1 < n_2 < n_3 < \dots$  such that  $\lim_{i \rightarrow \infty} \phi^{n_i}(\alpha) = \alpha$ , so either  $\alpha$  is periodic, or it eventually returns arbitrarily close to itself.

*Example 0.5. (Metric Spaces).* Let  $(S, \rho)$  be a compact metric space. For example,  $S$  could be the unit sphere sitting inside  $\mathbb{R}^3$  and  $\rho(\alpha, \beta)$  the usual Euclidean distance from  $\alpha$  to  $\beta$  in  $\mathbb{R}^3$ . The fundamental question in this setting is whether points that start off close to a given point  $\alpha$  continue to remain close to one another under repeated iteration of  $\phi$ . If this is true, we say that  $\phi$  is *equicontinuous* at  $\alpha$ , otherwise we say that  $\phi$  is *chaotic* at  $\alpha$ . (See Section 1.4 for the formal definition of equicontinuity.) Thus if  $\phi$  is equicontinuous at  $\alpha$ , we can approximate  $\phi^n(\alpha)$  quite well by computing  $\phi^n(\beta)$  for any point  $\beta$  that is close to  $\alpha$ . But if  $\phi$  is chaotic at  $\alpha$ , then no matter how close we choose  $\alpha$  and  $\beta$ , eventually  $\phi^n(\alpha)$  and  $\phi^n(\beta)$  move away from each other.

*Example 0.6. (Arithmetic Sets).* An arithmetic set is a set such as  $\mathbb{Z}$  or  $\mathbb{Q}$  or a number field that is of number theoretic interest, but doesn't have a natural underlying topology. More precisely, an arithmetic set tends to have a variety of interesting topologies; for example,  $\mathbb{Q}$  has the archimedean topology induced by the inclusion  $\mathbb{Q} \subset \mathbb{R}$  and the  $p$ -adic topologies induced by the inclusions  $\mathbb{Q} \subset \mathbb{Q}_p$ . In the arithmetic setting, the map  $\phi$  is generally a polynomial or a rational map. Here are some typical arithmetical-dynamical questions, where we take  $\phi(z) \in \mathbb{Q}(z)$  to be a rational function of degree  $d \geq 2$  with rational coefficients:

- Let  $\alpha \in \mathbb{Q}$  be a rational number. Under what conditions can the orbit  $\mathcal{O}_\phi(\alpha)$  contain infinitely many integer values? In other words, when can  $\mathcal{O}_\phi(\alpha) \cap \mathbb{Z}$  be an infinite set?
- Is the set  $\text{Per}(\phi, \mathbb{Q})$  of rational periodic points finite or infinite? If finite, how large can it be?
- Let  $\alpha \in \text{Per}(\phi)$  be a periodic point for  $\phi$ . It is clear that  $\alpha$  is an algebraic number. What are the arithmetic properties of the field  $\mathbb{Q}(\alpha)$ , or more generally of the field generated by all of the periodic points of a given period?

**What is in this Book:** We provide a brief summary of the material that is covered.

1. *An Introduction to Classical Dynamics*

We begin in Chapter 1 with a short self-contained overview, without proofs, of classical complex dynamics on the projective line.

2. *Dynamics Over Local Fields: Good Reduction*

Chapter 2, which starts our study of arithmetic dynamics, considers rational maps  $\phi(z)$  with coefficients in a local field  $K$ , for example,  $K = \mathbb{Q}_p$ . The emphasis in Chapter 2 is on maps that have “good reduction modulo  $p$ .” The good

reduction property implies that many of the geometric properties of  $\phi$  acting on the points of  $K$  are preserved under reduction modulo  $p$ . In particular, the map  $\phi$  is  $p$ -adically nonexpanding, and periodic points behave well when reduced modulo  $p$ . The remainder of the chapter gives applications exploiting these two key properties of good reduction.

3. *Dynamics Over Global Fields*

We move on in Chapter 3 to arithmetic dynamics over global fields such as  $\mathbb{Q}$  and its finite extensions. Just as in the study of Diophantine equations over global fields, the theory of height functions plays a key role, and we develop this theory, including the construction of the canonical height associated to a rational map. We discuss rationality of preperiodic points and formulate a general uniform boundedness conjecture. Using classical results from the theory of Diophantine approximation, we describe exactly which rational maps  $\phi$  can have orbits containing infinitely many integer points, and we give a more precise result saying that the numerator and denominator of  $\phi^n(\alpha)$  grow at approximately the same rate. We consider the extension fields generated by periodic points and describe their Galois groups, ramification, and units.

4. *Families of Dynamical Systems*

At this point we change our perspective and, rather than studying the dynamics of a single rational map, we consider families of rational maps and the variation of their dynamical properties. We construct various sorts of parameter and moduli spaces, including the space of quadratic polynomials with a point of exact period  $N$  (which are analogs of the classical modular curves  $X_1(N)$ ), the parameter space  $\text{Rat}_d$  of rational functions of degree  $d$ , and the moduli space  $\mathcal{M}_d$  of rational functions of degree  $d$  modulo the natural conjugation action by  $\text{PGL}_2$ . In particular, we prove that  $\mathcal{M}_2$  is isomorphic to the affine plane  $\mathbb{A}^2$ . We also study twists of rational maps, analogous to the classical theory of twists of varieties, and the field of moduli versus field of definition problem.

5. *Dynamics Over Local Fields: Bad Reduction*

Chapter 5 returns to arithmetic dynamics over local fields, but now in the case of “bad reduction.” It becomes necessary to work over an algebraically closed field, so we discuss the field  $\mathbb{C}_p$  and give a brief introduction to nonarchimedean analysis and Newton polygons. Using these tools, we define the nonarchimedean Julia and Fatou sets and prove a version of Montel’s theorem that is then used to study periodic points and wandering domains in the nonarchimedean setting. This is followed by the construction of  $p$ -adic Green functions and local canonical heights. The chapter concludes with a short introduction to dynamics on Berkovich space. The Berkovich projective line  $\mathbb{P}^{\mathcal{B}}$  is path connected, compact, and Hausdorff, yet it naturally contains the totally disconnected, non-locally compact, non-Hausdorff space  $\mathbb{P}^1(\mathbb{C}_p)$ .

6. *Dynamics Associated to Algebraic Groups*

There is a small collection of rational maps whose dynamics are much easier to understand than those of a general map. These special rational maps are associated to endomorphisms of algebraic groups. We devote Chapter 6 to the study of

these maps. The easiest ones are the power maps  $M_d(z) = z^d$  and the Chebyshev polynomials  $T_d(z)$  characterized by  $T_d(2 \cos \theta) = 2 \cos(d\theta)$ . They are associated to the multiplicative group. More interesting are the Lattès maps attached to elliptic curves. We give a short description, without proofs, of the theory of elliptic curve and then spend the remainder of the chapter discussing dynamical and arithmetic properties of Lattès maps.

#### 7. Dynamics in Dimension Greater Than One

With a few exceptions, the results in Chapters 1–6 all deal with iteration of maps on the one-dimensional space  $\mathbb{P}^1$ , i.e., they are dynamics of one variable. In Chapter 7 we consider some of the issues that arise when studying dynamics in higher dimensions. We first study a class of rational maps  $\phi : \mathbb{P}^N \rightarrow \mathbb{P}^N$  that are not everywhere defined. Even over  $\mathbb{C}$ , the geometry of dynamics of rational maps is imperfectly understood. We restrict attention to automorphisms  $\phi : \mathbb{A}^N \rightarrow \mathbb{A}^N$  and study height functions and rationality of periodic points for such maps. We next consider morphisms  $\phi : X \rightarrow X$  of varieties other than  $\mathbb{P}^N$ . In order to deal with higher dimensional dynamics, we use tools from basic algebraic geometry and Weil's height machine, which we describe without proof. We then study arithmetic dynamics, heights, and periodic points on K3 surfaces admitting two noncommuting involutions  $\iota_1$  and  $\iota_2$ . The composition  $\phi = \iota_1 \circ \iota_2$  provides an automorphism  $\phi : X \rightarrow X$  whose geometric and arithmetic dynamical properties are quite interesting.

**What's Missing:** A book necessarily reflects the author's interests and tastes, while space considerations limit the amount of material that can be included. There are thus many omitted topics that naturally fit into the purview of arithmetic dynamics. Some of these are active areas of current mathematical research with their own literature, including introductory and advanced textbooks. Others are younger areas that deserve books of their own. Examples of both sorts include the following, some of which overlap with one another:

- *Dynamics over finite fields*

This includes general iteration of polynomial and rational maps acting on finite fields, see for example [39, 40, 97, 165, 200, 202, 253, 286, 311, 324, 360, 359, 375, 410, 418], and more specialized topics such as permutation polynomials [253, Chapter 7] that are fields in their own right.

- *Dynamics over function fields*

The study of function fields over finite fields has long provided a parallel theory to the study of number fields, but inseparability and wild ramification often lead to striking differences, while function fields of characteristic 0 present their own arithmetic challenges, e.g., they have infinitely many points of bounded height. The study of arithmetic dynamics over function fields is in its infancy. For a handful of results, see [18, 57, 79, 98, 189, 257, 328, 330, 390].

- *Iteration of formal and  $p$ -adic power series*

There is an extensive literature, but no textbook, on the iteration properties of power series. Among the fundamental problems are the classification of nontrivial

commuting power series (to what extent do they come from formal groups) and the description of preperiodic points. See for example [244, 245, 246, 247, 248, 249, 250, 251, 252, 258, 259, 260, 261, 262, 364, 365].

- *Algebraic dynamics*  
There is no firm line between arithmetic dynamics and algebraic dynamics, and indeed much of the material in this book is quite algebraic. Some topics of an algebraic nature that we do not cover include irreducibility of iterates [5, 15, 103, 105, 320, 321, 399], formal transformations and algebraic identities such as [51, 73, 74], and various results of an algebro-geometric nature [137, 151, 367, 411].
- *Lie groups and homogeneous spaces, ergodic theory and entropy*  
This is a beautiful and much studied area of mathematics in which geometry, analysis, and algebra interact. There are many results of a global arithmetic nature, including for example hard problems of Diophantine approximation, as well as an extensive  $p$ -adic theory. For an introduction to some of the main ideas and theorems in this area, see [43, 225, 282, 397], and for other arithmetic aspects of ergodic theory and entropy, including relations with height functions, ergodic theory in a nonarchimedean setting, and arithmetic properties of dynamics on solenoids, see for example [8, 29, 95, 120, 135, 148, 171, 209, 218, 220, 229, 255, 256, 325, 414, 415, 416, 421].
- *Equidistribution in arithmetic dynamics*  
There are many ways to measure (arithmetic) equidistribution, including via canonical heights,  $p$ -adic measures, and invariant measures on projective and Berkovich spaces. In Section 3.10 we summarize some basic equidistribution conjectures and theorems (without proof) For additional material, see [14, 22, 26, 91, ?, 156, 166, 191, 402, 405, 424].
- *Topology and arithmetic dynamics on foliated spaces*  
This surprising connection between these diverse areas of mathematics has been investigated by Deninger in a series of papers [114, 115, 116, 117, 118].
- *Dynamics on Drinfeld modules*  
It is natural to study local and global arithmetic dynamics in the setting of Drinfeld modules, although only a small amount of work has yet been done. See for example [166, 370].
- *Number theoretic iteration problems not arising as maps on varieties*  
A famous example of this type of problem is the notorious  $3x + 1$  problem, see [231] for an extensive bibliography. Another problem that people have studied is iteration of arithmetic functions such as Euler's  $\varphi$  function, see for example [139, 317].
- *Realizability of integer sequences*  
A sequence  $(a_n)$  of nonnegative integers is said to be realizable if there is a set  $S$  and a function  $\phi : S \rightarrow S$  with the property that for all  $n$ , the map  $\phi$  has  $a_n$  periodic points of order  $n$ . See [146] for an overview and [10, 132, 145, 336, 337, 394] for further material on realizable sequences.

**Prerequisites:** The principal prerequisite for reading this book is basic algebraic number theory (rings of integers, ideals and ideal class groups, units, valuations and absolute values, completions, ramification, etc.) as covered, for example, in the first section of Lang's *Algebraic Number Theory* [236]. We also assume some knowledge of elementary complex analysis as typically covered in an undergraduate course in the subject. No background in dynamics or algebraic geometry is required; we summarize and give references as necessary. In particular, to help make the book reasonably self-contained, we have included introduction/overview material on Nonarchimedean Analysis in Section 5.2, Elliptic Curves in Section 6.3, and Algebraic Geometry in Section 7.2. However, previous familiarity with basic algebraic geometry will certainly be helpful in reading some parts of the book, especially Chapters 4 and 7.

**Cross References and Exercises:** Theorems, propositions, examples, etc. are numbered consecutively within each chapter and cross-references are given in full, for example Proposition 3.2 refers to the second labeled item in Chapter 3. Exercises appear at the end of each chapter and are also numbered consecutively, so Exercise 5.7 is the seventh exercise in Chapter 5. There is an extensive bibliography, with reference numbers in the text given in square brackets.

This book contains a large number of exercises. Some of the exercises are marked with a single asterisk  $*$ , which indicates a hard problem. Others exercises are marked with a double asterisk  $**$ , which means that the author does not know how to solve them. However, it should be noted that these “unsolved” problems are of varying degrees of difficulty, and in some cases their designation reflects only the author's lack of perspicacity. On the other hand, some of the unsolved problems are undoubtedly quite difficult. The author solicits solutions to the  $**$  marked problems, as well as solutions to the exercises that are posed as questions, for inclusion in later editions. The reader will find additional notes and references for the exercises on page 436.

**Standard Notation:** Throughout this book we use the standard symbols

$$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_q, \mathbb{Z}_p, \mathbb{A}^N, \text{ and } \mathbb{P}^N$$

to represent the integers, rational numbers, real numbers, complex numbers, field with  $q$  elements, ring of  $p$ -adic integers,  $N$ -dimensional affine space, and  $N$ -dimensional projective space, respectively. Additional notation is defined as it is introduced in the text. A detailed list of notation may be found on page 440.

## Exercises

**0.1.** Let  $S$  be a set and  $\phi : S \rightarrow S$  a function.

- (a) If  $S$  is a finite set, prove that  $\phi$  is bijective if and only if  $\text{Per}(\phi, S) = S$ .
- (b) In general, prove that if  $\text{Per}(\phi, S) = S$ , then  $\phi$  is bijective.
- (c) Give an example of an infinite set  $S$  and map  $\phi$  with the property that  $\phi$  is bijective and  $\text{Per}(\phi, S) \neq S$ .

(d) If  $\phi$  is injective, prove that  $\text{PrePer}(\phi, S) = \text{Per}(\phi, S)$ .

**0.2.** Let  $S$  be a set, let  $\phi : S \rightarrow S$  and  $\psi : S \rightarrow S$  be two maps of  $S$  to itself, and suppose that  $\phi$  and  $\psi$  commute, i.e., assume that  $\phi \circ \psi = \psi \circ \phi$ .

(a) Prove that  $\psi(\text{PrePer}(\phi)) \subset \text{PrePer}(\phi)$ .

(b) Assume further that  $\psi$  is a finite-to-one surjective map, i.e.,  $\psi(S) = S$ , and for every  $x \in S$ , the inverse image  $\psi^{-1}(x)$  is finite. Prove that  $\psi(\text{PrePer}(\phi)) = \text{PrePer}(\phi)$ .

(c) We say that a point  $P \in S$  is an *isolated preperiodic point* of  $\phi$  if there are integers  $n > m$  such that  $\phi^n(P) = \phi^m(P)$  and such that the set

$$\{Q \in S : \phi^n(Q) = \phi^m(Q)\}$$

is finite. Suppose that every preperiodic point of  $\phi$  is isolated. Prove that

$$\text{PrePer}(\phi) \subset \text{PrePer}(\psi).$$

Conclude that if the commuting maps  $\phi$  and  $\psi$  both have isolated preperiodic points, then  $\text{PrePer}(\phi) = \text{PrePer}(\psi)$ .

**0.3.** Let  $\phi(z) = z^d + a \in \mathbb{Z}[z]$  and let  $p$  be a prime. Prove that  $\text{Per}(\phi, \mathbb{F}_p) = \mathbb{F}_p$  if and only if  $\gcd(d, p-1) = 1$ .

**0.4.** Let  $G$  be a group and let  $\phi : G \rightarrow G$  be a homomorphism.

(a) Prove that  $\text{Per}(\phi, G)$  is a subgroup of  $G$ .

(b) Is  $\text{PrePer}(\phi, G)$  a subgroup of  $G$ ? Either prove that it is a subgroup or give a counterexample.

**0.5.** Let  $G$  be a topological group, that is,  $G$  is a topological space with a group structure such that the group composition and inversion laws are continuous maps. Let  $\phi : G \rightarrow G$  be a continuous homomorphism. Exercise 0.4 says that  $\text{Per}(\phi, G)$  is a subgroup of  $G$ , so its topological closure  $\overline{\text{Per}(\phi, G)}$  is also a subgroup of  $G$ . Compute this topological closure for each of the following examples. (In each example,  $d \geq 2$  is a fixed integer.)

(a)  $G = \mathbb{C}^*$  and  $\phi(\alpha) = \alpha^d$ .

(b)  $G = \mathbb{R}^*$  and  $\phi(\alpha) = \alpha^d$ .

(c)  $G = \mathbb{R}^N / \mathbb{Z}^N$  and  $\phi(\alpha) = d\alpha \bmod \mathbb{Z}^N$ .

**0.6.** (a) Describe  $\text{Per}(\phi, \mathbb{Q})$  for the function  $\phi(z) = z^2 + 1$ .

(b) Describe  $\text{Per}(\phi, \mathbb{Q})$  for the function  $\phi(z) = z^2 - 1$ .

(c) Let  $\phi(z) \in \mathbb{Z}[z]$  be a monic polynomial of degree at least two. Prove that  $\text{Per}(\phi, \mathbb{Q})$  is finite. (*Hint.* First prove that  $\text{Per}(\phi, \mathbb{Q}) \subset \mathbb{Z}$ .)

(d) Same question as (c), but now  $\phi(z) \in \mathbb{Q}[z]$  has rational coefficients and is not assumed to be monic.

**0.7.** Let  $\phi(z) = z + 1/z$  and let  $\alpha \in \mathbb{Q}^*$ . Prove that  $\mathcal{O}_\phi(\alpha) \cap \mathbb{Z}$  is finite. What is the largest number of points that it can contain?