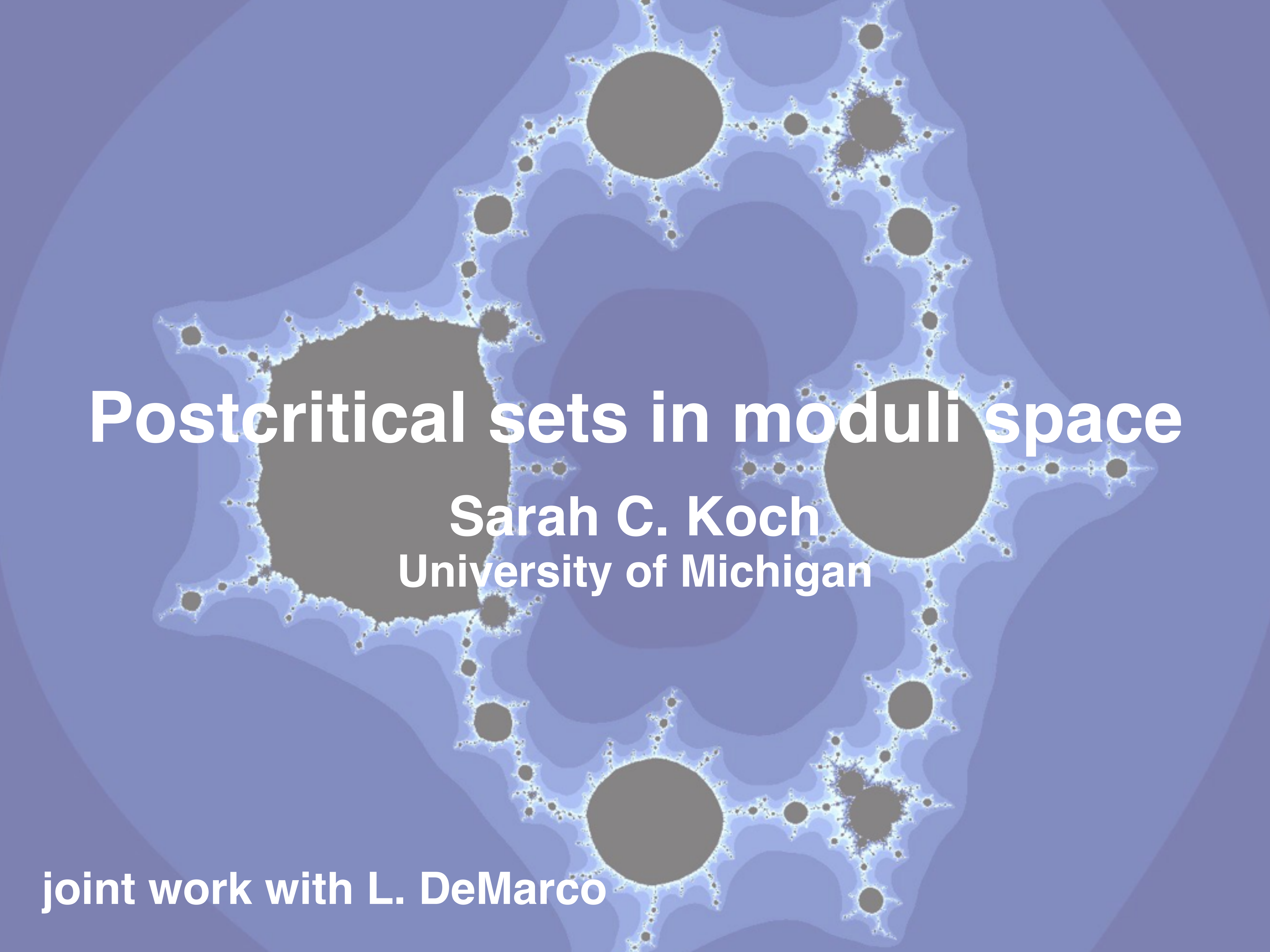

Postcritical sets in moduli space

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joint work with L. DeMarco

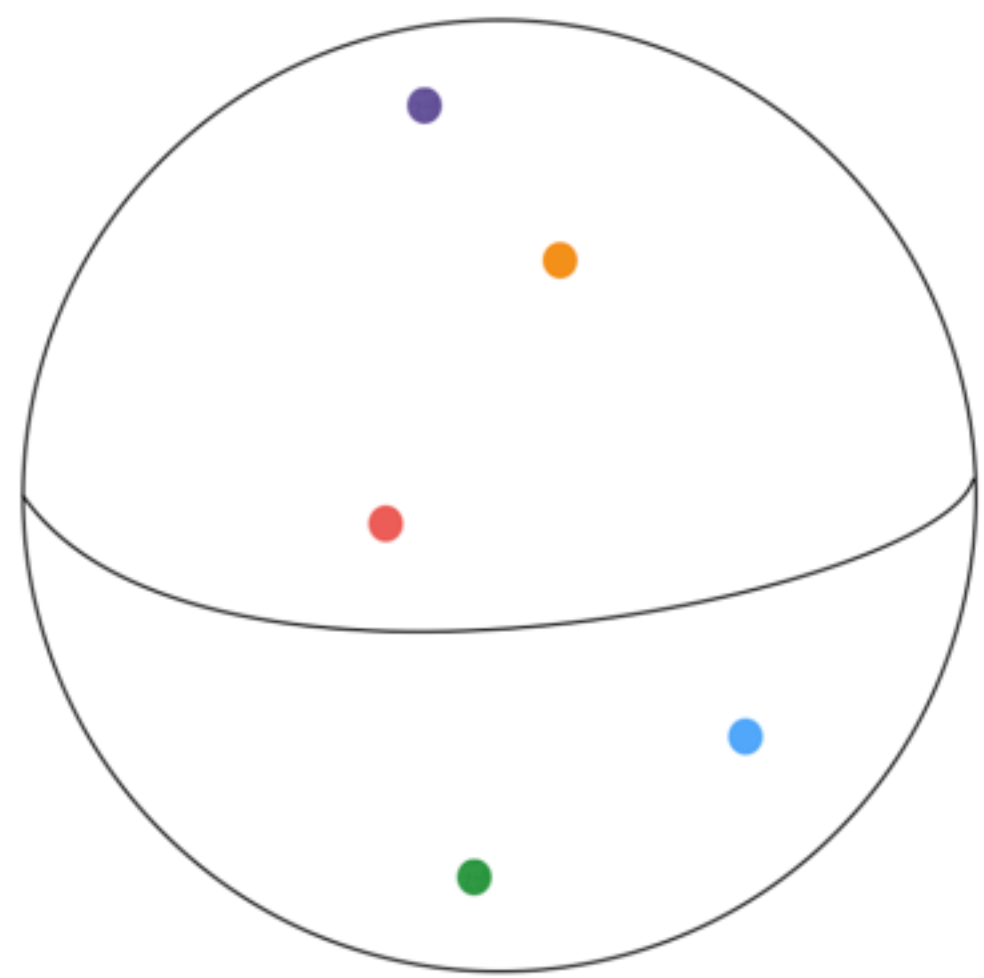
$f : \mathbb{P}^1 \to \mathbb{P}^1 \quad d \geqslant 2 \qquad f$ has $2d-2$ critical points

$P := \cup_{n>0} f^n(\text{crit pts})$

f is *postcritically finite* if $|P| < \infty$.

$P = \{p_1, \ldots, p_n\}$

f determines a point in the *moduli space* $\mathcal{M}_{0,n}$



Laura's Question:

Which configurations arise as postcritical sets for postcritically finite rational maps with n postcritical points?

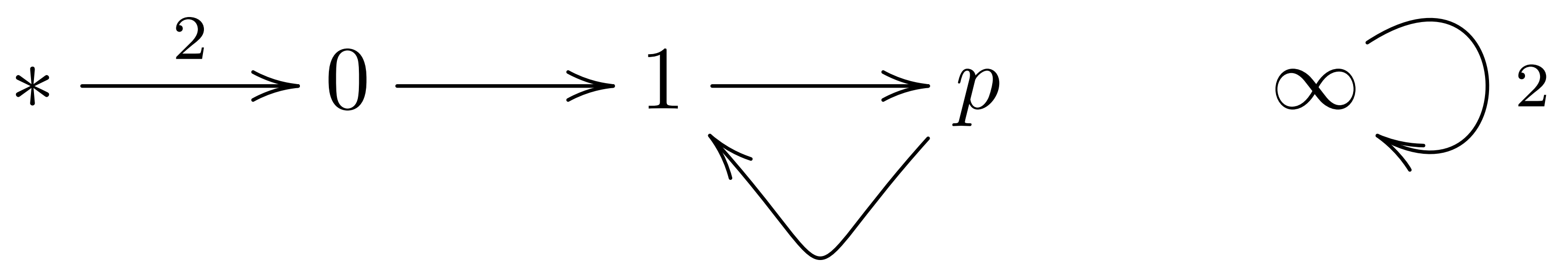
One answer:

These configurations are dense in the analytic topology.

Example $n = 4$

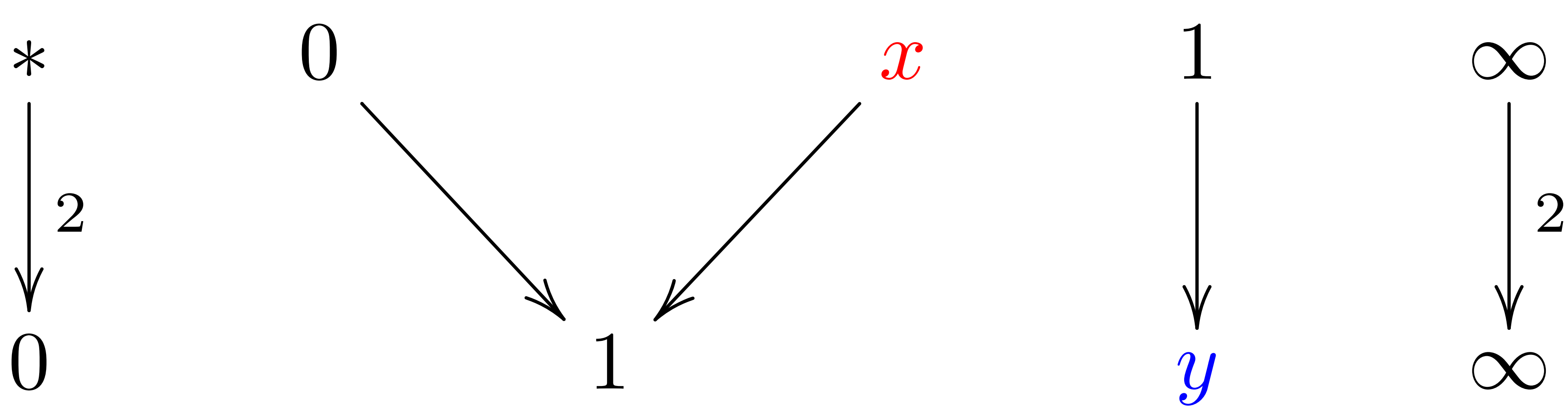
Exercise:

Find all quadratic polynomials with the following portrait:



Method 1: solving dynamical equations

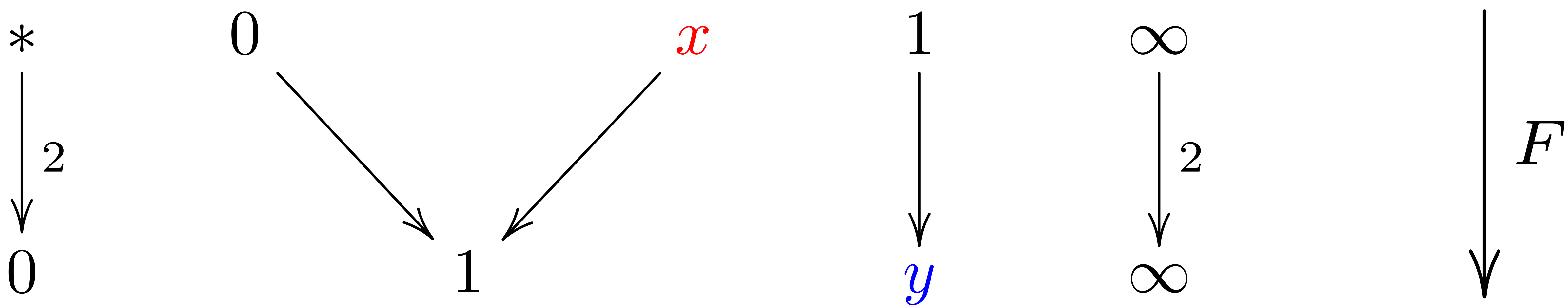
Method 2: recast the question as a fixed point problem



$$g : \mathbb{P}^1 \to \mathbb{P}^1$$

$$g : x \mapsto y$$

Method 2: recast the question as a fixed point problem



F quadratic polynomial

$* = x/2$

critical value 0

$F(0) = 1$

$$F_x(t) = \frac{(2t - x)^2}{x^2}$$

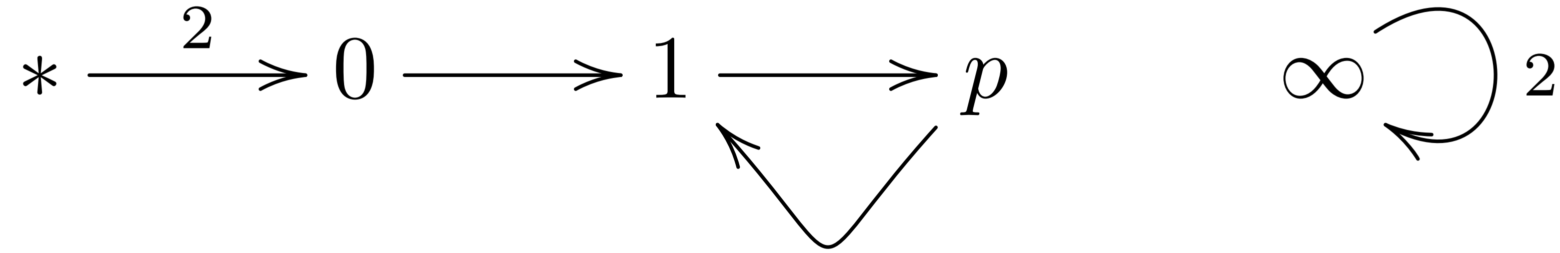
$$F_x(1) = \frac{(2 - x)^2}{x^2} = y$$

$$\mathcal{M}_{0,4} \approx \mathbb{P}^1 - \{0, 1, \infty\}$$

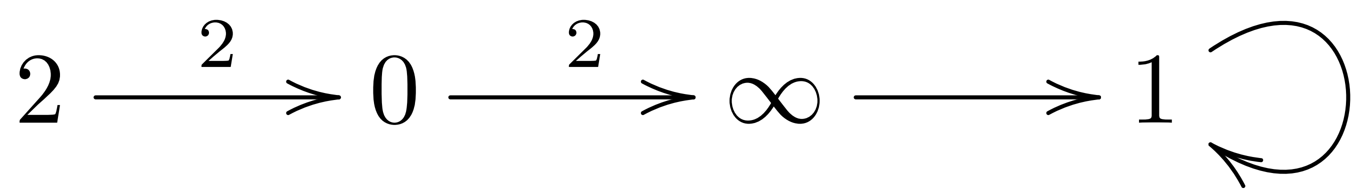
$$g : \mathbb{P}^1 \to \mathbb{P}^1$$

$$g : x \mapsto y$$

$$g : x \mapsto \frac{(2 - x)^2}{x^2}$$



$$g : \mathbb{P}^1 \to \mathbb{P}^1 \quad g : x \mapsto \frac{(2-x)^2}{x^2}$$



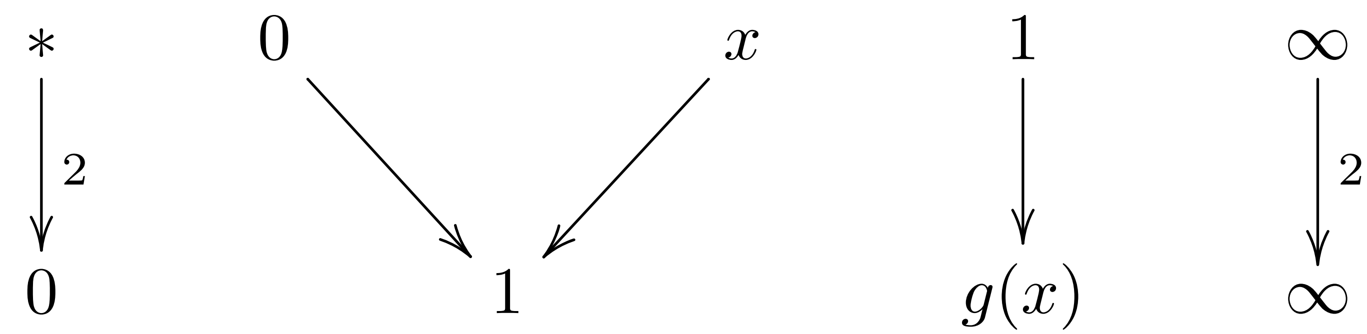
Theorem. The Julia set $J_g = \mathbb{P}^1$.

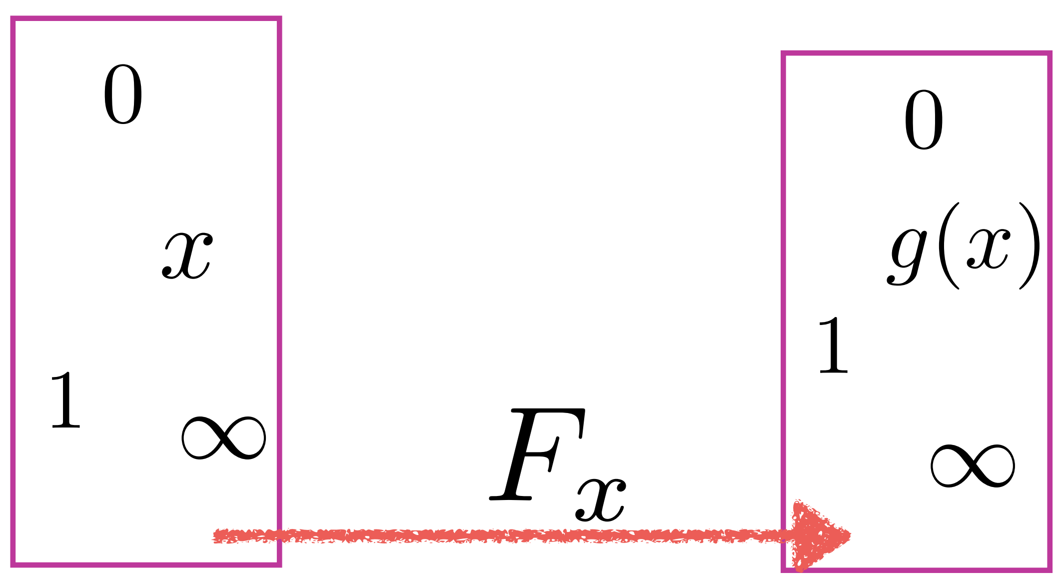
Corollary. Repelling periodic points of g are dense in $\mathbb{P}^1$.

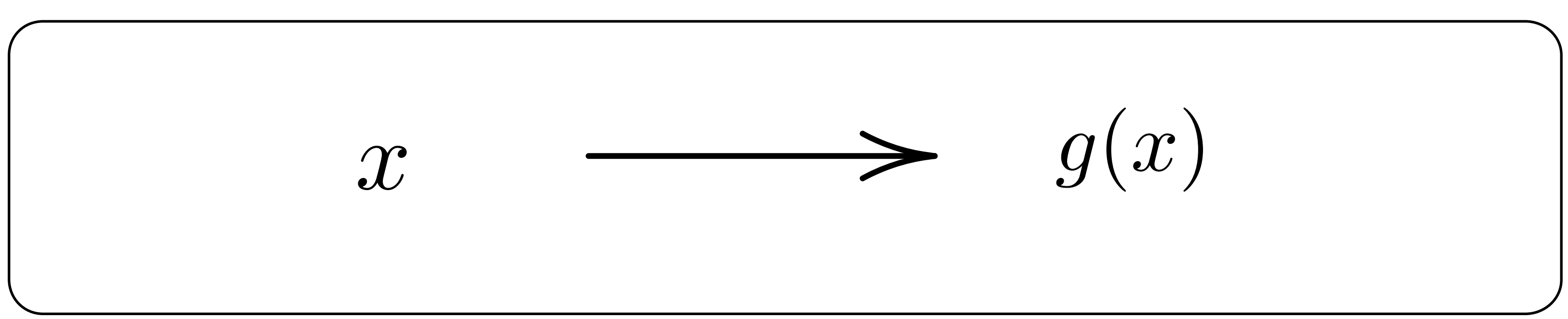
Periodic cycles

$$g : \mathbb{P}^1 \to \mathbb{P}^1 \qquad g : x \mapsto \frac{(2-x)^2}{x^2}$$

$$F_x(t) = \frac{(2t-x)^2}{x^2}$$







$\mathcal{M}_{0,4}$

Periodic cycles

$$g : \mathbb{P}^1 \to \mathbb{P}^1 \qquad g : x \mapsto \frac{(2-x)^2}{x^2}$$

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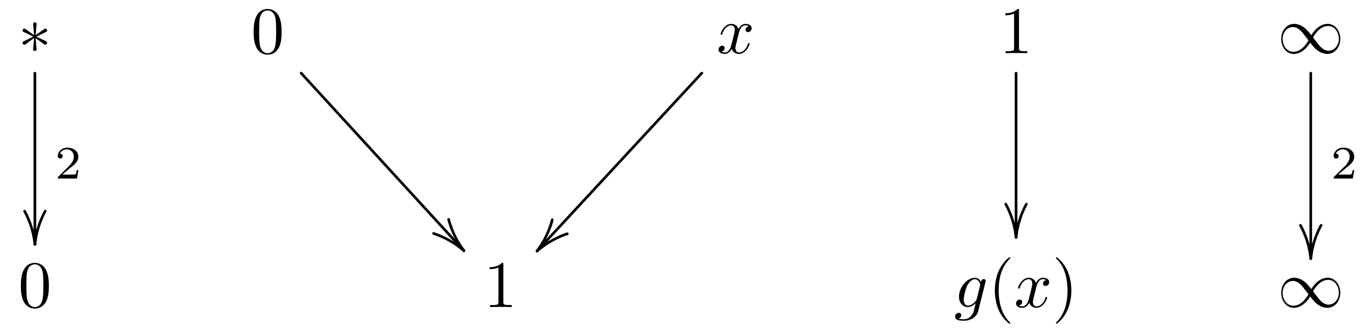
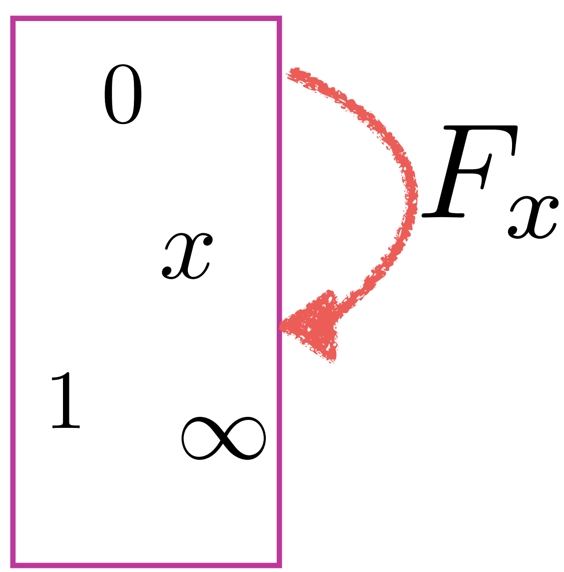
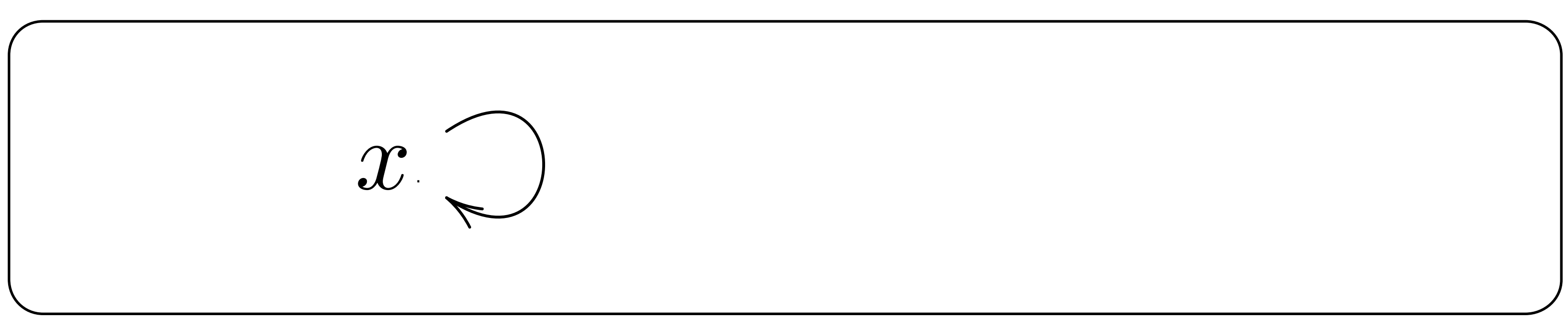




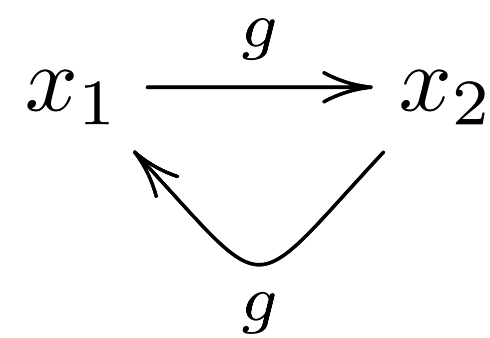
Fixed fiber



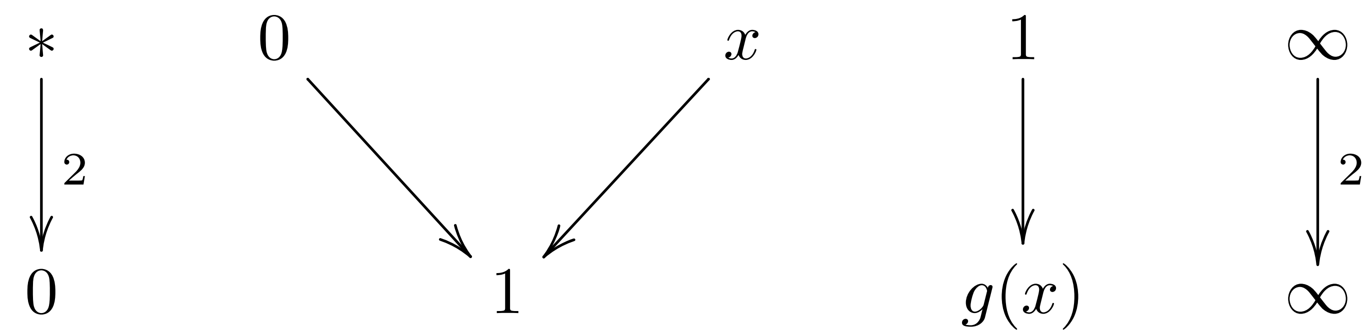
$\mathcal{M}_{0,4}$

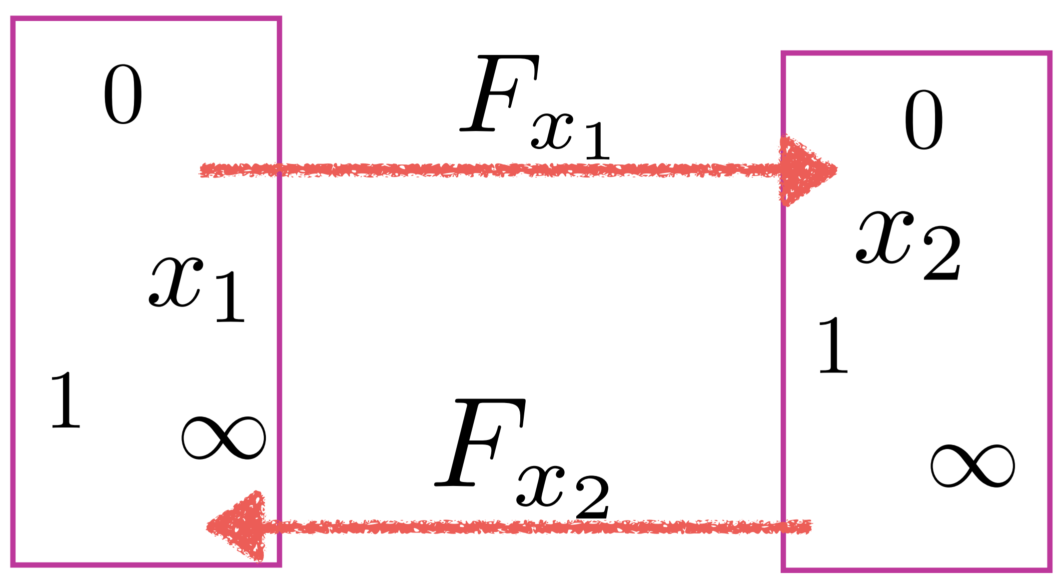
Periodic cycles

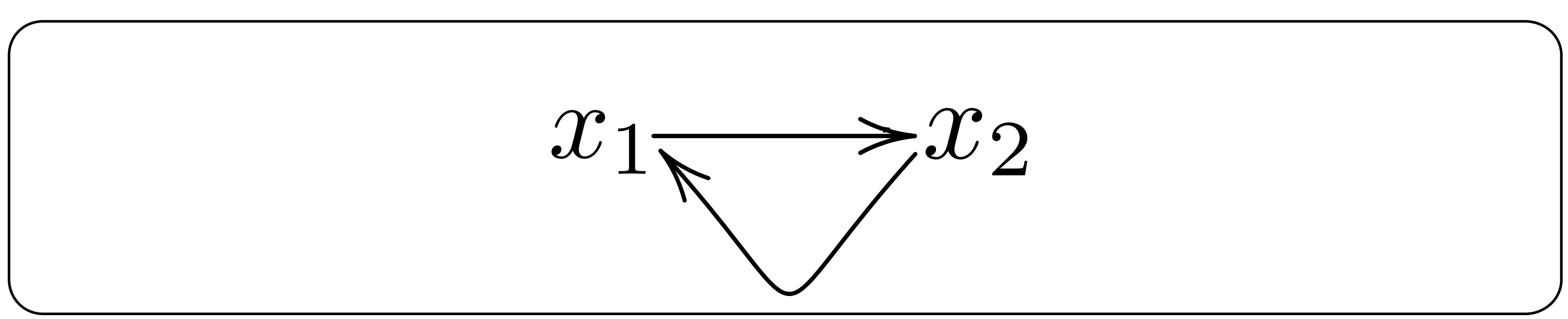
$$g : \mathbb{P}^1 \to \mathbb{P}^1 \qquad g : x \mapsto \frac{(2-x)^2}{x^2}$$



$$F_x(t) = \frac{(2t-x)^2}{x^2}$$







$\mathcal{M}_{0,4}$

If $x_1, \dots, x_m$ is a periodic cycle of period m for $g : \mathbb{P}^1 \to \mathbb{P}^1$ then

$$F_{x_m} \circ \cdots \circ F_{x_1} : (\mathbb{P}^1, \{0, 1, \infty, x_1\}) \to (\mathbb{P}^1, \{0, 1, \infty, x_1\})$$

is a postcritically finite polynomial of degree 2^m.

$$F_x(t) = \frac{(2t - x)^2}{x^2}$$

$$F_{x_{i+1}} \circ \cdots \circ F_{x_m} \circ F_{x_1} \cdots \circ F_{x_i} : (\mathbb{P}^1, \{0, 1, \infty, x_i\}) \to (\mathbb{P}^1, \{0, 1, \infty, x_i\})$$

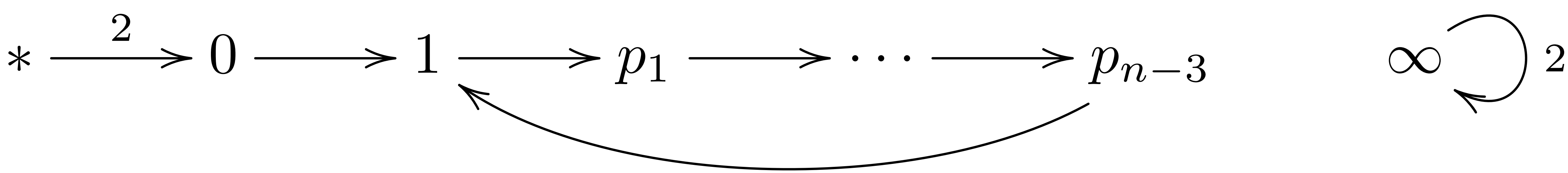
For every periodic cycle, $x_1, \dots, x_m$ for every x_i, there is a postcritically finite polynomial of degree 2^m with postcritical set equal to $\{0, 1, \infty, x_i\}$.

Since periodic cycles of g are dense in $\mathbb{P}^1$ the postcritical configurations are dense in $\mathcal{M}_{0,4}$ **!!**

The general case, n points

Strategy:

- Identify $\mathcal{M}_{0,n}$ with an open subset of $\mathbb{P}^{n-3}$
- Define a map $g : \mathbb{P}^{n-3} \to \mathbb{P}^{n-3}$ so that for every periodic cycle $x_1, \ldots, x_m$, for every x_i, there is a a postcritically finite polynomial of degree 2^m with postcritical set $\{0, 1, \infty, \text{the coordinates of } x_i\}$.
- Prove that the periodic cycles of g are dense in $\mathbb{P}^{n-3}$.



Theorem. This portrait determines a postcritically finite $g : \mathbb{P}^{n-3} \to \mathbb{P}^{n-3}$ with periodic cycles as above.

$$g : \mathbb{P}^{n-3} \to \mathbb{P}^{n-3}$$

Periodic cycles are dense in $\mathbb{P}^{n-3}$?

Critical components are strictly preperiodic. (not enough!)

We must check the places where the critical locus hits the periodic components of the postcritical locus...

all the way down.

If none of these intersections is periodic, then this map $g : \mathbb{P}^{n-3} \to \mathbb{P}^{n-3}$ is *strictly critically finite.*

Theorem. (Ueda). If $G : \mathbb{P}^N \to \mathbb{P}^N$ is SCF, then

- repelling periodic cycles are dense in $\mathbb{P}^N$
- the Julia set is equal to $\mathbb{P}^N$

$$g : \mathbb{P}^{n-3} \to \mathbb{P}^{n-3}$$

is SCF, so repelling cycles are therefore dense.

Theorem. (DeMarco - K.) Postcritical configurations are dense in $\mathcal{M}_{0,n}$.

These configurations can be achieved by compositions of *quadratic polynomials.*

Proof: relies on Ueda's theorem, analytic.

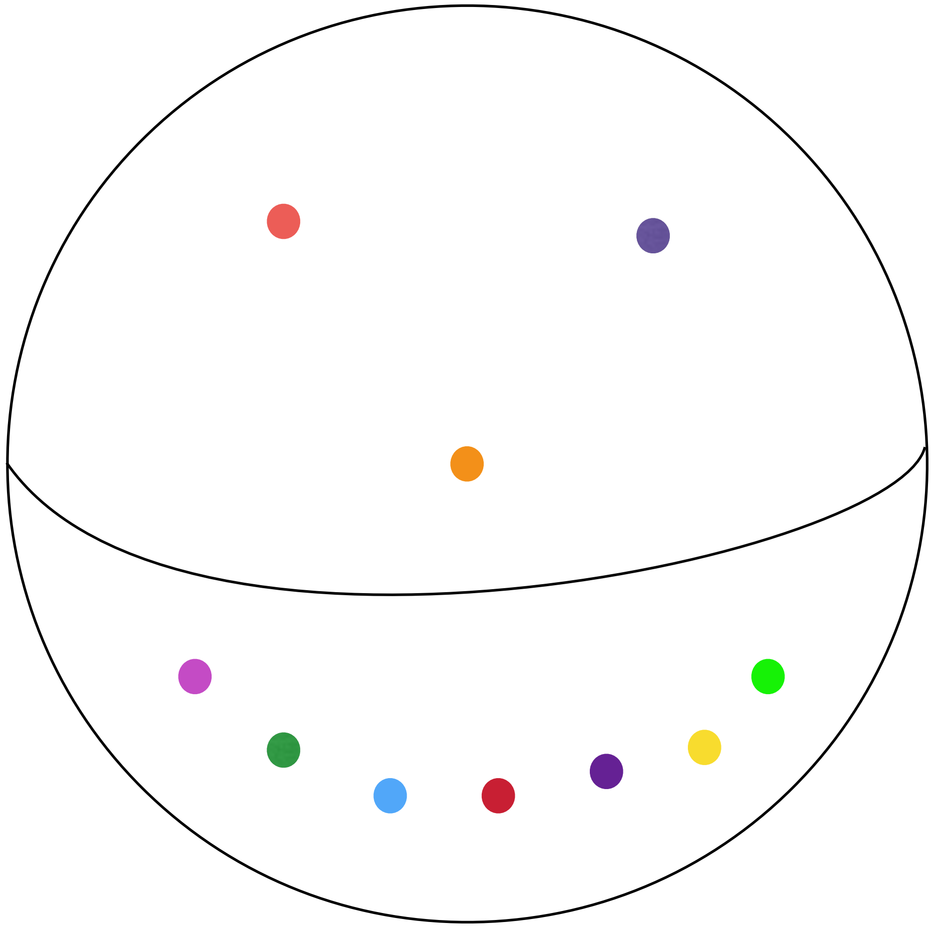
Flashforward... November 2015

Theorem. (McMullen) A point in $\mathcal{M}_{0,n}$ is a postcritical configuration if and only if it is defined over $\overline{\mathbb{Q}}$.

Theorem. (McMullen) A point in $\mathcal{M}_{0,n}$ is a postcritical configuration if and only it is defined over $\overline{\mathbb{Q}}$.

Proof: Thurston rigidity, Belyi's theorem

algebraic?



Thanks for
your attention!