A Very Elementary Proof of a Conjecture of B. and M. Shapiro for Cubic Rational Functions

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- $K =$ field with characteristic 0
- $\overline{K} =$ algebraic closure of K
- $f, g \in \overline{K}(z)$ are *equivalent* if there exists a linear fractional transformation $\sigma \in \overline{K}$ such that $f = \sigma \circ g$.

A Case of the B. and M. Shapiro Conjecture

Theorem (Eremenko-Gabrielov)

If $f \in \mathbb{C}(z)$ is a rational function with only real critical points, then f is equivalent to a rational function with real coefficients.

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- **Eremenko and Gabrielov**: Using topological, combinatorial, and complex analytic techniques construct exactly $\rho(d)$ real rational functions with a given set of real critical points.
- **But!** The relationship between a rational function and its critical points is purely algebraic, via the roots of the derivative.
- This leads to the following question:

Question:

Is there a truly elementary proof of the Eremenko and Gabrielov's result?

Corollary (Faber, T.)

Using only algebraic techniques we can show, if $f \in \mathbb{C}(z)$ is a degree 3 rational function with only real critical points, then f is equivalent to a rational function with real coefficients.

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- The quadratic case is trivial over any field.
- If $f \in \overline{K}(z)$ has critical points $c_1, c_2 \in \mathbb{P}^1(K)$, $c_1 \neq \infty$, then either $f = \left(\frac{z-c_1}{z-c_2}\right)^2$ or $f = (z - c_1)^2$.

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- f has 4 distinct critical points.

Normal form for cubic

- We begin with a normal form for cubic functions. For $u \in \overline{K} \setminus \{-1, -2\}$, define

$$f_u(z) = \frac{z^2(z+u)}{(2u+3)z - (u+2)}. \tag{1}$$

- This function has the property that it fixes $0, 1$, and ∞, and each of these three points is critical.

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Lemma

A cubic rational function that is critical at $0, 1$, and ∞ is equivalent to a unique f_u, and the fourth critical point is $\phi(u) = -u\frac{u+2}{2u+3}$.

Proposition

If $f_u \in \overline{K}(z)$ is equivalent to a rational function with K-coefficients, then $u \in K$.

Algebraic Condition

Definition

*For a field K and rational function $\phi \in K(z)$, we say K is ϕ-**perfect** if the map $\phi : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ is surjective.*

Theorem (Faber, T.)

Let K be a field of characteristic zero with algebraic closure $\overline{K}$. The following statements are equivalent:

1. *Any cubic rational function $f \in \overline{K}(z)$ with K-rational critical points is equivalent to a rational function in $K(z)$.*
2. *K is ϕ-perfect, where $\phi(z) = -z\frac{z+2}{2z+3}$.*

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(1) $\Rightarrow$ (2).

- Take $y \in K$. Solve the equation $\phi(u) = y$ with $u \in K$. If $y = \infty$, then we may take $u = -3/2$

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- Otherwise, choose $u \in \bar{K}$ such that $\phi(u) = y$.
- Then the function f_u has K-rational critical points $\{0, 1, \infty, y\}$.
- Since f_u is equivalent to a rational function with K-coefficients, the proposition implies that $u \in K$.

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- Suppose that f has at least three critical points. WLOG, assume that $0, 1$, and ∞ are among them.
- By the lemma, f is equivalent to f_u for some $u \in \bar{K}$.
- The remaining critical point is $\phi(u)$. By assumption, both solutions of $\phi(z) = \phi(u)$ lie in $\mathbb{P}^1(K)$, so that $u \in K$. That is, f is equivalent to a rational function with K-coefficients.

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Corollary (Faber, T.)

Using only algebraic techniques, we can show if $f \in \mathbb{C}(z)$ is a cubic rational function with only real critical points, then f is equivalent to a real rational function.

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- $\phi(-3/2) = \infty$, and if $y \in \mathbb{R}$, then the equation $\phi(z) = y$ is equivalent to a quadratic equation with discriminant $4(y^2 - y + 1) = (2y - 1)^2 + 3 > 0$.

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- Hence $\phi(z) = y$ has a real solution.

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Question:

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Are there other fields K for which our corollary will hold? That is can we show there are other fields K which are ϕ-perfect, where $\phi(z) = -z\frac{z+2}{2z+3}$?

- Number fields are not ϕ-perfect for any ϕ with $\deg(\phi) \geq 2$. We can show this using a canonical height argument.
- If $\phi(z) = -z\frac{z+2}{2z+3}$, the field $\mathbb{Q}_p$ is ϕ-perfect iff $p = 3$.

Proof for $p > 3$.

- The resultant of $\phi(z) = -z\frac{z+2}{2z+3}$ is 3 $\Rightarrow$ reduced modulo p to yield a quadratic function $\tilde{\phi} \in \mathbb{F}_p(z)$.

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- Since this is a finite set, $\tilde{\phi}$ also fails to be surjective. Choose $\tilde{y} \in \mathbb{F}_p$ such that $\tilde{\phi}(z) = \tilde{y}$ has no solution in $\mathbb{F}_p$.
- By Hensel's lemma, it follows that $\phi(z) = y$ has no solution in $\mathbb{Z}_p$ for any $y \in \mathbb{Z}_p$ such that $y \equiv \tilde{y} \pmod{p}$.

□

Proof continued.

- It remains to show that $\phi(z) = y$ has no solution in $\mathbb{Q}_p \setminus \mathbb{Z}_p$.
- If $\phi(x) = y$ with $|x|_p > 1$, then

$$|\phi(x)|_p = |x|_p \cdot \left|\frac{1 + 2/x}{2 + 3/x}\right|_p = |x|_p > 1,$$

which contradicts $y \in \mathbb{Z}_p$. Hence $\phi(z) = y$ has no solution in $\mathbb{P}^1(\mathbb{Q}_p)$, and we have proved that $\mathbb{Q}_p$ is not ϕ-perfect.

□

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- $d=3$: $2d-5=1$. Express the remaining critical point as a function of the free parameter.
- Is it possible to solve for the critical points as explicit functions of parameters for $d>3$?

Further Thoughts continued

- Bézout's Theorem gives an upper bound of 2^{2d-5} solutions for a general system of $2d-5$ conics, while Goldberg bounds the number of distinct solutions by the smaller quantity

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- This suggests a substantial amount of extra structure in our system of equations, which may make progress possible.

Quartic

Example

$$f(z) = \frac{(z^4 + az^3 + bz^2)}{cz^2 + dz + 1 + a + b - c - d}$$

where $d = \frac{(3a^2+5ab+2b^2-2ac-2bc+7a+6b-2c+4)}{a+b+1}$.
The critical points are:

$$t_1 = -(ac + 9a + 6b - 4c + 12)/(2c)$$
$$t_2 = (6a^2 + 4ab - 3ac + 9a + 2b)/(2c)$$
$$t_3 = -b(2a + b - c + 3)/c$$

THANK YOU!