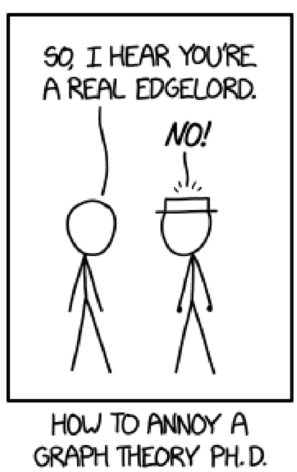
Name: ______________________________

Combinatorics/Graph Theory
Math 750 — Silverman
Unit 2 Exam — Tues Nov 6, 2018

INSTRUCTIONS—Read Carefully

- Time: 80 minutes
- There are 6 problems.
- Write your name *neatly* at the top of this page.
- Write your final answer in the answer box, if one is provided.
- **Show all your work**. Partial credit will be given for substantial progress towards the solution. **No credit** will be given for answers with no explanation. Feel free to continue writing on the back of the page if you need more room.

Problem	Value	Points
1	20	
2	20	
3	20	
4	20	
5	20	
6	15	
Total	115	



XKCD #2036: Edge Lord

Problem 1. (20 points)

It's Election Day!!

(a) There are 10 candidates running for city council, and 3 of them will be elected. How many different city councils are possible?

(b) Suppose that the candidate with the most votes becomes president of the council, and the other two winning candidates form the rest of the council. Now how many different city councils are possible?

Solution. (a) There are 10 candidates, and we need to know the number of ways of choosing 3 of them. This is the combinatorial symbol

$$\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2} = \boxed{120}.$$

(b) Now it matters who gets the most votes, since they become president. So there are 10 possibilities for the president, and then there are $\binom{9}{2} = 36$ ways to select the other two council members. Hence

$$\begin{aligned}
&\text{Number of city councils} \\
&\quad = \begin{pmatrix}\text{Number of ways to} \\ \text{select a president}\end{pmatrix} \cdot \begin{pmatrix}\text{Number of ways to select} \\ \text{two more council members after} \\ \text{having selected a president}\end{pmatrix} \\
&\quad = \binom{10}{1} \cdot \binom{9}{2} \\
&\quad = 10 \cdot \frac{9 \cdot 8}{2} = \boxed{360}.
\end{aligned}$$

Problem 2. (20 points) Recall that a *permutation of* $[n]$ is a rearrangement of the numbers from 1 to n, while a *derangement of* $[n]$ is a permutation in which no number remains at its original location.

Definition. An *almost-derangement of* $[n]$ is a permutation of $[n]$ in which <u>at most</u> one number remains at its original location

Let

$$A_n = \text{the number of almost-derangements of } [n].$$

Find a formula for A_n, similar to the formula that we found for the number of derangements D_n. Use your formula to compute the values of A_4 and A_5.

Solution. Every derangement is an almost-derangement, so

$$A_n = D_n + (\#\text{ of permutations that have exactly one fixed element}).$$

Let

$$X_i = \{\text{permutations that fix } i\text{, but don't fix any other element of } [n]\}.$$

Then

$$A_n = D_n + |X_1 \cup X_2 \cup \cdots \cup X_n|.$$

But $X_1, X_2, \ldots, X_n$ are pairwise disjoint, i.e., if $i \neq j$, then $X_i \cap X_j = \emptyset$, so

$$|X_1 \cup X_2 \cup \cdots \cup X_n| = |X_1| + |X_2| + \cdots + |X_n|.$$

This gives us the formula

$$A_n = D_n + |X_1| + |X_2| + \cdots + |X_n|.$$

How big is X_i? We need a permutation that fixes i, and is a derangement of the other $n-1$ numbers in $[n]$. So

$$|X_i| = D_{n-1},$$

which gives the appealing formula

$$\boxed{A_n = D_n + nD_{n-1}}.$$

Using the formula that we know for D_n, we can write this in various ways, such as

$$\begin{aligned}
A_n &= D_n + nD_{n-1} \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + n \cdot (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \\
&= n! \left(\sum_{k=0}^{n} \frac{(-1)^k}{k!} + \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right) \\
&= n! \left(2 \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + \frac{(-1)^n}{n!} \right) \\
&= 2n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + (-1)^n.
\end{aligned}$$

So for example:

$$\begin{aligned} A_4 &= 2 \cdot 4! \left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) + 1 \\ &= 24 - 8 + 1 = \boxed{17}. \\ A_5 &= 2 \cdot 5! \left(1 - 1 + \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!}\right) - 1 \\ &= 120 - 40 + 10 - 1 = \boxed{89}. \end{aligned}$$

Problem 3. (20 points) Let $n \geq k \geq 1$ be integers.
(If you do not see you how to solve this problem in general, you can get partial credit by solving the special case $n = 4$ and $k = 2$.)

(a) A function $f : [k] \to [n]$ is said to be *strictly increasing* if

$$f(1) < f(2) < \cdots < f(k-1) < f(k).$$

How many strictly increasing functions are there from $[k]$ to $[n]$?

(b) A function $f : [k] \to [n]$ is said to be *non-decreasing* if

$$f(1) \leq f(2) \leq \cdots \leq f(k-1) \leq f(k).$$

How many non-decreasing functions are there from $[k]$ to $[n]$?

Solution. (a) Let's give the set of values of the function f a name, say

$$R_f = \{f(1), f(2), \ldots, f(k)\}.$$

Since we want f to be strictly increasing, this set completely determines f, since we know that $f(1)$ is the smallest value in the set, and $f(2)$ is the second smallest value, and so on. In other words, each f is associated to a subset of $[n]$ consisting of k elements, and similarly any subset of $[n]$ consisting of k elements will give exactly one strictly increasing function from $[n]$ to $[k]$. Hence

$$\begin{pmatrix}\text{Number of strictly} \\ \text{increasing functions} \\ \text{from } [k] \text{ to } [n]\end{pmatrix} = \begin{pmatrix}\text{Number of subsets} \\ \text{of } [n] \text{ containing} \\ \text{exactly } k \text{ elements}\end{pmatrix} = \boxed{\binom{n}{k}}.$$

For partial credit, the strictly increasing functions for $n = 4$ and $k = 2$ are listed in the following table:

	$f(1)$	$f(2)$
f_1	1	2
f_2	1	3
f_3	1	4
f_4	2	3
f_5	2	4
f_6	3	4

There are $\binom{4}{2} = 6$ of them.

(b) This is similar, except now the values $f(1), f(2), \ldots, f(k)$ are allowed to contain repetitions. So we need to choose k elements from $[n]$, but now we're allowed to choose the same element of $[n]$ more than once. Thus we need to choose some 1's, and then choose some 2's, and so on. Say we choose

$$\{\underbrace{1, 1, \ldots, 1}_{m_1 \text{ copies}}, \underbrace{2, 2, \ldots, 2}_{m_2 \text{ copies}}, \cdots, \underbrace{n, n, \ldots, n}_{m_n \text{ copies}}, \}.$$

which corresponds to the non-decreasing function

$$\begin{gathered} f(1) = f(2) = \cdots = f(m_1) = 1 \quad \text{and} \\ f(m_1 + 1) = f(m_1 + 2) = \cdots + f(m_1 + m_2) = 2, \quad \text{etc.} \end{gathered}$$

Of course, we need

$$m_1 + m_2 + \cdots + m_n = k.$$

Counting these is a classic "stars-and-bars" type problem. We take $k+n-1$ locations, and we need to select k of them to be stars and $n-1$ of them to be bars. Then m_i, which is the number of copies of i, is the number of stars between the $i - 1$'st bar and the i'th bar. Hence

$$\begin{pmatrix} \text{Number of non-} \\ \text{decreasing functions} \\ \text{from } [k] \text{ to } [n] \end{pmatrix} = \begin{pmatrix} \text{Number of ways to} \\ \text{place } k \text{ stars and} \\ n-1 \text{ bars into} \\ k+n-1 \text{ locations} \end{pmatrix} = \boxed{\binom{k+n-1}{k}}.$$

For partial credit, the non-decreasing functions for $n = 4$ and $k = 2$ are listed in the following table:

	$f(1)$	$f(2)$
f_1	1	1
f_2	1	2
f_3	1	3
f_4	1	4
f_5	2	2
f_6	2	3
f_7	2	4
f_8	3	3
f_9	3	4
f_{10}	4	4

There are $\binom{2+4-1}{2} = \binom{5}{2} = 10$ of them.

Problem 4. (20 points) In this problem, when I ask you to count graphs, you should view two graphs as being the same if they are isomorphic. In other words, the way that the vertices are labeled does not matter. Also, for this problem, your proofs can consist of drawing all of the graphs.

(a) How many graphs are there with exactly 4 vertices such that the graph contain a 3-cycle?

(b) How many graphs are there with all of the following properties:
- G has exactly 6 vertices.
- G contains a 4-cycle, and no other cycles.
- G is connected.

Solution. (a) Three of the vertices need to be used to form the 3-cycle, let's call those vertices A, B, and C. That leaves one more vertex D. It can be isolated, i.e., D might not connect to any other vertex, or it can connect to one of A, B, C, or it can connect to 2 of A, B, C, or it can connect to all three of A, B, C. (Note that if D connects to one of A, B, C, the picture is the same regardless of which one D connects to, so we only get one picture.) Hence

$$\#\text{ of 4-vertex graphs with a 3-cycle} = \boxed{4}.$$

See Figure 1 for the pictures.

(b) We start with a 4-cycle, i.e., a square, which we label A, B, C, D. Since the graph is connected, the next vertex E must connect to one of A, B, C, D, and we get the same picture for any choice. So at this

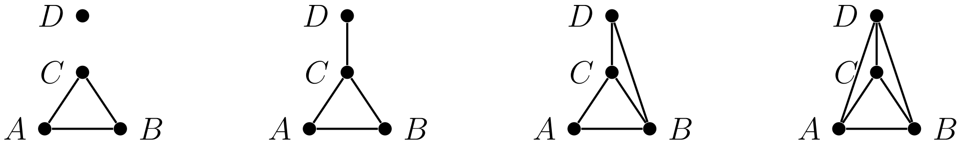


FIGURE 1. 4-vertex graphs with a 3-cycle

stage we only have one picture, namely A, B, C, D form a square, and E is connected to A. It remains to connect the sixth vertex F to the picture. One possibility is to connect F it to E, so we get a tail of length 2 hanging off the square. The other possiblities come from connecting F to one of the vertices in the square. So we can connect F to one of A, B, C, or D, but the graph using B or D are isomorphic, since then E and F are connected to adjacent vertices of the square. So we get three different graphs in this way.

# of 6-vertex connected graphs with a 4-cycle and no other cycles = $\boxed{4}$.

See Figure 2 for the pictures.

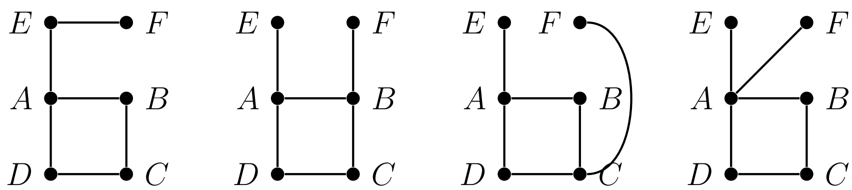


FIGURE 2. Connected 6-vertex graphs with a 4-cycle and no other cycles

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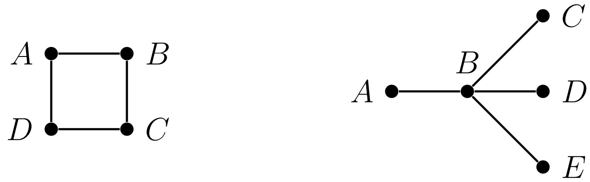
Problem 5. (20 points) Let G be a graph on n vertices. The *complement* of G, denoted $\overline{G}$, is the graph obtained from G by changing edges to non-edges and changing non-edges to edges. In other words, $ij \in E(G)$ if and only if $ij \notin E(\overline{G})$.

(a) For each of the following graphs G, draw the graph $\overline{G}$.



(b) Recall that $\chi(G)$ denotes the chromatic number of G. Prove that

$$\chi(G) \cdot \chi(\overline{G}) \geq n.$$

> *Hint*. Take a coloring of G and a coloring of $\overline{G}$ and use them to build a coloring of K_n.

Solution. (a)

(b) Let $k = \chi(G)$ be the chromatic number of G, and let $\ell = \chi(\overline{G})$ be the chromatic number of $\overline{G}$. So we have proper colorings

$$c : V(G) \longrightarrow [k] \quad \text{and} \quad \overline{c} : V(\overline{G}) \longrightarrow [\ell].$$

As the hint indicates, we want to use c and $\overline{c}$ to construct a proper $k\ell$-coloring of K_n, the complete graph on n vertices. We'll use as our colors the pairs of integers (a, b) with $a \in [k]$ and $b \in [\ell]$, and for each vertex $i \in V(G)$, we'll assign the color $\bigl(c(i), \overline{c}(i)\bigr)$. In other words, we color G using

$$C : V(G) \longrightarrow [k] \times [\ell], \quad C(i) = \bigl(c(i), \overline{c}(i)\bigr).$$

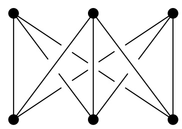
Then given any distinct vertices $i, j \in V(G)$, we have $C(i) \neq C(j)$, since if the edge ij is in $V(G)$, then $c(i) \neq c(j)$, while if ij is not in $V(G)$, then by definition ij is in $V(\overline{G})$, so $\overline{c}(i) \neq \overline{c}(j)$. This shows that if we connect every pair of vertices in $V(G)$ with an edge, then C is a proper coloring, so C is a proper coloring of the complete graph on $\#V(G)$ edges. Hence

$$\#\text{ of colors used by } C \geq \chi(\text{complete graph on } \#V(G) \text{ vertices}).$$

The coloring C using $k\ell$ colors, i.e., it uses $\chi(G)\chi(\overline{G})$ colors. On the other hand, $\#V(G) = n$ and we know that the complete graph on n vertices needs n colors, i.e., $\chi(K_n) = n$. Hence

$$\chi(G)\chi(\overline{G}) \geq n.$$

Problem 6. (15 points) Let $K_{3,3}$ be the usual bipartite graph:

$K_{3,3}$ 

(a) If $K_{3,3}$ were a planar graph, how many faces would it have?
(b) If $K_{3,3}$ were a planar graph, what is the minimum number of edges that each of its faces could it have?
(c) Using your answers to (a) and (b), prove that $K_{3,3}$ is not a planar graph!

Solution. (a) We can count the number of vertices and edges of $K_{3,3}$ by looking at the picture,

$$v = 6 \quad \text{and} \quad e = 9.$$

Euler's VEF formula says that $v - e + f = 2$, so if $K_{3,3}$ were a planar graph, then it would have $\boxed{f = 5}$ faces.
(b) There are graphs that have faces bounded by 3 edges, but that's not possible for $K_{3,3}$, since you can't start at a vertex and get back to that vertex by using only 3 edges. (This is true of any bipartite graph.) The point is that if you start at a top vertex and travel along three edges, you'll end up at a bottom vertex, so you can't close your triangle. Hence if $K_{3,3}$ is bipartite, then every face must have at least $\boxed{\text{4 edges}}$.
(c) We redo the double counting argument that we did in class (and in the notes). We look at the set of *edge-face incidences*, i.e. the set of ordered pairs

$$EF := \big\{(b, c) : b \text{ is an edge bounding the face } c\big\}.$$

We note that each edge bounds exactly 2 faces, and from (b) we know that every face of $K_{3,3}$ would be bounded by at least 4 edges. This allows us to double-count the set EF:

$$|EF| = \sum_{b \in E(G)} \big|\{c \in F(G) : b \text{ is an edge of } c\}\big| = 2\big|E(G)\big| = 2e.$$

$$|EF| = \sum_{c \in F(G)} \big|\{b \in E(G) : b \text{ is an edge of } c\}\big| \geq 4\big|F(G)\big| = 4f.$$

Hence $2e \geq 4f$. But from (a) and the picture of $K_{3,3}$, we know that $e = 9$ and $f = 5$, so this is a contradiction. Hence $K_{3,3}$ is not planar, i.e., it cannot be drawn in the plane with not crossings.