Geometry and Arithmetic of Dominant Rational Self-Maps of Projective Space

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Algebraic Dynamics

Morphisms, Rational Maps, and Dominant Maps

Iteration of a **morphism**

$$\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

is well-defined, as is the orbit $\mathcal{O}_\phi(P)$ of a point.

Iteration of a **dominant rational map**

$$\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$$

is well-defined, but the orbit of P is undefined if some iterate $\phi^n(P)$ is in the indeterminacy locus $I(\phi)$ of ϕ.

We set the notation

$$\mathbb{P}^N_\phi = \bigl\{ P \in \mathbb{P}^N : \mathcal{O}_\phi(P) \cap I(\phi) = \emptyset \bigr\}.$$

The Space of Rational Maps

A degree d rational map

$$\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$$

is described by degree d homogeneous polynomials

$$\phi = [\phi_0, \ldots, \phi_N], \qquad \phi_0, \ldots, \phi_N \in k[X_0, \ldots, X_N],$$

having no common factors.

Replacing $\phi_0, \ldots, \phi_N$ with $c\phi_0, \ldots, c\phi_N$ gives the same map, so ϕ corresponds to a point

$$\phi \in \mathbb{P}^M, \quad \text{where } M = \binom{N+d}{d}(N+1) - 1.$$

We write

$$\begin{aligned} \mathrm{Rat}_d^N &= \{\text{degree } d \text{ rational maps } \mathbb{P}^N \dashrightarrow \mathbb{P}^N\} \subset \mathbb{P}^M, \\ \mathrm{Hom}_d^N &= \{\text{degree } d \text{ morphisms } \mathbb{P}^N \to \mathbb{P}^N\} \subset \mathbb{P}^M. \end{aligned}$$

Then Hom_d^N and Rat_d^N are Zariski open subsets of $\mathbb{P}^M$.

Morphisms and Dominant Rational Maps

Classical Theorem. There is a polynomial $\mathcal{R}(\phi)$ in the coefficients of ϕ such that

$$\mathrm{Hom}_d^N = \mathbb{P}^M \setminus \{\mathcal{R}(\phi) = 0\}.$$

$\mathcal{R}(\phi)$ is called the Macaulay resultant of ϕ.

Define

$$\mathrm{Dom}_d^N = \{\text{dominant degree } d \text{ maps } \mathbb{P}^N \dashrightarrow \mathbb{P}^N\} \subset \mathbb{P}^M.$$

Clearly

$$\mathrm{Hom}_d^N \subset \mathrm{Dom}_d^N \subset \mathrm{Rat}_d^N \subset \mathbb{P}^M.$$

Classical(?) Theorem. The set Dom_d^N of dominant rational maps is a Zariski open subset of $\mathbb{P}^M$.

Arithmetic Dynamics

Height Functions

Let $K/\mathbb{Q}$ be a number field. The **height** of a point $P \in \mathbb{P}^N(K)$ is

$$h(P) = \sum_{v \in M_K} \log \max_{0 \le i \le N} \left\| x_i(P) \right\|_v,$$

where the absolute values v on K are appropriately normalized.

Intuition The height $h(P)$ satisfies

$$h(P) \asymp \text{number of bits to store } P \text{ on a computer.}$$

Example For $P \in \mathbb{P}^N(\mathbb{Q})$, write

$$P = [\alpha_0, \ldots, \alpha_N] \text{ with } \alpha_i \in \mathbb{Z} \text{ and } \gcd(\alpha_0, \ldots, \alpha_N) = 1.$$

Then

$$h(P) = \log \max\{|\alpha_0|, \ldots, |\alpha_N|\}.$$

Basic Properties of Height Functions

Height functions are a fundamental tool in arithmetic geometry and arithmetic dynamics. Two important properties:

Finiteness. For all A and B, the set

$$\bigl\{P \in \mathbb{P}^N(\bar{\mathbb{Q}}) : h(P) \le A \text{ and } [\mathbb{Q}(P) : \mathbb{Q}] \le B\bigr\}$$

is finite.

Functoriality. Let

$$\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

be a *morphism* defined over $\bar{\mathbb{Q}}$. Then

$$h\bigl(\phi(P)\bigr) = (\deg \phi) h(P) + O(1) \quad \text{for all } P \in \mathbb{P}^N(\bar{\mathbb{Q}}).$$

An Application to Periodic Points

Let

$$\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$$

be a *morphism* defined over $\bar{\mathbb{Q}}$ of degree $d \geq 2$. It's an exercise using functorialty to prove that

$$\{P \in \mathbb{P}^N : P \text{ is preperiodic for } \phi\}$$

is a set of bounded height.

In particular, for any number field K,

$$\text{PrePer}(\phi) \cap \mathbb{P}^N(K) \text{ is finite.}$$

These results are due to Northcott (1950).

Functoriality for Rational Maps?

If $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is a rational map of degree d, we always have an upper bound

$$h\bigl(\phi(P)\bigr) \le dh(P) + O(1).$$

The proof is elementary, using the triangle inequality and a lot of algebra.

There is no corresponding lower bound.

Example For the degree 2 map $\phi\bigl([x,y,z]\bigr) = [x^2, y^2, xz]$, there is a Zariski dense set of points P such that

$$h\bigl(\phi(P)\bigr) \approx h(P).$$

However, for this map we do have

$$h\bigl(\phi(P)\bigr) \ge h(P) \quad \text{for all } P \text{ with } x(P) \ne 0.$$

A Lower Bound for Dominant Rational Maps

Theorem. Let

$$\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$$

be a dominant rational map of degree $d \geq 2$ defined over $\bar{\mathbb{Q}}$. There are constants $c_1 > 0$ and c_2 and a nonempty Zariski open set $U_\phi \subset \mathbb{P}^N$ such that

$$h\bigl(\phi(P)\bigr) \geq c_1 h(P) - c_2 \quad \text{for all } P \in U_\phi(\bar{\mathbb{Q}}).$$

So for dominant maps there is a nontrivial lower bound for the height of $\phi(P)$ if we omit a closed subset. The optimal value of $c_1 = c_1(\phi)$ is naturally of interest.

The Height Expansion Coefficient

Definition The **height expansion coefficient of ϕ** is the quantity

$$\mu(\phi) = \sup_{\emptyset \ne U \subset \mathbb{P}^N} \liminf_{\substack{P \in U(\bar{\mathbb{Q}}) \\ h(P) \to \infty}} \frac{h\bigl(\phi(P)\bigr)}{h(P)}.$$

The theorem on the previous slide says that

$$\mu(\phi) > 0 \quad \text{for all } \phi \in \operatorname{Dom}_d^N(\bar{\mathbb{Q}}),$$

and that for all $\epsilon > 0$ there is a $\emptyset \ne U_\epsilon \subset \mathbb{P}^N$ such that

$$h\bigl(\phi(P)\bigr) \ge \bigl(\mu(\phi) - \epsilon\bigr) h(P) \quad \text{for all } P \in U_\epsilon(\bar{\mathbb{Q}}).$$

Of course, if ϕ is a morphism, i.e., if $\phi \in \operatorname{Hom}_d^N$, then we have $\mu(\phi) = d$.

Examples of Height Expansion Ratios

Example 1 The map

$$\phi\bigl([x_0,\ldots,x_N]\bigr) = [x_0^{-1},\ldots,x_N^{-1}]$$

has height expansion ratio

$$\mu(\phi) = \frac{1}{N} = \frac{1}{\deg\phi}.$$

Example 2 Let $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a regular affine automorphism of degree d with $\dim I(\phi) = 0$. Then

$$\mu(\phi) = \frac{1}{d^{N-1}}.$$

(This example follows from results of Kawaguchi.)

The Universal Height Expansion Ratio

The theorem says that $\mu(\phi) > 0$ for every $\phi \in \mathrm{Dom}_N^d(\bar{\mathbb{Q}})$.

Definition The **universal height expansion ratio** for degree d dominant maps of $\mathbb{P}^N$ is

$$\overline{\mu}_d(\mathbb{P}^N) \stackrel{\text{def}}{=} \inf_{\phi \in \mathrm{Dom}_d^N(\bar{\mathbb{Q}})} \mu(\phi).$$

Theorem. For all $N \geq 1$ and $d \geq 2$,

$$\overline{\mu}_d(\mathbb{P}^N) > 0.$$

The proof is a double induction on dimension using a general height result for dominant rational maps of varieties, applied to the universal family of degree d dominant rational maps of $\mathbb{P}^N$.

We have $\overline{\mu}_d(\mathbb{P}^1) = d$, while Example 2 shows that

$$\overline{\mu}_d(\mathbb{P}^N) \leq d^{-(N-1)} \quad \text{for all } N \geq 2.$$

Dynamical Degree

Dynamical Degree

The (first) **dynamical degree** of a dominant rational map $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is

$$\operatorname{DynDeg}(\phi) = \lim_{n\to\infty} (\deg \phi^n)^{1/n}.$$

(An elementary argument shows that the limit exists)

The **algebraic entropy of ϕ** is $\log \operatorname{DynDeg}(\phi)$.

Conjecture A. (Bellon–Viallet) The dynamical degree is always an algebraic integer.

Conjecture B. The quantity

$$\ell_\phi = \inf \left\{ \ell \ge 0 : \sup_{n\ge 1} \frac{\deg(\phi^n)}{n^\ell \operatorname{DynDeg}(\phi)^n} < \infty \right\}.$$

is an integer satisfying $0 \le \ell_\phi \le N$.

Monomial Maps

A **monomial map** $\phi_A : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ is an endomorphism of the torus $\mathbb{G}_m^N$ defined by a matrix $A = (a_{ij})$ with integer coefficients and $\det(A) \neq 0$:

$$\begin{aligned}\phi_A(x_1, \ldots, x_N) \\ &= \left(x_1^{a_{11}} x_2^{a_{12}} \cdots x_N^{a_{1N}}, \ldots, x_1^{a_{N1}} x_2^{a_{N2}} \cdots x_N^{a_{NN}}\right).\end{aligned}$$

The map ϕ_A is **semisimple** if A is diagonalizable.

Theorem. Let $\phi_A : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a monomial map.
(a) (Hasselblatt–Propp)

$$\operatorname{DynDeg}(\phi_A) = \text{spectral radius of } A.$$

(b) (Lin, Jonsson–Wulcan)

$$\deg(\phi_A^n) \asymp n^\ell \operatorname{DynDeg}(\phi_A)^n$$

for an integer $0 \leq \ell < N$.

Thus Conjectures A and B are true for monomial maps.

Arithmetic Degree
and
Arithmetic Entropy

The Arithmetic Degree of a Rational Map at a Point

If ϕ is a *morphism* of degree d and P is not preperiodic, then $h\bigl(\phi^n(P)\bigr) \asymp d^n$, so

$$\lim_{n\to\infty} h\bigl(\phi^n(P)\bigr)^{1/n} = d.$$

For $\phi \in \operatorname{Dom}_d^N(\bar{\mathbb{Q}})$ and $P \in \mathbb{P}_\phi^N(\bar{\mathbb{Q}})$, the **arithmetic degree of ϕ at P** is

$$\operatorname{ArithDeg}(\phi, P) = \limsup_{n\to\infty} h\bigl(\phi^n(P)\bigr)^{1/n}.$$

We call $\log\bigl(\operatorname{ArithDeg}(\phi, P)\bigr)$ the **arithmetic entropy** of the orbit $\mathcal{O}_\phi(P)$.

Since $h\bigl(\phi^n(P)\bigr) \ll d^n$, we have

$$1 \le \operatorname{ArithDeg}(\phi, P) \le \deg(\phi).$$

Dynamical Degree and Arithmetic Degree

Elementary Theorem.

$$\operatorname{ArithDeg}(\phi, P) \le \operatorname{DynDeg}(\phi).$$

Conjecture C. Let $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational map defined over $\bar{\mathbb{Q}}$.
(a) The set

$$\left\{\operatorname{ArithDeg}(\phi, P) : P \in \mathbb{P}^N_\phi(\bar{\mathbb{Q}})\right\}$$

is a finite set of algebraic integers.
(b) If $\mathcal{O}_\phi(P)$ is Zariski dense in $\mathbb{P}^N$, then

$$\operatorname{ArithDeg}(\phi, P) = \operatorname{DynDeg}(\phi).$$

Theorem. (JS) Conjecture C is true for (semisimple) monomial maps.

Canonical Heights

Canonical Height for Dominant Rational Maps

Let $\phi \in \mathrm{Dom}_d^N(\bar{\mathbb{Q}})$ and assume (Conjecture B) that

$$\deg(\phi^n) \asymp n^\ell \cdot \mathrm{DynDeg}(\phi)^n.$$

The **ϕ-canonical height of $P \in \mathbb{P}_\phi^N(\bar{\mathbb{Q}})$** is

$$\hat{h}_\phi(P) = \limsup_{n\to\infty} \frac{h\bigl(\phi^n(P)\bigr)}{n^\ell \cdot \mathrm{DynDeg}(\phi)^n}.$$

This definition generalizes Kawaguchi's canonical heights for regular affine automorphisms.

The height is "canonical" in the sense that

$$\hat{h}_\phi\bigl(\phi(P)\bigr) = \mathrm{DynDeg}(\phi) \cdot \hat{h}_\phi(P).$$

Conjecture D. $\mathrm{DynDeg}(\phi) > 1 \Longrightarrow \hat{h}_\phi(P) < \infty.$

Conjecture D is true for monomial maps.

Two Conjectures Relating Degrees and Heights

Conjecture E. For $\phi \in \mathrm{Dom}_d^N(\bar{\mathbb{Q}})$ and $P \in \mathbb{P}^N(\bar{\mathbb{Q}})_\phi$,

$$\hat{h}_\phi(P) > 0 \quad \Longleftrightarrow \quad \mathrm{ArithDeg}(\phi, P) = \mathrm{DynDeg}(\phi).$$

(The implication $\Longrightarrow$ is easy.)

Conjecture F. Let $\phi \in \mathrm{Dom}_d^N(\bar{\mathbb{Q}})$ be a map with $\mathrm{DynDeg}(\phi) > 1$, and let $P \in \mathbb{P}^N(\bar{\mathbb{Q}})_\phi$ be a point whose orbit $\mathcal{O}_\phi(P)$ is Zariski dense. Then

$$\hat{h}_\phi(P) > 0.$$

Conjectures C, D, E and F are true for (semisimple) monomial maps. The proof uses:

- A local non-limit description of $\hat{h}_\phi(P) = 0$.
- A lemma describing when $\bar{\mathbb{Q}}$-linear relations among transcendental numbers descend to $\mathbb{Q}$-linear relations.
- Baker's theorem on linear-forms-in-logarithms.

Rational Automorphisms

Rational Automorphisms

For a birational map $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$, we look at

$$\mathcal{O}^{\pm}(P) = \{\phi^n(P) : n \in \mathbb{Z}\}.$$

The **total height expansion coefficient** is

$$\mu^{\pm}(\phi) = \sup_{\emptyset \ne U \subset \mathbb{P}^N} \liminf_{\substack{P \in U(\bar{\mathbb{Q}}) \\ h(P) \to \infty}} \frac{1}{h(P)} \left(\frac{h\big(\phi(P)\big)}{\deg(\phi)} + \frac{h\big(\phi^{-1}(P)\big)}{\deg(\phi^{-1})} \right).$$

It is easy to see that $0 \le \mu^{\pm}(\phi) \le 2$.

Theorem. (Kawaguchi, Lee) For regular affine automorphisms,

$$\mu^{\pm}(\phi) = 1 + \frac{1}{\deg(\phi)\deg(\phi^{-1})}.$$

Question. What are the possible values of $\mu^{\pm}(\phi)$? For algebraically stable maps? For affine automorphisms?

Rational Automorphisms

A birational map $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ has two dynamical degrees and two canonical heights:

$$\delta_+ = \operatorname{DynDeg}(\phi) \quad \text{and} \quad \delta_- = \operatorname{DynDeg}(\phi^{-1}),$$
$$\hat{h}^+ = \hat{h}_\phi \quad \text{and} \quad \hat{h}^- = \hat{h}_{\phi^{-1}}.$$

Following Kawaguchi, we define the total canonical height

$$\hat{h} = \hat{h}^+ + \hat{h}^-.$$

The canonical property of $\hat{h}^+$ and $\hat{h}^-$ give

$$\frac{1}{\delta_+}\hat{h}\bigl(\phi(P)\bigr) + \frac{1}{\delta_-}\hat{h}\bigl(\phi^{-1}(P)\bigr) = \left(1 + \frac{1}{\delta_+\delta_-}\right)\hat{h}(P).$$

Question. For which ϕ is it true that $\hat{h} \asymp h$?

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