Dynamical Degrees, Arithmetic Degrees, and Canonical Heights: History, Conjectures, and Future Directions

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Algebraic, Complex and Arithmetic Dynamics
Simons Symposium, Schloss Elmau, Germany

Monday, 19 May 2019, 11:30–12:30

Measuring Complexity of Iteration

Let X be an object in some category, and let

$$h : \mathrm{End}(X) \longrightarrow \mathbb{R}_{\geq 0}$$

be a function that measures the complexity of endomorphisms of X.

The Endomorphism Complexity Problem. Describe the growth rate of $h(f^n)$ as $n \to \infty$.

Suppose that our objects are sets, and that for every object X we have a function

$$h_X : X \longrightarrow \mathbb{R}_{\geq 0}$$

that measures the complexity of the elements of X.

The Orbit Complexity Problem. For $x \in X$, describe the growth rate of $h_X\bigl(f^n(x)\bigr)$ as $n \to \infty$. Classify the subsets of X exhibiting various growth rates.

That's all very abstract. On to some classical examples.

The Endomorphism Complexity Problem for $\mathbb{P}^N$

Let's start with projective space:

$$f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N \quad \text{a dominant rational map.}$$

Measure complexity by degree,

$$\deg : \operatorname{End}(\mathbb{P}^N) \longrightarrow \mathbb{N}_{\ge 1}.$$

If f is a morphism, then $\deg(f^n) = \deg(f)^n$. In general:
Definition: The **dynamical degree of f** is

$$\delta(f) := \lim_{n\to\infty} \Bigl(\deg(f^n)\Bigr)^{1/n}.$$

Inuition: $\deg(f^n)$ is roughly $\delta(f)^n$.

Conjecture. (Bellon–Viallet) $\delta(f) \in \overline{\mathbb{Z}}$.

True for various cases for $\mathbb{P}^2$ (Diller, Favre, Jonsson, ...).

The Endomorphism Complexity Problem for Varieties

Let X be a smooth projective variety of dimension N, let H be an ample divisor on X, and measure the complexity of $f \in \operatorname{End}(X)$ by

$$\deg_H : \operatorname{End}(X) \longrightarrow \mathbb{N}_{\geq 1}, \quad \deg_H(f) := f^*H \cdot H^{N-1}.$$

Definition: The **dynamical degree of f** is

$$\delta(f) := \lim_{n\to\infty} \Bigl(\deg_H(f^n)\Bigr)^{1/n}.$$

- N.B. For rational maps f, in general $(f^n)^* \neq (f^*)^n$ as maps on $\operatorname{Pic}(X)$.
- The limit $\delta(f)$ exists and is independent of H.
- It is enough to take X normal and H to be a nef and big Cartier divisor.

Variation of the Dynamical Degree in Families

Let $f : \mathbb{P}^N_{\mathbb{Q}} \dashrightarrow \mathbb{P}^N_{\mathbb{Q}}$. For each prime p, we can reduce to obtain a map $\tilde{f}_p : \mathbb{P}^N_{\mathbb{F}_p} \dashrightarrow \mathbb{P}^N_{\mathbb{F}_p}$. Note that $\delta(\tilde{f}_p) \le \delta(f)$.

Conjecture. $\lim_{p\to\infty} \delta(\tilde{f}_p) = \delta(f)$?

Let $f : X/T \dashrightarrow X/T$ be a family of dominant rational maps parameterized by a variety T. This gives a dynamical degree $\delta(f_\eta)$ of f on the generic fiber, i.e., over $k(T)$, and also for each $t \in T$, a dynamical degree $\delta(f_t)$.

Conjecture. For all $\epsilon > 0$, we have

$$\overline{\{t \in T : \delta(f_t) \le \delta(f_\eta) - \epsilon\}} \ne T.$$

Theorem. (Xie) Both conjectures are true for $\mathbb{P}^2$.

Refined Estimates for Degree Growth

For

$$f : X \dashrightarrow X,$$

the definition of $\delta(f)$ is equivalent to

$$\log \deg_H(f^n) = n \log \delta(f) + o(n).$$

Question: For which f is it true that

$$\log \deg_H(f^n) = n \log \delta(f) + O(n^\epsilon) \quad \text{for an } 0 < \epsilon < 1?$$

Question: For which f and H is it true that

$$\log \deg_H(f^n) = n \log \delta(f) + O(\log n)?$$

And for those who want to the stars and the moon!!

Question: For which $f : X \dashrightarrow X$ and H does the limit

$$\lim_{n\to\infty} \frac{\deg_H(f^n)}{\delta(f)^n \cdot n^{\ell(f)}} \quad \text{exist for some } \ell(f) \in \mathbb{Z}_{\geq 0}?$$

The Arithmetic Degree of an Orbit

For $X/\bar{\mathbb{Q}}$ and $f : X \dashrightarrow X$, let

$$X(\bar{\mathbb{Q}})_f := \{Q \in X(\bar{\mathbb{Q}}) : f^n(Q) \text{ is defined for all } n \geq 1\},$$

and let

$$h_X : X(\bar{\mathbb{Q}}) \longrightarrow [1, \infty)$$

be a height function relative to an ample line divisor.

Intuition: $h_X(P) = \#$ of bits to describe P.

Definition: The **arithmetic degree** of $P \in X(\bar{\mathbb{Q}})_f$ is

$$\alpha(f, P) := \lim_{n\to\infty} h_X\bigl(f^n(P)\bigr)^{1/n}.$$

Theorem. (Kawaguchi–Silverman, Matsuzawa)

$$\overline{\alpha}(f, P) := \limsup_{n\to\infty} h_X\bigl(f^n(P)\bigr)^{1/n} \leq \delta(f)$$

$$\begin{pmatrix}\text{Arithmetic complexity} \\ \text{of an orbit}\end{pmatrix} \leq \begin{pmatrix}\text{Dynamical complexity} \\ \text{of the map}\end{pmatrix}$$

Arithmetic Degree Versus Dynamical Degree

Conjecture. (Kawaguchi–Silverman)
The limit defining $\alpha(f,P)$ converges.

The convergence is known in many situations, including for morphisms and for many types of maps of surfaces.

Density Conjecture. (Kawaguchi–Silverman)

$$\mathcal{O}_f(P) \text{ Zariski dense in } X \Longrightarrow \alpha(f,P) = \delta(f).$$

$$\begin{pmatrix}\text{Maximal geometric}\\ \text{complexity of an orbit}\end{pmatrix} \Longrightarrow \begin{pmatrix}\text{Maximal arithmetic}\\ \text{complexity of the orbit}\end{pmatrix}$$

The density conjecture is known in some cases, including:

(1) Monomial maps of $\mathbb{P}^N$.

(2) Many classes of rational maps of $\mathbb{P}^2$.

(3) Maps of abelian varieties. More generally, translated isogenies of semi-abelian varieties.

(4) Morphisms of surfaces.

(5) Morphisms of certain higher dimensional varieties having additional structure.

(6) Dominant rational maps of large topological degree.

Canonical Heights for Polarized Morphisms

Let $f : X \to X$ be a morphism and $D \in \operatorname{Div}(X) \otimes \mathbb{R}$ a divisor, and suppose that

$$f^*D \sim \delta D \quad \text{for some } \delta > 1.$$

The associated **canonical height of $P \in X(\bar{\mathbb{Q}})$** is

$$\hat{h}_{f,D}(P) := \lim_{n\to\infty} \frac{1}{\delta^n} h_D\bigl(f^n(P)\bigr).$$

Properties of $\hat{h}_{f,D}$:

$$\hat{h}_{f,D}(P) = h_D(P) + O(1); \quad \hat{h}_{f,D}\bigl(f(P)\bigr) = \delta \hat{h}_{f,D}(P);$$
$$D \text{ ample} : \hat{h}_{f,D}(P) = 0 \iff P \in \operatorname{PrePer}(f).$$

Dynamical Lehmer Conjecture. For D ample, $\exists$ $C(f,D) > 0$ so that for all $P \in X(\bar{\mathbb{Q}}) \smallsetminus \operatorname{PrePer}(f)$,

$$\hat{h}_{f,D}(P) \ge \frac{C(f,D)}{[\mathbb{Q}(P) : \mathbb{Q}]}.$$

Shibata's Ample Canonical Height

Let $f : X \to X$ be a dominant morphism with $\delta(f) > 1$, let $h_X : X(\bar{\mathbb{Q}}) \to [1, \infty)$ be a height relative to an ample divisor, and let $\ell(f)$ be the smallest non-negative integer such that

$$\sup_{n \geq 1} \frac{h_X\bigl(f^n(P)\bigr)}{n^{\ell(f)} \cdot \delta(f)^n} < \infty \quad \text{for all } P \in X(\bar{\mathbb{Q}}).$$

Definition. The (*lower*) *ample canonical height* is

$$\underline{\hat{h}}_f : X(\bar{\mathbb{Q}}) \to [0, \infty), \quad \underline{\hat{h}}_f(P) := \liminf_{n \to \infty} \frac{h_X\bigl(f^n(P)\bigr)}{n^{\ell(f)} \cdot \delta(f)^n}.$$

Conjecture. (Shibata) For every number field $K/\mathbb{Q}$,

$$\bigl\{ P \in X(K) : \underline{\hat{h}}_f(P) = 0 \bigr\} \qquad (*)$$

is not Zariski dense in X.

The set $(*)$ is independent of the choice of h_X.

Shibata Conjecture $\Longrightarrow$ K-S Density Conjecture

If $\alpha(f,P) < \delta(f)$, then for sufficiently large n,

$$\begin{aligned} h_X\big(f^n(P)\big)^{1/n} &\le \alpha(f,P) + \tfrac{1}{2}\big(\delta(f) - \alpha(f,P)\big) \\ &= \delta(f) - \underbrace{\tfrac{1}{2}\big(\delta(f) - \alpha(f,P)\big)}_{\text{call this } \epsilon(f,P)}. \end{aligned}$$

Hence

$$\underline{\hat{h}}_f(P) \le \liminf_{n\to\infty} \frac{\big(\delta(f) - \epsilon(f,P)\big)^n}{n^{\ell(f)} \cdot \delta(f)^n} = 0.$$

Since $\alpha\big(f, f^n(P)\big) = \alpha(f,P)$, we see that

$$\alpha(f,P) < \delta(f) \quad \Longrightarrow \quad \mathcal{O}_f(P) \subseteq \underbrace{\big\{Q \in X(K) : \underline{\hat{h}}_f(Q) = 0\big\}}_{\text{Shibata} \Rightarrow \text{not Zariski dense}}.$$

Therefore Shibata's conjecture implies the K-S density conjecture (for morphisms).

Other Types of Growth Rates for $h_X(f^n(P))$?

The map

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \qquad f(x,y,z) = [xy + xz, yz + z^2, z^2]$$

is interesting. It satisfies $\deg(f^n) = n + 1$, so $\delta(f) = 1$, and the point $P = [1,0,1]$ satisfies

$$f^n(P) = [n!, n, 1],$$

so

$$h\bigl(f^n(P)\bigr) = \log(n!) \sim n \log n.$$

Questions: For rational maps $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$, is it possible to have:

(1) $h\bigl(f^n(P)\bigr) \sim n^i (\log n)^j$ for some $j \geq 2$?

(2) $h\bigl(f^n(P)\bigr) \sim \delta(f)^n n^i (\log n)^j$ with $j \geq 1$ and $\delta(f) > 1$?

(3) what other sorts of growth rates?

Canonical Heights and Critical Heights

Let $\mathcal{M}_d^N := \operatorname{End}_d(\mathbb{P}^N)/\!/\operatorname{PGL}_{N+1}$, and fix an ample height $h_{\mathcal{M}}$ on $\mathcal{M}_d^N(\bar{\mathbb{Q}})$.

Dynamical Lang Height Conjecture. Let $K/\mathbb{Q}$. There are $C_i(K,N,d) > 0$ so that for all $f \in \mathcal{M}_d^N(K)$ and all $P \in \mathbb{P}^N(K)$ with Zariski dense orbit,

$$\hat{h}_f(P) \ge C_1 h_{\mathcal{M}}(f) - C_2.$$

Restricting to $\mathbb{P}^1$, the **critical height** of $f \in \mathcal{M}_d^1(\bar{\mathbb{Q}})$ is

$$\hat{h}^{\text{crit}}(f) := \sum_{P \in \operatorname{Crit}(f)} \hat{h}_f(P).$$

$$\hat{h}^{\text{crit}}(f) = 0 \iff \operatorname{Crit}(f) \subset \operatorname{PrePer}(f) \iff f \text{ is PCF}.$$

Theorem. (Ingram) $\hat{h}^{\text{crit}}(f) \asymp h_{\mathcal{M}}(f)$.

Problem: Generalize to $\mathcal{M}_d^N$. What replaces $\hat{h}^{\text{crit}}$? TBC. . .

Higher Order Dynamical Degrees

Let X be a smooth projective variety of dimension N, let $f : X \dashrightarrow X$ be a dominant rational map, and let H be an ample divisor.

Definition. The ***k*'th dynamical degree of *f*** is

$$\delta_k(f) := \lim_{n\to\infty} \Bigl((f^n)^*(H^k) \cdot H^{N-k}\Bigr)^{1/n}.$$

Theorem. (Guedj) Dynamical degrees form a log concave sequence, i.e., $\delta_{i-1}\delta_{i+1} \le \delta_i^2$. In particular, for some k we have

$$\delta_1(f) \le \delta_2(f) \le \cdots \le \delta_k(f),$$
$$\delta_k(f) \ge \delta_{k+1}(f) \ge \cdots \ge \delta_N(f).$$

Favre–Wulcan & Lin describe $\delta_k(f)$ for monomial maps.

Question. Are all $\delta_k(f)$ algebraic integers?

Question. Generalize to arithmetic degree?

Higher Order Arithmetic Degree

Let $X/\bar{\mathbb{Q}}$ be a smooth projective variety of dimension N, and let $f : X \to X$ be a morphism. For $k \geq 2$, the "natural" way to define the k'th arithmetic degree $\alpha_k(f, P)$ of a point P (conjecturally) yields

$$\alpha_k(f, P) = 1 \quad \text{for all } P,$$

so that's not very interesting.

The problem is that scheme-theoretically, a point P has dimension 1 and a divisor H has codimension 1, so their arithmetic intersection is often large; but if we replace H with something of higher codimension, then the intersection with P is likely to be small.

One solution is to replace the point P with a higher dimensional subvariety:

$$\alpha_k(f, \cdot\,) : \{k - 1 \text{ dim'l subvarieties}\} \longrightarrow \mathbb{R}_{\geq 0}.$$

Higher Order Arithmetic Degrees

There is a theory that assigns a height to each subvariety

$$Z \subseteq X,$$

especially for $X = \mathbb{P}^N$. Indeed, there are several formulations, including a height for $Z \subset \mathbb{P}^N$ using the Chow coordinates of Z, and heights using metrized line bundles $\overline{L}$ and Arakelov theory due to Faltings, Zhang, Bost–Gillet–Soulé, . . . ; see [BGS *JAMS* 1994].

Higher Order Arithmetic Degree

Definition. Let $f : X \to X$ be a morphism. The **arithmetic degree** of $Z \subseteq X$ is

$$\alpha(f, Z) := \lim_{n\to\infty} h_{X,\overline{L}}\bigl(f^n(Z)\bigr)^{1/n}.$$

Questions. (1) Does the limit $\alpha(f, Z)$ converge?

(2) Is there a natural upper bound for $\alpha(f, Z)$ in terms of $\delta_1(f), \ldots, \delta_{1+\dim Z}(f)$?

(3) When is this upper bound attained?

Example. (K-S unpublished) Let $f : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant monomial map, and let $Z \subset \mathbb{P}^N$ be an irreducible hypersurface $\neq$ a coordinate hyperplane. Then

$$\overline{\alpha}(f, Z) \le \min\bigl\{\delta_{N-1}(f), \delta_N(f)\bigr\}.$$

Further, there are examples with $N = 2$ satisfying:

$$(1)\ \alpha(f, Z) = \delta_1(f) < \delta_2(f);$$
$$(2)\ \alpha(f, Z) = \delta_2(f) < \delta_1(f).$$

Height Lower Bounds and the Bogomolov Property

We fix a polarized dynamical system (X, f, D), i.e.,

$$f : X \to X, \quad D \text{ ample}, \quad f^*D \sim \delta D \text{ for some } \delta > 1.$$

Definition. A subvariety $Z \subseteq X$ has the **Bogomolov property** (relative to f and D) if there is an $\epsilon > 0$ s.t.

$$\overline{Z_{f,D}(\epsilon)} := \overline{\left\{P \in Z(\bar{\mathbb{Q}}) : \hat{h}_{X,f,D}(P) < \epsilon\right\}} \neq Z$$

Examples.
(1) X an abelian variety, Z not a translate of an abelian subvariety by a torsion point (Ullmo, S. Zhang, David–Philippon).
(2) $X = (\mathbb{P}^1)^N$, f a dominant endomorphism (Ghioca–Nguyen–Ye).

N.B. As shown by the construction of Ghioca–Tucker,

$$\overline{Z \cap \operatorname{PrePer}(f)} = Z \quad \not\Longrightarrow \quad Z \text{ is } f\text{-preperiodic.}$$

The Bogomolov Canonical Height of a Subvariety

This suggests defining the Bogomolov height of Z to be the largest ϵ that gives the Bogomolov property.

Definition The **Bogomolov height of Z** (relative to f and D) is

$$\hat{h}^{\mathcal{B}}_{X,f,D}(Z) := \sup_{\emptyset \ne U \subseteq Z} \inf_{P \in U(\bar{\mathbb{Q}})} \hat{h}_{X,f,D}(P),$$

where U ranges over Zariski open subsets of Z. And if $Z = \sum n_i Z_i$ is a formal sum of equidimensional subvarieties, we extend linearly, $\hat{h}^{\mathcal{B}}(Z) = \sum n_i \hat{h}^{\mathcal{B}}(Z_i)$.

Since $$\hat{h}^{\mathcal{B}}_{X,f,D}(Z) = \sup\left\{\epsilon > 0 : \overline{Z_{f,D}(\epsilon)} \ne Z\right\},$$

we see that

$$Z \text{ has the Bogomolov property} \iff \hat{h}^{\mathcal{B}}_{X,f,D}(Z) > 0.$$

Maybe: Z is **formally preperiodic** if $\hat{h}^{\mathcal{B}}_{X,f,D}(Z) = 0$?

Addendum

Shouwu Zhang proved that

$$\hat{h}^{\mathcal{Z}}_{X,f,D}(Z) := \lim_{n\to\infty} \frac{1}{\delta^n} h_{X,D}\bigl(f^n(Z)\bigr)$$

converges, and that the Zhang height and the Bogomolov heights,

$$\hat{h}^{\mathcal{Z}}_{X,f,D}(Z) \quad \text{and} \quad \hat{h}^{\mathcal{B}}_{X,f,D}(Z),$$

are commensurate.

The Critical Height of an Endomorphism of $\mathbb{P}^N$

Let $f : \mathbb{P}^N \to \mathbb{P}^N$ be an endomorphism, and let

$$\mathrm{Crit}_f := (\text{the critical locus of } f) \in \mathrm{Div}(\mathbb{P}^N).$$

Definition. The **critical height of $f : \mathbb{P}^N \to \mathbb{P}^N$** is

$$\hat{h}^{\mathrm{crit}}(f) := \hat{h}^{\mathcal{B}}_{X,f,\mathcal{O}(1)}(\mathrm{Crit}_f).$$

This gives a well-defined function

$$\hat{h}^{\mathrm{crit}} : \mathcal{M}_d^N(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}_{\ge 0}.$$

We might say that f is "formally PCF" if $\hat{h}^{\mathrm{crit}}(f) = 0$.

Conjecture. As maps $\mathcal{M}_d^N(\bar{\mathbb{Q}}) \to \mathbb{R}_{\ge 0}$, we have

$$\hat{h}^{\mathrm{crit}} \gg\ll h_{\mathcal{M}}.$$

The upper bound $\hat{h}^{\mathrm{crit}} \ll h_{\mathcal{M}}$ is probably not too hard; the lower bound, generalizing Ingram, seems harder.

I want to thank the organizers

Laura DeMarco and Mattias Jonsson

for inviting me to speak, the

Simons Foundation

for funding this Symposium, and all of you for your attention.

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