Chapter 47

The Topsy-Turvy World of Continued Fractions [online]

The other night, from cares exempt, I slept—and what d'you think I dreamt?
I dreamt that somehow I had come, To dwell in Topsy-Turveydom!—
Where babies, much to their surprise, Are born astonishingly wise;
With every Science on their lips, And Art at all their fingertips.
For, as their nurses dandle them, They crow binomial theorem,
With views (it seems absurd to us), On differential calculus.
But though a babe, as I have said, Is born with learning in his head,
He must forget it, if he can, Before he calls himself a man.

W.S. Gilbert, *My Dream*, 1870 (one of the *Bab Ballads*).

The famous number π has a never-ending, never-repeating decimal expansion

$$\pi = 3.141592653589793238462643\ldots.$$

If we are willing to sacrifice accuracy for brevity, we might say that

$$\pi = 3 + \text{"a little bit more."}$$

The "little bit more" is a number between 0 and 1. We take Gilbert's advice and turn that "little bit more" topsy-turvy. Turning the small number $0.14159\ldots$ upside down, we obtain the reciprocal of a number that is larger than 1. Thus

$$\begin{aligned}\pi &= 3 + 0.14159265358979323846264 33\ldots \\ &= 3 + \cfrac{1}{\cfrac{1}{0.14159265358979323846264 33\ldots}} \\ &= 3 + \frac{1}{7.0625133059310457697930051\ldots} \\ &= 3 + \frac{1}{7 + 0.0625133059310457697930051\ldots} \\ &= 3 + \frac{1}{7 + \text{“a little bit more”}}.\end{aligned}$$

Notice that if we ignore the "little bit more" in this last equation, we find that π is approximately equal to $3 + \frac{1}{7}$, that is, to $\frac{22}{7}$. You may have learned in high school that $\frac{22}{7}$ is a fairly good approximation to π.

Let's repeat the process. We take the "little bit more" in the last equation and turn it topsy-turvy,

$$\begin{aligned}0.0625133059310457697930051\ldots &= \cfrac{1}{\cfrac{1}{0.0625133059310457697930051\ldots}} \\ &= \frac{1}{15.9965944066857198889230 60\ldots}.\end{aligned}$$

Now we substitute this into the earlier formula, which gives a double-decker fraction,

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + 0.9965944066857198889230 60\ldots}}$$

The bottom level of this fraction is $15.99659\ldots$, which is very close to 16. If we replace it with 16, we get a rational number that is quite close to π,

$$3 + \cfrac{1}{7 + \cfrac{1}{16}} = \frac{355}{113} = 3.14159292035398230088495 57\ldots.$$

The fraction $\frac{355}{113}$ agrees with π to six decimal places.

Continuing on our merry way, we compute

$$0.9965944066857198889230 60\ldots = \frac{1}{1.0034172310133726034641468\ldots}$$

to get the triple-decker fraction

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + 0.0034172310133726034641468\ldots}}},$$

and then

$$0.0034172310133726034641468\ldots = \frac{1}{292.63459101439547237857177\ldots}$$

to add yet another layer to our fraction,

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + 0.63459101439547237857177\ldots}}}}$$

Now let's see what happens if we round that last denominator to 293. We get the rational number

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{293}}}} = \frac{104348}{33215} = 3.14159265392142104470871\,59\cdots.$$

So the fraction $\frac{104348}{33215}$ agrees with the value of π to nine decimal places.

Just how accurate is nine decimal places? Suppose that we are given that the distance from Earth to the Sun is approximately 145,000,000 kilometers and that we want to calculate the length of Earth's orbit using the formula[1]

$$\text{Circumference} = 2 \times \pi \times \text{Radius}.$$

Then the error in the circumference if you use $\frac{104348}{33215}$ instead of π will amount to a little under 100 meters. So, unless you've managed to measure the distance to the Sun to within a fraction of a kilometer, it's fine to use the approximation $\pi \approx \frac{104348}{33215}$.

These multistory, topsy-turvy fractions have a name. They are called

[1]All right, all right, you caught me, Earth's orbit is an ellipse, not a circle. So we're really calculating the circumference of an invisible circle whose radius is approximately 145,000,000 kilometers.

Continued Fractions

We can form the continued fraction for any number by repeatedly flipping and separating off the whole integer part. The first few steps in the computation of the continued fraction for the cube root of 2 are given in full in Figure 47.1. In a similar fashion, we compute the continued fraction of $\sqrt{2}$,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}}$$

and the continued fraction of $e = 2.7182818\ldots$ (the base of the natural logarithms),

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{8 + \cfrac{1}{\ddots}}}}}}}}}}}}$$

Continued fractions are visually striking as they slide down to the right, but writing them as fractions takes a lot of ink and a lot of space. There must be a more convenient way to describe a continued fraction. All the numerators are 1's, so all we need to do is list the denominators. We write

$$[a_0, a_1, a_2, a_3, a_4, \ldots]$$

$$
\begin{aligned}
\sqrt[3]{2} &= 1.259921\ldots \\
&= 1 + \cfrac{1}{3.847322\ldots} \\
&= 1 + \cfrac{1}{3 + \cfrac{1}{1.180189\ldots}} \\
&= 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5.549736\ldots}}} \\
&= 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1.819053\ldots}}}} \\
&= 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1.220922\ldots}}}}} \\
&= 1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4.526491\ldots}}}}}}
\end{aligned}
$$

Figure 47.1: The Continued Fraction Expansion of $\sqrt[3]{2}$

as shorthand for the continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_5 + \cfrac{1}{a_6 + \cfrac{1}{\ddots}}}}}}}.$$

Using this new notation, our earlier continued fractions expansions (extended a bit further) can be written succinctly as

$$\begin{aligned}
\pi &= [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, \ldots],\\
\sqrt[3]{2} &= [1, 3, 1, 5, 1, 1, 4, 1, 1, 8, 1, 14, 1, 10, 2, 1, 4, 12, 2, 3, 2, \ldots],\\
\sqrt{2} &= [1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots],\\
e &= [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, \ldots].
\end{aligned}$$

Now that we've looked at several examples of continued fractions, it's time to work out some of the general theory. If a number α has a continued fraction expansion

$$\alpha = [a_0, a_1, a_2, a_3, \ldots],$$

then we have seen that cutting off after a few terms gives a rational number that is quite close to α. The n^{th} *convergent* to α is the rational number

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

obtained by using the terms up to a_n. For example, the first few convergents to

$\sqrt{2} = [1, 2, 2, 2, 2, \ldots]$ are

$$\frac{p_0}{q_0} = 1,$$

$$\frac{p_1}{q_1} = 1 + \frac{1}{2} = \frac{3}{2},$$

$$\frac{p_2}{q_2} = 1 + \cfrac{1}{2 + \cfrac{1}{2}} = 1 + \frac{2}{5} = \frac{7}{5},$$

$$\frac{p_3}{q_3} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = 1 + \cfrac{1}{2 + \cfrac{2}{5}} = 1 + \frac{5}{12} = \frac{17}{12}.$$

A longer list of convergents to $\sqrt{2}$ is given in Table 47.1.

n	0	1	2	3	4	5	6	7	8	9	10
$\frac{p_n}{q_n}$	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{17}{12}$	$\frac{41}{29}$	$\frac{99}{70}$	$\frac{239}{169}$	$\frac{577}{408}$	$\frac{1393}{985}$	$\frac{3363}{2378}$	$\frac{8119}{5741}$

Table 47.1: Convergents to $\sqrt{2}$

Staring at the list of convergents to $\sqrt{2}$ is not particularly enlightening, but it would certainly be useful to figure out how successive convergents are generated from the earlier ones. It is easier to spot the pattern if we look at $[a_0, a_1, a_2, a_3, \ldots]$ using symbols, rather than looking at any particular example.

$$\frac{p_0}{q_0} = \frac{a_0}{1},$$

$$\frac{p_1}{q_1} = \frac{a_1 a_0 + 1}{a_1},$$

$$\frac{p_2}{q_2} = \frac{a_2 a_1 a_0 + a_2 + a_0}{a_2 a_1 + 1},$$

$$\frac{p_3}{q_3} = \frac{a_3 a_2 a_1 a_0 + a_3 a_2 + a_3 a_0 + a_1 a_0 + 1}{a_3 a_2 a_1 + a_3 + a_1}.$$

Let's concentrate for the moment on the numerators $p_0, p_1, p_2, \ldots$. Table 47.2 gives the values of $p_0, p_1, p_2, p_3, p_4, p_5$.

At first glance, the formulas in Table 47.2 look horrible, but you might notice that p_0 appears at the tail end of p_2, that p_1 appears at the tail end of p_3, that p_2

n	p_n
p_0	a_0
p_1	a_1a_0+1
p_2	$a_2a_1a_0+a_2+a_0$
p_3	$a_3a_2a_1a_0+a_3a_2+a_3a_0+a_1a_0+1$
p_4	$a_4a_3a_2a_1a_0+a_4a_3a_2+a_4a_3a_0+a_4a_1a_0+a_2a_1a_0+a_4+a_2+a_0$
p_5	$a_5a_4a_3a_2a_1a_0+a_5a_4a_3a_2+a_5a_4a_3a_0+a_5a_4a_1a_0+a_5a_2a_1a_0$ $+a_3a_2a_1a_0+a_5a_4+a_5a_2+a_3a_2+a_5a_0+a_3a_0+a_1a_0+1$

Table 47.2: Numerator of the Continued Fraction $[a_0, a_1, \ldots, a_n]$

appears at the tail end of p_4, and that p_3 appears in the tail end of p_5. In other words, it seems that p_n is equal to p_{n-2} plus some other stuff. Here's a list of the "other stuff" for the first few values of n:

$$\begin{aligned}
p_2 - p_0 &= a_2a_1a_0 + a_2 \\
&= a_2(a_1a_0 + 1) \\
p_3 - p_1 &= a_3a_2a_1a_0 + a_3a_2 + a_3a_0 \\
&= a_3(a_2a_1a_0 + a_2 + a_0) \\
p_4 - p_2 &= a_4a_3a_2a_1a_0 + a_4a_3a_2 + a_4a_3a_0 + a_4a_1a_0 + a_4 \\
&= a_4(a_3a_2a_1a_0 + a_3a_2 + a_3a_0 + a_1a_0 + 1).
\end{aligned}$$

Looking back at Table 47.2, it seems that the "other stuff" for $p_n - p_{n-2}$ is simply a_n multiplied by the quantity p_{n-1}. We can describe this observation by the formula

$$p_n = a_np_{n-1} + p_{n-2}.$$

This is an example of a *recursion formula*, because it gives the successive values of $p_0, p_1, p_2, \ldots$ recursively in terms of the previous values. It is very similar to the recursion formula for the Fibonacci numbers that we investigated in Chapter 39.[2] Of course, this recursion formula needs two initial values to get started,

$$p_0 = a_0 \qquad \text{and} \qquad p_1 = a_1a_0 + 1.$$

A similar investigation of the denominators $q_0, q_1, q_2, \ldots$ reveals an analogous recursion. In fact, if you make a table for $q_0, q_1, q_2, \ldots$ similar to Table 47.2, you

[2] Indeed, if all the a_n's are equal to 1, then the sequence of p_n's is precisely the Fibonacci sequence. You can study the connection between continued fractions and Fibonacci numbers by doing Exercise 47.8.

will find that $q_0, q_1, q_2, \ldots$ seem to obey exactly the same recursion formula as the one obeyed by $p_0, p_1, p_2, \ldots$, but $q_0, q_1, q_2, \ldots$ use different starting values for q_0 and q_1. We summarize our investigations in the following important theorem.

Theorem 47.1 (Continued Fraction Recursion Formula)**.** *Let*

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}},$$

where we treat $a_0, a_1, a_2, \ldots$ as variables, rather than as specific numbers. Then the numerators $p_0, p_1, p_2, \ldots$ are given by the recursion formula

$$p_0 = a_0, \qquad p_1 = a_1 a_0 + 1, \qquad \textit{and} \qquad p_n = a_n p_{n-1} + p_{n-2} \quad \textit{for } n \geq 2,$$

and the denominators $q_0, q_1, q_2, \ldots$ are given by the recursion formula

$$q_0 = 1, \qquad q_1 = a_1, \qquad \textit{and} \qquad q_n = a_n q_{n-1} + q_{n-2} \quad \textit{for } n \geq 2,$$

Proof. When a sequence is defined by a recursive formula, it is often easiest to use induction to prove facts about the sequence. To get our induction started, we need to check that

$$\frac{p_0}{q_0} = [a_0] \qquad \text{and} \qquad \frac{p_1}{q_1} = [a_0, a_1].$$

We are given that $p_0 = a_0$ and $q_0 = 1$, so $p_0/q_0 = a_0$, which verifies the first equation. Similarly, we are given that $p_1 = a_1 a_0 + 1$ and $q_1 = a_1$, so

$$[a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1} = \frac{p_1}{q_1},$$

which verifies the second equation.

Now we assume that the theorem is true when $n = N$ and use that assumption to prove that it is also true when $n = N+1$. A key observation is that the continued fraction

$$[a_0, a_1, a_2, \ldots, a_N, a_{N+1}]$$

can be written as a continued fraction with one less term by combining the last two terms,[3]

$$[a_0, a_1, a_2, \ldots, a_N, a_{N+1}] = \left[a_0, a_1, a_2, \ldots, a_N + \frac{1}{a_{N+1}}\right].$$

[3]Don't let the complicated last term confuse you. If you think about writing everything out as a fraction, you'll see immediately that both sides are equal. Try it for $N = 2$ and for $N = 3$ if it's still not clear.

To simplify the notation, let's use different letters for the terms in the continued fraction on the right, say

$$[b_0, b_1, \ldots, b_N]$$
$$\text{with} \quad b_0 = a_0, \quad b_1 = a_1, \ldots, b_{N-1} = a_{N-1}, \quad \text{and} \quad b_N = a_N + \frac{1}{a_{N+1}}.$$

Notice that $[b_0, b_1, \ldots, b_N]$ is a continued fraction with one fewer term than the continued fraction $[a_0, a_1, \ldots, a_{N+1}]$, so our induction hypothesis says that the theorem is true for $[b_0, b_1, \ldots, b_N]$. To avoid confusion, we use capital letters P_n/Q_n for the convergents of $[b_0, b_1, \ldots, b_N]$. Then the induction hypothesis tells us that the P_n's and the Q_n's satisfy the recursion formulas

$$P_n = b_n P_{n-1} + P_{n-2} \qquad \text{and} \qquad Q_n = b_n Q_{n-1} + Q_{n-2} \qquad \text{for all } 2 \le n \le N.$$

Therefore

$$[b_0, b_1, \ldots, b_N] = \frac{P_N}{Q_N} = \frac{b_N P_{N-1} + P_{N-2}}{b_N Q_{N-1} + Q_{N-2}}. \qquad (*)$$

How are the convergents for $[a_0, a_1, \ldots, a_{N+1}]$ and $[b_0, b_1, \ldots, b_N]$ related? We know that $b_n = a_n$ for all $0 \le n \le N-1$, so the n^{th} convergents are the same for all $0 \le n \le N-1$. This means that we can make the following substitutions into the formula $(*)$:

$$P_{N-1} = p_{N-1}, \quad P_{N-2} = p_{N-2}, \quad Q_{N-1} = q_{N-1}, \quad Q_{N-2} = q_{N-2}.$$

Since we also know that

$$[b_0, b_1, \ldots, b_N] = [a_0, a_1, \ldots, a_{N+1}] \qquad \text{and} \qquad b_N = a_N + \frac{1}{a_{N+1}},$$

we find that

$$\begin{aligned} [a_0, a_1, \ldots, a_{N+1}] &= \frac{b_N P_{N-1} + P_{N-2}}{b_N Q_{N-1} + Q_{N-2}} \\ &= \frac{\left(a_N + \dfrac{1}{a_{N+1}}\right) p_{N-1} + p_{N-2}}{\left(a_N + \dfrac{1}{a_{N+1}}\right) q_{N-1} + q_{N-2}} \\ &= \frac{a_{N+1}(a_N p_{N-1} + p_{N-2}) + p_{N-1}}{a_{N+1}(a_N q_{N-1} + q_{N-2}) + q_{N-1}}. \qquad (**) \end{aligned}$$

The induction hypothesis applied to the continued fraction $[a_0, a_1, \ldots, a_N]$ tells us that its convergents satisfy

$$p_N = a_N p_{N-1} + p_{N-2} \qquad \text{and} \qquad q_N = a_N q_{N-1} + q_{N-2},$$

which allows us to simplify the formula $(**)$ to read

$$[a_0, a_1, \ldots, a_{N+1}] = \frac{a_{N+1}p_N + p_{N-1}}{a_{N+1}q_N + q_{N-1}}.$$

But by definition, the $(N+1)^{\text{st}}$ convergent is

$$[a_0, a_1, \ldots, a_{N+1}] = \frac{p_{N+1}}{q_{N+1}}.$$

Comparing these two expressions for $[a_0, a_1, \ldots, a_{N+1}]$, we see that

$$p_{N+1} = a_{N+1}p_N + p_{N-1} \qquad \text{and} \qquad q_{N+1} = a_{N+1}q_N + q_{N-1}.$$

(We are using the fact that both fractions are already written in lowest terms.) We have now shown that if the recursion relations are true for $n = N$ they are also true for $n = N + 1$. This completes our induction proof that they are true for all values of n. □

We expect that the convergents to a number such as $\sqrt{2}$ should get closer and closer to $\sqrt{2}$, so it might be interesting to see how close the convergents are to one another:

$$\frac{p_0}{q_0} - \frac{p_1}{q_1} = \frac{1}{1} - \frac{3}{2} = -\frac{1}{2}$$
$$\frac{p_1}{q_1} - \frac{p_2}{q_2} = \frac{3}{2} - \frac{7}{5} = \frac{1}{10}$$
$$\frac{p_2}{q_2} - \frac{p_3}{q_3} = \frac{7}{5} - \frac{17}{12} = -\frac{1}{60}$$
$$\frac{p_3}{q_3} - \frac{p_4}{q_4} = \frac{17}{12} - \frac{41}{29} = \frac{1}{348}$$
$$\frac{p_4}{q_4} - \frac{p_5}{q_5} = \frac{41}{29} - \frac{99}{70} = -\frac{1}{2030}$$
$$\frac{p_5}{q_5} - \frac{p_6}{q_6} = \frac{99}{70} - \frac{239}{169} = \frac{1}{11830}$$

The Difference Between Successive Convergents of $\sqrt{2}$

The difference between successive convergents does indeed seem to be getting smaller and smaller, but an even more interesting pattern has emerged. It seems that all the numerators are equal to 1 and that the values alternate between positive and negative.

Let's try another example, say the continued fraction expansion of π. We find that

$$\frac{p_0}{q_0} - \frac{p_1}{q_1} = \frac{3}{1} - \frac{22}{7} = -\frac{1}{7}$$
$$\frac{p_1}{q_1} - \frac{p_2}{q_2} = \frac{22}{7} - \frac{333}{106} = \frac{1}{742}$$
$$\frac{p_2}{q_2} - \frac{p_3}{q_3} = \frac{333}{106} - \frac{355}{113} = -\frac{1}{11978}$$
$$\frac{p_3}{q_3} - \frac{p_4}{q_4} = \frac{355}{113} - \frac{103993}{33102} = \frac{1}{3740526}$$
$$\frac{p_4}{q_4} - \frac{p_5}{q_5} = \frac{103993}{33102} - \frac{104348}{33215} = -\frac{1}{1099482930}$$
$$\frac{p_5}{q_5} - \frac{p_6}{q_6} = \frac{104348}{33215} - \frac{208341}{66317} = \frac{1}{2202719155}$$

The Difference Between Successive Convergents of π

The exact same pattern has appeared. So let's buckle down and prove a theorem.

Theorem 47.2 (Difference of Successive Convergents Theorem)**.** *As usual, let* $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ *be the convergents to the continued fraction* $[a_0, a_1, a_2, \ldots]$. *Then*

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n \qquad \textit{for all } n = 1, 2, 3, \ldots.$$

Equivalently, dividing both sides by $q_{n-1}q_n$*,*

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_{n-1}q_n} \qquad \textit{for all } n = 1, 2, 3, \ldots.$$

Proof. This theorem is quite easy to prove using induction and the Continued Fraction Recursion Formula (Theorem 47.1). First we check that it is true for $n = 1$:

$$p_0q_1 - p_1q_0 = a_0 \cdot a_1 - (a_1a_0 + 1) \cdot 1 = -1.$$

This gets our induction started.

Now we assume that the theorem is true for $n = N$, and we need to prove that it is true for $n = N + 1$. We compute

$$\begin{aligned} p_N q_{N+1} - p_{N+1} q_N &= p_N \left(a_{N+1} q_N + q_{N-1}\right) - \left(a_{N+1} p_N + p_{N-1}\right) q_N \\ &\qquad \text{using the Continued Fraction Recursion Theorem,} \\ &= p_N q_{N-1} - p_{N-1} q_N \quad \text{since the } a_{N+1} p_N q_N \text{ terms cancel,} \\ &= -\left(p_{N-1} q_N - p_N q_{N-1}\right) \\ &= -(-1)^N \quad \text{from the induction hypothesis with } n = N, \\ &= (-1)^{N+1}. \end{aligned}$$

We have now shown that the desired formula is true for $n = 1$ and that if it is true for $n = N$, then it is also true for $n = N + 1$. Therefore, by induction it is true for all values of $n \geq 1$, which completes the proof of the theorem. □

Exercises

47.1. **(a)** Compute the first ten terms in the continued fractions of $\sqrt{3}$ and $\sqrt{5}$.

(b) Do the terms in the continued fraction of $\sqrt{3}$ appear to follow a repetitive pattern? If so, prove that they really do repeat.

(c) Do the terms in the continued fraction of $\sqrt{5}$ appear to follow a repetitive pattern? If so, prove that they really do repeat.

47.2. The continued fraction of π^2 is

$$[\_\_, \_\_, \_\_, 1, 2, 47, 1, 8, 1, 1, 2, 2, 1, 1, 8, 3, 1, 10, 5, 1, 3, 1, 2, 1, 1, 3, 15, 1, 1, 2, \ldots].$$

(a) Fill in the three initial missing entries.

(b) Do you see any sort of pattern in the continued fraction of π^2?

(c) Use the first fiveterms in the continued fraction to find a rational number that is close to π^2. How close do you come?

(d) Same question as (c), but use the first six terms.

47.3. The continued fraction of $\sqrt{2} + \sqrt{3}$ is

$$[\_\_, \_\_, \_\_, 5, 7, 1, 1, 4, 1, 38, 43, 1, 3, 2, 1, 1, 1, 1, 2, 4, 1, 4, 5, 1, 5, 1, 7, \ldots].$$

(a) Fill in the three initial missing entries.

(b) Do you see any sort of pattern in the continued fraction of $\sqrt{2} + \sqrt{3}$?

(c) For each $n = 1, 2, 3, \ldots, 7$, compute the n^{th} convergent

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n]$$

to $\sqrt{2} + \sqrt{3}$.

(d) The fractions that you computed in (b) should give more and more accurate approximations to $\sqrt{2}+\sqrt{3}$. Verify this by making a table of values

$$\left|\sqrt{2}+\sqrt{3}-\frac{p_n}{q_n}\right| = \frac{1}{10^{\text{some power}}}$$

for $n = 1, 2, 3, \ldots, 7$.

47.4. Let p_n/q_n be the n^{th} convergent to α. For each of the following values of α, make a table listing the value of the quantity

$$q_n\,|p_n - q_n\alpha| \qquad \text{for } n = 1, 2, 3, \ldots, N.$$

(The continued fraction expansions of $\sqrt{2}$, $\sqrt[3]{2}$, and π are listed on page 415, so you can use that information to compute the associated convergents.)

(a) $\alpha = \sqrt{2}$ up to $N = 8$.
(b) $\alpha = \sqrt[3]{2}$ up to $N = 7$.
(c) $\alpha = \pi$ up to $N = 5$.
(d) Your data from (a) suggest that not only is $|p_n - q_n\sqrt{2}|$ bounded, but it actually approaches a limit as $n \to \infty$. Try to guess what that limit equals, and then prove that your guess is correct.
(e) Recall that Dirichlet's Diophantine Approximation Theorem (Theorem 33.2) says that for any irrational number α, there are infinitely many pairs of positive integers x and y satisfying

$$|x - y\alpha| < 1/y. \tag{47.1}$$

Your data from (a), (b), and (c) suggest that if p_n/q_n is a convergent to α then (p_n, q_n) provides a solution to the inequality (47.1). Prove that this is true.

47.5. Suppose that we use the recursion for p_n backwards in order to define p_n for negative values of n. What are the values of p_{-1} and p_{-2}? Same question for q_{-1} and q_{-2}.

47.6. The Continued Fraction Recursion Formula (Theorem 47.1) gives a procedure for generating two lists of numbers $p_0, p_1, p_2, p_3, \ldots$ and $q_0, q_1, q_2, q_3, \ldots$ from two initial values a_0 and a_1. The fraction p_n/q_n is then the n^{th} convergent to some number α. Prove that the fraction p_n/q_n is already in lowest terms; that is, prove that $\gcd(p_n, q_n) = 1$. [*Hint*. Use the Difference of Successive Convergents Theorem (Theorem 47.2).]

47.7. We proved that successive convergents p_{n-1}/q_{n-1} and p_n/q_n satisfy

$$p_{n-1}q_n - p_nq_{n-1} = (-1)^n.$$

In this exercise you will figure out what happens if instead we take every other convergent.

(a) Compute the quantity

$$p_{n-2}q_n - p_nq_{n-2} \tag{$*$}$$

for the convergents of the partial fraction $\sqrt{2} = [1, 2, 2, 2, 2, \ldots]$. Do this for $n = 2, 3, \ldots, 6$.

(b) Compute the quantity $(*)$ for $n = 2, 3, \ldots, 6$ for the convergents of the partial fraction

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, \ldots].$$

(c) Using your results from (a) and (b) (and any other data that you want to collect), make a conjecture for the value of the quantity $(*)$ for a general continued fraction $[a_0, a_1, a_2, \ldots]$.

(d) Prove that your conjecture in (c) is correct. [*Hint*. The Continued Fraction Recursion Formula may be useful.]

47.8. The "simplest" continued fraction is the continued fraction $[1, 1, 1, \ldots]$ consisting entirely of 1's.

(a) Compute the first 10 convergents of $[1, 1, 1, \ldots]$.

(b) Do you recognize the numbers appearing in the numerators and denominators of the fractions that you computed in (a)? (If not, look back at Chapter 39.)

(c) What is the exact value of the limit

$$\lim_{n\to\infty} \frac{p_n}{q_n}$$

of the convergents for the continued fraction $[1, 1, 1, \ldots]$?

47.9. In Table 47.2 we listed the numerator p_n of the continued fraction $[a_0, a_1, \ldots, a_n]$ for the first few values of n.

(a) How are the numerators of $[a, b]$ and $[b, a]$ related to one another?

(b) How are the numerators of $[a, b, c]$ and $[c, b, a]$ related to one another?

(c) More generally, how do the numerators of

$$[a_0, a_1, a_2, \ldots, a_{n-1}, a_n] \qquad \text{and} \qquad [a_n, a_{n-1}, \ldots, a_2, a_1, a_0]$$

seem to be related to one another?

(d) Prove that your conjecture in (c) is correct.

47.10. Write a program that takes as input a decimal number A and an integer n and returns the following values:

(a) the first $n + 1$ terms $[a_0, a_1, \ldots, a_n]$ of the continued fraction of A;

(b) the n^{th} convergent p_n/q_n of A, as a fraction;

(c) the difference between A and p_n/q_n, as a decimal.

47.11. Use your program from Exercise 47.10 to make a table of (at least) the first 10 terms of the continued fraction expansion of $\sqrt{D}$ for $2 \le D \le 30$. What sort of pattern(s) can you find? (You can check your output by comparing with Table 48.1 in the next chapter.)

47.12. Same question as Exercise 47.11, but with cube roots. In other words, make a table of (at least) the first 10 terms of the continued fraction expansion of $\sqrt[3]{D}$ for each value of D satisfying $2 \le D \le 20$. Do you see any patterns?

47.13. (**Advanced Calculus Exercise**) Let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of real numbers satisfying $a_i \geq 1$. Then, for each $n = 0, 1, 2, 3, \ldots$, we can compute the real number

$$u_n = [a_0, a_1, a_2, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}.$$

Prove that the limit $\lim_{n\to\infty} u_n$ exists. [*Hint*. Use Theorems 47.1 and 47.2 to prove that the sequence $u_1, u_2, u_3, \ldots$ is a Cauchy sequence.]

Chapter 48

Continued Fractions, Square Roots, and Pell's Equation [online]

The continued fraction for $\sqrt{2}$,

$$[1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, \ldots],$$

certainly appears to be quite repetitive. Let's see if we can prove that the continued fraction of $\sqrt{2}$ really does consist of the number 1 followed entirely by 2's. Since $\sqrt{2} = 1.414\ldots$, the first step in the continued fraction algorithm is to write

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{1/(\sqrt{2} - 1)}.$$

Next we simplify the denominator,

$$\frac{1}{\sqrt{2} - 1} = \frac{1}{\sqrt{2} - 1} \cdot \frac{\sqrt{2} + 1}{\sqrt{2} + 1} = \frac{\sqrt{2} + 1}{\sqrt{2}^2 - 1} = \sqrt{2} + 1.$$

Substituting this back in above yields

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2} + 1}.$$

The number $\sqrt{2} + 1$ is between 2 and 3, so we write it as

$$\sqrt{2} + 1 = 2 + (\sqrt{2} - 1) = 2 + \frac{1}{1/(\sqrt{2} - 1)}.$$

But we already checked that $1/(\sqrt{2}-1)$ is equal to $\sqrt{2}+1$, so we find that

$$\sqrt{2}+1 = 2 + \frac{1}{\sqrt{2}+1}. \tag{$*$}$$

and hence that

$$\sqrt{2} = 1 + \frac{1}{\sqrt{2}+1} = 1 + \cfrac{1}{2 + \cfrac{1}{\sqrt{2}+1}}.$$

Now we can use the formula $(*)$ again to obtain

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\sqrt{2}+1}}},$$

and yet again to obtain

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\sqrt{2}+1}}}}.$$

Continuing to employ the formula $(*)$, we find that the continued fraction of $\sqrt{2}$ does indeed consist of a single 1 followed entirely by 2's.

Emboldened by our success with $\sqrt{2}$, can we find other numbers whose continued fractions are similarly repetitive (or, to employ proper mathematical terminology, whose continued fractions are *periodic*)? If you have done Exercise 47.11, which asks you to compute the continued fraction of $\sqrt{D}$ for $D = 2, 3, 4, \ldots, 20$, you found some examples. Collecting further data of this sort, Table 48.1 lists the continued fractions for $\sqrt{p}$ for each prime p less than 40.

Let's turn the question on its head and ask what we can deduce about a continued fraction that happens to be repetitive. We start with a simple example. Suppose that the number A has as its continued fraction

$$A = [a, b, b, b, b, b, b, b, \ldots].$$

Table 48.1 includes several numbers of this sort, including $\sqrt{2}$, $\sqrt{5}$, and $\sqrt{37}$. We can write A in the form

$$A = a + \frac{1}{[b, b, b, b, b, \ldots]},$$

D	Continued fraction of $\sqrt{D}$	Period
2	$[1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,\cdots]$	1
3	$[1,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,2,1,\cdots]$	2
5	$[2,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,4,\cdots]$	1
7	$[2,1,1,1,4,1,1,1,4,1,1,1,4,1,1,1,4,1,\cdots]$	4
11	$[3,3,6,3,6,3,6,3,6,3,6,3,6,3,6,3,6,3,6,\cdots]$	2
13	$[3,1,1,1,1,6,1,1,1,1,6,1,1,1,1,6,1,1,\cdots]$	5
17	$[4,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,\cdots]$	1
19	$[4,2,1,3,1,2,8,2,1,3,1,2,8,2,1,3,1,2,8,\cdots]$	6
23	$[4,1,3,1,8,1,3,1,8,1,3,1,8,1,3,1,8,1,\cdots]$	4
29	$[5,2,1,1,2,10,2,1,1,2,10,2,1,1,2,10,2,1,1,\cdots]$	5
31	$[5,1,1,3,5,3,1,1,10,1,1,3,5,3,1,1,10,1,1,\cdots]$	8
37	$[6,12,12,12,12,12,12,12,12,12,12,12,\cdots]$	1

Table 48.1: Continued Fractions of Square Roots

so we really need to determine the value of the continued fraction

$$B = [b, b, b, b, b, b, b, b, \ldots].$$

Just as we did for A, we can pull off the first entry of B and write B as

$$B = b + \frac{1}{[b, b, b, b, b, \ldots]}.$$

But "lo and behold," the denominator $[b, b, b, b, \ldots]$ is simply B itself, so we find that

$$B = b + \frac{1}{B}.$$

Now we can multiply through by B to get $B^2 = bB + 1$ and then use the quadratic formula to solve for B,

$$B = \frac{b + \sqrt{b^2 + 4}}{2}.$$

(Note that we use the plus sign, since we need B to be positive.) Finally, we compute the value of A,

$$\begin{aligned} A = a + \frac{1}{B} &= a + \frac{2}{b + \sqrt{b^2+4}} \\ &= a + \frac{2}{b + \sqrt{b^2+4}} \cdot \left(\frac{b - \sqrt{b^2+4}}{b - \sqrt{b^2+4}} \right) \\ &= a - \frac{b - \sqrt{b^2+4}}{2} \\ &= \frac{2a-b}{2} + \frac{\sqrt{b^2+4}}{2}. \end{aligned}$$

We summarize our calculations, including two very interesting special cases.

Proposition 48.1. *For any positive integers a and b, we have the continued fraction formula*

$$\frac{2a-b}{2} + \frac{\sqrt{b^2+4}}{2} = [a, b, b, b, b, b, b, b, \ldots].$$

In particular, taking $a = b$ gives the formula

$$\frac{b + \sqrt{b^2+4}}{2} = [b, b, b, b, b, b, b, \ldots]$$

and taking $b = 2a$ gives the formula

$$\sqrt{a^2+1} = [a, 2a, 2a, 2a, 2a, 2a, 2a, 2a, \ldots].$$

What happens if we have a continued fraction that repeats in a more complicated fashion? Let's do an example to try to gain some insight. Suppose that A has the continued fraction

$$A = [1, 2, 3, 4, 5, 4, 5, 4, 5, 4, 5, \ldots],$$

where the subsequent terms continue to alternate 4 and 5. The first thing to do is to pull off the nonrepetitive part,

$$A = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{[4, 5, 4, 5, 4, 5, 4, 5, \ldots]}}}.$$

So now we need to figure out the value of the *purely periodic* continued fraction

$$B = [4, 5, 4, 5, 4, 5, 4, 5, 4, 5, \ldots].$$

(A continued fraction is called *purely periodic* if it repeats from the very beginning.) We can write B as

$$B = 4 + \cfrac{1}{5 + \cfrac{1}{[4, 5, 4, 5, 4, 5, 4, 5, 4, 5, \ldots]}}.$$

As in our earlier example, we recognize that the bottommost denominator is equal to B, so we have shown that

$$B = 4 + \cfrac{1}{5 + \cfrac{1}{B}}.$$

Now we simplify this complicated fraction to get an equation for B,

$$B = 4 + \cfrac{1}{5 + \cfrac{1}{B}} = 4 + \cfrac{1}{\cfrac{5B+1}{B}} = 4 + \frac{B}{5B+1} = \frac{21B+4}{5B+1}.$$

Cross-multiplying by $5B + 1$, moving everything to one side, and doing a little bit of algebra, we find the equation

$$5B^2 - 20B - 4 = 0,$$

and then the good old quadratic formula yields

$$B = \frac{20 + \sqrt{400 + 80}}{10} = \frac{10 + 2\sqrt{30}}{5}.$$

Next we find the value of A by substituting the value of B into our earlier formula and using elementary algebra to repeatedly flip, combine, and simplify.

$$A = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{B}}} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\frac{10+2\sqrt{30}}{5}}}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{5}{10+2\sqrt{30}}}} = 1 + \cfrac{1}{2 + \cfrac{1}{\frac{35+6\sqrt{30}}{10+2\sqrt{30}}}}$$

$$= 1 + \cfrac{1}{2 + \frac{10+2\sqrt{30}}{35+6\sqrt{30}}} = 1 + \cfrac{1}{\frac{80+14\sqrt{30}}{35+6\sqrt{30}}}$$

$$= 1 + \frac{35+6\sqrt{30}}{80+14\sqrt{30}} = \frac{115+20\sqrt{30}}{80+14\sqrt{30}}.$$

Finally, we rationalize the denominator of A by multiplying the numerator and denominator by $80 - 14\sqrt{30}$,

$$A = \frac{115+20\sqrt{30}}{80+14\sqrt{30}} \cdot \left(\frac{80-14\sqrt{30}}{80-14\sqrt{30}}\right) = \frac{800-10\sqrt{30}}{520} = \frac{80-\sqrt{30}}{52}.$$

If you have done Exercise 47.10, try running your program with the input

$$\frac{80-\sqrt{30}}{52} = 1.4331302774028526704 89\ldots$$

and check that you indeed get the continued fraction $[1, 2, 3, 4, 5, 4, 5, 4, 5, \ldots]$.

A continued fraction is called *periodic* if it looks like

$$[\underbrace{a_1, a_2, \ldots, a_\ell}_{\text{initial part}}, \underbrace{b_1, b_2, \ldots, b_m}_{\text{periodic part}}, \underbrace{b_1, b_2, \ldots, b_m}_{\text{periodic part}}, \underbrace{b_1, b_2, \ldots, b_m}_{\text{periodic part}}, \ldots].$$

In other words, it is periodic if, after some initial terms, it consists of a finite list of terms that are repeated over and over again. The number of repeated terms m is called the *period*. For example, $\sqrt{2} = [1, 2, 2, 2, \ldots]$ has period 1 and $\sqrt{23} =$

$[4, 1, 3, 1, 8, 1, 3, 1, 8, \ldots]$ has period 4. Other examples are given in Table 48.1. A convenient notation, which makes the periodicity more visible, is to place a bar over the repeating part to indicate that it repeats indefinitely. For example

$$\sqrt{2} = [1, \overline{2}\,], \qquad \sqrt{23} = [4, \overline{1, 3, 1, 8}\,], \qquad \frac{80 - \sqrt{30}}{52} = [1, 2, 3, \overline{4, 5}\,].$$

Similarly, a general periodic continued fraction is written as

$$[a_1, a_2, \ldots, a_\ell, \overline{b_1, b_2, \ldots, b_m}\,].$$

The examples that we have done suggest that the following theorem might be true. We prove the first part and leave the second part as a (challenging) exercise.

Theorem 48.2 (Periodic Continued Fraction Theorem)**.**
(a) *Suppose that the number A has a periodic continued fraction*

$$A = [a_1, a_2, \ldots, a_\ell, \overline{b_1, b_2, \ldots, b_m}\,].$$

Then A is equal to a number of the form

$$A = \frac{r + s\sqrt{D}}{t} \qquad \textit{with } r, s, t, D \textit{ integers and } D > 0.$$

(b) *Let r, s, t, D be integers with $D > 0$. Then the number*

$$\frac{r + s\sqrt{D}}{t}$$

has a periodic continued fraction.

Proof. (a) Let's start with the purely periodic continued fraction

$$B = [\,\overline{b_1, b_2, \ldots, b_m}\,].$$

If we write out the first m steps, we find that

$$B = b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_m + \cfrac{1}{[\,\overline{b_1, b_2, \ldots, b_m}\,]}}}} = b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_m + \cfrac{1}{B}}}}.$$

We now simplify the right-hand side by repeatedly combining terms and flipping fractions, where we treat B as a variable and the quantities $b_1, \dots, b_m$ as numbers. After much algebra, our equation eventually simplifies to

$$B = \frac{uB + v}{wB + z}, \qquad (*)$$

where u, v, w, z are certain integers that depend on $b_1, b_2, \dots, b_m$. Furthermore, it is clear that u, v, w, z are all positive numbers, since $b_1, b_2, \dots$ are positive.

To illustrate this procedure, we do the case $m = 2$.

$$B = b_1 + \cfrac{1}{b_2 + \cfrac{1}{B}} = b_1 + \cfrac{1}{\cfrac{b_2 B + 1}{B}} = b_1 + \frac{B}{b_2 B + 1} = \frac{(b_1 b_2 + 1)B + b_1}{b_2 B + 1}.$$

Returning to the general case, we cross-multiply equation $(*)$ and move everything to one side, which gives the equation

$$wB^2 + (z - u)B - v = 0.$$

Now the quadratic formula yields

$$B = \frac{-(z-u) + \sqrt{(z-u)^2 + 4vw}}{2w},$$

so B has the form

$$B = \frac{i + j\sqrt{D}}{k} \qquad \text{with } i, j, k, D \text{ integers and } D > 0.$$

Returning to our original number $A = [a_1, a_2, \dots, a_\ell, \overline{b_1, b_2, \dots, b_m}\,]$, we can write A as

$$A = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_\ell + \cfrac{1}{B}}}} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_\ell + \cfrac{1}{\cfrac{i + j\sqrt{D}}{k}}}}}.$$

Again we repeatedly flip, combine, and simplify, which eventually yields an expression for A of the form

$$A = \frac{e + f\sqrt{D}}{g + h\sqrt{D}}, \qquad \text{where } e, f, g, h, D \text{ are integers and } D > 0.$$

Finally, we multiply both numerator and denominator by $g - h\sqrt{D}$. This rationalizes the denominator and expresses A as a number of the form

$$A = \frac{r + s\sqrt{D}}{t}, \qquad \text{where } r, s, t, D \text{ are integers and } D > 0.$$

We have completed the proof of part (a) of the Periodic Continued Fraction Theorem (Theorem 48.2). The proof of part (b) is left to you as (challenging) Exercise 48.10. □

The Continued Fraction of $\sqrt{D}$ and Pell's Equation

The convergents to a continued fraction form a list of rational numbers that get closer and closer to the original number. For example, the number $\sqrt{71}$ has continued fraction

$$\sqrt{71} = [8, \overline{2, 2, 1, 7, 1, 2, 2, 16}\,]$$

and its first few convergents are

$$\frac{17}{2}, \frac{42}{5}, \frac{59}{7}, \frac{455}{54}, \frac{514}{61}, \frac{1483}{176}, \frac{3480}{413}, \frac{57163}{6784}, \frac{117806}{13981}, \frac{292775}{34746}.$$

If p/q is a convergent to $\sqrt{D}$, then

$$\frac{p}{q} \approx \sqrt{D}, \qquad \text{so} \qquad \frac{p^2}{q^2} \approx D.$$

Multiplying by q^2, this means that we would expect p^2 to be fairly close to Dq^2. Table 48.2 lists the values of the differences $p^2 - Dq^2$ for the first few convergents to $\sqrt{71}$.

Among the many striking features of the data in Table 48.2, we pick out the seemingly mundane appearance of the number 1 in the final column. This occurs on the seventh row and reflects that fact that

$$3480^2 - 71 \cdot 413^2 = 1.$$

Thus the convergent $3480/413$ to the number $\sqrt{71}$ provides a solution $(3480, 413)$ to the Pell equation

$$x^2 - 71y^2 = 1.$$

This suggests a connection between the convergents to $\sqrt{D}$ and Pell's equation $x^2 - Dy^2 = 1$.

p	q	$p^2 - 71q^2$
17	2	5
42	5	-11
59	7	2
455	54	-11
514	61	5
1483	176	-7
3480	413	1
57163	6784	-7
117806	13981	5
292775	34746	-11

Table 48.2: Convergents p/q to $\sqrt{71}$

In Chapters 32 and 34 we carefully and completely proved that Pell's equation

$$x^2 - Dy^2 = 1$$

always has a solution. But if you look back at Chapter 34, you will see that our proof does not provide an efficient way to actually find a solution. It would thus be very useful if the convergents to $\sqrt{D}$ could be used to efficiently compute a solution to Pell's equation.

The continued fraction of $\sqrt{71}$,

$$\sqrt{71} = [8, \overline{2, 2, 1, 7, 1, 2, 2, 16}\,],$$

has period 8, and the convergent that gives the solution to Pell's equation is

$$[8, 2, 2, 1, 7, 1, 2, 2] = \frac{3480}{413}.$$

A brief examination of Table 48.1 shows that the continued fractions of square roots $\sqrt{D}$ have many special features.[1] Here are some further examples with moderately large periods.

$$\begin{aligned}
\sqrt{73} &= [8, \overline{1, 1, 5, 5, 1, 1, 16}\,], && \text{Period} = 7.\\
\sqrt{89} &= [9, \overline{2, 3, 3, 2, 18}\,], && \text{Period} = 5.\\
\sqrt{97} &= [9, \overline{1, 5, 1, 1, 1, 1, 1, 1, 5, 1, 18}\,], && \text{Period} = 11.
\end{aligned}$$

[1] Exercise 48.9 describes various special properties of the continued fraction for $\sqrt{D}$, but before you look at that exercise, you should try to discover some for yourself.

For $D = 71$, the convergent that solved Pell's equation was the one obtained by removing the overline and dropping the last entry. Let's try doing the same for $D = 73$, $D = 89$, and $D = 97$. The results are shown in Table 48.3.

$\sqrt{D}$	$[a, b_1, b_2, \ldots, b_{m-1}] = \dfrac{p}{q}$	$p^2 - Dq^2$
$\sqrt{71}$	$[8, 2, 2, 1, 7, 1, 2, 2] = \dfrac{3480}{413}$	$3480^2 - 71 \cdot 413^2 = 1$
$\sqrt{73}$	$[8, 1, 1, 5, 5, 1, 1] = \dfrac{1068}{125}$	$1068^2 - 73 \cdot 125^2 = -1$
$\sqrt{79}$	$[8, 1, 7, 1] = \dfrac{80}{9}$	$80^2 - 79 \cdot 9^2 = 1$
$\sqrt{97}$	$[9, 1, 5, 1, 1, 1, 1, 1, 1, 5, 1] = \dfrac{5604}{569}$	$5604^2 - 97 \cdot 569^2 = -1$

Table 48.3: Convergents to $\sqrt{D}$ and Pell's Equation

This looks very promising. We did not get solutions to Pell's equation in all cases, but we found either a solution to Pell's equation $p^2 - Dq^2 = 1$ or a solution to the similar equation $p^2 - Dq^2 = -1$. Furthermore, we obtain a plus sign when the period of $\sqrt{D}$ is even and a minus sign when the period of $\sqrt{D}$ is odd. We summarize our observations in the following wonderful theorem.

Theorem 48.3. *Let D be a positive integer that is not a perfect square. Write the continued fraction of $\sqrt{D}$ as*

$$\sqrt{D} = [a, \overline{b_1, b_2, \ldots, b_{m-1}, b_m}\,] \qquad \textit{and let} \qquad \frac{p}{q} = [a, b_1, b_2, \ldots, b_{m-1}].$$

Then (p, q) is the smallest solution in positive integers to the equation

$$p^2 - Dq^2 = (-1)^m.$$

We do not give the proof of Theorem 48.3, since it is time to wrap up our discussion of continued fractions and move on to other topics. If you are interested in reading the proof, you will find it in Chapter 4 of Davenport's *The Higher Arithmetic* and in many other number theory textbooks. Instead, we conclude with one final observation and one Brobdingnagian[2] example.

[2]"The Learning of this People [the Brobdingnags] is very defective, consisting only in Morality, History, Poetry, and Mathematicks, wherein they must be allowed to excel." (*Gulliver's Travels*, Chapter II:7, Jonathan Swift)

Our observation has to do with the problem of solving $x^2 - Dy^2 = 1$ when Theorem 48.3 happens to give a solution to $x^2 - Dy^2 = -1$. In other words, what can we do when $\sqrt{D} = [a, \overline{b_1, b_2, \ldots, b_{m-1}, b_m}\,]$ and m is odd? The answer is provided by our earlier work. Recall that Pell's Equation Theorem (Theorem 32.1) says that if (x_1, y_1) is the smallest solution to $x^2 - Dy^2 = 1$ in positive integers, then every other solution (x_k, y_k) can be computed from the smallest solution via the formula

$$x_k + y_k\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^k, \qquad k = 1, 2, 3, \ldots. \tag{$*$}$$

The following computation shows why this formula works:

$$\begin{aligned} x_k^2 - Dy_k^2 &= \left(x_k + y_k\sqrt{D}\right)\left(x_k - y_k\sqrt{D}\right) \\ &= \left(x_1 + y_1\sqrt{D}\right)^k \left(x_1 - y_1\sqrt{D}\right)^k \\ &= (x_1^2 - Dy_1^2)^k \\ &= 1 \qquad \text{since we have assumed that } x_1^2 - Dy_1^2 = 1. \end{aligned}$$

Suppose instead that (x_1, y_1) is a solution to $x^2 - Dy^2 = -1$ and that we compute (x_k, y_k) using formula $(*)$. Then we get

$$x_k^2 - Dy_k^2 = (x_1^2 - Dy_1^2)^k = (-1)^k.$$

So if k is even, then we get a solution to Pell's equation $x^2 - Dy^2 = 1$.

Do you see how this solves our problem? Suppose that m is odd in Theorem 48.3, so (p, q) satisfies $p^2 - Dq^2 = -1$. Then we simply compute the square

$$(p + q\sqrt{D})^2 = (p^2 + q^2D) + 2pq\sqrt{D}$$

to find the desired solution $(p^2 + q^2D, 2pq)$ to $x^2 - Dy^2 = 1$. This finally gives us an efficient way to solve Pell's equation in all cases, and in fact one can show that it provides a method for finding the smallest solution.

Theorem 48.4 (Continued Fractions and Pell's Equation Theorem)**.** *Write the continued fraction of $\sqrt{D}$ as*

$$\sqrt{D} = [a, \overline{b_1, b_2, \ldots, b_{m-1}, b_m}\,] \qquad \textit{and let} \qquad \frac{p}{q} = [a, b_1, b_2, \ldots, b_{m-1}].$$

Then the smallest solution in positive integers to Pell's equation

$$x^2 - Dy^2 = 1 \qquad \textit{is given by} \qquad (x_1, y_1) = \begin{cases} (p, q) & \textit{if } m \textit{ is even,} \\ (p^2 + q^2D, 2pq) & \textit{if } m \textit{ is odd.} \end{cases}$$

All other solutions are given by the formula

$$x_k + y_k\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)^k, \qquad k = 1, 2, 3, \ldots.$$

We conclude our exploration of the world of continued fractions by solving the seemingly innocuous Pell equation

$$x^2 - 313y^2 = 1.$$

The continued fraction of $\sqrt{313}$ is

$$\sqrt{313} = [17, \overline{1, 2, 4, 11, 1, 1, 3, 2, 2, 3, 1, 1, 11, 4, 2, 1, 34}\,].$$

Following the procedure laid out by Theorem 48.4, we discard the last number in the periodic part, which in this case is the number 34, and compute the fraction

$$\frac{126862368}{7170685} = [17, 1, 2, 4, 11, 1, 1, 3, 2, 2, 3, 1, 1, 11, 4, 2, 1].$$

The period m is equal to 17, so the pair $(p, q) = (126862368, 7170685)$ gives a solution to

$$126862368^2 - 313 \cdot 7170685^2 = -1.$$

To find the smallest solution to Pell's equation, Theorem 48.4 tells us to compute

$$\begin{aligned} p^2 + q^2 D &= 126862368^2 + 7170685^2 \cdot 313 = 32188120829134849 \\ 2pq &= 2 \cdot 126862368 \cdot 7170685 = 1819380158564160 \end{aligned}$$

Thus the smallest solution[3] to

$$x^2 - 313y^2 = 1 \quad \text{is} \quad (x, y) = (32188120829134849, 1819380158564160).$$

And if we desire the next smallest solution, we simply square

$$32188120829134849 + 1819380158564160\sqrt{313}$$

and read off the answer

$$(x, y) = (2072150245021969438104715652505601, \\ 117124856755987405647781716823680).$$

[3]As noted in Chapter 32, this is the solution found by Brouncker in 1657. Now you know how someone could find such a large solution back in the days before computers!

Exercises

48.1. Find the value of each of the following periodic continued fractions. Express your answer in the form $\frac{r+s\sqrt{D}}{t}$, where r, s, t, D are integers, just as we did in the text when we computed the value of $[1, 2, 3, \overline{4, 5}\,]$ to be $\frac{80-\sqrt{30}}{52}$.

(a) $[\,\overline{1, 2, 3}\,] = [1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots]$
(b) $[1, 1, \overline{2, 3}\,] = [1, 1, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, \ldots]$
(c) $[1, 1, 1, \overline{3, 2}\,] = [1, 1, 1, 3, 2, 3, 2, 3, 2, 3, 2, 3, 2, \ldots]$
(d) $[3, \overline{2, 1}\,] = [3, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, \ldots]$
(e) $[\overline{1, 3, 5}\,] = [1, 3, 5, 1, 3, 5, 1, 3, 5, 1, 3, 5, 1, 3, 5, \ldots]$
(f) $[1, 2, \overline{1, 3, 4}\,] = [1, 2, 1, 3, 4, 1, 3, 4, 1, 3, 4, 1, 3, 4, \ldots]$

48.2. For each of the following numbers, find their (periodic) continued fraction. What is the period?

(a) $\dfrac{16-\sqrt{3}}{11}$ **(b)** $\dfrac{1+\sqrt{293}}{2}$ **(c)** $\dfrac{3+\sqrt{5}}{7}$ **(d)** $\dfrac{1+2\sqrt{5}}{3}$

48.3. During the proof of the Periodic Continued Fraction Theorem (Theorem 48.2), we simplified the continued fraction $[b_1, b_2, B]$ and found that it equals

$$\frac{(b_1b_2+1)B+b_1}{b_2B+1}.$$

(a) Do a similar calculation for $[b_1, b_2, b_3, B]$ and write it as

$$[b_1, b_2, b_3, B] = \frac{uB+v}{wB+z},$$

where u, v, w, z are given by formulas that involve b_1, b_2, and b_3.

(b) Repeat (a) for $[b_1, b_2, b_3, b_4, B]$.

(c) Look at your answers in (a) and (b). Do the expressions for u, v, w, z look familiar? [*Hint*. Compare them to the fractions $[b_1, b_2]$, $[b_1, b_2, b_3]$, and $[b_1, b_2, b_3, b_4]$. These are convergents to $[b_1, b_2, b_3, \ldots]$. Also look at Table 47.2.]

(d) More generally, when the continued fraction $[b_1, b_2, \ldots, b_m, B]$ is simplified as

$$[b_1, b_2, b_3, \ldots, b_m, B] = \frac{u_mB+v_m}{w_mB+z_m},$$

explain how the numbers u_m, v_m, w_m, z_m can be described in terms of the convergents $[b_1, b_2, b_3, \ldots, b_{m-1}]$ and $[b_1, b_2, b_3, \ldots, b_m]$. Prove that your description is correct.

48.4. Proposition 48.1 describes the number with continued fraction expansion $[a, \overline{b}\,]$.

(a) Do a similar computation to find the number whose continued fraction expansion is $[a, \overline{b, c}\,]$.

(b) If you let $b = c$ in your formula, do you get the same result as described in Proposition 48.1? [If your answer is "No," then you made a mistake in (a)!]

(c) For which values of a, b, c does the number in (a) have the form $\frac{s\sqrt{D}}{t}$ for integers s, t, D?

(d) For which values of a, b, c is the number in (a) equal to the square root $\sqrt{D}$ of some integer D?

48.5. Theorem 48.3 tells us that if the continued fraction of $\sqrt{D}$ has odd period we can find a solution to $x^2 - Dy^2 = -1$.

(a) Among the numbers $2 \le D \le 20$ with D not a perfect square, which $\sqrt{D}$ have odd period and which have even period? Do you see a pattern?

(b) Same question for $\sqrt{p}$ for primes $2 \le p \le 40$. (See Table 48.1.)

(c) Write down infinitely many positive integers D such that $\sqrt{D}$ has odd period. For each of your D values, give a solution to the equation $x^2 - Dy^2 = -1$. [*Hint*. Look at Proposition 48.1.]

(d) Write down infinitely many positive integers D so that $\sqrt{D}$ has even period. [*Hint*. Use your solution to Exercise 48.4(d).]

48.6. **(a)** Write a program that takes as input a positive integer D and returns as output a list of numbers $[a, b_1, \ldots, b_m]$ so that the continued fraction expansion of $\sqrt{D}$ is $[a, \overline{b_1, \ldots, b_m}\,]$. Use your program to print a table of continued fractions of $\sqrt{D}$ for all nonsquare D between 2 and 50.

(b) Generalize (a) by writing a program that takes as input integers r, s, t, D with $t > 0$ and $D > 0$ and returns as output a list of numbers

$$[a_1, \ldots, a_\ell, b_1, \ldots, b_m] \qquad \text{satisfying} \qquad \frac{r + s\sqrt{D}}{t} = [a_1, \ldots, a_\ell, \overline{b_1, \ldots, b_m}\,].$$

Use your program to print a table of continued fractions of $(3 + 2\sqrt{D})/5$ for all nonsquare D between 2 and 50.

48.7. **(a)** Write a program that takes as input a list $[b_1, \ldots, b_m]$ and returns the value of the purely periodic continued fraction $[\,\overline{b_1, b_2, \ldots, b_m}\,]$. The output should be in the form (r, s, t, D), where the value of the continued fraction is $(r + s\sqrt{D})/t$.

(b) Use your program from (a) to compute the values of each of the following continued fractions:

$$[\,\overline{1}\,], \quad [\,\overline{1, 2}\,], \quad [\,\overline{1, 2, 3}\,], \quad [\,\overline{1, 2, 3, 4}\,], \quad [\,\overline{1, 2, 3, 4, 5}\,], \quad [\,\overline{1, 2, 3, 4, 5, 6}\,].$$

(c) Extend your program in (a) to handle periodic continued fractions that are not purely periodic. In other words, take as input two lists $[a_1, \ldots, a_\ell]$ and $[b_1, \ldots, b_m]$ and return the value of $[a_1, \ldots, a_\ell, \overline{b_1, b_2, \ldots, b_m}\,]$.

(d) Use your program from (c) to compute the values of each of the following continued fractions:

$$[6, 5, 4, 3, 2, \overline{1}\,], \quad [6, 5, 4, 3, \overline{1, 2}\,], \quad [6, 5, 4, \overline{1, 2, 3}\,], \quad [6, 5, \overline{1, 2, 3, 4}\,], \quad [6, \overline{1, 2, 3, 4, 5}\,].$$

48.8. Write a program to solve Pell's equation $x^2 - Dy^2 = 1$ using the method of continued fractions. If it turns out that there is a solution to $x^2 - Dy^2 = -1$, list a solution to this equation also.

(a) Use your program to solve Pell's equation for all nonsquare values of D between 2 and 20. Check your answers against Table 32.1 (page 249).

(b) Use your program to extend the table by solving Pell's equation for all nonsquare values of D between 76 and 99.

48.9. (**hard problem**) Let D be a positive integer that is not a perfect square.

(a) Prove that the continued fraction of $\sqrt{D}$ is periodic.

(b) More precisely, prove that the continued fraction of $\sqrt{D}$ looks like

$$\sqrt{D} = [a, \overline{b_1, b_2, \ldots, b_m}\,].$$

(c) Prove that $b_m = 2a$.

(d) Prove that the list of numbers $b_1, b_2, \ldots, b_{m-1}$ is symmetric; that is, it's the same left to right as it is right to left.

48.10. (**hard problem**) Let r, s, t, D be integers with $D > 0$ and $t \neq 0$ and let

$$A = \frac{r + s\sqrt{D}}{t}.$$

Prove that the continued fraction of A is periodic. [This is part (b) of the Periodic Continued Fraction Theorem (Theorem 48.2).]

Chapter 49

Generating Functions [online]

Aptitude tests, intelligence tests, and those ubiquitous grade school math worksheets teem with questions such as:[1]

What is the next number in the sequence 23, 27, 28, 32, 36, 37, 38, 39, 41, 43, 47, 49, 50, 51, 52, 53, 56, 58, 61, 62, 77, 78, ___?

Number theory abounds with interesting sequences. We've seen lots of them in our excursions, including for example[2]

Natural Numbers	$0, 1, 2, 3, 4, 5, 6, \ldots$
Square Numbers	$0, 1, 4, 9, 16, 25, 36, \ldots$
Fibonacci Numbers	$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

We have also seen that sequences can be described in various ways, for example by a formula such as

$$s_n = n^2$$

or by a recursion such as

$$F_n = F_{n-1} + F_{n-2}.$$

Both of these methods of describing a sequence are useful, but since a sequence consists of an infinitely long list of numbers, it would be nice to have a way of bundling the entire sequence into a single package. We will build these containers out of power series.

[1]This problem is for baseball fans. The answer is given at the end of the chapter.

[2]For this chapter, it is convenient to start these interesting sequences with 0, rather than with 1.

For example, we can package the sequence 0, 1, 2, 3, ... of natural numbers into the power series

$$0 + 1 \cdot x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots ,$$

and we can package the sequence 0, 1, 1, 2, 3, 5, 8, ... of Fibonacci numbers into the power series

$$0 + 1 \cdot x + 1 \cdot x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots .$$

In general, any sequence

$$a_0, a_1, a_2, a_3, a_4, a_5, \ldots$$

can be packaged into a power series

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

that is called the *generating function* for the sequence $a_0, a_1, a_2, a_3, \ldots$.

What good are generating functions, other than providing a moderately inconvenient way to list the terms of a sequence? The answer lies in that powerful word *function*. A generating function $A(x)$ is a function of the variable x; that is, we can substitute in a value for x and (if we're lucky) get back a value for $A(x)$. We say "if we're lucky" because, as you know if you have studied calculus, a power series need not converge for every value of x.

To illustrate these ideas, we start with the seemingly uninteresting sequence

$$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$$

consisting of all ones.[3] Its generating function, which we call $G(x)$, is

$$G(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots .$$

The ratio test[4] from calculus says that this series converges provided that

$$1 > \rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = |x|.$$

[3]If intelligence tests asked for the next term in sequences like this one, we could all have an IQ of 200!

[4]Recall that the ratio test says that a series $b_0 + b_1 + b_2 + \cdots$ converges if the limiting ratio $\rho = \lim_{n\to\infty} |b_{n+1}/b_n|$ satisfies $\rho < 1$.

You've undoubtedly already recognized that $G(x)$ is the *geometric series* and you probably also remember its value, but in case you've forgotten, here is the elegant method used to evaluate the geometric series.

$$\begin{aligned} xG(x) &= x(1 + x + x^2 + x^3 + x^4 + x^5 + \cdots) \\ &= x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots \\ &= (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots) - 1 \\ &= G(x) - 1. \end{aligned}$$

Thus $xG(x) = G(x) - 1$, and we can solve this equation for $G(x)$ to obtain the formula

$$G(x) = \frac{1}{1-x}.$$

This proves the following formula.

Geometric Series Formula

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1-x} \quad \text{valid for } |x| < 1$$

The sequence 1, 1, 1, 1,... is rather dull, so let's move on to the sequence of natural numbers 0, 1, 2, 3,... whose generating function is

$$N(x) = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + \cdots .$$

The ratio test tells us that $N(x)$ converges provided that

$$1 > \rho = \lim_{n\to\infty} \left| \frac{(n+1)x^{n+1}}{nx^n} \right| = |x|.$$

We would like to find a simple formula for $N(x)$, similar to the formula we found for $G(x)$. The way we do this is to start with the Geometric Series Formula

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1-x}$$

and use a little bit of calculus. If we differentiate both sides of this formula, we get

$$\underbrace{0 + 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots}_{\text{When multiplied by } x\text{, this becomes } N(x).} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

Multiplying both sides of this equation by x gives us the formula

$$N(x) = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + \cdots = \frac{x}{(1-x)^2}.$$

If we differentiate again and multiply both sides by x, we get a formula for the generating function

$$S(x) = x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + 36x^6 + \cdots$$

for the sequence of squares 0, 1, 4, 9, 16, 25, ... Thus

$$\begin{aligned} x\frac{dN(x)}{dx} &= x\frac{d}{dx}(x + 2x^2 + 3x^3 + 4x^4 + \cdots) &&= x\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) \\ S(x) &= x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \cdots &&= \frac{x + x^2}{(1-x)^3}. \end{aligned}$$

We now turn to the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, ... and its generating function

$$\begin{aligned} F(x) &= F_1x + F_2x^2 + F_3x^3 + F_4x^4 + F_5x^5 + F_6x^6 + \cdots \\ &= x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \cdots . \end{aligned}$$

How can we find a simple expression for $F(x)$? The differentiation trick we used earlier doesn't seem to help, so instead we make use of the recursive formula

$$F_n = F_{n-1} + F_{n-2}.$$

Thus we can replace F_3 with $F_2 + F_1$, and we can replace F_4 with $F_3 + F_2$, and so on, which means we can write $F(x)$ as

$$\begin{aligned} F(x) &= F_1x + F_2x^2 + \quad F_3x^3 \quad + \quad F_4x^4 \quad + \quad F_5x^5 \quad + \cdots \\ &= F_1x + F_2x^2 + (F_2 + F_1)x^3 + (F_3 + F_2)x^4 + (F_4 + F_3)x^5 + \cdots . \end{aligned}$$

Ignoring the first two terms for the moment, we regroup the other terms in the following manner:

Group these terms together

$$(\underset{\downarrow}{F_2} + \underset{\uparrow}{F_1})x^3 + (\underset{\downarrow}{F_3} + \underset{\uparrow}{F_2})x^4 + (\underset{\downarrow}{F_4} + \underset{\uparrow}{F_3})x^5 + (\underset{\downarrow}{F_5} + \underset{\uparrow}{F_4})x^6 + \ldots .$$

Group these terms together

This gives the formula

$$\begin{aligned} F(x) = F_1x + F_2x^2 &+ \{F_1x^3 + F_2x^4 + F_3x^5 + F_4x^6 + \cdots\} \\ &+ \{F_2x^3 + F_3x^4 + F_4x^5 + F_5x^6 + \cdots\}. \end{aligned}$$

Now observe that the series between the first set of braces is almost equal to the generating function $F(x)$ with which we started; more precisely, it is equal to $x^2F(x)$. Similarly, the series between the second set of braces is equal to $xF(x)$ except that it is missing the initial F_1x^2 term. In other words,

$$\begin{aligned} F(x) &= F_1x + F_2x^2 + \{F_1x^3 + F_2x^4 + \cdots\} + \{F_2x^3 + F_3x^4 + \cdots\} \\ &= F_1x + F_2x^2 + x^2\{\underbrace{F_1x + F_2x^2 + \cdots}_{\text{equals } F(x)}\} + x\{\underbrace{F_2x^2 + F_3x^3 + \cdots}_{\text{equals } F(x) - F_1x}\}. \end{aligned}$$

If we use the values $F_1 = 1$ and $F_2 = 1$, this gives us the formula

$$\begin{aligned} F(x) &= x + x^2 + x^2F(x) + x\big(F(x) - x\big) \\ &= x + x^2F(x) + xF(x). \end{aligned}$$

This gives us an equation that we can solve for $F(x)$ to obtain the following beautiful formula.

Fibonacci Generating Function Formula

$$x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots = \frac{x}{1 - x - x^2}$$

We can use the formula for the Fibonacci generating function together with the method of partial fractions that you learned in calculus to rederive Binet's formula for the n^{th} Fibonacci number (Theorem 39.1). The first step is to use the quadratic formula to find the roots of the polynomial $1 - x - x^2$. The roots are $\frac{-1\pm\sqrt{5}}{2}$, which are the reciprocals of the two numbers[5]

$$\alpha = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1-\sqrt{5}}{2}.$$

This lets us factor the polynomial as

$$1 - x - x^2 = (1 - \alpha x)(1 - \beta x).$$

The idea of partial fractions is to take the function $\dfrac{x}{1 - x - x^2}$ and split it up into the sum of two pieces

$$\frac{x}{1 - x - x^2} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x},$$

[5]Notice how the Golden Ratio (page 330) has suddenly appeared! This should not be surprising, since we saw in Chapter 39 that the Fibonacci sequence and the Golden Ratio are intimately related to one another.

where we need to find the correct values for A and B. To do this, we clear denominators by multiplying both sides by $1 - x - x^2$ to get

$$x = A(1 - \beta x) + B(1 - \alpha x).$$

[Remember that $1 - x - x^2 = (1 - \alpha x)(1 - \beta x)$.] Rearranging this relation yields

$$x = (A + B) - (A\beta + B\alpha)x.$$

We're looking for values of A and B that make the polynomial x on the left equal to the polynomial on the right, so we must choose A and B to satisfy

$$\begin{aligned} 0 &= A + B \\ 1 &= -A\beta - B\alpha. \end{aligned}$$

It is easy to solve these two equations for the unknown quantities A and B (remember that α and β are particular numbers). We find that

$$A = \frac{1}{\alpha - \beta} \qquad \text{and} \qquad B = \frac{1}{\beta - \alpha},$$

and using the values $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$ gives

$$A = \frac{1}{\sqrt{5}} \qquad \text{and} \qquad B = -\frac{1}{\sqrt{5}}.$$

To recapitulate, we have found the partial fraction decomposition

$$\frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \alpha x}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \beta x}\right).$$

This may not seem like progress, but it is, because we have replaced the complicated function $x/(1 - x - x^2)$ with the sum of two simpler expressions. If this were a calculus textbook, I would now ask you to compute the indefinite integral $\int \frac{x}{1-x-x^2}\,dx$ and you would use the partial fraction formula to compute

$$\begin{aligned} \int \frac{x}{1 - x - x^2}\,dx &= \frac{1}{\sqrt{5}} \int \frac{dx}{1 - \alpha x} - \frac{1}{\sqrt{5}} \int \frac{dx}{1 - \beta x} \\ &= \frac{-1}{\sqrt{5}\,\alpha} \log|1 - \alpha x| + \frac{1}{\sqrt{5}\,\beta} \log|1 - \beta x| + C. \end{aligned}$$

However, our subject is not calculus, it is number theory, so we instead observe that the two pieces of the partial fraction decomposition can be expanded using the geometric series as

$$\frac{1}{1-\alpha x} = 1 + \alpha x + (\alpha x)^2 + (\alpha x)^3 + (\alpha x)^4 + \cdots,$$
$$\frac{1}{1-\beta x} = 1 + \beta x + (\beta x)^2 + (\beta x)^3 + (\beta x)^4 + \cdots.$$

This lets us write the function $x/(1-x-x^2)$ as a power series

$$\frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha x}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1-\beta x}\right)$$
$$= \frac{\alpha-\beta}{\sqrt{5}}x + \frac{\alpha^2-\beta^2}{\sqrt{5}}x^2 + \frac{\alpha^3-\beta^3}{\sqrt{5}}x^3 + \cdots.$$

But we know that $x/(1-x-x^2)$ is the generating function for the Fibonacci sequence,

$$\frac{x}{1-x-x^2} = F_1x + F_2x^2 + F_3x^3 + F_4x^4 + F_5x^5 + F_6x^6 + \cdots,$$

so matching the two series for $x/(1-x-x^2)$, we find that

$$F_1 = \frac{\alpha-\beta}{\sqrt{5}}, \quad F_2 = \frac{\alpha^2-\beta^2}{\sqrt{5}}, \quad F_3 = \frac{\alpha^3-\beta^3}{\sqrt{5}}, \ldots.$$

Substituting the values of α and β, we again obtain Binet's formula (Theorem 39.1) for the n^{th} Fibonacci number.

Binet's Formula

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

The two numbers appearing in the Binet's Formula are approximately equal to

$$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618034 \qquad \text{and} \qquad \beta = \frac{1-\sqrt{5}}{2} \approx -0.618034.$$

Notice that $|\beta| < 1$, so if we raise β to a large power, it becomes very small. In particular,

$$F_n = \text{Closest integer to } \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$
$$\approx (0.447213\ldots) \times (1.61803\ldots)^n.$$

For example,

$$F_{10} \approx 55.003636\ldots \qquad \text{and} \qquad F_{25} \approx 75024.999997334\ldots,$$

which are indeed extremely close to the correct values $F_{10} = 55$ and $F_{25} = 75025$.

Exercises

49.1. **(a)** Find a simple formula for the generating function $E(x)$ for the sequence of even numbers 0, 2, 4, 6, 8, ...

(b) Find a simple formula for the generating function $J(x)$ for the sequence of odd numbers 1, 3, 5, 7, 9, ...

(c) What does $E(x^2) + xJ(x^2)$ equal? Why?

49.2. Find a simple formula for the generating function of the sequence of numbers

$$a, \quad a+m, \quad a+2m, \quad a+3m, \quad a+4m, \ldots.$$

(If $0 \le a < m$, then this is the sequence of nonnegative numbers that are congruent to a modulo m.)

49.3. **(a)** Find a simple formula for the generating function of the sequence whose n^{th} term is n^3, that is, the sequence 0, 1, 8, 27, 64, ...

(b) Repeat (a) for the generating function of the sequence 0, 1, 16, 81, 256, ... (This is the sequence whose n^{th} term is n^4.)

(c) If you have access to a computer that does symbolic differentiation or if you enjoy length calculations with paper and pencil, find the generating function for the sequence whose n^{th} term is n^5.

(d) Repeat (c) for the sequence whose n^{th} term is n^6.

49.4. Let $G(x) = 1 + x + x^2 + x^3 + \cdots$ be the generating function of the sequence $1, 1, 1, \ldots$.

(a) Compute the first five coefficients of the power series $G(x)^2$.

(b) Prove that the power series $G(x)^2 - G(x)$ is equal to some other power series that we studied in this chapter.

49.5. Let $T(x) = x + 3x^2 + 6x^3 + 10x^4 + \cdots$ be the generating function for the sequence $0, 1, 3, 6, 10, \ldots$ of triangular numbers. Find a simple expression for $T(x)$.

49.6. This question investigates the generating functions of certain sequences whose terms are binomial coefficients (see Chapter 38).

(a) Find a simple expression for the generating function of the sequence whose n^{th} term is $\binom{n}{1}$.

(b) Same question for the sequence whose n^{th} term is $\binom{n}{2}$.

(c) Same question for the sequence whose n^{th} term is $\binom{n}{3}$.

(d) For a fixed number k, make a conjecture giving a simple expression for the generating function of the sequence whose n^{th} term is $\binom{n}{k}$.

(e) Prove that your conjecture in (d) is correct.

49.7. Let $k \ge 0$ be an integer and let $D_k(x)$ be the generating function of the sequence $0^k, 1^k, 2^k, 3^k, 4^k, \ldots$. In this chapter we computed

$$D_0(x) = \frac{1}{1-x}, \qquad D_1(x) = \frac{x}{(1-x)^2}, \qquad D_2(x) = \frac{x+x^2}{(1-x)^3},$$

and in Exercise 49.3 you computed further examples. These computations suggest that $D_k(x)$ looks like

$$D_k(x) = \frac{P_k(x)}{(1-x)^{k+1}}$$

for some polynomial $P_k(x)$.

(a) Prove that there is a polynomial $P_k(x)$ such that $D_k(x)$ can be written in the form $P_k(x)/(1-x)^{k+1}$. [*Hint.* Use induction on k.]

(b) Make a list of values of $P_k(0)$ for $k = 0, 1, 2, \ldots$ and make a conjecture. Prove that your conjecture is correct.

(c) Same as (b) for the values of $P_k(1)$.

(d) Repeat (b) and (c) for the values of the derivative $P_k'(0)$ and $P_k'(1)$.

(e) What other patterns can you find in the $P_k(x)$ polynomials?

49.8. Let ϕ be Euler's phi function (see Chapter 11), and let p be a prime number. Find a simple formula for the generating function of the sequence $\phi(1), \phi(p), \phi(p^2), \phi(p^3), \ldots$.

49.9. The *Lucas sequence* is the sequence of numbers L_n given by the rules $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$.

(a) Write down the first 10 terms of the Lucas sequence.

(b) Find a simple formula for the generating function of the Lucas sequence.

(c) Use the partial fraction method to find a simple formula for L_n, similar to Binet's Formula for the Fibonacci number F_n.

49.10. Write down the first few terms in each of the following recursively defined sequences, and then find a simple formula for the generating function.

(a) $a_1 = 1$, $a_2 = 2$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for $n = 3, 4, 5, \ldots$

(b) $b_1 = 1$, $b_2 = 3$, and $b_n = 2b_{n-1} - 2b_{n-2}$ for $n = 3, 4, 5, \ldots$

(c) $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, and $c_n = 4c_{n-1} + 11c_{n-2} - 30c_{n-3}$ for $n = 4, 5, 6, \ldots$

49.11. Use generating functions and the partial fraction method to find a simple formula for the n^{th} term of each of the following sequences similar to the formula we found in the text for the n^{th} term of the Fibonacci sequence. (Note that these are the same sequences as in the previous exercise.) Be sure to check your answer for the first few values of n.

(a) $a_1 = 1$, $a_2 = 2$, and $a_n = 5a_{n-1} - 6a_{n-2}$ for $n = 3, 4, 5, \ldots$

(b) $b_1 = 1$, $b_2 = 3$, and $b_n = 2b_{n-1} - 2b_{n-2}$ for $n = 3, 4, 5, \ldots$ [*Hint.* You may need to use complex numbers!]

(c) $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, and $c_n = 4c_{n-1} + 11c_{n-2} - 30c_{n-3}$ for $n = 4, 5, 6, \ldots$

49.12. **(a)** Fix an integer $k \geq 0$, and let $H(x)$ be the generating function of the sequence whose n^{th} term is $h_n = n^k$. Use the ratio test to find the interval of convergence of the generating function $H(x)$.

(b) Use the ratio test to find the interval of convergence of the generating function $F(x)$ of the Fibonacci sequence $0, 1, 1, 2, 3, 5, \ldots$.

49.13. Sequences $a_0, a_1, a_2, a_3, \ldots$ are also sometimes packaged in an *exponential generating function*

$$a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + a_4\frac{x^4}{4!} + a_5\frac{x^5}{5!} + \cdots.$$

(a) What is the exponential generating function for the sequence 1, 1, 1, 1, ...? [*Hint*. Your answer explains why the word *exponential* is used in the name of this type of generating function.]

(b) What is the exponential generating function for the sequence 0, 1, 2, 3, ... of natural numbers?

49.14. Let $f(x)$ be the exponential generating function of the Fibonacci sequence

$$f(x) = F_0 + F_1\frac{x}{1!} + F_2\frac{x^2}{2!} + F_3\frac{x^3}{3!} + F_4\frac{x^4}{4!} + F_5\frac{x^5}{5!} + \cdots.$$

(a) Find a simple relation satisfied by $f(x)$ and its derivatives $f'(x)$ and $f''(x)$.

(b) Find a simple formula for $f(x)$.

49.15. Fix an integer N and create a sequence of numbers $a_0, a_1, a_2, \ldots$ in the following way:

$$\begin{aligned}
a_0 &= 1^0 + 2^0 + 3^0 + \cdot + N^0 \\
a_1 &= 1^1 + 2^1 + 3^1 + \cdot + N^1 \\
a_2 &= 1^2 + 2^2 + 3^2 + \cdot + N^2 \\
a_3 &= 1^3 + 2^3 + 3^3 + \cdot + N^3 \\
&\vdots \qquad\qquad \vdots
\end{aligned}$$

Compute the exponential generating function of this sequence. (We will study these power sums further in Chapter 50.)

Solution to Sequence on page 442. The next five terms in the sequence

$$23, 27, 28, 32, 36, 37, 38, 39, 41, 43, 47, 49, 50, 51, 52, 53, 56, 58, 61, 62, 77, 78$$

given at the beginning of this chapter are 96, 98, 99, 00, and 09, as is obvious to those who know that the New York Yankees won the World Series in the years 1923, 1927, 1928, ..., 1977, 1978, 1996, 1998, 1999, 2000, and 2009. Those who are not Yankee fans might prefer to complete the shorter sequence 03, 12, 15, 16, 18, ___. [*Hint*. There is a gap of 86 years before the next entry.]

Chapter 50

Sums of Powers [online]

The n^{th} triangular number

$$T_n = 1 + 2 + 3 + 4 + \cdots + n$$

is the sum of the first n natural numbers. In Chapter 1 we used geometry to find a simple formula for T_n,

$$T_n = \frac{n(n+1)}{2}.$$

This formula was extremely useful in Chapter 31, where we described all numbers that are simultaneously triangular and square.

The reason that the formula for T_n is so helpful is that it expresses a sum of n numbers as a simple polynomial in the variable n. To say this another way, let $F(X)$ be the polynomial

$$F(X) = \frac{1}{2}X^2 + \frac{1}{2}X.$$

Then the sum

$$1 + 2 + 3 + 4 + \cdots + n,$$

which at first glance requires us to add n numbers, can be computed very simply as the value $F(n)$.

Now suppose that rather than adding the first n integers, we instead add the first n squares,

$$R_n = 1 + 4 + 9 + 16 + \cdots + n^2.$$

We make a short table of the first few values and look for patterns.

n	1	2	3	4	5	6	7	8	9	10
R_n	1	5	14	30	55	91	140	204	285	385

The numbers $R_1, R_2, R_3, \ldots$ are increasing fairly rapidly, but they don't seem to obey any simple pattern. It isn't easy to see how we might get our hands on these numbers. We use a tool called the method of *telescoping sums*. To illustrate this technique, we first look at the following easier problem. Suppose we want to compute the value of the sum

$$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n-1)\cdot n}.$$

For this sum, if we compute the first few values, it's easy to see the pattern:

n	2	3	4	5	6	7	8	9	10
S_n	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{8}{9}$	$\frac{9}{10}$

So we guess that S_n is probably equal to $\frac{n-1}{n}$, but how can we prove that this is true? The key is to observe that the first few terms of the sum can be written as

$$\frac{1}{1\cdot 2} = 1 - \frac{1}{2} \qquad \text{and} \qquad \frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3} \qquad \text{and} \qquad \frac{1}{3\cdot 4} = \frac{1}{3} - \frac{1}{4}$$

and so on. More generally, the i^{th} term of the sum is equal to

$$\frac{1}{i\cdot(i+1)} = \frac{1}{i} - \frac{1}{i+1}.$$

Hence the sum S_n is equal to

$$\begin{aligned} S_n = &\ \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)\cdot n} \\ = &\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right). \end{aligned}$$

Now look what happens when we add the terms on this last line. We start with 1. Next we get $-\frac{1}{2}$ followed by $+\frac{1}{2}$, so these two terms cancel. Then we get $-\frac{1}{3}$, which is followed by $+\frac{1}{3}$, so these two terms also cancel. Notice how the sum "telescopes" (imagine how the tubes of a telescope fold into one another), with only the first term and the last term remaining at the end. This proves the formula

$$S_n = 1 - \frac{1}{n} = \frac{n-1}{n}.$$

Now we return to the problem of computing the sum of squares

$$R_n = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

For reasons that will become apparent in a moment, we look at the following telescoping sum involving cubes:

$$(n+1)^3 = 1^3 + (2^3 - 1^3) + (3^3 - 2^3) + \cdots + \bigl((n+1)^3 - n^3\bigr).$$

Using summation notation, we can write this as

$$(n+1)^3 = 1 + \sum_{i=1}^{n} \bigl((i+1)^3 - i^3\bigr).$$

Next we expand the expression $(i+1)^3$ using the binomial formula (Theorem 38.2)

$$(i+1)^3 = i^3 + 3i^2 + 3i + 1.$$

Substituting this into the telescoping sum gives (notice that the i^3 terms cancel)

$$(n+1)^3 = 1 + \sum_{i=1}^{n} (3i^2 + 3i + 1).$$

Now we split the sum into three pieces and add each piece individually,

$$\begin{aligned}(n+1)^3 &= 1 + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ &= 1 + 3R_n + 3T_n + n.\end{aligned}$$

But we already know that $T_n = \sum_{i=1}^{n} i$ is equal to $(n^2+n)/2$, so we can solve for R_n,

$$\begin{aligned}R_n &= \frac{(n+1)^3 - n - 1}{3} - T_n \\ &= \frac{n^3 + 3n^2 + 2n}{3} - \frac{n^2+n}{2} \\ &= \frac{2n^3 + 3n^2 + n}{6}.\end{aligned}$$

That was a lot of algebra, but we are amply rewarded for our efforts by the beautiful formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

Notice how nifty this formula is. If we want to compute the value of

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + 9999^2 + 10000^2,$$

we could add 10000 terms, but using the formula for R_n, we only need to compute

$$R_{10000} = \frac{2 \cdot 10000^3 + 3 \cdot 10000^2 + 10000}{6} = 333{,}383{,}335{,}000.$$

Now take a deep breath, because we next tackle the problem of adding sums of k^{th} powers for higher values of k. We write

$$F_k(n) = 1^k + 2^k + 3^k + \cdots + n^k$$

for the sum of the first n numbers, each raised to the k^{th} power.

The telescoping sum method that worked so well computing sums of squares works just as well for higher powers. We begin with the telescoping sum

$$(n+1)^k = 1^k + (2^k - 1^k) + (3^k - 2^k) + \cdots + \bigl((n+1)^k - n^k\bigr),$$

which, using summation notation, becomes

$$(n+1)^k = 1 + \sum_{i=1}^{n} \bigl((i+1)^k - i^k\bigr).$$

Just as before, we expand $(i+1)^k$ using the binomial formula (Theorem 38.2),

$$(i+1)^k = \sum_{j=0}^{k} \binom{k}{j} i^j.$$

The last term (i.e., the $j = k$ term) is i^k, so it cancels the i^k in the telescoping sum, leaving

$$(n+1)^k = 1 + \sum_{i=1}^{n} \sum_{j=0}^{k-1} \binom{k}{j} i^j.$$

Now switch the order of the two sums, and lo and behold, we find the power sums $F_0(n), F_1(n), \ldots, F_{k-1}(n)$ appearing,

$$(n+1)^k = 1 + \sum_{j=0}^{k-1} \binom{k}{j} \sum_{i=1}^{n} i^j = 1 + \sum_{j=0}^{k-1} \binom{k}{j} F_j(n).$$

What good is a formula like this, which seems to involve all sorts of quantities that we don't know? The answer is that it relates each of $F_0(n), F_1(n), F_2(n), \ldots$ with the previous ones. To make this clearer, we pull off the last term in the sum, that is, the term with $j = k-1$, and we move all the other terms to the other side,

$$\binom{k}{k-1} F_{k-1}(n) = (n+1)^k - 1 - \sum_{j=0}^{k-2} \binom{k}{j} F_j(n).$$

Now $\binom{k}{k-1} = k$, so dividing by k gives the recursive formula

$$F_{k-1}(n) = \frac{(n+1)^k - 1}{k} - \frac{1}{k}\sum_{j=0}^{k-2}\binom{k}{j}F_j(n).$$

We call this a recursive formula for the F_k's, because it expresses each F_k in terms of the previous ones. It is thus similar in some ways to the recursive formula used to describe the Fibonacci sequence (Chapter 39), although this formula is obviously much more complicated than the Fibonacci formula.

Let's use the recursive formula to find a new power-sum formula. Taking $k = 4$ in the recursive formula gives

$$F_3(n) = \frac{(n+1)^4 - 1}{4} - \frac{1}{4}\left\{\binom{4}{0}F_0(n) + \binom{4}{1}F_1(n) + \binom{4}{2}F_2(n)\right\}.$$

We already know from our earlier work that

$$F_1(n) = T_n = \frac{n^2+n}{2} \qquad \text{and} \qquad F_2(n) = R_n = \frac{2n^3+3n^2+n}{6},$$

while the value of $F_0(n)$ is clearly equal to

$$F_0(n) = 1^0 + 2^0 + \cdots + n^0 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n.$$

Substituting in these values for $F_0(n)$, $F_1(n)$, and $F_2(n)$ yields

$$\begin{aligned} F_3(n) &= \frac{(n+1)^4 - 1}{4} - \frac{1}{4}\left\{\binom{4}{0}F_0(n) + \binom{4}{1}F_1(n) + \binom{4}{2}F_2(n)\right\} \\ &= \frac{n^4+4n^3+6n^2+4n+1-1}{4} \\ &\qquad\qquad - \frac{1}{4}\left\{n + 4\frac{n^2+n}{2} + 6\frac{2n^3+3n^2+n}{6}\right\} \\ &= \frac{n^4+2n^3+n^2}{4}. \end{aligned}$$

Thus

$$F_3(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^4+2n^3+n^2}{4}.$$

For example,

$$\begin{aligned} 1^3 + 2^3 + 3^3 + 4^3 + \cdots + 10000^3 &= \frac{10000^4 + 2 \cdot 10000^3 + 10000^2}{4} \\ &= 2{,}500{,}500{,}025{,}000{,}000. \end{aligned}$$

The recursive formula for power sums is very beautiful, so we record our discovery in the form of a theorem.

Theorem 50.1 (Sum of Powers Theorem)**.** *Let $k \geq 0$ be an integer. There is a polynomial $F_k(X)$ of degree $k+1$ such that*

$$F_k(n) = 1^k + 2^k + 3^k + \cdots + n^k \quad \textit{for every value of } n = 1, 2, 3, \ldots.$$

These polynomials can be computed using the recurrence formula

$$F_{k-1}(X) = \frac{(X+1)^k - 1}{k} - \frac{1}{k}\sum_{i=0}^{k-2}\binom{k}{i}F_i(X).$$

Proof. We proved above that the power sums can be computed by the recurrence formula. It is also clear from the recurrence formula that the power sums are polynomials, since each successive power sum is simply the polynomial $\frac{(X+1)^k-1}{k}$ adding to some multiples of the previous power sums.

All that remains is to prove that $F_k(X)$ has degree $k+1$. We use induction. To start the induction, we observe that $F_0(X) = X$ has the correct degree. Now suppose that we know that $F_k(X)$ has degree $k+1$ for $k = 0, 1, 2, \ldots, m-1$. In other words, suppose we've finished the proof for all values of k less than m. We use the recurrence formula with $k = m+1$ to compute

$$F_m(X) = \frac{(X+1)^{m+1} - 1}{m+1} - \frac{1}{m+1}\sum_{i=0}^{m-1}\binom{m+1}{i}F_i(X).$$

The first part looks like

$$\frac{(X+1)^{m+1} - 1}{m+1} = \frac{1}{m+1}X^{m+1} + \cdots.$$

On the other hand, by the induction hypothesis we know that $F_i(X)$ has degree $i+1$ for each $i = 0, 1, \ldots, m-1$, so the polynomials in the sum have degree at most m. This proves that the X^{m+1} coming from the first part isn't canceled by any of the other terms, so $F_m(X)$ has degree $m+1$. This completes our induction proof. □

We've computed the power-sum polynomials

$$\begin{aligned}
F_1(X) &= \frac{1}{2}\bigl(X^2 + X\bigr),\\
F_2(X) &= \frac{1}{6}\bigl(2X^3 + 3X^2 + X\bigr),\\
F_3(X) &= \frac{1}{4}\bigl(X^4 + 2X^3 + X^2\bigr).
\end{aligned}$$

Now it's your turn to compute the next few power-sum polynomials, so turn to Exercise 50.2 and use the recursive formula to compute $F_4(X)$ and $F_5(X)$. Be sure to check your answers.

Lest you feel that I've done the easy computations and you've been stuck with the hard ones, here are the next few power-sum polynomials.

$$\begin{aligned}
F_6(X) &= \frac{1}{42}\big(6X^7 + 21X^6 + 21X^5 - 7X^3 + X\big)\\
F_7(X) &= \frac{1}{24}\big(3X^8 + 12X^7 + 14X^6 - 7X^4 + 2X^2\big)\\
F_8(X) &= \frac{1}{90}\big(10X^9 + 45X^8 + 60X^7 - 42X^5 + 20X^3 - 3X\big)\\
F_9(X) &= \frac{1}{20}\big(2X^{10} + 10X^9 + 15X^8 - 14X^6 + 10X^4 - 3X^2\big)\\
F_{10}(X) &= \frac{1}{66}\big(6X^{11} + 33X^{10} + 55X^9 - 66X^7 + 66X^5 - 33X^3 + 5X\big)
\end{aligned}$$

You can use this list to look for patterns and to test conjectures.

Three-Dimensional Number Shapes

In our work on number theory and geometry, we have studied various sorts of number shapes, such as triangular numbers and square numbers (Chapters 1 and 31) and even pentagonal numbers (Exercise 31.4). Triangles, squares, and pentagons are plane figures; that is, they lie on a flat surface. We, on the other hand, live in three-dimensional space, so it's about time we looked at three-dimensional number shapes. We'll build pyramids with triangular bases, as illustrated in Figure 50.1.

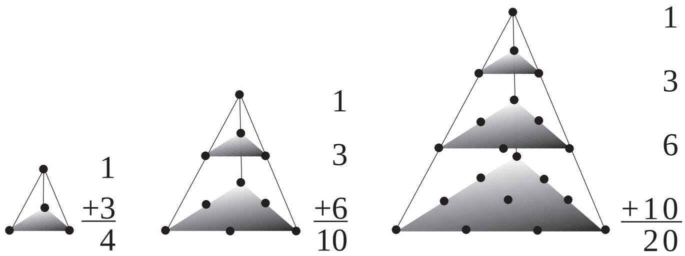

Figure 50.1: The Tetrahedral Numbers $\mathbb{T}_2 = 4$, $\mathbb{T}_3 = 10$, and $\mathbb{T}_4 = 20$

The fancy mathematical term for this sort of solid shape is a *tetrahedron*. We define the n^{th} *Tetrahedral Number* to be the number of dots in a tetrahedron with

n layers, and we let

$$\mathbb{T}_n = \text{ the } n^{\text{th}} \text{ Tetrahedral Number.}$$

Looking at Figure 50.1, we see that

$$\mathbb{T}_1 = 1, \qquad \mathbb{T}_2 = 4, \qquad \mathbb{T}_3 = 10, \qquad \text{and} \qquad \mathbb{T}_4 = 20.$$

The pictures illustrate how tetrahedral numbers are formed,

$$\begin{aligned}
\mathbb{T}_1 &= 1, \\
\mathbb{T}_2 &= 4 \ \ = 1 + 3, \\
\mathbb{T}_3 &= 10 = 1 + 3 + 6, \\
\mathbb{T}_4 &= 20 = 1 + 3 + 6 + 10.
\end{aligned}$$

To form the fifth tetrahedral number, we need to add another triangle onto the bottom of the previous tetrahedron. In other words, we need to add the next triangular number to the previous tetrahedral number. If this isn't clear, notice how $\mathbb{T}_4$ is formed by adding the first four triangular numbers: 1, 3, 6, and 10. So to get $\mathbb{T}_5$, we take $\mathbb{T}_4$ and add on the fifth triangular number $T_5 = 15$ to get

$$\mathbb{T}_5 = \mathbb{T}_4 + T_5 = (1 + 3 + 6 + 10) + 15 = 35.$$

In general, the n^{th} tetrahedral number is equal to the sum of the first n triangular numbers,

$$\mathbb{T}_n = T_1 + T_2 + T_3 + \cdots + T_n.$$

We know that the n^{th} triangular number T_n is given by

$$T_n = \frac{n^2 + n}{2},$$

so we can find a formula for $\mathbb{T}_n$ by adding

$$\mathbb{T}_n = \sum_{j=1}^{n} T_j = \sum_{j=1}^{n} \frac{j^2 + j}{2} = \frac{1}{2}\sum_{j=1}^{n} j^2 + \frac{1}{2}\sum_{j=1}^{n} j.$$

To finish the calculation, we use our power-sum formulas, in particular our formula for the sum of the first n squares, to compute

$$\mathbb{T}_n = \frac{1}{2}\left(\frac{2n^3 + 3n^2 + n}{6}\right) + \frac{1}{2}\left(\frac{n^2 + n}{2}\right) = \frac{n^3 + 3n^2 + 2n}{6}.$$

It is interesting to observe that the tetrahedral polynomial factors as

$$\mathbb{T}_n = \frac{n(n+1)(n+2)}{6}.$$

Thus the n^{th} triangular number and the n^{th} tetrahedral number can be expressed using binomial coefficients

$$T_n = \binom{n+1}{2} \qquad \text{and} \qquad \mathbb{T}_n = \binom{n+2}{3}.$$

In other words, the n^{th} two-dimensional pyramid (triangle) has $\binom{n+1}{2}$ dots, and the n^{th} three-dimensional pyramid (tetrahedron) has $\binom{n+2}{3}$ dots. How many dots do you think it takes to fill up a four-dimensional pyramid?

Exercises

50.1. In the text we used a telescoping sum to prove that the quantity $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n-1)\cdot n}$ is equal to $\frac{n-1}{n}$. Use induction to give a different proof of this formula.

50.2. **(a)** Use the recursive formula to compute the polynomial $F_4(X)$. Be sure to check your answer by computing $F_4(1)$, $F_4(2)$, and $F_4(3)$ and verifying that they equal 1, $1 + 2^4 = 17$, and $1 + 2^4 + 3^4 = 98$, respectively.

(b) Find the polynomial $F_5(X)$ and check your answer as in (a).

50.3. **(a)** Prove that the leading coefficient of $F_k(X)$ is $\frac{1}{k+1}$. In other words, prove that $F_k(X)$ looks like

$$F_k(X) = \frac{1}{k+1}X^{k+1} + aX^k + bX^{k-1} + \cdots .$$

(b) Try to find a similar formula for the next coefficient (i.e., the coefficient of X^k) in the polynomial $F_k(X)$.

(c) Find a formula for the coefficient of X^{k-1} in the polynomial $F_k(X)$.

50.4. **(a)** What is the value of $F_k(0)$?

(b) What is the value of $F_k(-1)$?

(c) If p is a prime number and if $p - 1 \nmid k$, prove that

$$1^k + 2^k + \cdots + (p-1)^k \equiv 0 \pmod{p}.$$

What is the value when $p - 1$ divides k?

(d) What is the value of $F_k(-1/2)$? More precisely, try to find a large collection of k's for which you can guess (and prove correct) the value of $F_k(-1/2)$.

50.5. Prove the remarkable fact that

$$(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3.$$

50.6. The coefficients of the polynomial $F_k(X)$ are rational numbers. We would like to multiply by some integer to clear all the denominators. For example,

$$F_1(X) = \tfrac{1}{2}X^2 + \tfrac{1}{2}X \quad \text{and} \quad F_2(X) = \tfrac{1}{3}X^3 + \tfrac{1}{2}X^2 + \tfrac{1}{6}X,$$

so $2 \cdot F_1(X)$ and $6 \cdot F_2(X)$ have coefficients that are integers.

(a) Prove that

$$(k+1)! \cdot F_k(X)$$

has integer coefficients.

(b) It is clear from the examples in this chapter that $(k+1)!$ is usually much larger than necessary for clearing the denominators of the coefficients of $F_k(X)$. Can you find any sort of patterns in the actual denominator?

50.7. A pyramid with a square base of side n requires $F_2(n)$ dots, so $F_2(n)$ is the n^{th} *Square Pyramid Number*. In Chapter 31 we found infinitely many numbers that are both triangular and square. Search for numbers that are both tetrahedral and square pyramid numbers. Do you think there are finitely many, or infinitely many, such numbers?

50.8. **(a)** Find a simple expression for the sum

$$\mathbb{T}_1 + \mathbb{T}_2 + \mathbb{T}_3 + \cdots + \mathbb{T}_n$$

of the first n tetrahedral numbers.

(b) Express your answer in (a) as a single binomial coefficient.

(c) Try to understand and explain the following statement: "The number $\mathbb{T}_1 + \mathbb{T}_2 + \cdots + \mathbb{T}_n$ is the number of dots needed to form a pyramid shape in four-dimensional space."

50.9. The n^{th} triangular number T_n equals the binomial coefficient $\binom{n+1}{2}$, and the n^{th} tetrahedral number $\mathbb{T}_n$ equals the binomial coefficient $\binom{n+2}{3}$. This means that the formula $\mathbb{T}_n = T_1 + T_2 + \cdots + T_n$ can be written using binomial coefficients as

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n+1}{2} = \binom{n+2}{3}.$$

(a) Illustrate this formula for $n = 5$ by taking Pascal's Triangle (see Chapter 38), circling the numbers $\binom{2}{2}, \binom{3}{2}, \ldots, \binom{6}{2}$, and putting a box around their sum $\binom{7}{3}$.

(b) Write the formula $1+2+3+\cdots+n = T_n$ using binomial coefficients and illustrate your formula for $n = 5$ using Pascal's Triangle as in (a). [*Hint.* $\binom{n}{1} = n$.]

(c) Generalize these formulas to write a sum of binomial coefficients $\binom{r}{r}, \binom{r+1}{r}, \ldots$ in terms of a binomial coefficient.

(d) Prove that your formula in (c) is correct.

50.10. This exercise and the next one give an explicit formula for the sum of k^{th} powers that was studied in this chapter. *Stirling numbers* (*of the second kind*) are defined to be the integers $S(k,j)$ that make the following polynomial equation true:

$$x^k = \sum_{j=0}^{k} S(k,j)x(x-1)(x-2)\cdots(x-j+1).$$

For example, taking $k = 1$ gives

$$x = S(1,0) + S(1,1)x, \quad \text{so } S(1,0) = 0 \text{ and } S(1,1) = 1.$$

Similarly, taking $k = 2$ gives

$$\begin{aligned} x^2 &= S(2,0) + S(2,1)x + S(2,2)x(x+1) \\ &= S(2,0) + (S(2,1) + S(2,2))x + S(2,2)x^2, \end{aligned}$$

so

$$S(2,0) = 0 \text{ and } S(2,2) = 1 \text{ and } S(2,1) = -1.$$

(a) Compute the value of $S(3,j)$ for $j = 0,1,2,3$ and $S(4,j)$ for $j = 0,1,2,3,4$.

(b) Prove that the Stirling numbers satisfy the recurrence

$$S(k,j) = S(k,j-1) + jS(k,j).$$

(c) Prove that the Stirling numbers are given by the following formula:

$$S(k,j) = \frac{1}{j!}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i}i^k.$$

50.11. Prove that the sum of k^{th} powers is given by the following explicit formula using the Stirling numbers $S(k,j)$ defined in the previous exercise.

$$1^k + 2^k + \cdots + n^k = \sum_{j=0}^{k} \frac{S(k,j)}{j+1}(n+1)n(n-1)(n-2)\cdots(n-j+1).$$

50.12 (For students who know calculus)**.** Let $P_0(x)$ be the polynomial

$$P_0(x) = 1 + x + x^2 + x^3 + \cdots + x^{n-1}.$$

Next let

$$P_1(x) = \frac{d}{dx}\big(xP_0(x)\big), \quad \text{and} \quad P_2(x) = \frac{d}{dx}\big(xP_1(x)\big), \quad \text{and so on.}$$

(a) What does $P_k(x)$ look like? What is the value of $P_k(1)$? [*Hint*. The answer has something to do with the material in this chapter.]

(b) The polynomial $P_0(x)$ is the geometric sum that we used in Chapter 14. Recall that the formula for the geometric sum is $P_0(x) = (x^n - 1)/(x - 1)$, at least provided that $x \neq 1$. Compute the limit

$$\lim_{x \to 1} \frac{x^n - 1}{x - 1}$$

and check that it gives the same value as $P_0(1)$. [*Hint*. Use L'Hôpital's rule.]

(c) Find a formula for $P_1(x)$ by differentiating,

$$P_1(x) = \frac{d}{dx}\left(x\frac{x^n - 1}{x - 1}\right).$$

(d) Compute the limit of your formula in (c) as $x \to 1$. Explain why this gives a new proof for the value of $1 + 2 + \cdots + n$.

(e) Starting with your formula in (c), repeat (c) and (d) to find a formula for $P_2(x)$ and for the limit of $P_2(x)$ as $x \to 1$.

(f) Starting with your formula in (e), repeat (c) and (d) to find a formula for $P_3(x)$ and for the limit of $P_3(x)$ as $x \to 1$.

50.13. Fix an integer $k \geq 0$ and let $F_k(n) = 1^k + 2^k + \cdots + n^k$ be the sum of powers studied in this chapter. Let

$$\begin{aligned} A(x) &= \text{generating function of the sequence } F_k(0), F_k(1), F_k(2), F_k(3), \ldots \\ &= F_k(1)x + F_k(2)x^2 + F_k(3)x^3 + \cdots, \\ B(x) &= \text{generating function of the sequence } 0^k, 1^k, 2^k, 3^k, \ldots \\ &= x + 2^k x^2 + 3^k x^3 + 4^k x^4 + \cdots. \end{aligned}$$

Find a simple formula relating $A(x)$ and $B(x)$.

Appendix A

Factorization of Small Composite Integers [online]

The following table gives the factorization of small composite integers that are not divisible by 2, 3, or 5. To use this table, first divide your number by powers of 2, 3, and 5 until no such factors remain; then look it up in the table.

$49 = 7^2$	$77 = 7 \cdot 11$	$91 = 7 \cdot 13$	$119 = 7 \cdot 17$
$121 = 11^2$	$133 = 7 \cdot 19$	$143 = 11 \cdot 13$	$161 = 7 \cdot 23$
$169 = 13^2$	$187 = 11 \cdot 17$	$203 = 7 \cdot 29$	$209 = 11 \cdot 19$
$217 = 7 \cdot 31$	$221 = 13 \cdot 17$	$247 = 13 \cdot 19$	$253 = 11 \cdot 23$
$259 = 7 \cdot 37$	$287 = 7 \cdot 41$	$289 = 17^2$	$299 = 13 \cdot 23$
$301 = 7 \cdot 43$	$319 = 11 \cdot 29$	$323 = 17 \cdot 19$	$329 = 7 \cdot 47$
$341 = 11 \cdot 31$	$343 = 7^3$	$361 = 19^2$	$371 = 7 \cdot 53$
$377 = 13 \cdot 29$	$391 = 17 \cdot 23$	$403 = 13 \cdot 31$	$407 = 11 \cdot 37$
$413 = 7 \cdot 59$	$427 = 7 \cdot 61$	$437 = 19 \cdot 23$	$451 = 11 \cdot 41$
$469 = 7 \cdot 67$	$473 = 11 \cdot 43$	$481 = 13 \cdot 37$	$493 = 17 \cdot 29$
$497 = 7 \cdot 71$	$511 = 7 \cdot 73$	$517 = 11 \cdot 47$	$527 = 17 \cdot 31$
$529 = 23^2$	$533 = 13 \cdot 41$	$539 = 7^2 \cdot 11$	$551 = 19 \cdot 29$
$553 = 7 \cdot 79$	$559 = 13 \cdot 43$	$581 = 7 \cdot 83$	$583 = 11 \cdot 53$
$589 = 19 \cdot 31$	$611 = 13 \cdot 47$	$623 = 7 \cdot 89$	$629 = 17 \cdot 37$
$637 = 7^2 \cdot 13$	$649 = 11 \cdot 59$	$667 = 23 \cdot 29$	$671 = 11 \cdot 61$
$679 = 7 \cdot 97$	$689 = 13 \cdot 53$	$697 = 17 \cdot 41$	$703 = 19 \cdot 37$
$707 = 7 \cdot 101$	$713 = 23 \cdot 31$	$721 = 7 \cdot 103$	$731 = 17 \cdot 43$
$737 = 11 \cdot 67$	$749 = 7 \cdot 107$	$763 = 7 \cdot 109$	$767 = 13 \cdot 59$
$779 = 19 \cdot 41$	$781 = 11 \cdot 71$	$791 = 7 \cdot 113$	$793 = 13 \cdot 61$
$799 = 17 \cdot 47$	$803 = 11 \cdot 73$	$817 = 19 \cdot 43$	$833 = 7^2 \cdot 17$
$841 = 29^2$	$847 = 7 \cdot 11^2$	$851 = 23 \cdot 37$	$869 = 11 \cdot 79$
$871 = 13 \cdot 67$	$889 = 7 \cdot 127$	$893 = 19 \cdot 47$	$899 = 29 \cdot 31$
$901 = 17 \cdot 53$	$913 = 11 \cdot 83$	$917 = 7 \cdot 131$	$923 = 13 \cdot 71$
$931 = 7^2 \cdot 19$	$943 = 23 \cdot 41$	$949 = 13 \cdot 73$	$959 = 7 \cdot 137$

$961 = 31^2$	$973 = 7 \cdot 139$	$979 = 11 \cdot 89$	$989 = 23 \cdot 43$
$1001 = 7 \cdot 11 \cdot 13$	$1003 = 17 \cdot 59$	$1007 = 19 \cdot 53$	$1027 = 13 \cdot 79$
$1037 = 17 \cdot 61$	$1043 = 7 \cdot 149$	$1057 = 7 \cdot 151$	$1067 = 11 \cdot 97$
$1073 = 29 \cdot 37$	$1079 = 13 \cdot 83$	$1081 = 23 \cdot 47$	$1099 = 7 \cdot 157$
$1111 = 11 \cdot 101$	$1121 = 19 \cdot 59$	$1127 = 7^2 \cdot 23$	$1133 = 11 \cdot 103$
$1139 = 17 \cdot 67$	$1141 = 7 \cdot 163$	$1147 = 31 \cdot 37$	$1157 = 13 \cdot 89$
$1159 = 19 \cdot 61$	$1169 = 7 \cdot 167$	$1177 = 11 \cdot 107$	$1183 = 7 \cdot 13^2$
$1189 = 29 \cdot 41$	$1199 = 11 \cdot 109$	$1207 = 17 \cdot 71$	$1211 = 7 \cdot 173$
$1219 = 23 \cdot 53$	$1241 = 17 \cdot 73$	$1243 = 11 \cdot 113$	$1247 = 29 \cdot 43$
$1253 = 7 \cdot 179$	$1261 = 13 \cdot 97$	$1267 = 7 \cdot 181$	$1271 = 31 \cdot 41$
$1273 = 19 \cdot 67$	$1309 = 7 \cdot 11 \cdot 17$	$1313 = 13 \cdot 101$	$1331 = 11^3$
$1333 = 31 \cdot 43$	$1337 = 7 \cdot 191$	$1339 = 13 \cdot 103$	$1343 = 17 \cdot 79$
$1349 = 19 \cdot 71$	$1351 = 7 \cdot 193$	$1357 = 23 \cdot 59$	$1363 = 29 \cdot 47$
$1369 = 37^2$	$1379 = 7 \cdot 197$	$1387 = 19 \cdot 73$	$1391 = 13 \cdot 107$
$1393 = 7 \cdot 199$	$1397 = 11 \cdot 127$	$1403 = 23 \cdot 61$	$1411 = 17 \cdot 83$
$1417 = 13 \cdot 109$	$1421 = 7^2 \cdot 29$	$1441 = 11 \cdot 131$	$1457 = 31 \cdot 47$
$1463 = 7 \cdot 11 \cdot 19$	$1469 = 13 \cdot 113$	$1477 = 7 \cdot 211$	$1501 = 19 \cdot 79$
$1507 = 11 \cdot 137$	$1513 = 17 \cdot 89$	$1517 = 37 \cdot 41$	$1519 = 7^2 \cdot 31$
$1529 = 11 \cdot 139$	$1537 = 29 \cdot 53$	$1541 = 23 \cdot 67$	$1547 = 7 \cdot 13 \cdot 17$
$1561 = 7 \cdot 223$	$1573 = 11^2 \cdot 13$	$1577 = 19 \cdot 83$	$1589 = 7 \cdot 227$
$1591 = 37 \cdot 43$	$1603 = 7 \cdot 229$	$1631 = 7 \cdot 233$	$1633 = 23 \cdot 71$
$1639 = 11 \cdot 149$	$1643 = 31 \cdot 53$	$1649 = 17 \cdot 97$	$1651 = 13 \cdot 127$
$1661 = 11 \cdot 151$	$1673 = 7 \cdot 239$	$1679 = 23 \cdot 73$	$1681 = 41^2$
$1687 = 7 \cdot 241$	$1691 = 19 \cdot 89$	$1703 = 13 \cdot 131$	$1711 = 29 \cdot 59$
$1717 = 17 \cdot 101$	$1727 = 11 \cdot 157$	$1729 = 7 \cdot 13 \cdot 19$	$1739 = 37 \cdot 47$
$1751 = 17 \cdot 103$	$1757 = 7 \cdot 251$	$1763 = 41 \cdot 43$	$1769 = 29 \cdot 61$
$1771 = 7 \cdot 11 \cdot 23$	$1781 = 13 \cdot 137$	$1793 = 11 \cdot 163$	$1799 = 7 \cdot 257$
$1807 = 13 \cdot 139$	$1813 = 7^2 \cdot 37$	$1817 = 23 \cdot 79$	$1819 = 17 \cdot 107$
$1829 = 31 \cdot 59$	$1837 = 11 \cdot 167$	$1841 = 7 \cdot 263$	$1843 = 19 \cdot 97$
$1849 = 43^2$	$1853 = 17 \cdot 109$	$1859 = 11 \cdot 13^2$	$1883 = 7 \cdot 269$
$1891 = 31 \cdot 61$	$1897 = 7 \cdot 271$	$1903 = 11 \cdot 173$	$1909 = 23 \cdot 83$
$1919 = 19 \cdot 101$	$1921 = 17 \cdot 113$	$1927 = 41 \cdot 47$	$1937 = 13 \cdot 149$
$1939 = 7 \cdot 277$	$1943 = 29 \cdot 67$	$1957 = 19 \cdot 103$	$1961 = 37 \cdot 53$
$1963 = 13 \cdot 151$	$1967 = 7 \cdot 281$	$1969 = 11 \cdot 179$	$1981 = 7 \cdot 283$
$1991 = 11 \cdot 181$	$2009 = 7^2 \cdot 41$	$2021 = 43 \cdot 47$	$2023 = 7 \cdot 17^2$
$2033 = 19 \cdot 107$	$2041 = 13 \cdot 157$	$2047 = 23 \cdot 89$	$2051 = 7 \cdot 293$
$2057 = 11^2 \cdot 17$	$2059 = 29 \cdot 71$	$2071 = 19 \cdot 109$	$2077 = 31 \cdot 67$
$2093 = 7 \cdot 13 \cdot 23$	$2101 = 11 \cdot 191$	$2107 = 7^2 \cdot 43$	$2117 = 29 \cdot 73$
$2119 = 13 \cdot 163$	$2123 = 11 \cdot 193$	$2147 = 19 \cdot 113$	$2149 = 7 \cdot 307$
$2159 = 17 \cdot 127$	$2167 = 11 \cdot 197$	$2171 = 13 \cdot 167$	$2173 = 41 \cdot 53$
$2177 = 7 \cdot 311$	$2183 = 37 \cdot 59$	$2189 = 11 \cdot 199$	$2191 = 7 \cdot 313$
$2197 = 13^3$	$2201 = 31 \cdot 71$	$2209 = 47^2$	$2219 = 7 \cdot 317$
$2227 = 17 \cdot 131$	$2231 = 23 \cdot 97$	$2233 = 7 \cdot 11 \cdot 29$	$2249 = 13 \cdot 173$
$2257 = 37 \cdot 61$	$2261 = 7 \cdot 17 \cdot 19$	$2263 = 31 \cdot 73$	$2279 = 43 \cdot 53$
$2291 = 29 \cdot 79$	$2299 = 11^2 \cdot 19$	$2303 = 7^2 \cdot 47$	$2317 = 7 \cdot 331$

Appendix B

A List of Primes [online]

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601
607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997	1009	1013
1019	1021	1031	1033	1039	1049	1051	1061	1063	1069
1087	1091	1093	1097	1103	1109	1117	1123	1129	1151
1153	1163	1171	1181	1187	1193	1201	1213	1217	1223
1229	1231	1237	1249	1259	1277	1279	1283	1289	1291
1297	1301	1303	1307	1319	1321	1327	1361	1367	1373
1381	1399	1409	1423	1427	1429	1433	1439	1447	1451
1453	1459	1471	1481	1483	1487	1489	1493	1499	1511
1523	1531	1543	1549	1553	1559	1567	1571	1579	1583
1597	1601	1607	1609	1613	1619	1621	1627	1637	1657
1663	1667	1669	1693	1697	1699	1709	1721	1723	1733
1741	1747	1753	1759	1777	1783	1787	1789	1801	1811
1823	1831	1847	1861	1867	1871	1873	1877	1879	1889
1901	1907	1913	1931	1933	1949	1951	1973	1979	1987

1993	1997	1999	2003	2011	2017	2027	2029	2039	2053
2063	2069	2081	2083	2087	2089	2099	2111	2113	2129
2131	2137	2141	2143	2153	2161	2179	2203	2207	2213
2221	2237	2239	2243	2251	2267	2269	2273	2281	2287
2293	2297	2309	2311	2333	2339	2341	2347	2351	2357
2371	2377	2381	2383	2389	2393	2399	2411	2417	2423
2437	2441	2447	2459	2467	2473	2477	2503	2521	2531
2539	2543	2549	2551	2557	2579	2591	2593	2609	2617
2621	2633	2647	2657	2659	2663	2671	2677	2683	2687
2689	2693	2699	2707	2711	2713	2719	2729	2731	2741
2749	2753	2767	2777	2789	2791	2797	2801	2803	2819
2833	2837	2843	2851	2857	2861	2879	2887	2897	2903
2909	2917	2927	2939	2953	2957	2963	2969	2971	2999
3001	3011	3019	3023	3037	3041	3049	3061	3067	3079
3083	3089	3109	3119	3121	3137	3163	3167	3169	3181
3187	3191	3203	3209	3217	3221	3229	3251	3253	3257
3259	3271	3299	3301	3307	3313	3319	3323	3329	3331
3343	3347	3359	3361	3371	3373	3389	3391	3407	3413
3433	3449	3457	3461	3463	3467	3469	3491	3499	3511
3517	3527	3529	3533	3539	3541	3547	3557	3559	3571
3581	3583	3593	3607	3613	3617	3623	3631	3637	3643
3659	3671	3673	3677	3691	3697	3701	3709	3719	3727
3733	3739	3761	3767	3769	3779	3793	3797	3803	3821
3823	3833	3847	3851	3853	3863	3877	3881	3889	3907
3911	3917	3919	3923	3929	3931	3943	3947	3967	3989
4001	4003	4007	4013	4019	4021	4027	4049	4051	4057
4073	4079	4091	4093	4099	4111	4127	4129	4133	4139
4153	4157	4159	4177	4201	4211	4217	4219	4229	4231
4241	4243	4253	4259	4261	4271	4273	4283	4289	4297
4327	4337	4339	4349	4357	4363	4373	4391	4397	4409
4421	4423	4441	4447	4451	4457	4463	4481	4483	4493
4507	4513	4517	4519	4523	4547	4549	4561	4567	4583
4591	4597	4603	4621	4637	4639	4643	4649	4651	4657
4663	4673	4679	4691	4703	4721	4723	4729	4733	4751
4759	4783	4787	4789	4793	4799	4801	4813	4817	4831
4861	4871	4877	4889	4903	4909	4919	4931	4933	4937
4943	4951	4957	4967	4969	4973	4987	4993	4999	5003
5009	5011	5021	5023	5039	5051	5059	5077	5081	5087
5099	5101	5107	5113	5119	5147	5153	5167	5171	5179
5189	5197	5209	5227	5231	5233	5237	5261	5273	5279
5281	5297	5303	5309	5323	5333	5347	5351	5381	5387
5393	5399	5407	5413	5417	5419	5431	5437	5441	5443
5449	5471	5477	5479	5483	5501	5503	5507	5519	5521
5527	5531	5557	5563	5569	5573	5581	5591	5623	5639
5641	5647	5651	5653	5657	5659	5669	5683	5689	5693