

# Contents

Preface .....	v
Flowchart of Chapter Dependencies .....	ix
Introduction .....	1
1 What Is Number Theory? .....	6
2 Pythagorean Triples .....	13
3 Pythagorean Triples and the Unit Circle .....	21
4 Sums of Higher Powers and Fermat's Last Theorem .....	26
5 Divisibility and the Greatest Common Divisor .....	30
6 Linear Equations and the Greatest Common Divisor .....	37
7 Factorization and the Fundamental Theorem of Arithmetic .....	46
8 Congruences .....	55
9 Congruences, Powers, and Fermat's Little Theorem .....	65
10 Congruences, Powers, and Euler's Formula .....	71
11 Euler's Phi Function and the Chinese Remainder Theorem .....	75
12 Prime Numbers .....	83
13 Counting Primes .....	90
14 Mersenne Primes .....	96
15 Mersenne Primes and Perfect Numbers .....	101
16 Powers Modulo $m$ and Successive Squaring .....	111
17 Computing $k^{\text{th}}$ Roots Modulo $m$ .....	118
18 Powers, Roots, and "Unbreakable" Codes .....	123
19 Primality Testing and Carmichael Numbers .....	129
20 Squares Modulo $p$ .....	141
21 Is $-1$ a Square Modulo $p$ ? Is $2$ ? .....	148
22 Quadratic Reciprocity .....	159

23	Proof of Quadratic Reciprocity . . . . .	171
24	Which Primes Are Sums of Two Squares? . . . . .	181
25	Which Numbers Are Sums of Two Squares? . . . . .	193
26	As Easy as One, Two, Three . . . . .	199
27	Euler's Phi Function and Sums of Divisors . . . . .	206
28	Powers Modulo $p$ and Primitive Roots . . . . .	211
29	Primitive Roots and Indices . . . . .	224
30	The Equation $X^4 + Y^4 = Z^4$ . . . . .	231
31	Square-Triangular Numbers Revisited . . . . .	236
32	Pell's Equation . . . . .	245
33	Diophantine Approximation . . . . .	251
34	Diophantine Approximation and Pell's Equation . . . . .	260
35	Number Theory and Imaginary Numbers . . . . .	267
36	The Gaussian Integers and Unique Factorization . . . . .	281
37	Irrational Numbers and Transcendental Numbers . . . . .	297
38	Binomial Coefficients and Pascal's Triangle . . . . .	313
39	Fibonacci's Rabbits and Linear Recurrence Sequences . . . . .	324
40	Oh, What a Beautiful Function . . . . .	339
41	Cubic Curves and Elliptic Curves . . . . .	353
42	Elliptic Curves with Few Rational Points . . . . .	366
43	Points on Elliptic Curves Modulo $p$ . . . . .	373
44	Torsion Collections Modulo $p$ and Bad Primes . . . . .	384
45	Defect Bounds and Modularity Patterns . . . . .	388
46	Elliptic Curves and Fermat's Last Theorem . . . . .	394
	Further Reading . . . . .	396
	Index . . . . .	397
47	The Topsy-Turvy World of Continued Fractions [online] . . . . .	410
48	Continued Fractions and Pell's Equation [online] . . . . .	426
49	Generating Functions [online] . . . . .	442
50	Sums of Powers [online] . . . . .	452
A	Factorization of Small Composite Integers [online] . . . . .	464
B	A List of Primes [online] . . . . .	466

# Preface

The 1990s saw a wave of calculus reform whose aim was to teach students to think for themselves and to solve substantial problems, rather than merely memorizing formulas and performing rote algebraic manipulations. This book has a similar, albeit somewhat more ambitious, goal: to lead you to think mathematically and to experience the thrill of independent intellectual discovery. Our chosen subject, Number Theory, is particularly well suited for this purpose. The natural numbers  $1, 2, 3, \dots$  satisfy a multitude of beautiful patterns and relationships, many of which can be discerned at a glance; others are so subtle that one marvels they were noticed at all. Experimentation requires nothing more than paper and pencil, but many false alleys beckon to those who make conjectures on too scanty evidence. It is only by rigorous demonstration that one is finally convinced that the numerical evidence reflects a universal truth. This book will lead you through the groves wherein lurk some of the brightest flowers of Number Theory, as it simultaneously encourages you to investigate, analyze, conjecture, and ultimately prove your own beautiful number theoretic results.

This book was originally written to serve as a text for Math 42, a course created by Jeff Hoffstein at Brown University in the early 1990s. Math 42 was designed to attract nonscience majors, those with little interest in pursuing the standard calculus sequence, and to convince them to study some college mathematics. The intent was to create a course similar to one on, say, “The Music of Mozart” or “Elizabethan Drama,” wherein an audience is introduced to the overall themes and methodology of an entire discipline through the detailed study of a particular facet of the subject. Math 42 has been extremely successful, attracting both its intended audience and also scientifically oriented undergraduates interested in a change of pace from their large-lecture, cookbook-style courses.

The prerequisites for reading this book are few. Some facility with high school algebra is required, and those who know how to program a computer will have fun generating reams of data and implementing assorted algorithms, but in truth the reader needs nothing more than a simple calculator. Concepts from calculus are mentioned in passing, but are not used in an essential way. However, and the reader

is hereby forewarned, it is not possible to truly appreciate Number Theory without an eager and questioning mind and a spirit that is not afraid to experiment, to make mistakes and profit from them, to accept frustration and persevere to the ultimate triumph. Readers who are able to cultivate these qualities will find themselves richly rewarded, both in their study of Number Theory and their appreciation of all that life has to offer.

### **Acknowledgments for the First Edition**

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### **Changes in the Fourth Edition**

There are a number of major changes in the fourth edition.

- There is a new chapter on mathematical induction (Chapter 26).
- Some material on proof by contradiction has been moved forward to Chapter 8. It is used in the proof that a polynomial of degree  $d$  has at most  $d$  roots modulo  $p$ . This fact is then used in place of primitive roots as a tool to prove Euler's quadratic residue formula in Chapter 21. (In earlier editions, primitive roots were used for this proof.)
- The chapters on primitive roots (Chapters 28–29) have been moved to follow the chapters on quadratic reciprocity and sums of squares (Chapters 20–25).

The rationale for this change is the author's experience that students find the Primitive Root Theorem to be among the most difficult in the book. The new order allows the instructor to cover quadratic reciprocity first, and to omit primitive roots entirely if desired.

- Chapter 22 now includes a proof of part of quadratic reciprocity for Jacobi symbols, with the remaining parts included as exercises.
- Quadratic reciprocity is now proved in full. The proofs for  $\left(\frac{-1}{p}\right)$  and  $\left(\frac{2}{p}\right)$  remain as before in Chapter 21, and there is a new chapter (Chapter 23) that gives Eisenstein's proof for  $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)$ . Chapter 23 is significantly more difficult than the chapters that precede it, and it may be omitted without affecting the subsequent chapters.
- As an application of primitive roots, Chapter 28 discusses the construction of Costas arrays.
- Chapter 39 includes a proof that the period of the Fibonacci sequence modulo  $p$  divides  $p - 1$  when  $p$  is congruent to 1 or 4 modulo 5.
- There are many new exercises scattered throughout the text.
- A flowchart giving chapter dependencies is included on page ix.
- Number theory is a vast and sprawling subject, and over the years this book has acquired many new chapters. In order to keep the length of this edition to a reasonable size, Chapters 47–50 have been removed from the printed version of the book. These omitted chapters are freely available online at  
<http://www.math.brown.edu/~jhs/frint.html>  
<http://www.pearsonhighered.com/mathstatsresources>  
The online chapters are included in the index.

### **Email and Electronic Resources**

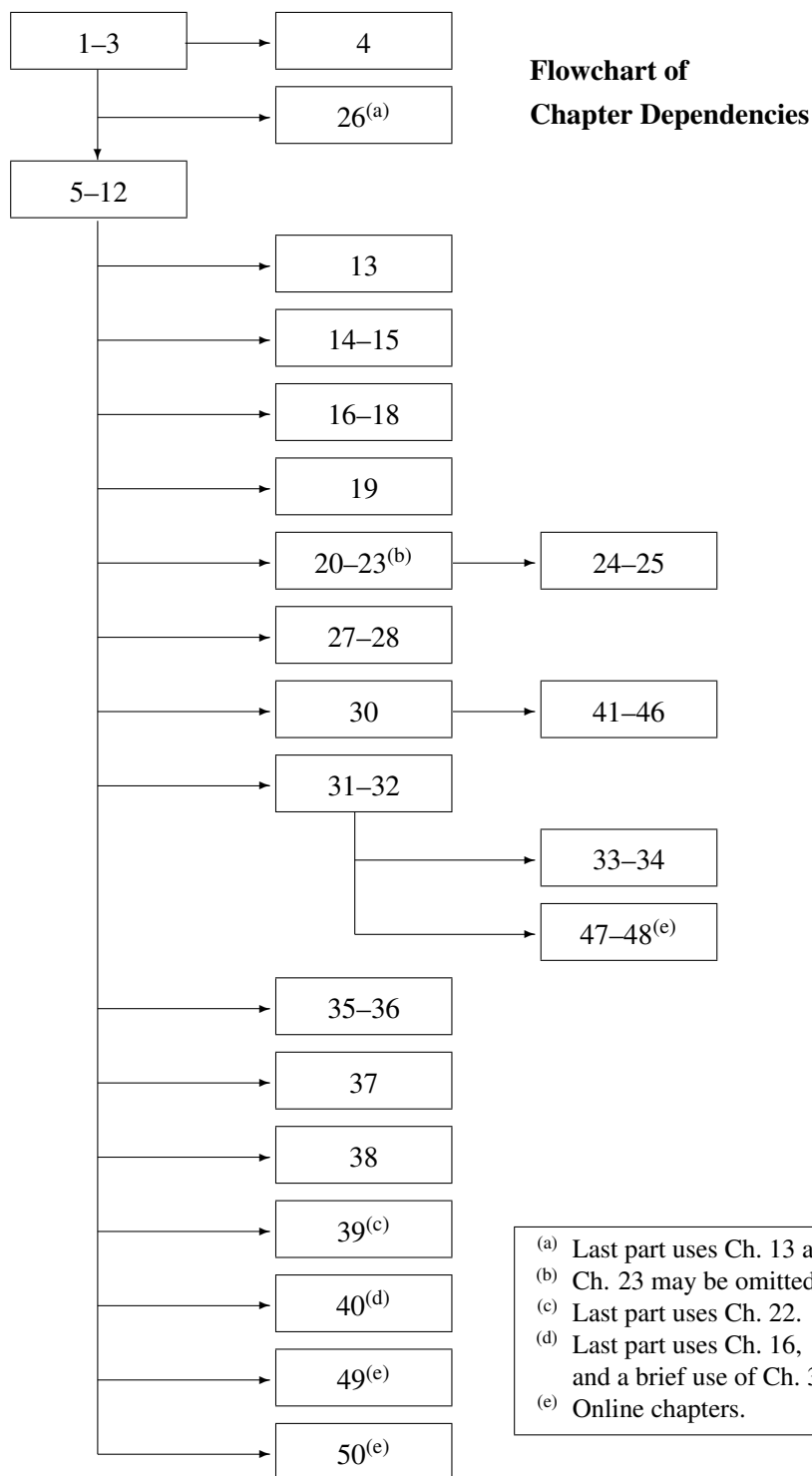
All the people listed above have helped me to correct numerous mistakes and to greatly refine the exposition, but no book is ever free from error or incapable of being improved. I would be delighted to receive comments, good or bad, and corrections from my readers. You can send mail to me at

`jhs@math.brown.edu`

Additional material, including extra chapters, an errata sheet, links to interesting number theoretic sites, and downloadable versions of various computer exercises, are available on the *Friendly Introduction to Number Theory* Home Page:

`www.math.brown.edu/~jhs/frint.html`

Joseph H. Silverman







# Introduction

*Euclid alone  
Has looked on Beauty bare. Fortunate they  
Who, though once only and then but far away,  
Have heard her massive sandal set on stone.*

Edna St. Vincent Millay (1923)

The origins of the natural numbers 1, 2, 3, 4, 5, 6, ... are lost in the mists of time. We have no knowledge of who first realized that there is a certain concept of “threeness” that applies equally well to three rocks, three stars, and three people. From the very beginnings of recorded history, numbers have inspired an endless fascination—mystical, aesthetic, and practical as well. It is not just the numbers themselves, of course, that command attention. Far more intriguing are the relationships that numbers exhibit, one with another. It is within these profound and often subtle relationships that one finds the Beauty<sup>1</sup> so strikingly described in Edna St. Vincent Millay’s poem. Here is another description by a celebrated twentieth-century philosopher.

Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. (Bertrand Russell, 1902)

The Theory of Numbers is that area of mathematics whose aim is to uncover the many deep and subtle relationships among different sorts of numbers. To take a simple example, many people through the ages have been intrigued by the square numbers 1, 4, 9, 16, 25, ... If we perform the experiment of adding together pairs

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<sup>1</sup>Euclid, indeed, has looked on Beauty bare, and not merely the beauty of geometry that most people associate with his name. Number theory is prominently featured in Books VII, VIII, and IX of Euclid’s famous *Elements*.

of square numbers, we will find that occasionally we get another square. The most famous example of this phenomenon is

$$3^2 + 4^2 = 5^2,$$

but there are many others, such as

$$5^2 + 12^2 = 13^2, \quad 20^2 + 21^2 = 29^2, \quad 28^2 + 45^2 = 53^2.$$

Triples like  $(3, 4, 5)$ ,  $(5, 12, 13)$ ,  $(20, 21, 29)$ , and  $(28, 45, 53)$  have been given the name Pythagorean triples. Based on this experiment, anyone with a lively curiosity is bound to pose various questions, such as “Are there infinitely many Pythagorean triples?” and “If so, can we find a formula that describes all of them?” These are the sorts of questions dealt with by number theory.

As another example, consider the problem of finding the remainder when the huge number

$$32478543^{743921429837645}$$

is divided by 54817263. Here’s one way to solve this problem. Take the number 32478543, multiply it by itself 743921429837645 times, use long division to divide by 54817263, and take the remainder. In principle, this method will work, but in practice it would take far longer than a lifetime, even on the world’s fastest computers. Number theory provides a means for solving this problem, too. “Wait a minute,” I hear you say, “Pythagorean triples have a certain elegance that is pleasing to the eye, but where is the beauty in long division and remainders?” The answer is not in the remainders themselves, but in the use to which such remainders can be put. In a striking turn of events, mathematicians have shown how the solution of this elementary remainder problem (and its inverse) leads to the creation of simple codes that are so secure that even the National Security Agency<sup>2</sup> is unable to break them. So much for G.H. Hardy’s singularly unprophetic remark that “no one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years.”<sup>3</sup>

The land of Number Theory is populated by a variety of exotic flora and fauna. There are square numbers and prime numbers and odd numbers and perfect numbers (but no square-prime numbers and, as far as anyone knows, no odd-perfect numbers). There are Fermat equations and Pell equations, Pythagorean triples and

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<sup>2</sup>The National Security Agency (NSA) is the arm of the United States government charged with data collection, code making, and code breaking. The NSA, with a budget larger than that of the CIA, is supposedly the single largest employer of mathematicians in the world.

<sup>3</sup>*A Mathematician’s Apology*, §28, G.H. Hardy, Camb. Univ. Press, 1940.

elliptic curves, Fibonacci's rabbits, unbreakable codes, and much, much more. You will meet all these creatures, and many others, as we journey through the Theory of Numbers.

### Guide for the Instructor

This book is designed to be used as a text for a one-semester or full-year course in undergraduate number theory or for an independent study or reading course. It contains approximately two semesters' worth of material, so the instructor of a one-semester course will have some flexibility in the choice of topics. The first 11 chapters are basic, and probably most instructors will want to continue through the RSA cryptosystem in Chapter 18, since in my experience this is one of the students' favorite topics.

There are now many ways to proceed. Here are a few possibilities that seem to fit comfortably into one semester, but feel free to slice-and-dice the later chapters to fit your own tastes.

**Chapters 20–26, 31–34, and 47–48.** Quadratic Reciprocity, sums of squares, induction, Pell's equation, Diophantine approximation, and continued fractions.

**Chapters 30–34 and 41–46.** Fermat's equation for exponent 4, Pell's equation, Diophantine approximation, elliptic curves, and Fermat's Last Theorem.

**Chapters 26, 31–39 and 47–48.** Induction, Pell's equation, Diophantine approximation, Gaussian integers, transcendental numbers, binomial coefficients, linear recurrences, and continued fractions.

**Chapters 19–22, 26–29, and 38–40.** Primality testing, quadratic reciprocity, induction, primitive roots, binomial coefficients, linear recurrences, big-Oh notation. (This syllabus is designed in particular for students planning further work in computer science or cryptography.)

In any case, a good final project is to have the students read a few of the omitted chapters and do the exercises.

Most of the nonnumerical nonprogramming exercises in this book are designed to foster discussion and experimentation. They do not necessarily have "correct" or "complete" answers. Many students will find this extremely disconcerting at first, so it must be stressed repeatedly. You can make your students feel more at ease by prefacing such questions with the phrase "Tell me as much as you can about . . ." Tell your students that accumulating data and solving special cases are not merely acceptable, but encouraged. On the other hand, tell them that there is

no such thing as a complete solution, since the solution of a good problem always raises additional questions. So if they can fully answer the specific question given in the text, their next task is to look for generalizations and for limitations on the validity of their solution.

Aside from a few clearly marked exercises, calculus is required only in two late chapters (Big-Oh notation in Chapter 40 and Generating Functions in Chapter 49). If the class has not taken calculus, these chapters may be omitted with no harm to the flow of the material.

Number theory is not easy, so there's no point in trying to convince the students that it is. Instead, this book will show your students that they are capable of mastering a difficult subject and experiencing the intense satisfaction of intellectual discovery. Your reward as the instructor is to bask in the glow of their endeavors.


### **Computers, Number Theory, and This Book**

At this point I would like to say a few words about the use of computers in conjunction with this book. I neither expect nor desire that the reader make use of a high-level computer package such as Maple, Mathematica, PARI, or Derive, and most exercises (except as otherwise indicated) can be done with a simple pocket calculator. To take a concrete example, studying greatest common divisors (Chapter 5) by typing `GCD [M, N]` into a computer is akin to studying electronics by turning on a television set. Admittedly, computers allow one to do examples with large numbers, and you will find such computer-generated examples scattered through the text, but our ultimate goal is always to understand concepts and relationships. So if I were forced to make a firm ruling, yea or nay, regarding computers, I would undoubtedly forbid their use.

However, just as with any good rule, certain exceptions will be admitted. First, one of the best ways to understand a subject is to explain it to someone else; so if you know a little bit of how to write computer programs, you will find it extremely enlightening to explain to a computer how to perform the algorithms described in this book. In other words, don't rely on a canned computer package; do the programming yourself. Good candidates for such treatment are the Euclidean algorithm (Chapters 5–6), the RSA cryptosystem (Chapters 16–18), primality testing (Chapter 19), Quadratic Reciprocity (Chapter 22), writing numbers as sums of two squares (Chapters 24–25), continued fractions and solving Pell's equation (Chapters 47–48), and generating rational points on elliptic curves (Chapter 41).

The second exception to the "no computer rule" is generation of data. Discovery in number theory is usually based on experimentation, which may involve examining reams of data to try to distinguish underlying patterns. Computers are well suited to generating such data and also sometimes to assist in searching for

patterns, and I have no objection to their being used for these purposes.

I have included a number of computer exercises and computer projects to encourage you to use computers properly as tools to help understand and investigate the theory of numbers. Some of these exercises can be implemented on a small computer (or even a programmable calculator), while others require more sophisticated machines and/or programming languages. Exercises and projects requiring a computer are marked by the symbol .

For many of the projects I have not given a precise formulation, since part of the project is to decide exactly what the user should input and exactly what form the output should take. Note that a good computer program must include all the following features:

- Clearly written documentation explaining what the program does, how to use it, what quantities it takes as input, and what quantities it returns as output.
- Extensive internal comments explaining how the program works.
- Complete error handling with informative error messages. For example, if  $a = b = 0$ , then the  $\text{gcd}(a, b)$  routine should return the error message “ $\text{gcd}(0, 0)$  is undefined” instead of going into an infinite loop or returning a “division by zero” error.

As you write your own programs, try to make them user friendly and as versatile as possible, since ultimately you will want to link the pieces together to form your own package of number theoretic routines.

The moral is that computers are useful as a tool for experimentation and that you can learn a lot by teaching a computer how to perform number theoretic calculations, but when you are first learning a subject, a prepackaged computer program merely provides a crutch that prevents you from learning to walk on your own.