HW (2) Solution

1. Let $S = \{\sigma_1, \dots, \sigma_{d'}\}$ where $\sigma_i \in \{1,-1\}^d$. $d' = 2^d$

define $f: X \to l_\infty^{d'}$

$$x \longmapsto \{\langle x, \sigma_i \rangle\}_{i=1}^{d'}$$

want to show $d(x,y) = d'(f(x), f(y))$

$$d(x,y) = \sum_{k=1}^{d} |x_k - y_k| = \#\{k, x_k \neq y_k\}$$

$$d'(f(x), f(y)) = \sup_k |\langle x, \sigma_k \rangle - \langle y, \sigma_k \rangle|$$

$$= \sup_k |\langle x-y, \sigma_k \rangle|$$

Since x, y both $\in \{0,1\}^d$, so $(x-y)_k = \begin{cases} 0 & \text{if } x_k = y_k \\ 1 & \text{if } x_k = 1, y_k = 0 \\ -1 & \text{if } x_k = 0\ y_k = 1 \end{cases}$

consider the σ_J with $(\sigma_J)_k = \begin{cases} 1 & \text{if } (x-y)_k = 1 \\ -1 & \text{if } (x-y)_k = -1 \end{cases}$

then this σ_J will maximize $|\langle x-y, \sigma_k \rangle|$.

i.e $d'(f(x), f(y)) = \sup_k |\langle x-y, \sigma_k \rangle| = |\langle x-y, \sigma_J \rangle| = \#\{k : x_k \neq y_k\}$

$$= d(x,y)$$

therefore f is isometric

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2.11. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$

we have $2^n > n$, then $\sqrt[n]{n} < 2$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}} > \sum_{n=1}^{\infty} \frac{1}{2n}$ diverge.

$\therefore \sum \frac{1}{n\sqrt[n]{n}}$ diverge.

2.17. if $\sum |a_n - a_{n+1}| < \infty$,

then $\forall \varepsilon > 0$, $\exists N$. s.t $m > N$, $\sum_{n=m}^{\infty} |a_n - a_{n+1}| < \varepsilon$

for $m, l > N$, then

$|a_m - a_l| = |a_m - a_{m+1} + a_{m+1} - a_{m+2} \cdots - a_l|$

$\leq \sum_{n=m}^{l-1} |a_n - a_{n+1}| < \sum_{n=m}^{\infty} |a_n - a_{n+1}| < \varepsilon$

$\therefore \{a_n\}$ Cauchy $\therefore \{a_n\}$ converge.

Conversely, consider. $a_n = \frac{(-1)^n}{n}$. which converge to 0,

while, $|a_n - a_{n+1}| = \frac{2n+1}{n(n+1)} > \frac{1}{n+1}$

$\therefore \sum |a_n - a_{n+1}|$ diverge

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3. since $ab \le \frac{a^p}{p} + \frac{b^q}{q}$

$$\frac{\sum x_k y_k}{\|x\|_p \|y\|_q} = \sum \frac{x_k y_k}{\|x\|_p \|y\|_q} \le \sum \left(\frac{1}{p}\left[\frac{x_k}{\|x\|_p}\right]^p + \frac{1}{q}\left[\frac{y_k}{\|y\|_q}\right]^q \right)$$

$$= \frac{1}{p} + \frac{1}{q} = 1$$

so $\|x+y\|_p^p = \sum |x_k + y_k|^p$

$$\le \sum \left(|x_k| |x_k + y_k|^{p-1} \right) + \sum \left(|y_k| |x_k + y_k|^{p-1} \right)$$

$$\le \left(\sum |x_k|^p\right)^{\frac{1}{p}} \cdot \left(\sum |x_k + y_k|^{\frac{p}{p-1} \cdot (p-1)} \right)^{\frac{p-1}{p}} + \left(\sum |y_k|^p \right)^{\frac{1}{p}} \cdot \left(\sum |x+y|^{\frac{p}{p-1} \cdot (p-1)} \right)^{p-1/p}$$

$\Rightarrow$

$$\|x+y\|_p^p \le \left(\|x\|_p + \|y\|_p \right) \cdot \|x+y\|_p^{p-1}$$

$\therefore \|x+y\|_p \le \|x\|_p + \|y\|_p$

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4. (i) Let $y \in B(x,r)$, Let ε s.t $0 < \varepsilon < r - d(x,y)$

then $d(x,z) \le d(x,y) + d(y,z)$

$\forall z \in B(y,\varepsilon)$ $< d(x,y) + \varepsilon < r$

$\therefore z \in B(x,r)$

$\therefore B(y,\varepsilon) \subset B(x,r)$ $\therefore B(x,r)$ open

(ii) Let $V = \bigcup A_i$ where A_i is open set for all i

$\forall x \in V$, exist k, s.t $x \in A_k$,

and since A_k open, there exist r, s.t $B(x,r) \subset A_k \subset V$

$\therefore V$ is open

(iii) Let $V = \bigcap_{i=1}^{n} A_i$ A_i open.

$\forall x \in V$, $x \in A_i$ for all i

$\therefore$ for each i, there exist r_i, s.t $B(x, r_i) \subset A_i$

Now take $r = \min r_i$,

$\therefore B(x,r) \subset B(x,r_i)$ $\forall i$.

$\therefore B(x,r) \subset A_i$ $\forall i$ $\Rightarrow$ $B(x,r) \subset \bigcap A_i = V$

$\therefore V$ is open.

if countable, not true.

consider: $A_n = (0, 1 + \frac{1}{n})$, $\bigcap_{n=1}^{\infty} A_n = (0,1]$ not open

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5. (i) Let $\{a_n\}$ be Cauchy sequence in l_1^d,

write $a_n = (a_1^n, a_2^n, \cdots, a_d^n)$, then $\forall \varepsilon > 0$, $\exists N$, s.t

$m, n \geq N \Rightarrow d(a_n, a_m) < \varepsilon$.

i.e $\sum_{k=1}^{d} |a_k^n - a_k^m| < \varepsilon$

Since $\sum_{k=1}^{d} |a_k^n - a_k^m|$ can only be integer.

Then we must have $\sum_{k=1}^{d} |a_k^n - a_k^m| = 0$

i.e $a_n = a_m$ for $n, m \geq N$

Then we have $\lim_{n\to\infty} a_n = a_N$ and $d(a_n, a_N) \xrightarrow{n\to\infty} 0$

$\therefore$ l_1^d is complete.

(ii) $d(f, g) = \sup_x \{|f(x) - g(x)|\}$

Let $\{f_n\}$ be Cauchy sequence in $B(X)$, then $\forall \varepsilon > 0$, $\exists N$, s.t

$m, n \geq N$, we have $\sup_x |f_n(x) - f_m(x)| < \varepsilon$

then we must have $\forall x$, $|f_n(x) - f_m(x)| < \varepsilon$,

so for any fixed x, $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$. So ~~we~~ ~~fix~~ $\{f_n(x)\}$ converge to a value in $\mathbb{R}$ since $\mathbb{R}$ is complete.

define this value is our $f(x)$.

So we may define

$$f(x) = \lim_{x\to\infty} f_n(x)$$

Now, need to show $f_n \to f$ in $B(X)$.

note again that $\forall \varepsilon > 0$, $\exists N$, which depend only on ε s.t

for all $x \in X$, $m, n \geq N \Rightarrow |f_n(x) - f_m(x)| < \varepsilon$.

Let $m \to \infty$, we get $|f_n(x) - f(x)| < \varepsilon$ for $n \geq N$. for all x

Thus, we have $\sup_x |f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$

$\therefore$ $f_n \to f$ in $B(X)$

Finally need to check $f \in B(X)$, i.e f is bounded.

~~Since ... ...~~ $\forall \varepsilon > 0$, $\exists n$, s.t $n \geq N$. $\|f_n - f\| < \varepsilon$

then $\|f\| \leq \|f - f_N\| + \|f_N\|$

$< \varepsilon + \|f_N\|$

since f_N is bounded, certaintly we have f is bounded

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